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# Integer invariants of joint action of Boolean variables and their properties

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**Abstract**

The sufficient causes theory is a generally accepted way to describe causality in the biomedical sciences. This theory can be adequately formalized in the Boolean algebra framework. An important question of biomedical research is determination of a joint (combined) action type of the acting factors. In Boolean interpretation of sufficient causes theory, this question is treated as computation of the orbits of the action of a certain group of automorphisms on the algebra of Boolean functions. Boolean formalization allows us to introduce additional concepts that help to study in more detail various questions of combined action of binary factors. In particular, the article provides a definition of combined action of a set of Boolean variables in a Boolean function and its geometric interpretation. The invariance of this concept is proved under the action of the automorphism group of the Boolean cube. These automorphisms are formal representation of the experimental symmetries. An integer number is introduced, called the degree of joint action. This number can be considered as a characteristic of the strength of the joint action of a number of variables in a Boolean function. This number is proved to be an invariant under the action of the Boolean cube automorphism group. The notion of a spectrum of combined action of a set of variables in a Boolean function is proposed. This concept allows us put all types of joint action in an order. An upper bound

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is found for the number of all possible spectra of joint action of variables in Boolean functions depending on a given number of variables.

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## 1. Introduction

A concept of joint action (interaction) of acting agents widely used in various applied sciences for describing of how these factors affect the change in the resulting variable. For example, in the study of variables whose change is subject to random influences, this is implemented in regression analysis, where the average value of the response is approximated by a linear function of the acting variables [28].

At the same time, the notion of interaction is not self-evident and implicitly presupposes comparison with some model of joint action for which it is assumed that there is no interaction (zero interaction model). For example, in regression analysis such a model is linear regression with uncorrelated predictors. The concept of zero interaction model is not formally defined and depends on which mathematical model is used to describe joint action. In addition to the above mentioned statistical model with uncorrelated predictors, the zero interaction isobolographic model (additivity isobole) generally accepted in toxicology [1, 4, 5] can also be regarded as a zero interaction model.

Thus, the presence of a zero interaction model is a necessary prerequisite for determining presence of a nontrivial joint action of given factors. However, in this case we can make a comparison only in relation to the zero interaction model, and if there are two models with nontrivial joint action, there is no possibility to make a comparison between them, for example, to say that for one case there is a stronger joint action than for another.

A more subtle classification of the joint action types can be implemented with the notion of a transformation that preserves the type of joint action. Below we examine this construction using the Boolean model of the theory of sufficient causes, which is one of the generally accepted causality concept in epidemiology.

An important feature of this approach is that it does not introduce the intricate notion of zero interaction, but considers certain types of joint action at once. These types can be given some meaning through the joint action degree notion introduced below.

In the second half of the 20th century, a concept of causality appeared in philosophy, which proposed a causality mechanism based on the representation of a cause of a given event in the form of a set of simultaneously occurring conditions, which inevitably lead to the onset of this event (INUS-conditions, i. e. **I**nsufficient, but **N**ecessary part of an **U**nnecessary but **S**ufficient condition) [9, 10, 12, 13]. At the same time, a similar concept was presented in epidemiology, where revealing causes of a disease in population is one of the most important problem. [14, 23] The subsequent development of these ideas has mainly continued in epidemiological studies under the name *sufficient causes* or *sufficient causes component framework* [6, 7, 8, 15, 25, 26, 27]. The first formal models of this causality concept were proposed in [6, 15, 26].

From an epidemiological point of view, it is important to establish whether the joint presence of acting factors that lead to a given outcome has any additional effect on this outcome compared to the total isolated effect of these factors. The presence of such an additional effect in comparison with a certain reference model based on their isolated action is expressed in terms of synergism or antagonism depending on the sign of this effect.

A formalized presentation of two-factor sufficient causes model was suggested in [15]. There, all possible outcomes were organized in classes according to the presence in them of a specific type of joint action, and to each of which some term was assigned expressing the medical meaning of such an effect (preventive antagonism, causal synergism etc.). The [6] notes the importance of accounting for data symmetry in analyzing types of joint action. In [27] this idea has been clarified and elaborated.

The importance of data symmetries for identification the type of joint action of factors is as follows. In epidemiology, factors and a resulting variable (response or outcome) are often used, the values of which may or may not have any natural order. Typical examples are gender, race, religion, type of work, region of residence etc. Since there is no natural order of values that such a factor takes, a choice of their encoding for numerical data analysis is completely arbitrary. At the same time, it is obvious that a type of combined action (understood in a certain sense) should not be dependent on an encoding method. In other words, a joint action type must be invariant with regard to an encoding method.

In epidemiology, the most elaborated theory of sufficient causes is one for binary factors. It only deals with two-level factors and a two-level response. It is easy to see that the semantic construction of that theory is quite clearly conforms to the algebraic structure of Boolean algebra. For instance, in [20, 21] this two-factor binary theory was considered in terms of Boolean functions of two variables. It was demonstrated that Boolean formalization adequately represents those logical propositions that underlie the epidemiological understanding of the sufficient causes theory, as well as the concepts introduced into this theory later. Besides, Boolean framework allows us to use a lot of mathematical concepts in the sufficient causes theory. [17, 22]

In the discussed Boolean formalization, epidemiological symmetries are automorphisms on the algebra of all Boolean functions. Consequently, responses (Boolean functions) having the same type of combined action fall into the same orbit of the action of a given automorphism group. This formalism avoids the discussion of what the reference model should be to determine the presence of an additional effect of joint action. The identification of possible types of joint action is then reduced to the algebraic calculation of the orbits of a given automorphism group action on the algebra of Boolean functions. [17, 18, 20, 21, 22]

Summary of the article. In the Section 2 Boolean framework for sufficient causes theory is outlined. Main results are presented in Section 3. As an extension of the concept of joint action of variables in a Boolean function introduced in [17], the concept of joint action of a certain number of variables in a Boolean function is introduced. It is shown that this notion correctly generalizes the concept of joint action of all variables on which a function depends. In addition, the notion of the degree of joint action of a given number of variables is proposed, which also generalizes a similar notion for all variables in a given function from [17]. Below the Boolean domain $\{0, 1\}$ is denoted by $\mathbb{B}$; the end of each proof is marked with a sign $\square$. For necessary concepts of Boolean algebras and functions, see e.g. [3, 11, 16].

