Upper-modular elements of the lattice of semigroup varieties

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Dedicated to Georg Grätzer and E. Tamás Schmidt on their 70th birthdays

Abstract. We completely determine all commutative semigroup varieties that are upper-modular elements of the lattice of all semigroup varieties. It is verified that if a semigroup variety is an upper-modular element of this lattice and different from the variety of all semigroups then it is a periodic variety and every nilsemigroup in the variety is commutative and satisfies the identity $x^2y = xy^2$.

1. Introduction and summary

The class of all varieties of semigroups forms a lattice under the following naturally defined operations: for varieties $\mathbf{X}$ and $\mathbf{Y}$, their *join* $\mathbf{X} \vee \mathbf{Y}$ is the variety generated by the set-theoretical union of $\mathbf{X}$ and $\mathbf{Y}$ (as classes of semigroups), while their *meet* $\mathbf{X} \wedge \mathbf{Y}$ coincides with the set-theoretical intersection of $\mathbf{X}$ and $\mathbf{Y}$. Special elements of different types in lattices of varieties of semigroups or universal algebras have been examined in several articles (see [5, 6, 15, 19, 22], for instance). The present article continues these investigations.

An element x of a lattice $\langle L; \vee, \wedge\rangle$ is called *modular* if

$$\forall\, y, z \in L\colon\ y \le z \longrightarrow (x \vee y) \wedge z = (x \wedge z) \vee y,$$

and *upper-modular* if

$$\forall\, y, z \in L\colon\ y \le x \longrightarrow (z \vee y) \wedge x = (z \wedge x) \vee y.$$

Lower-modular elements are defined dually to upper-modular ones.

For convenience, we call a semigroup variety *modular* (*upper-modular*, *lower-modular*) if it is a modular (respectively upper-modular, lower-modular) element of the lattice $\mathbb{SEM}$ of all semigroup varieties. A number of results about varieties of these three types have been obtained in [6, 16–19, 22].

To formulate the main results of this article, we need some definitions and notation. We denote by $\mathbf{SEM}$ the variety of all semigroups. A semigroup S with 0 is said to be a *nilsemigroup* if, for every $s \in S$, there exists a positive integer n with $s^n = 0$. A semigroup variety $\mathbf{V}$ is called a *nil-variety* if each member of $\mathbf{V}$ is a nilsemigroup. A semigroup is called *periodic* if every cyclic subsemigroup is finite. As is well known, every semigroup variety is either *periodic* (that is, consists of periodic semigroups) or *overcommutative* (that is, contains the variety of all commutative semigroups). It is easy to see that an arbitrary periodic semigroup variety $\mathbf{V}$ contains a greatest nil-subvariety. We denote this subvariety by $\mathrm{Nil}(\mathbf{V})$. It is proved in [19, Propositions 2.4 and 2.6] that an upper-modular nil-variety satisfies the identities $xy = yx$ and $x^2y = xy^2$. The following theorem generalizes this claim.

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Theorem 1.1. *If a semigroup variety* $\mathbf{V}$ *is an upper-modular element of the lattice* $\mathbb{SEM}$ *then either* $\mathbf{V} = \mathbf{SEM}$ *or* $\mathbf{V}$ *is a periodic variety and the variety* $\mathrm{Nil}(\mathbf{V})$ *is commutative and satisfies the identity*

$$x^2y = xy^2. \tag{1.1}$$

A semigroup variety $\mathbf{V}$ is called *proper* if $\mathbf{V} \neq \mathbf{SEM}$. By Theorem 1.1 a proper upper-modular variety is periodic. Note that the analogous claim is true for modular varieties and for lower-modular ones (see [6, Proposition 1.6] and [16, Theorem 1] respectively).

