CANCELLABLE ELEMENTS OF VARIETAL LATTICES

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The joint work with Vyacheslav Shaprynskiĭ and Dmitry Skokov

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An element x of a lattice L is called *neutral* if

$$\forall y, z \in L: \quad \text{the sublattice of all generated by } x, y \text{ and } z \text{ is distributive}$$

or, equivalently, if

$$\forall y, z \in L: \quad (x \vee y) \wedge (y \vee z) \wedge (z \vee x) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x).$$

a is neutral in L if and only if L is a subdirect product of $(a] = \{x \in L \mid x \leq a\}$ and $[a) = \{x \in L \mid x \geq a\}$,

L embeds in $(a] \times [a)$ by the rule

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Consider a weaker restricion: the mapping from L to $(a] \times [a)$ given by the rule

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is one-to-one.

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We examine cancellable elements in three varietal lattices:

the lattice of all semigroup varieties;

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We denote by $\mathbb{SEM}$ the lattice of all semigroup varieties.

To determine all cancellable elements of $\mathbb{SEM}$, we need notation for the following concrete varieties:

$\mathbf{T}$ is the trivial variety,

$\mathbf{SEM}$ is the variety of all semigroups,

$\mathbf{SL}$ is the variety of semilattices,

$\mathbf{X}_{\infty,\infty} = \mathrm{var}\{x^2y \approx xyx \approx yx^2 \approx 0\}$

($\mathbf{w} \approx 0$ means $\mathbf{w}x \approx x\mathbf{w} \approx \mathbf{w}$ where x is a letter that does not occur in $\mathbf{w}$),

$\mathbf{X}_{m,\infty} = \mathbf{X}_{\infty,\infty} \wedge \mathrm{var}\{x_1x_2\cdots x_m \approx x_{1\pi}x_{2\pi}\cdots x_{m\pi} \mid \pi \in S_m\}$ where $2 \le m < \infty$

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Theorem (Shaprynskiĭ, Skokov and ∼)

A semigroup variety $\mathbf{V}$ is a cancellable element of $\mathbb{SEM}$ if and only if either $\mathbf{V} = \mathbf{SEM}$ or $\mathbf{V} = \mathbf{N}$ or $\mathbf{V} = \mathbf{SL} \vee \mathbf{N}$ where $\mathbf{N}$ is one of the varieties $\mathbf{T}$, $\mathbf{X}_{m,n}$ or $\mathbf{Y}_{m,n}$ with $2 \le m \le n \le \infty$.

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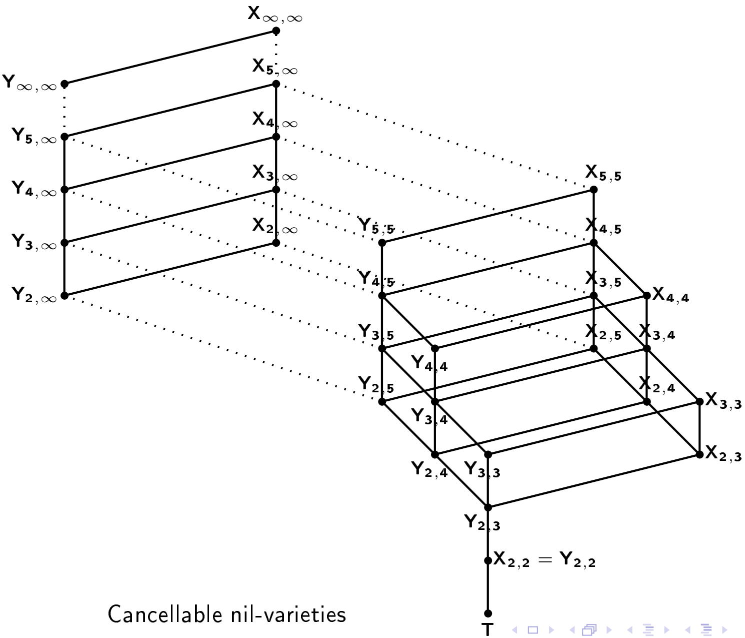
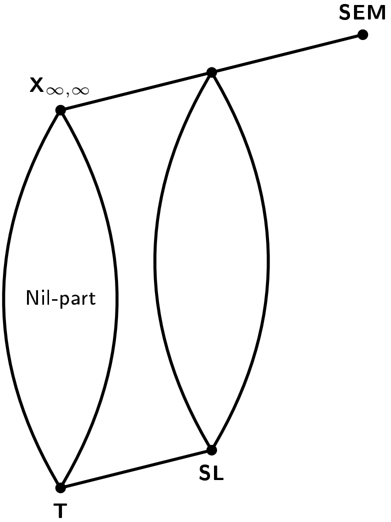
Corollary

Cancellable elements of the lattice $\mathbb{SEM}$ *form a countably infinite distributive sublattice of* $\mathbb{SEM}$.