## 2. A concept of joint action of Boolean variables

The questions discussed below concern only the case of the binary theory of sufficient causes, which deals with a finite number of two-level factors and two-level response. The main points of Boolean framework of the binary sufficient causes theory are stated in [20-22]. Briefly, one can say that a response studied in that theory is formally represented by a Boolean function which depends on the same number of Boolean variables as the number of acting factors is, and each acting factor is represented by a certain Boolean variable. Therefore, below we say about a response that depends on factors and a Boolean function that depends on Boolean variables as equivalent concepts. Here and below, Boolean functions are conveniently represented in the DNF (disjunctive normal form). Let us denote $\mathbb{B}(x_1,\ldots,x_n)$ the Boolean algebra of all Boolean functions of variables $x_1,\ldots,x_n$, $n\geqslant 2$. In accordance with common ideas of that formalism the following definition of joint action of Boolean variables (also called interaction) was proposed (see, also [26] for similar epidemiological notion).

**Definition 2.1:** [17]. Let $\boldsymbol{x}=x_1,\ldots,x_n$ be a set of Boolean variables, and $f\in\mathbb{B}\ (x_1,\ \ldots,\ x_n)$ a Boolean function. There is joint (combined) action of these variables in the function $f$ if there exists such a vector $\boldsymbol{\alpha}\in\mathbb{B}^n$ that a conjunction $\boldsymbol{x}^{\alpha}=x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$ presents in every irredundant representation of the Boolean function *f*. In this case we say that joint action in the function *f* attains at $\boldsymbol{x}=\boldsymbol{\alpha}$.

One can verify presence of the combined action of variables $x_1,\ldots,x_n$ in a Boolean function $f\in\mathbb{B}\ (x_1,\ldots,x_n)$ using the following statement.

**Theorem 2.2:** [17]. *Joint action of variables* $\boldsymbol{x}=x_1,\ldots,x_n$ *which attained at* $\boldsymbol{x}=\boldsymbol{\alpha}$, $\boldsymbol{\alpha}\in\mathbb{B}^n$, *is present in a Boolean function* $f\in\mathbb{B}\ (x_1,\ldots,x_n)$ *if and only if the conjunction* $\boldsymbol{x}^{\alpha}$ *is a prime implicant of the Boolean function* $f$.

It is important to note that the Definition 2.1 refers to joint action of all the variables involved in a given Boolean function which offers a logical representation for a given experimental binary response. Thus, any Boolean function that does not meet this definition does not represent the joint action of all the variables involved. For example, a Boolean function $f=x_1x_2\vee x_2x_3$ doesn't meet the Definition 2.1 although this function is clearly dependent on three variables in some way. It is obvious that a certain degree of joint action is present in this function, though it cannot be taken into account with Definition 2.1. This can be done using the following notion.

**Definition 2.3:** There is joint action of *k* variables, $2\leqslant k\leqslant n$ in a Boolean function $f\in\mathbb{B}\ (x_1,\ldots,x_n)$, if there are an ordered *k*-element subset $I\subseteq\{1,2,\ldots,n\}$ and a vector $\boldsymbol{\beta}\in\mathbb{B}^{n-k}$ such that joint action of *k* variables $\boldsymbol{x}_I = \{x_i\}_{i\in I}$ is present in the function $f_{I,\beta}$, where the Boolean function $f_{I,\beta}$ is defined as follows. For $\boldsymbol{y}=(y_1,\ y_2,\ \ldots,\ y_k)$, $f_{I,\beta}(\boldsymbol{y})=1$ iff $f(\boldsymbol{x})=1$ and $\boldsymbol{x}_I=\boldsymbol{y}$, $\boldsymbol{x}_{\overline{I}}=\boldsymbol{\beta}$, where $\overline{I}$ is the ordered complement of the set $I$ relative to $\{1,2,\ldots,n\}$. We say then that combined action of variables $\boldsymbol{x}_I$ for $\boldsymbol{x}_I=\boldsymbol{\alpha}$

under $\boldsymbol{x}_{\bar{I}} = \boldsymbol{\beta}$ is present in the function *f* if joint action attains at $y = \alpha$ in $f_{I,\beta}(y)$ for some $\alpha \in \mathbb{B}^k$.

Definition 2.3 is a mathematically rigorous formulation of a similar concept considered in [26]. In short, in a function *f* there is combined action of *k* variables $\boldsymbol{x}_I = \{x_i\}_{i \in I}$ if in the function *f* of variables $\boldsymbol{x}_I$ joint action presents under fixed values $\boldsymbol{x}_{\bar{I}} = \boldsymbol{\beta}$ of other variables.

Thus, this Definition generalizes the Definition 2.1 and allows us to define a notion of joint action for a number of variables that is less than or equal to the number of variables on which a given Boolean function depends. In this regard, it is necessary to examine the fulfillment for the Definition 2.3 of those important properties that were proved in [17] for the concept of combined action of *n* variables. In particular, it is necessary to build criteria for joint action of *k* factors in a function that depends on *n* variables, where $2 \leqslant k \leqslant n$, similar to Theorems 4, 8 from [17], and also check the invariance of the introduced concept with regard to the action of the group of automorphisms of Boolean cube as shown for the concept of the joint action of *n* variables in [17].

An important property of the concept of joint action of all variables (Definition 2.1) is that it does not depend on how the values of independent variables are encoded. Obviously, this property is quite natural and obligatory for the joint action concept. Mathematically, that means the invariance of joint action with respect to the $G_n$-action of the on the Boolean algebra $\mathbb{B}(x_1, \ldots, x_n)$, there $G_n$ is the group of all automorphisms of the Boolean cube $\mathbb{B}^n$. [17, 22] $G_n$-action on $\mathbb{B}(x_1, \ldots, x_n)$ generates a natural partition of that algebra into disjoined orbits, each of which is a class of Boolean functions that have the same type of joint action. [17, 22] Below the class of all Boolean functions which are $G_n$-equivalent to a given Boolean function *f* is denoted by $\langle f \rangle$.

We denote by $C_f$ support of a function *f*, i.e. the set $\{\alpha \in \mathbb{B}^n \mid f(\alpha) = 1\}$. The Boolean cube can be regarded as a graph of which two vertices are connected by an edge iff the Hamming distance between them is 1. As in [17], denote $\Gamma_f$ a graph vertex-generated by the support $C_f$. In [17] a following criterion for the presence of joint action in terms of graphs is given.

**Theorem 2.4:** [17]. *Joint action of n variables in a function* $f \in \mathbb{B}\ (x_1, \ldots, x_n)$ *is present iff the graph* $\Gamma_f$ *contains an isolated vertex.*

A useful consequence of the Boolean formalization of the binary sufficient causes theory is the possibility to define an integer characteristic

$\mu_f$ for any function $f$ from $\mathbb{B}(x_1,\dots,x_n)$. This number is an invariant under $G_n$-action on $\mathbb{B}(x_1,\dots,x_n)$ and is called the degree of joint action. It expresses, in a sense, the strength of joint action of variables in a given function (for more information, see [17]). Let $f \in \mathbb{B}\,(x_1,\dots,x_n)$ be a nonzero Boolean function and $\alpha \in C_f$.

