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SHORT NOTE

Completely Regular Semigroup Varieties Whose Free Objects Have Weakly Permutable Fully Invariant Congruences*

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Abstract

We prove that a variety of completely regular semigroups has the property from the title if and only if it consists of either completely simple semigroups or semilattices of groups.

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Recall that congruences α and β on some algebra are called [*weakly*] *permutable* if $\alpha\beta = \beta\alpha$ [respectively, $\alpha\beta\alpha = \beta\alpha\beta$]. For brevity, we call a semigroup variety $\mathcal{V}$ [*weakly*] *fi-permutable* if, on every $\mathcal{V}$-free object, any two fully invariant congruences [weakly] permute. A complete description of fi-permutable semigroup varieties was given in [6]. In [5] we announced a description of weakly fi-permutable varieties whose groups are trivial. In this article we prove that any weakly fi-permutable semigroup variety is either a completely regular variety (i. e. a variety of completely regular semigroups) or a nilvariety (i. e. a variety of nilsemigroups) and give a description of completely regular weakly fi-permutable varieties. Weakly fi-permutable nilvarieties also are described by the author; the corresponding result has the following form: a nilvariety is weakly fi-permutable if and only if it satisfies one of the 237 concrete systems of identities. This result will be presented elsewhere.

Let us introduce some notation. By F we denote the free semigroup over a countable alphabet. The equality relation and the universal relation on F will be denoted by $\equiv$ and ∇, respectively. For a word $u \in F$ we denote by $c(u)$ the set of all letters occuring in u and by $\ell(u)$ the length of u. A

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congruence α on F is called a *group* [*semilattice*] *congruence* if F/α is a group [a semilattice]. As usual, the lattice of subvarieties of a variety $\mathcal{V}$ is denoted by $L(\mathcal{V})$. As well known, any completely regular variety $\mathcal{V}$ contains the greatest group subvariety. We denote the latter by $\mathrm{Gr}(\mathcal{V})$. Let us fix notation for some individual varieties. By $\mathcal{LZ}$ [respectively, $\mathcal{RZ}$] we denote the variety of all left zero [right zero] semigroups, $\mathcal{SL}$ stands for the variety of all semilattices, and $\mathcal{ZM}$ denotes the variety of all semigroups with zero multiplication. It is well known that varieties $\mathcal{LZ}$, $\mathcal{RZ}$, $\mathcal{SL}$ and $\mathcal{ZM}$ are atoms of the lattice of all semigroup varieties. We will use the following well-known facts: an identity $u = v$ holds in $\mathcal{SL}$ if and only if $c(u) = c(v)$; if a variety $\mathcal{ZM}$ satisfies an identity of the kind $x = u$ where x is a letter, then $u \equiv x$.

We need the following general remark which can be straightforwardly checked.

Lemma 1. *Let α, β and ν be equivalences on a set S such that $\alpha, \beta \supseteq \nu$. Then α and β [weakly] permute if and only if the equivalences α/ν and β/ν on the quotient set S/ν do so.*

The following proposition generalizes Lemma 1.6 of [6].

Proposition. *A weakly fi-permutable semigroup variety $\mathcal{V}$ is either a completely regular variety or a nilvariety.*

Proof. Suppose that $\mathcal{V}$ is not completely regular. It suffices to verify that $\mathcal{ZM}$ is a unique atom of the lattice $L(\mathcal{V})$. Arguing by contradiction, assume that this lattice contains $\mathcal{ZM}$ and some other atom $\mathcal{A}$. Let α and ζ be fully invariant congruences on F corresponding to the varieties $\mathcal{A}$ and $\mathcal{ZM}$, respectively. The variety $\mathcal{ZM} \wedge \mathcal{A}$ is trivial, therefore $\zeta \vee \alpha = \nabla$. By Lemma 1 congruences ζ and α weakly permute. Hence $\zeta\alpha\zeta = \nabla$. In particular $(x, y) \in \zeta\alpha\zeta$ for any two letters x, y. This means that $x \,\zeta\, u \,\alpha\, v \,\zeta\, y$ for some words u and v. Since $x \zeta u$ and $v \zeta y$, we have that $u \equiv x$ and $v \equiv y$. Therefore $\mathcal{A}$ satisfies the identity $x = y$ that contradicts the choice of $\mathcal{A}$. ∎