Cancellable nil-varieties

Diagram



All cancellable varieties

An *epigroup* is a semigroup S with the following property: for any $x \in S$ there is n such that x^n lies in some subgroup of S.

All periodic semigroups as well as all completely regular semigroups are epigroups.

Epigroups may be considered as *unary semigroups*, that is semigroups with an additional unary operation.

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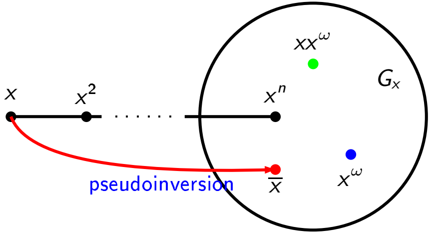
Let S be an epigroup, $x \in S$, G_x is the maximum subgroup of S containing x.

Let x^ω be an identity element of G_x. Then $xx^\omega = x^\omega x \in G_x$. Put

$$\overline{x} = (xx^\omega)^{-1} \text{ in } G_x.$$

$\overline{x}$ is called *pseudoinverse* to x



Every periodic semigroup variety can be considered as a variety of epigroups.

If an epigroup variety **V** consists of periodic semigroups then the operation of pseudoinversion may be defined by multiplication. Namely, if **V** satisfies the identity $x^m = x^{m+n}$ then $\overline{x} = x^{(m+1)n-1}$. Thus a variety of periodic epigroups can be considered as a semigroup variety.

Periodic varieties of epigroups may be identified with periodic varieties of semigroups.

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Let $\mathbb{EPI}$ be the lattice of all epigroup varieties.

Theorem (Shaprynskiĭ, Skokov and $\sim$)

An epigroup variety $\mathbf{V}$ *is a cancellable element of* $\mathbb{EPI}$ *if and only if either* $\mathbf{V} = \mathbf{N}$ *or* $\mathbf{V} = \mathbf{SL} \vee \mathbf{N}$ *where* $\mathbf{N}$ *is one of the varieties* $\mathbf{T}$, $\mathbf{X}_{m,n}$ *or* $\mathbf{Y}_{m,n}$ *with* $2 \leq m \leq n \leq \infty$.

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Cancellable elements of the lattice $\mathbb{EPI}$ *form a countably infinite distributive sublattice of* $\mathbb{EPI}$.

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For a periodic epigroup variety $\mathbf{V}$, *the following are equivalent:*

a) $\mathbf{V}$ *is a cancellable element of the lattice* $\mathbb{EPI}$;
b) $\mathbf{V}$ *is a cancellable element of the lattice* $\mathbb{SEM}$.

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A semigroup variety is called *overcommutative* if it contains the variety of all commutative semigroups.

Every semigroup variety is either periodic or overcommutative. Therefore, the lattice $\mathbb{SEM}$ is the disjoint union of the lattice of all periodic varieties and the lattice $\mathbb{OC}$ of all overcommutative varieties.

All cancellable elements of $\mathbb{SEM}$ *except* **SEM** *are periodic varieties.*

It is interesting to examine cancellable elements of $\mathbb{OC}$.

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An element x of a lattice L is called *distributive* if

$$\forall y, z:\quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Codistributive elements are defined dually.

Proposition (Shaprynskiĭ and ∼, 2011)

For an overcommutative semigroup variety $\mathbf{V}$*, the following are equivalent:*

a) $\mathbf{V}$ *is a neutral element of* $\mathbb{OC}$*;*
b) $\mathbf{V}$ *is a distributive element of* $\mathbb{OC}$*;*
c) $\mathbf{V}$ *is a codistributive element of* $\mathbb{OC}$*.*

Varieties with the properties a)–c) are completely determined in that work.

Theorem (Shaprynskiĭ and ∼)

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Cancellable elements in $\mathbb{OC}$

An element x of a lattice L is called *distributive* if

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Codistributive elements are defined dually.

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For an overcommutative semigroup variety $\mathbf{V}$*, the following are equivalent:*

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