**Definition 2.5:** [17]. Degree of joint action of variables $x_1,\dots,x_n$ in a function f at values $\boldsymbol{x} = \alpha$ is a number defined as follows

$$\mu_f(\alpha) = \begin{cases} \min\{d(\alpha,\beta) \mid \beta \in C_f, \beta \neq \alpha\} - 1, & \text{if } |C_f| > 1 \\ n, & \text{if } |C_f| = 1 \end{cases}$$

where $d(\alpha,\beta)$ is the Hamming distance between vectors $\alpha$ and $\beta$, and $|C_f|$ is the cardinality of $C_f$.

From Definition 2.5 the natural equality $0 \leqslant \mu_f(\alpha) \leqslant n$ follows.

**Definition 2.6:** [17]. Degree of joint action of variables $x_1 \dots, x_n$ in a Boolean function $f \in \mathbb{B}\,(x_1,\dots,x_n)$ is a number defined by equalities: $\mu_f = \max\{\mu_f(\alpha) \mid \alpha \in C_f\}$ if $f$ is a nonzero Boolean function and $\mu_f = 0$ for zero function $f$.

The degree of joint action allows us to obtain the following criterion.

**Theorem 2.7:** [17]. *Joint action of n variables is present in a function $f \in \mathbb{B}\,(x_1,\dots,x_n)$ iff the inequality $\mu_f \geqslant 1$ holds.*

Before presenting main results we introduce some notations. For an arbitrary positive integer $m$ we denote $\mathbb{N}_m$ the integer interval $\mathbb{N}_m = \{1,2,\dots,m\}$. For a fixed $k \in \mathbb{N}_n$ we denote by $I = \{i_1, i_2, \dots, i_k\}$, $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ an ordered $k$-element set of numbers from $\mathbb{N}_n$, and by $\overline{I}$ its ordered complement in $\mathbb{N}_n$ : $\overline{I} = \{i_{k+1},\dots,i_n\}$, $1 \leqslant i_{k+1} < i_{k+2} < \cdots < i_n \leqslant n$, $I \cup \overline{I} = \mathbb{N}_n$. For a vector $\delta \in \mathbb{B}^k$, $\mathbb{B}_I^\delta$ denotes an $(n-k)$-face $\{\xi \in \mathbb{B}^n \mid \xi_I = \delta\}$ of the Boolean cube $\mathbb{B}^n$. For a given $k$-element set $I$ denote by $pr_I$ a projection of $n$-dimensional Boolean cube $\mathbb{B}^n$ which maps a Boolean vector $\boldsymbol{x} \in \mathbb{B}^n$ to a vector $\boldsymbol{x}_I \in \mathbb{B}^k$ with Boolean coordinates $\boldsymbol{x}_I = \{\xi_i\}_{i \in I}$. The restriction of the projection $pr_I$ to the $k$-face $\mathbb{B}_{\overline{I}}^\beta \subseteq \mathbb{B}^n$ which is obviously a bijection of $\mathbb{B}_{\overline{I}}^\beta$ onto $\mathbb{B}^k$, is denoted by $p_{I,\beta}$. $\mathbf{0}_k$ is a $k$-dimensional zero-vector in $\mathbb{B}^k$, $\mathbf{0}$ and $\mathbf{1}$ are constant Boolean functions. Hereafter, the sets $I$ and its complement $\overline{I}$ are considered ordered.

## 3. Main results

Before formulating the main statements (Theorems 3.2, 3.6, 3.10, 3.13, 3.15) we need some simple auxiliary lemmas. Consider Boolean cube $\mathbb{B}^n$ as a metric space whose metric is given by the Hamming distance. Then any face $\mathbb{B}_{\bar{I}}^{\beta}$ of the Boolean cube $\mathbb{B}^n$ is a metric subspace with induced metric.

**Lemma 3.1:** *Let $f \in \mathbb{B}(x_1,\dots,x_n), I$ is a k-element subset of $\mathbb{N}_n$, $\beta \in \mathbb{B}^{n-k}$, $2 \leqslant k \leqslant n$. Then the following statements hold*

(1) *The map $p_{I,\beta} : \mathbb{B}_{\bar{I}}^{\beta} \to \mathbb{B}^k$ is an isometry of metric spaces with respect to the Hamming distance.*

(2) *Consider graph $\mathbb{B}_{\bar{I}}^{\beta}$ as a section graph defined by the vertices of the graph $\mathbb{B}_{\bar{I}}^{\beta}$ in the graph $\mathbb{B}^n$. Then the function $p_{I,\beta}$ is an isomorphism of $\mathbb{B}_{\bar{I}}^{\beta}$ onto $\mathbb{B}^k$ as graphs.*

(3) *The equality $f_{I,\beta} = f \circ (p_{I,\beta})^{-1}$ holds.*

(4) *Let $C_f, C_{f_{I,\beta}}$ are supports of f and $f_{I,\beta}$ respectively. Then the equality $C_{f_{I,\beta}} = p_{I,\beta}(C_f \cap \mathbb{B}_{\bar{I}}^{\beta})$ holds.*

(5) *Metric spaces $C_f \cap \mathbb{B}_{\bar{I}}^{\beta}$ and $C_{f_{I,\beta}}$ are isometric with respect to the Hamming distance.*

(6) *The graph $\Gamma_{f_{I,\beta}}$ is isomorphic to a section graph of the graph $\Gamma_f$ defined by vertices of $C_f \cap \mathbb{B}_{\bar{I}}^{\beta}$.*

***Proof:***

(1) Statement is obvious.

(2) The proof follows from (1) and from the fact that edges in the Boolean cube connect only those vertices which Hamming distance is equal to 1.

(3) The proof follows from the equalities $(f_{I,\beta} \circ p_{I,\beta})\,(\eta) = f_{I,\beta}\,(p_{I,\beta}(\eta)) = f_{I,\beta}(\eta_I) = f(\eta)$, where $\eta \in \mathbb{B}_{\bar{I},\beta}$, i. e. $\eta_{\bar{I}} = \beta$.

(4) By Lemma 3.1 (3) a point $\xi \in \mathbb{B}^k$ belongs to the support $C_{f_{I,\beta}}$ if and only if the equality $(f \circ (p_{I,\beta})^{-1})(\xi) = 1.$ Hence, taking $\eta = p_{I,\beta}^{-1}(\xi)$ we get $\eta_I = \xi$, $\eta_{\bar{I}} = \beta$, and $f(\eta) = 1$, i. e. $\eta \in C_f \cap \mathbb{B}_{\bar{I}}^{\beta}$ or, which is the same, $\xi \in p_{I,\beta}\,(C_f \cap \mathbb{B}_{\bar{I}}^{\beta})$.