The main result of this note is the following

Theorem. *A completely regular semigroup variety is weakly fi-permutable if and only if it is either a variety of completely simple semigroups or a variety of semilattices of groups.*

Proof. **Necessity.** Let $\mathcal{V}$ be a weakly fi-permutable completely regular semigroup variety. It is sufficient to verify that if $\mathcal{V} \supseteq \mathcal{SL}$ then $\mathcal{V}$ does not contain neither $\mathcal{LZ}$ nor $\mathcal{RZ}$. Suppose that $\mathcal{V}$ contains both $\mathcal{SL}$ and a variety $\mathcal{X}$ coinciding with one of the varieties $\mathcal{LZ}$ and $\mathcal{RZ}$. Let σ and χ be fully invariant congruences on F corresponding to the varieties $\mathcal{SL}$ and $\mathcal{X}$, respectively. Since the variety $\mathcal{SL} \wedge \mathcal{X}$ is trivial, we have $\sigma \vee \chi = \nabla$. By Lemma 1 congruences σ and χ weakly permute. Hence $\sigma\chi\sigma = \nabla$. Therefore $(x, y) \in \sigma\chi\sigma$ for any two letters x, y. This means that $x \,\sigma\, u \,\chi\, v \,\sigma\, y$ for some words u and v. Therefore,

$u \equiv x^m$ and $v \equiv y^n$ for some m and n. We obtain that $\mathcal{X}$ satisfies the identity $x^m = y^n$. But this identity false both in $\mathcal{LZ}$ and $\mathcal{RZ}$.

Sufficiency. It is known that any variety of completely simple semigroups is fi-permutable [3, 4]. It remains to verify that any variety of semilattices of groups is weakly fi-permutable. Let $\mathcal{V}$ be a variety of semilattices of groups and $\mathcal{G} = \mathrm{Gr}(\mathcal{V})$. It is well known that either $\mathcal{V} = \mathcal{G}$ or $\mathcal{V} = \mathcal{G} \vee \mathcal{SL}$ (see, for instance, [1]). In the former case $\mathcal{V}$ is congruence permutable. Let now $\mathcal{V} = \mathcal{G} \vee \mathcal{SL}$. Then $L(\mathcal{V}) \cong L(\mathcal{G}) \times L(\mathcal{SL})$ (it easily follows from [2], for instance). Let us denote by n the exponent of the variety $\mathcal{G}$ and by σ and ν the fully invariant congruences on F corresponding to the varieties $\mathcal{SL}$ and $\mathcal{V}$, respectively. Clearly, any fully invariant congruence on F containing ν either is a group congruence or coincides with σ or equals $\gamma \wedge \sigma$, where γ is some group fully invariant congruence on F. Let now α and β be fully invariant congruences on F such that $\alpha, \beta \supseteq \nu$. By Lemma 1 it suffices to verify that α and β weakly permute. Clearly, we may assume that α and β are non-comparable in the lattice of all fully invariant congruences on F. Up to symmetry, one has to consider the following 4 cases.

Case 1: *α and β are group congruences.* Clearly, the group variety corresponding to the congruence $\alpha \wedge \beta$ is congruence permutable. By Lemma 1 α and β permute.

Case 2: *α is a group congruence, and $\beta = \sigma$.* Let x and y be any two letters. Then $x \,\alpha\, xy^n \,\beta\, x^n y \,\alpha\, y$ and $x \,\beta\, x^n \,\alpha\, y^n \,\beta\, y$. Hence $\alpha\beta\alpha = \nabla = \beta\alpha\beta$.

Case 3: *α is a group congruence, and $\beta = \gamma \wedge \sigma$, where γ is a group congruence.* Here we need the following result obtained in [3, 4].