Clearly, if w is a semigroup word then a semigroup S satisfies the identity system $wu = uw = w$ where u runs over the set of all words if and only if S contains a zero element 0 and all values of the word w in S equal 0. We adopt the usual convention of writing $w = 0$ as a short form of such a system and referring to the expression $w = 0$ as to a single identity. If Σ is a system of identities then $\mathrm{var}\,\Sigma$ stands for the variety of all semigroups satisfying Σ. We denote by $\mathbf{T}$ the trivial variety and by $\mathbf{SL}$ the variety of all semilattices. Furthermore, put $\mathbf{C} = \mathrm{var}\{x^2 = x^3,\, xy = yx\}$. As is well known, the variety $\mathbf{SL}$ is generated by the 2-element semilattice (which can be considered as the singleton semigroup with the unit element adjoined), while the variety $\mathbf{C}$ is generated by the 3-element semigroup $\{0, c, 1\}$ where $\{0, c\}$ is the 2-element semigroup with zero multiplication and 1 is a unit. Our second main result is the following

Theorem 1.2. *A commutative semigroup variety* $\mathbf{V}$ *is an upper-modular element of the lattice* $\mathbb{SEM}$ *if and only if one of the following holds:*

(i) $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$ *or* $\mathbf{SL}$*, and* $\mathbf{N}$ *is a commutative nil-variety satisfying the identity* (1.1)*;*

(ii) $\mathbf{V} = \mathbf{G} \vee \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{G}$ *is an abelian periodic group variety,* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$, $\mathbf{SL}$ *or* $\mathbf{C}$*, and the variety* $\mathbf{N}$ *is commutative and satisfies the identity*

$$x^2y = 0. \tag{1.2}$$

Note that commutative lower-modular varieties and commutative modular varieties were completely determined in [16] and [17] respectively. In particular, it turns out that every commutative lower-modular variety is modular, while every commutative modular variety is upper-modular — the facts that were hard to predict apriory.

Theorems 1.1 and 1.2 immediately imply the description of upper-modular nil-varieties obtained earlier in [19, Theorem 2] (see Corollary 4.2 below).

The article is structured as follows. It contains 5 sections. Section 2 contains preliminary information about lattices and semigroup varieties. In Sections 3 and 4 we prove Theorems 1.1 and 1.2 respectively. Section 4 contains some corollaries of Theorem 1.2. In Section 5 we formulate several open questions.

2. Preliminaries

For an element x of a lattice L, we denote by $(x]$ the set $\{y \in L \mid y \leq x\}$, that is the principal ideal of L generated by x. We start with the following lattice-theoretical observation.

Lemma 2.1. *Let* L *be a lattice and* w *an upper-modular element of* L*. The lattice* $(w]$ *is modular if and only if every element of this lattice is an upper-modular element of* L*.*

Proof. It suffices to verify the 'only if' part because the 'if' part is evident. Let $x \in (w]$, $y \le x$, and $z \in L$. Then

$$\begin{aligned}
(z \vee y) \wedge x &= \big((z \vee y) \wedge x\big) \wedge w && \text{because } (z \vee y) \wedge x \le w\\
&= \big((z \vee y) \wedge w\big) \wedge x &&\\
&= \big((z \wedge w) \vee y\big) \wedge x && \text{because } y \le w \text{ and } w \text{ is}\\
& && \text{an upper-modular element of } L\\
&= \big((z \wedge w) \wedge x\big) \vee y && \text{because } z \wedge w, x, y \in (w],\ y \le x,\\
& && \text{and } (w] \text{ is a modular lattice}\\
&= \big((z \wedge x) \wedge w\big) \vee y &&\\
&= (z \wedge x) \vee y && \text{because } z \wedge x \le w.
\end{aligned}$$

Thus $(z \vee y) \wedge x = (z \wedge x) \vee y$, that is x is an upper-modular element of L. □

We need a description of the identities of a few concrete semigroup varieties. For a positive integer $k > 1$, we denote by $\mathbf{A}_k$ the variety of all abelian groups of exponent dividing k. Furthermore, let **COM** be the variety of all commutative semigroups and $\mathbf{P} = \mathrm{var}\{xy = x^2y,\ x^2y^2 = y^2x^2\}$. As is well known, the variety $\mathbf{P}$ is generated by the 3-element semigroup $\{e, a, 0\}$ where $e^2 = e$, $ea = a$, and all other products equal 0. If u is a word and x is a letter, then we denote by $c(u)$ the set of all letters occurring in u, by $\ell(u)$ the length of u, by $\ell_x(u)$ the number of occurrences of x in u, and by $t(u)$ the last letter of u. The symbol $\equiv$ stands for the equality relation on the absolutely free semigroup over a countably infinite alphabet. The claims (i)–(iv) of the following lemma are well known and can be easily verified. The claim (v) was proved in [3, Lemma 7].