(5) follows directly from (1) and (4).

(6) As it follows from the definition of the graph $\Gamma_f$ for a subset $I \subseteq \mathbb{N}_n$, $|I| = k$ and a vector $\beta \in \mathbb{B}^{n-k}$ the graph $\Gamma_{f_{I,\beta}}$ is a section graph of the Boolean cube $\mathbb{B}^k$ which is generated by the set $C_{f_{I,\beta}}$ [17]. Thus, (6) follows from (4) and (2).

To formulate the next statement we denote by $\delta_f$ the minimum degree of vertices in $\Gamma_f$. Then the following geometric criterion can be obtained.

**Theorem 3.2:** *Joint action of k variables is present in a function* $f \in \mathbb{B}(x_1, \ldots, x_n)$, $2 \leqslant k \leqslant n$, *iff the inequality* $k \leqslant n - \delta_f$ *holds.*

***Proof:*** It follows from the Definition 2.3 and Theorem 2.4 that in a function $f \in \mathbb{B}\ (x_1, \ldots, x_n)$ there is combined action of *k* variables iff there are *k*-element subset *I* in $\mathbb{N}_n$, vectors $\alpha \in \mathbb{B}^k$, $\beta \in \mathbb{B}^{n-k}$, such that the vertex $\alpha$ is an isolated vertex of $\Gamma_{f_{I,\beta}}$. The graph $\Gamma_{f_{I,\beta}}$ is defined by the support $C_{I,\beta}$ as a section graph of the graph $\mathbb{B}^k$. By the Lemma 3.1 (6) and (4) this means that the degree of the vertex $\gamma = (p_{I,\beta})^{-1}(\alpha)$ in the section graph of the graph $\mathbb{B}^n$ defined by the set $C_f \cap \mathbb{B}_I^\beta$ is isolated. As the degree of each vertex of a *k*-face of $\mathbb{B}_I^\beta$ as a section graph of $\mathbb{B}^n$ is equal to *k*, the degree of the vertex $\gamma$ in $\Gamma_f$ is no greater than $n - k$. Thus, the presence of combined action of *k* variables in the function *f* is equivalent to the existence in the graph $\Gamma_f$ a vertex which degree is not greater than $n - k$, i.e. $k \leqslant n - \delta_f$.

Theorem 3.2 is a geometric criterion of joint action of *k* variables in a function *f* depending on a larger number of variables. It is a natural generalization of the Theorem 2.4.

From Theorem 3.2 we obtain

**Corollary 3.3:** *If there is combined action of k variables in a function* $f \in \mathbb{B}\ (x_1, \ldots, x_n)$, $2 < k \leqslant n$, *then there is combined action of a smaller number of variables in this function.*

Note that the restriction of any automorphism of the graph $\mathbb{B}^n$ to the support $C_f$ is an isomorphism of $\Gamma_f$ onto the image of $\Gamma_f$ under this automorphism. Thus, the following statement holds

**Corollary 3.4:** *The presence of joint action of k variables in a function f is invariant with respect to the* $G_n$*-action.*

**Example 1:** Let $n = 3$. The types of combined action of two variables which are not types of combined action of three variables are only the following classes of $G_n$-equivalent Boolean functions

$\langle x_1x_2\rangle, \langle x_1x_2 \vee x_3\rangle, \langle x_1x_2 \vee x_1x_3\rangle, \langle x_1x_2 \vee \overline{x}_1x_3\rangle, \langle x_1x_2 \vee x_2x_3 \vee x_1x_3\rangle,$
$\langle x_1x_2 \vee \overline{x}_1\overline{x}_2\rangle, \langle x_1x_2 \vee \overline{x}_1\overline{x}_2 \vee x_3\rangle, \langle x_1x_2 \vee \overline{x}_1\overline{x}_2 \vee x_1x_3\rangle$

Indeed, for any representative *f* of any of these classes there are no isolated vertices in $\Gamma_f$. Thus, by Theorem 2.4 there is not combined action of all three variables in those functions *f*. In order to check is there joint action of two variables let us take the set $I = \{1,2\}$, $\beta = \mathbf{0}$ in the Definition 2.3, i.e. a condition $x_3 = 0$ for the additional variable $x_3$ is imposed. Hence, for the first five classes the function $f_{I,\beta}$ for which joint action of two variables takes place is $f_{I,\beta} = x_1x_2$, so for these classes the Definition 2.3 is fulfilled for two variables in the function depending on three variables. For other three classes we get $f_{I,\beta} = x_1x_2 \vee \overline{x}_1\overline{x}_2$, which is also a type of two-factor joint action [17].

One can get the same conclusion using Theorem 3.2. Indeed, for any given representative *f* of these classes the vertex (1,1,0) in the Boolean cube $\mathbb{B}^3$ has degree 1 in the graph $\Gamma_f$, that means that $\delta_f = 1$, because there is no isolated vertex in $\Gamma_f$.

Thus, the introduced notion of join action of *k* variables, $2 \leqslant k \leqslant n$, in a function that depends on *n* variables is invariant under the $G_n$-action on Boolean cube $\mathbb{B}^n$. Now we consider an integer invariant for the introduced concept that generalizes the notion of the degree of combined action for *n* variables.

**Definition 3.5:** Degree of joint action of *k* variables in a function $f \in \mathbb{B}(x_1,\ldots,x_n), 2 \leqslant k \leqslant n$ is a number defined as follows

$$\mu_{f,k} = \max\{\mu_{f_{I,\beta}} \mid I \subseteq \mathbb{N}_n, |I| = k, \beta \in \mathbb{B}^{n-k}\}$$

We assume that $\mu_{f,0} = 0$ for any function *f*, $\mu_{f,1} = 1$ for any $f \not\in \{\mathbf{0},\mathbf{1}\}$ and $\mu_{\mathbf{0},1} = \mu_{\mathbf{1},1} = 0$. In addition, it is reasonable to assume that $\mu_{\mathbf{0},k} = 0$ for any $k \in \mathbb{N}_n$.

Clearly, the Definition 3.5 generalizes the Definition 2.6, and the equalities $\mu_{f,n} = \mu_f$ and $0 \leqslant \mu_{f,k} \leqslant k$ hold. The next statement is a generalization of the Theorem 2.7.