Lemma 2. *Let $\mathcal{V}$ be a completely regular semigroup variety, and S a $\mathcal{V}$-free object. Any two fully invariant congruences on S contained in the least semilattice congruence on S permute.*

Put $\alpha' = \alpha \wedge \sigma$. Since $\alpha', \beta \subseteq \sigma$, Lemmas 1 and 2 imply that congruences α' and β permute. It is well known that σ is a neutral element of the lattice of all fully invariant congruences on F (see [2]). Therefore

$$(\alpha \vee \beta) \wedge \sigma = (\alpha \wedge \sigma) \vee (\beta \wedge \sigma) = \alpha' \vee \beta = \alpha'\beta = \beta\alpha'.$$

Since $\alpha' \subseteq \alpha$, we have

$$(\alpha \vee \beta) \wedge \sigma \subseteq \alpha\beta \quad \text{and} \quad (\alpha \vee \beta) \wedge \sigma \subseteq \beta\alpha. \tag{1}$$

Suppose now that $w_1, w_2 \in F$, $(w_1, w_2) \in \alpha \vee \beta$ and $c(w_1) \subseteq c(w_2)$. Clearly, $w_1 w_2^n \,\alpha\, w_1$ whence $(w_1 w_2^n, w_2) \in \alpha \vee \beta$. Furthermore, $w_1 w_2^n \,\sigma\, w_2$ because $c(w_1 w_2^n) = c(w_2)$. Using (1), we have $(w_1 w_2^n, w_2) \in \alpha\beta$. Therefore $(w_1, w_2) \in \alpha\beta$. Thus,

$$\text{if } (w_1, w_2) \in \alpha \vee \beta \text{ and } c(w_1) \subseteq c(w_2) \text{ then } (w_1, w_2) \in \alpha\beta. \tag{2}$$

Let now $u, v \in F$ and $(u, v) \in \alpha \vee \beta$. It suffices to verify that $(u, v) \in \alpha\beta\alpha$ and $(u, v) \in \beta\alpha\beta$. Clearly, $v \,\alpha\, vu^n$. Hence $(u, vu^n) \in \alpha \vee \beta$. Furthermore,

$c(u) \subseteq c(vu^n)$. Using (2), we have $(u, vu^n) \in \alpha\beta$. Hence $u\,\alpha\,w\,\beta\,vu^n\,\alpha\,v$ for some word $w \in F$. We see that $(u, v) \in \alpha\beta\alpha$.

It remains to check that $(u, v) \in \beta\alpha\beta$. Suppose at first that $c(u) \cap c(v) = \varnothing$. Substitute the word u^n for each letter from $c(v)$ thus transforming the word v into the word $u^{n\ell}$, where $\ell = \ell(v)$. Since letters from $c(v)$ do not occur in u, and $(u, v) \in \alpha \vee \beta$, we conclude that $(u, u^{n\ell}) \in \alpha \vee \beta$. Furthermore, $u\,\sigma\,u^{n\ell}$ because $c(u) = c(u^{n\ell})$. By (1), we have $(u, u^{n\ell}) \in \beta\alpha$. Analogously, $(v, v^{nk}) \in \beta\alpha$, where $k = \ell(u)$. Clearly, $u^{n\ell}\,\alpha\,v^{nk}$. Hence there are words $u', v' \in F$ with $u\,\beta\,u'\,\alpha\,u^{n\ell}\,\alpha\,v^{nk}\,\alpha\,v'\,\beta\,v$. Therefore $(u, v) \in \beta\alpha\beta$.

Finally, let $c(u) \cap c(v) \neq \varnothing$. According to (2) we may assume that $c(u) \nsubseteq c(v)$. Now we substitute an arbitrary letter from $c(u) \cap c(v)$ for each letter from $c(u) \backslash c(v)$ thus transforming the word u into a new word w. Clearly, $(v, w) \in \alpha \vee \beta$. Since $(u, v) \in \alpha \vee \beta$, we see that $(u, w) \in \alpha \vee \beta$ too. Furthermore, $c(w) = c(u) \cap c(v)$. Using (2), we have $(w, u) \in \alpha\beta$ and $(w, v) \in \alpha\beta$. Hence $u\,\beta\,u'\,\alpha\,w\,\alpha\,v'\,\beta\,v$ for some words $u', v' \in F$. We prove that $(u, v) \in \beta\alpha\beta$ again.

Case 4: *$\alpha = \gamma_1 \wedge \sigma$ and $\beta = \gamma_2 \wedge \sigma$, where γ_1 and γ_2 are group congruences.* Here α and β permute by Lemmas 1 and 2. ∎

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