Lemma 2.2. *The identity $u = v$ holds in the variety:*

(i) $\mathbf{A}_k$ *if and only if $\ell_x(u) - \ell_x(v)$ is divisible by k for every letter x;*
(ii) **SL** *if and only if $c(u) = c(v)$;*
(iii) **C** *if and only if $c(u) = c(v)$ and, for every letter $x \in c(u)$, either $\ell_x(u) > 1$ and $\ell_x(v) > 1$ or $\ell_x(u) = \ell_x(v) = 1$;*
(iv) **COM** *if and only if $\ell_x(u) = \ell_x(v)$ for every letter x;*
(v) $\mathbf{P}$ *if and only if $c(u) = c(v)$ and either $\ell_{t(u)}(u) > 1$ and $\ell_{t(v)}(v) > 1$ or $\ell_{t(u)}(u) = \ell_{t(v)}(v) = 1$ and $t(u) \equiv t(v)$.* □

We need the following two well known and easily verified technical remarks about identities of nilsemigroups.

Lemma 2.3. *Let* $\mathbf{V}$ *be a nil-variety.*

(i) *If the variety* $\mathbf{V}$ *satisfies an identity $u = v$ with $c(u) \ne c(v)$ then* $\mathbf{V}$ *satisfies also the identity $u = 0$.*
(ii) *If the variety* $\mathbf{V}$ *satisfies an identity of the form $u = vuw$ where at least one the words v, w is non-empty then* $\mathbf{V}$ *satisfies also the identity $u = 0$.* □

It is well known that an arbitrary periodic semigroup variety $\mathbf{V}$ contains a greatest group subvariety. We denote this subvariety by $\mathrm{Gr}(\mathbf{V})$. If S is a semigroup then S^1 stands for the semigroup S with the new unit element adjoined. If a variety $\mathbf{V}$ contains semigroups of the form N^1 where N is a nilsemigroup then $\mathrm{Nil}^1(\mathbf{V})$ denotes the variety generated by all semigroups of such a form; otherwise $\mathrm{Nil}^1(\mathbf{V}) = \mathbf{T}$. The following lemma immediately follows from properties of the varieties **SL** and **C** mentioned in the Introduction.

Lemma 2.4. *For an arbitrary semigroup variety* $\mathbf{V}$*, either* $\mathrm{Nil}^1(\mathbf{V}) = \mathbf{T}$ *or* $\mathrm{Nil}^1(\mathbf{V}) = \mathbf{SL}$ *or* $\mathrm{Nil}^1(\mathbf{V}) \supseteq \mathbf{C}$. □

Results of the article [4] and the proof of [20, Proposition 1] imply the following

Lemma 2.5. *Let* $\mathbf{V}$ *be a periodic commutative semigroup variety. Then*
$$\mathbf{V} = \mathrm{Gr}(\mathbf{V}) \vee \mathrm{Nil}^1(\mathbf{V}) \vee \mathrm{Nil}(\mathbf{V}). \qquad \square$$

Recall that a semigroup variety is called *combinatorial* if all its groups are singleton. The following observation will be helpfull.

Lemma 2.6. *If* $\mathbf{V}$ *is a periodic semigroup variety and* $\mathbf{K}$ *is a combinatorial semigroup variety then* $\mathrm{Gr}(\mathbf{V} \vee \mathbf{K}) = \mathrm{Gr}(\mathbf{V})$.

Proof. Let $u = v$ be an arbitrary identity satisfied by the variety $\mathbf{V}$. Since the variety $\mathbf{K}$ is combinatorial, it satisfies an identity of the form $x^n = x^{n+1}$ for some positive integer n. Then the variety $\mathbf{V} \vee \mathbf{K}$ satisfies the identity $u^{n+1}v^n = u^n v^{n+1}$. Therefore the identity $u = v$ holds in every group from $\mathbf{V} \vee \mathbf{K}$. Thus $\mathrm{Gr}(\mathbf{V} \vee \mathbf{K}) \subseteq \mathbf{V}$, and therefore $\mathrm{Gr}(\mathbf{V} \vee \mathbf{K}) \subseteq \mathrm{Gr}(\mathbf{V})$. The opposite inclusion is evident. $\square$

Lemma 2.7. *Let* $\mathbf{V}$ *be a periodic commutative semigroup variety. If the variety* $\mathrm{Nil}(\mathbf{V})$ *satisfies the identity* (1.1) *then* $\mathrm{Nil}^1(\mathbf{V})$ *is one of the varieties* $\mathbf{T}$, $\mathbf{SL}$ *or* $\mathbf{C}$.