**Theorem 3.6:** *Joint action of k variables in a function* $f \in \mathbb{B}\,(x_1,\ldots,x_n)$, $2 \leqslant k \leqslant n$, *is present iff the inequality* $\mu_{f,k} \geqslant 1$ *holds.*

***Proof:*** If a function *f* which depends on Boolean variables $x_1,\ldots,x_n$ has combined action of *k* variables, then by Definition 2.3 there are a subset $I \subseteq \mathbb{N}_n$, $|I| = k$, and a vector $\beta \in \mathbb{B}^{n-k}$ such that the function $f_{I,\beta}$ has joint action of $k$ variables $x_I$. By Theorem 2.7 and Definition 3.5 we obtain that $\mu_{f,k} \geqslant 1$. The converse is proved similarly.

In order for the introduced degree of combined action of $k$ variables to be a generalization of the degree of combined action of $n$ variables, it remains to prove that the number $\mu_{f,k}$ is an invariant under the $G_n$-action. It is well known that the group $G_n$, being the automorphism group $Aut(\mathbb{B}^n)$ of graph $\mathbb{B}^n$, is isomorphic to the hyperoctahedral group $Oct_n$, i.e., the $n$-dimensional hypercube symmetry group (see, e.g. [22]). In addition, the following statements hold for this group.

**Lemma 3.7:**

(1) *The group $G_n$ is the isometry group of a Boolean cube $\mathbb{B}^n$ as a metric space with respect to the Hamming distance.*

(2) *For a fixed $k \in \mathbb{N}_n$ the group $G_n$ acts transitively on the set of all pairs $(B,\delta)$, where $B$ is a k-face of the Boolean cube $\mathbb{B}^n$ and $\delta \in B$.*

(3) *For a fixed $k \in \mathbb{N}_n$ the group $G_n$ acts transitively on the set of all k-faces of the Boolean cube $\mathbb{B}^n$.*

***Proof:***

(1) Proof is obvious, since any automorphism of Boolean cube $\mathbb{B}^n$ considered as a graph is an isometry of $\mathbb{B}^n$ considered as a metric space with respect to the Hamming distance.

(2) It is well known that group $Oct_n$ acts transitively and simply on the flags of hypercube $K_n = [0,1]^n$ [2]. As above, consider $k$-element set $I$ and its complement $\overline{I}$ as ordered ascending subsets from $\mathbb{N}_n$ : $I = \{i_1, i_2, \ldots, i_k\}$, $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n$, $J = \overline{I} = \{i_{k+1}, i_{k+2}, \ldots, i_n\}$, $1 \leqslant i_{k+1} < i_{k+2} < \ldots < i_n \leqslant n$. It is clear that for any $k$-face $B = \mathbb{B}_J^\beta$ and some of its vertex $\delta$ one can match one-to-one a system of nested $l$-faces $C_0 \subset C_1 \subset \ldots \subset C_n$, where $C_0 = \mathbb{B}_{\mathbb{N}_n}^\delta$, $C_l = \mathbb{B}_{J_l}^{\beta^l}$, $J_l$ is the ordered set $\{i_{l+1}, i_{l+2}, \ldots, i_n\}$, $\beta^l = \delta_{J_l}$ for $l \in \{0, 1, \ldots, n-1\}$ and $C_n = \mathbb{B}^n$. In particular, $C_k = \mathbb{B}_J^\beta$. It is obvious that the system $C_0 \subset C_1 \subset \ldots \subset C_n$ one-to-one corresponds to a flag of the hypercube $K_n$.

(3) follows from (2).

Let us take a subset $I$, $|I| = k$, from $\mathbb{N}_n$ and arbitrary tuple $\beta \in \mathbb{B}^{n-k}$. Let $t \in G_n$ be an automorphism. By Lemma 3.7 (2) the range $t(\mathbb{B}_{\overline{I}}^\beta)$ is a

$k$-face $\mathbb{B}_J^\gamma$ where $J$ is a $k$-element subset from $\mathbb{N}_n$, and $\gamma \in \mathbb{B}^{n-k}$. Then one can define a map $t_{I,\beta}$ by the equality $t_{I,\beta} = (p_{J,\gamma}) \circ t \circ (p_{I,\beta})^{-1}$.

**Lemma 3.8:** *The map $t_{I,\beta}$ is an isometry of the Boolean cube $\mathbb{B}^k$ with respect to the Hamming distance.*

***Proof:*** Proof follows from Lemmas 3.1 (1), 3.7 (1).

**Lemma 3.9:** *Let t be an automorphism from the group $G_n$ and $f \in \mathbb{B}(x_1,\ldots,x_n)$, $2 \leqslant k \leqslant n$. If the automorphism t maps a k-face $\mathbb{B}_I^\beta$ onto k-face $\mathbb{B}_J^\gamma$ then for the Boolean function $\tilde{f} = t(f)$ the following equalities hold*

(1) $t_{I,\beta}(C_{f_{I,\beta}}) = C_{\tilde{f}_{J,\gamma}}$

(2) $\mu_{f_{I,\beta}}(\alpha) = \mu_{\tilde{f}_{J,\gamma}}(\tilde{\alpha})$ *and* $\alpha \in \mathbb{B}^k, \tilde{\alpha} = t_{I,\beta}(\alpha)$

(3) $\mu_{f_{I,\beta}} = \mu_{\tilde{f}_{J,\gamma}}$.

***Proof:***

(1) follows from the the definition of bijection $t_{I,\beta}$ and straightforward equality $t(C_f) = C_{\tilde{f}}$ : $t_{I,\beta}(C_{f_{I,\beta}}) = (p_{J,\gamma} \circ t \circ (p_{I,\beta})^{-1})\ (C_{f,\beta}) = (p_{J,\gamma} \circ t)\ (C_f \cap \mathbb{B}_I^\beta) = p_{J,\gamma}(t(C_f) \cap t(\mathbb{B}_I^\beta)) = p_{J,\gamma}(C_{\tilde{f}} \cap \mathbb{B}_J^\gamma) = C_{\tilde{f}_{J,\gamma}}$.

(2) Let $\alpha \in C_{f_{I,\beta}}$. If $|C_{f_{I,\beta}}| > 1$, then by the Lemma 3.8 for any $\delta \in C_{f_{I,\beta}}$, $\delta \neq \alpha$, we have the equality $d(\alpha,\delta) = d(t_{I,\beta}(\alpha), t_{I,\beta}(\delta)) = d(\tilde{\alpha},\tilde{\delta})$, where $\tilde{\alpha} = t_{I,\beta}(\alpha)$, $\tilde{\delta} = t_{I,\beta}(\delta)$ and $\tilde{\alpha} \neq \tilde{\delta}$. As the bijection $t_{I,\beta}$ maps $C_{f_{I,\beta}}$ onto $C_{\tilde{f}_{J,\gamma}}$ (see, Lemma 3.9 (1)), we get equality of the sets $\{d(\alpha,\delta) \mid \delta \in C_{f_{I,\beta}}, \delta \neq \alpha\} = \{d(\tilde{\alpha},\tilde{\delta}) \mid \tilde{\delta} \in C_{\tilde{f}_{J,\gamma}}, \tilde{\delta} \neq \tilde{\alpha}\}$. If $|C_{f_{I,\beta}}| = 1$, then $|C_{\tilde{f}_{J,\gamma}}| = 1$ as well. Thus, by the Definition 2.5 we get the required equality $\mu_{f_{I,\beta}}(\alpha) = \mu_{\tilde{f}_{J,\gamma}}(\tilde{\alpha})$.

(3) By the Lemma 3.8 the map $t_{I,\beta}$ is a bijection of Boolean cube $\mathbb{B}^k$ onto itself. By the Lemma 3.9 (1) we obtain equality $C_{\tilde{f}_{J,\gamma}} = t_{I,\beta}(C_{f_{I,\beta}})$. By the Lemma 3.9 (2) the equality $\{\mu_{f_{I,\beta}}(\alpha) \mid \alpha \in C_{f_{I,\beta}}\} = \{\mu_{\tilde{f}_{J,\gamma}}(\tilde{\alpha}) \mid \tilde{\alpha} \in C_{\tilde{f}_{J,\gamma}}\}$ holds. Then by the Definition 3.5 we get equality $\mu_{f_{I,\beta}} = \mu_{\tilde{f}_{J,\gamma}}$.

We can now formulate and prove invariance of the degree $\mu_{f,k}$ of combined action with regard to the $G_n$-action.

**Theorem 3.10:** *For any function* $f \in \mathbb{B}(x_1,\ldots,x_n)$ *and* $k \in \mathbb{N}_n$ *the degree* $\mu_{f,k}$ *is invariant under the action of the automorphism group* $G_n$ *of Boolean cube* $\mathbb{B}^n$.

***Proof:*** For $k \geqslant 2$ it follows from the Definition 3.5, Lemma 3.7 (3) and Lemma 3.9 (3). For $k=1$ the statement is obvious.

**Example 2:** Let us present the values of the degree $\mu_{f,k}$ for Boolean functions f of three variables, for which $\mu_f = \mu_{f,3} = 0$, and $\mu_{f,2} \geqslant 1$ (see Example 1). In other words, consider the values of the degree of combined action of two variables for those Boolean functions for which there is no combined action of three variables. For the following classes

$$\langle x_1x_2\rangle, \langle x_1x_2 \vee x_3\rangle, \langle x_1x_2 \vee x_1x_3\rangle, \langle x_1x_2 \vee \overline{x}_1x_3\rangle, \langle x_1x_2 \vee x_2x_3 \vee x_1x_3\rangle$$

we have $I = \{1,2\}$, $f_{I,\beta} = x_1x_2$ and the condition $x_{\overline{I}} = \beta$ for which the equality $\mu_{f,2} = \mu_{f_{I,\beta}}$ holds is $x_3 = 0$. Hence, $\mu_{f,2} = 2$. For other classes

$$\langle x_1x_2 \vee \overline{x}_1\overline{x}_2\rangle, \langle x_1x_2 \vee \overline{x}_1\overline{x}_2 \vee x_3\rangle, \langle x_1x_2 \vee \overline{x}_1\overline{x}_2 \vee x_1x_3\rangle$$

we have $I = \{1,2\}, f_{I,\beta} = x_1x_2 \vee \overline{x}_1\overline{x}_2$ and the condition $x_{\overline{I}} = \beta$ is the same, but here $\mu_{f,2} = 1$.

Thus, the integer invariant $\mu_{f,k}$ enables us to classify the types of combined action more accurately than the similar invariant $\mu_f$ considered earlier in [17].

In order to characterize the strength of combined action in a given Boolean function as an entire, we introduce the following

**Definition 3.11:** [18]. A sequence $M_f = (\mu_{f,1}, \mu_{f,2}, \ldots, \mu_{f,n})$ is called spectrum of joint action of variables in a function $f \in \mathbb{B}(x_1,\ldots,x_n)$.

From the Theorem 3.10 it follows that the spectrum $M_f$ is invariant under the $G_n$-action. Thus, it is correctly defined on the $G_n$-orbit $\langle f\rangle$ of the function $f \in \mathbb{B}(x_1,\ldots,x_n)$.

All the combined action types $\langle f\rangle$ can be ordered by inverse lexicographical order $\succcurlyeq$ on the tuples $M_f$. This order is reasonable because of the greater importance of combined action of more variables than a smaller number of them.

Let us consider some properties of the degree $\mu_{f,k}$ (see also [18]). Effective algorithms for calculation of interaction spectrum are proposed in [19].

**Lemma 3.12:** *Let* $f \in \mathbb{B}(x_1,\ldots,x_n), f \notin \{\mathbf{0},\mathbf{1}\}$ *and* $k \in \mathbb{N}_n$. *Then*

(1) *if* $\mu_{f,k} = k$ *then* $\mu_{f,k-1} = k-1$;

(2) *if* $\mu_{f,k} < k$ *then* $\mu_{f,k-1} \geqslant \mu_{f,k}$.