Proof. Let N be a nilsemigroup such that $N^1 \in \mathbf{V}$ and $\mathbf{M}$ the variety generated by the semigroup N^1. It is verified in [20] that if N does not satisfy the identity $x^2 = 0$ then $\mathbf{M} \supseteq \mathrm{var}\{x^3 = x^4,\ xy = yx\}$. Since $\mathbf{V} \supseteq \mathbf{M}$, Lemma 2.3(ii) implies that $\mathrm{Nil}(\mathbf{V}) \supseteq \mathrm{var}\{x^3 = 0,\ xy = yx\}$. But this is impossible because the variety $\mathrm{var}\{x^3 = 0,\ xy = yx\}$ does not satisfy the identity (1.1). Therefore N satisfies the identity $x^2 = 0$, whence N^1 satisfies the identity $x^2 = x^3$. Since $\mathbf{V}$ is commutative, we have $\mathbf{M} \subseteq \mathbf{C}$, and therefore $\mathrm{Nil}^1(\mathbf{V}) \subseteq \mathbf{C}$. Now Lemma 2.4 applies. $\square$

Recall that a semigroup S is said to be *nilpotent* (*of index* n) if it satisfies the identity $x_1x_2\cdots x_n = 0$ for some n (and n is the least number with such a property). A semigroup variety is called *nilpotent* if all its members are nilpotent. A variety $\mathbf{V}$ is said to be a *variety of finite index* (a *variety of index* n) if there exists a positive integer n such that every nilsemigroup in $\mathbf{V}$ is nilpotent of index $\leq n$ (and n is the least number with such a property). A semigroup variety is called *completely regular* if it consists of *completely regular* semigroups (unions of groups). Put $\mathbf{ZM} = \mathrm{var}\{xy = 0\}$. The following lemma is well known.

Lemma 2.8. *Let* $\mathbf{V}$ *be a semigroup variety. The following are equivalent:*

(i) $\mathbf{V}$ *is completely regular;*
(ii) $\mathbf{V} \not\supseteq \mathbf{ZM}$*;*
(iii) $\mathbf{V}$ *is a variety of index* 1*;*
(iv) $\mathbf{V}$ *satisfies an identity of the form*
$$x = x^{r+1} \tag{2.1}$$
for some positive integer r. $\square$

The following lemma readily follows from the proof of [12, Lemma 1].

Lemma 2.9. *Let* n *be a positive integer. If a semigroup variety* $\mathbf{V}$ *satisfies an identity of the form* $x_1 \cdots x_n = w$ *for some word* w *with* $\ell(w) > n$ *then* $\mathbf{V}$ *is a variety of index* $\leq n$. $\square$

For a semigroup variety $\mathbf{V}$, we write $\mathrm{ind}(\mathbf{V}) = n$ if $\mathbf{V}$ is a variety of (finite) index n, and $\mathrm{ind}(\mathbf{V}) = \infty$ if $\mathbf{V}$ is not a variety of finite index. Lemmas 2.3(i) and 2.9 imply the following

Corollary 2.10. *If a semigroup variety* $\mathbf{V}$ *satisfies a non-trivial identity of the form* $xy = u$ *then* $\mathbf{V}$ *is either a commutative variety or a variety of index* ≤ 2.

Proof. If $c(u) \neq \{x, y\}$ then Lemma 2.3(i) applies with the conclusion that every nilsemigroup from the variety $\mathbf{V}$ satisfies the identity $xy = 0$. Therefore $\mathrm{ind}(\mathbf{V}) \leq 2$. Let now $c(u) = \{x, y\}$. Then $\ell(u) \geq 2$. If $\ell(u) > 2$ then Lemma 2.9 applies and we have $\mathrm{ind}(\mathbf{V}) \le 2$ again. Finally, if $\ell(u) = 2$ then $u \equiv yx$ and $\mathbf{V}$ satisfies the commutative law. $\square$

The following proposition gives an equational characterization of semigroup varieties of index n and seems to be of some independent interest. It was announced by A. P. Birjukov in 1981 (this is mentioned in [13, Section 8], for instance). But its proof was not published so far, as far as we know.