***Proof:*** By the Definition 3.5 both statements are valid for $k=1$ and $k=2$. Let $k>2$.

(1) Let $\mu_{f,k} = k$. By the Definition 3.5 there exist such a subset $I \subseteq \mathbb{N}_n$, $|I| = k$, and $\beta \in \mathbb{B}^{n-k}$ that $\mu_{f,k} = \mu_{f_{I,\beta}}$. From $\mu_{f_{I,\beta}} = k$ by the Definition 2.5 it follows that $|C_{f_{I,\beta}}| = 1$, and by Lemma 3.1 (5) that $|C_f \cap \mathbb{B}_{\bar{I}}^{\beta}| = 1$. Let $C_f \cap \mathbb{B}_{\bar{I}}^{\beta} = \{\alpha\}$ and denote $\tilde{\alpha} = \mathbf{0}_n$, $\gamma = \mathbf{0}_{n-k}$, $J = \mathbb{N}_k$. Then, obviously, $\tilde{\alpha} \in \mathbb{B}_{\bar{J}}^{\gamma}$. By Lemma 3.7 (2) there exists an automorphism $t \in G_n$, such that $t(\alpha) = \tilde{\alpha}$, $t(\mathbb{B}_{\bar{I}}^{\beta}) = t(\mathbb{B}_{\bar{J}}^{\gamma})$. By the Theorem 3.10 we have equalities $\mu_{\tilde{f},k} = \mu_{f,k} = k$ and $\mu_{\tilde{f},k-1} = \mu_{f,k-1}$, where $\tilde{f} = t(f)$. Thus, without loss of generality we can assume that $\alpha = \mathbf{0}_n$, $\beta = \mathbf{0}_{n-k}$, $I = \mathbb{N}_k$. Let $I' = \mathbb{N}_{k-1}$, $\beta' = \mathbf{0}_{n-k+1}$. Then obviously $\mathbb{B}_{\bar{I'}}^{\beta'} \subseteq \mathbb{B}_{\bar{I}}^{\beta}$, $\alpha \in \mathbb{B}_{\bar{I'}}^{\beta'}$, and hence $C_f \cap \mathbb{B}_{\bar{I'}}^{\beta'} = \{\alpha\}$. Then by Lemma 3.1 (5) we have $|C_{f_{I',\beta'}}| = 1$. Hence, by the Definitions 2.5 and 2.6 the equality $\mu_{f_{I',\beta'}} = k-1$ holds. Because of equality $|I'| = k-1$, $\beta' \in \mathbb{B}^{n-k+1}$, inequality $\mu_{f,k-1} \leqslant k-1$, and by Definition 3.5 we get $\mu_{f,k-1} = k-1$.

(2) Let $\mu_{f,k} < k$. By Definitions 2.6, 3.5 there exist a $k$-element subset $I$ in $\mathbb{N}_n$, a tuple $\beta \in \mathbb{B}^{n-k}$, and a point $\delta \in C_{f_{I,\beta}}$ such that the equalities $\mu_{f,k} = \mu_{f_{I,\beta}} = \mu_{f_{I,\beta}}(\delta)$ hold. Let us denote $\alpha = p_{I,\beta}^{-1}(\delta)$. Without loss of generality just as in (1), we can assume that $\alpha = \mathbf{0}_n$, $\beta = \mathbf{0}_{n-k}$, $I = \mathbb{N}_k$. Then $\delta = \mathbf{0}_k$. By Definitions 2.5, 2.6 and 3.5 it follows from $\mu_{f,k} < k$ that $|C_{f_{I,\beta}}| > 1$. Let us denote $m = \mu_{f,k}$. Then it follows from $\mu_{f_{I,\beta}}(\delta) = m$ and Definition 2.5 that for every point $\rho \in C_{f_{I,\beta}}$, $\rho \neq \delta$, Hamming's weight $w(\rho) > m$. It follows from Lemma 3.1 (5) that $|C_f \cap \mathbb{B}_{\bar{I}}^{\beta}| > 1$, and $w(\tau) > m$ for any $\tau \in C_f \cap \mathbb{B}_{\bar{I}}^{\beta}$, $\tau \neq \alpha$.

Let $I' = \mathbb{N}_{k-1}$ and let $\beta' = \mathbf{0}_{n-k+1}$. Then $\alpha \in \mathbb{B}_{\bar{I'}}^{\beta'}$, and $C_f \cap \mathbb{B}_{\bar{I'}}^{\beta'} \subseteq C_f \cap \mathbb{B}_{\bar{I}}^{\beta}$. Thus, for any point $\tau' \in C_f \cap \mathbb{B}_{\bar{I'}}^{\beta'}$, $\tau' \neq \alpha$ the inequality $w(\tau') > m$ holds. From Lemma 3.1 (5) it follows that $w(\tau'') > m$ for every $\tau'' \in C_{f_{I',\beta'}}$, if $\tau'' \neq \gamma$ for $\gamma = p_{I',\beta'}(\alpha) = \mathbf{0}_{k-1}$. If $|C_{f_{I',\beta'}}| > 1$, then $\mu_{f,k-1} \geqslant \mu_{f_{I',\beta'}} \geqslant \mu_{f_{I',\beta'}}(\gamma) \geqslant m$ by the Definitions 2.5, 2.6, 3.5. If $|C_{f_{I',\beta'}}| = 1$, then by the same

Definitions and from the inequality $\mu_{f,k-1} \leqslant k-1$, we get $\mu_{f,k-1} = k-1$, i.e. the inequality $\mu_{f,k-1} \geqslant m$ holds.

The following theorem describes arithmetic structure of the joint action spectrum of variables in a Boolean function.

**Theorem 3.13:** *For any function* $f \in \mathbb{B}\,(x_1,\dots,x_n)$ *there exists a unique integer* $m_f \in \{0,1,\dots,n\}$ *such that*

(1) $\mu_{f,k} = k$ *for* $k \in \{0,1,2,\dots,m_f\}$ *and*

(2) $\mu_{f,k} \leqslant \mu_{f,k-1}$ *for* $k \in \{m_f+1,\dots,n\}$ *if* $m_f < n$.

***Proof:*** For $f = \mathbf{0}$ or $f = \mathbf{1}$ we have $m_f = 0$. For other Boolean functions we have $\mu_{f,1} = 1$, and hence, we can set $m_f = \max\,\{k \in \mathbb{N}_n \mid \mu_{f,k} = k\}$. By Lemma 3.12 (1) we obtain (1). The inequality (2) follows from $0 \leqslant \mu_{f,k} \leqslant k$ and Lemma 3.12 (2).

**Example 3:** As it is noticed in [17], for the Boolean function $f = x_1x_2x_3$ we have $\mu_f = 3$. Thus, $\mu_{f,3} = \mu_f = 3$ and by the Theorem 3.13 (1) we obtain the equalities $\mu_{f,2} = 2$, $\mu_{f,1} = 1$, i.e. the spectrum of that function is $M_f =$ $(1,2,3)$. In the same paper it is shown that $\mu_f = 2$ for $f = x_1x_2x_3 \vee \overline{x_1}\,\overline{x_2}\,\overline{x_3}$. For this function with $I = \{1,2\}$, $\beta = 1$ we get $f_{I,\beta} = x_1x_2$ and $\mu_{f_{I,\beta}} = 2$ (see, Example 2). Hence, $\mu_{f,2} = 2$ and the spectrum of this function is $\mu_f = (1,2,2)$. In the Example 2 it is shown that $\mu_{f,2} = 2$ and $\mu_{f,3} = 0$ for the function $f = x_1x_2$. Hence, its spectrum is $M_f = (1,2,0)$. As the equalities $(1,2,3) \succcurlyeq (1,2,2) \succcurlyeq (1,2,0)$ hold, one can order the corresponding types of joint action as follows: $\langle x_1x_2x_3 \rangle$ is the most strong type of joint action, $\langle x_1x_2x_3 \vee \overline{x_1}\,\overline{x_2}\,\overline{x_3} \rangle$ is a weaker one, and $\langle x_1x_2 \rangle$ is the weakest one among these three types. Moreover, it is easy to see that the type $\langle x_1x_2x_3 \rangle$ is the strongest joint action type among all the joint action types for $n = 3$.

An important question is the estimation of how many spectra of joint action can be for Boolean functions depending on a given number of variables. Let's find an upper bound for this number.

From the Theorem 3.13 it follows that a set of interaction degrees possesses their own specific property which allows us to introduce

**Definition 3.14:** We say that a sequence of integers $M=(m_1,\dots,m_n)$, $1\leqslant m_i\leqslant n$, possesses a spectrum property if there exists such a unique number $K\in\{0,1,\dots,n\}$ that the the following properties hold

(1) $m_i=i$ for any $i\in\{0,1,\dots,K\}$ and

(2) $m_j\leqslant m_{j-1}$ for any $j\in\{K+1,\dots,n\}$ if $K<n$.

We can now formulate the Theorem 3.13 as a property that spectrum of joint action of $n$ Boolean variables for arbitrary Boolean function from $\mathbb{B}(x_1,\dots,n)$ is a sequence with the spectrum property. Therefore, the number of all spectra of joint action of $n$ variables does not exceed the number $S_n$ of sequences possessing the spectrum property. Let us calculate $S_n$.

**Theorem 3.15:** *For any integer $n\geqslant 2$ the equality $S_n=2^n$ holds.*

***Proof:*** The number $num(p,r)$ of all sequences $\{x_i\}_{i=1}^{p}$ of integers such that $1\leqslant x_1<x_2<\dots<x_p\leqslant r$ equals $\binom{r}{p}$ as the set $\{x_i\mid i\in\mathbb{N}_p\}$ is a $p$-element subset of $\mathbb{N}_r$. Let us denote $N(p,r)$ the number of all sequences $\{y_i\}_{i=1}^{p}$ of integers such that $1\leqslant y_1\leqslant y_2\leqslant\dots\leqslant y_p\leqslant r$.

One can put every sequence $\{y_j\}_{j=1}^{p}$ in a one-to-one correspondence with a sequence $\{z_j\}_{j=1}^{p}$, where $z_j = y_j+(j-1)$, $j\in\mathbb{N}_p$ and $1\leqslant z_1<z_2<\dots<z_p\leqslant s$, where $s=r+p-1$. Conversely, to every such a sequence $\{z_j\}_{j=1}^{p}$ one can establish one-to-one correspondence with a sequence $\{y_j\}_{j=1}^{p}$, where $y_j=z_j-(j-1)$, $j\in\mathbb{N}_p$, and the equalities $1\leqslant y_1\leqslant y_2\leqslant\dots\leqslant y_p\leqslant r$ hold. Thus, we obtain the equality $N(p,r) = num(p,r+p-1) = \binom{r+p-1}{p}$ (a similar problem was considered in [24]).

By Definition 3.14 there is a one-to-one correspondence between the set of all sequences $M=(m_1,\dots,m_n)$ possessing the spectrum property with $K<n$ and the set of all such non-increasing sequences $\{m_i\}_{i=K+1}^{n}$ that $K\geqslant m_{K+1}\geqslant m_{K+2}\geqslant\dots\geqslant m_n\geqslant 0$. It is clear that the number of that sequences is equal to

$$N(n-K,K+1)=\binom{n}{n-K}=\binom{n}{K}.$$

For $K=n$ there exist a unique sequence with the spectrum property, namely, $M=(1,2,\dots,n)$. Thus,

$$S_n=\sum_{K=0}^{n-1}N(n-K,K+1)+1=\sum_{K=0}^{n-1}\binom{n}{K}+1=2^n.$$

**Corollary 3.16:** *The number of combined action spectra of n Boolean variables in Boolean functions from* $\mathbb{B}(x_1,\dots,x_n)$ *does not exceed* $2^n$.

**Example 4:** Let $n=2$. In [17] it was shown that $\mu_f=2$ for $f=x_1x_2$, $\mu_f=1$ for $f=\bar{x}_1\bar{x}_2\vee x_1x_2$, and $\mu_f=0$ for $f=0$ or $f=x_1$, $f=x_2$. Then by Definitions 3.5 and 3.11 every pair $(1,2)$, $(1,1)$, $(1,0)$, $(0,0)$ is a spectrum of combined action of variables $x_1,x_2$ for a particular function from $\mathbb{B}(x_1,x_2)$. By Theorem 3.13 there are no other spectra for the case under consideration. Thus, there are exactly 4 different spectra of joint action of two Boolean variables. This is consistent with Corollary 3.16.

## 4. Conclusions

The theory of sufficient causes for two-level variables considered as one of the possible concept for describing causality in epidemiology has an adequate mathematical language that is the theory of Boolean functions and Boolean algebras. The foundations of this formalization laid down in [20]-[22] continues in the present paper where analogues of the concepts of joint action and degree of joint action of all variables in a given function introduced in [17] are built. These analogues consider the notion of joint action of $k\geqslant 2$ variables in a function that depends on $n\geqslant k$ variables and as shown above they satisfy the properties proved in [17] for the joint action of all variables in a given function. In particular,

1. A notion of joint (combined) action of *k* variables is introduced.
2. A criterion for the combined action of *k* variables is formulated and proved, which generalizes the corresponding criterion for the presence of combined action of *n* variables, in graph theory terms.
3. It is possible to define the notion of the degree of combined action of *k* variables which also generalizes the notion of degree of combined action of all variables from .
4. A criterion for the presence of combined action of *k* variables is given and proved, which generalizes the criterion for the presence of combined action of *n* variables.

5. The degree of combined action of $k$ variables, as well as the notion of combined action of $k$ variables, are invariant under the $G_n$-action there $G_n$ is automorphism group of Boolean cube $\mathbb{B}^n$ (i.e. the hyperoctahedral group $Oct_n$). This means that the concept of the combined action of $k$ variables and the degree of combined action of $k$ variables are correctly defined on the $G_n$-orbits. Thus, each such orbit (referred to as a class above in the text) represents some specific type of combined action.
6. Properties of degree of combined action of $k$ variables for a Boolean function which depends on $n$ variables are formulated and proved for different $k$.
7. The concept of the spectrum of joint action of variables in a Boolean function is introduced. This notion allows us to order the types of combined action according to their power.
8. An upper bound for the number of joint action spectra of variables in Boolean functions depending on a given number of variables is obtained.

In general, we can say that the Boolean framework for the binary sufficient causes theory is not only an adequate language for describing existing concepts, but also a mathematical tool allowing one to effectively develop this theory.

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