Proposition 2.11. *Let n be a positive integer. A semigroup variety is a variety of index $\le n$ if and only if it satisfies an identity of the form*

$$x_1 \cdots x_n = x_1 \cdots x_{i-1}(x_i \cdots x_j)^t x_{j+1} \cdots x_n \tag{2.2}$$

for some $t > 1$ and $1 \le i \le j \le n$.

Proof. The 'if' part immediately follows from Lemma 2.9. We prove the 'only if' part by induction on n.

Induction basis is evident. Indeed, if $n = 1$ then it suffices to refer to Lemma 2.8 because (2.1) is an identity of the form (2.2) with $i = j = n = 1$.

Induction step. Now let $n > 1$ and let $\mathbf{V}$ be a variety of index $\le n$. By $\overleftarrow{\mathbf{P}}$ we denote the semigroup variety dual to $\mathbf{P}$. It is verified in [14, Theorem 2] that if a variety of index $\le n$ does not contain the varieties $\mathbf{P}$ and $\overleftarrow{\mathbf{P}}$ then it satisfies an identity of the form (2.2) (with $i = 1$ and $j = n$). Suppose now that $\mathbf{V}$ contains one of the varieties $\mathbf{P}$ or $\overleftarrow{\mathbf{P}}$. By symmetry we may assume that $\mathbf{P} \subseteq \mathbf{V}$.

For a positive integer k, we put $\mathbf{F}_k = \mathrm{var}\{x^2 = x_1x_2 \cdots x_k = 0,\ xy = yx\}$. Obviously, $\mathrm{ind}(\mathbf{F}_k) = k$. Whence $\mathbf{V} \not\supseteq \mathbf{F}_{n+1}$. Therefore there is an identity $u = v$ which holds in $\mathbf{V}$ but fails in $\mathbf{F}_{n+1}$. Obviously, the variety $\mathbf{F}_{n+1}$ satisfies an identity $w_1 = w_2$ whenever either $\ell(w_1), \ell(w_2) \ge n+1$ or $\ell_x(w_1) \ge 2$ and $\ell_y(w_2) \ge 2$ for some letters x and y. Therefore we may assume without any loss that $u \equiv x_1 \cdots x_m$ for some $m \le n$. The identity $x_1 \cdots x_m = v$ holds in the variety $\mathbf{P}$. Now Lemma 2.2(v) applies and we conclude that $c(v) = \{x_1, \dots, x_m\}$, $t(v) \equiv x_m$, and $\ell_{x_m}(v) = 1$. In particular, if $m = 1$ then $v \equiv x_1$ and the identity $x_1 \cdots x_m = v$ is the trivial identity $x_1 = x_1$. But this is impossible because $x_1 \cdots x_m = v$ fails in $\mathbf{F}_{n+1}$. Hence $m > 1$. Therefore $v \equiv v'x_m$ for some word v' with $c(v') = \{x_1, \dots, x_{m-1}\}$. Clearly, $\ell(v) \ge m$. If $\ell(v) = m$ then $u = v$ is an identity of the form $x_1 \cdots x_m = x_{1\pi} \cdots x_{m\pi}$ where π is a permutation on the set $\{1, \dots, m\}$. But every identity of such a form holds in the variety $\mathbf{F}_{n+1}$ (because this variety is commutative) while $u = v$ fails in $\mathbf{F}_{n+1}$. Therefore $\ell(v) > m$, whence $\ell(v') > m - 1$. Put $\mathbf{V}' = \mathrm{var}\{x_1 \cdots x_{m-1} = v'\}$. By Lemma 2.9, $\mathbf{V}'$ is a variety of index $\le m - 1$. Since $m \le n$, the induction assumption applies with the conclusion that the variety $\mathbf{V}'$ satisfies an identity of the form

$$x_1 \cdots x_{m-1} = x_1 \cdots x_{i-1}(x_i \cdots x_j)^t x_{j+1} \cdots x_{m-1}$$

for some $t > 1$ and $1 \le i \le j \le m - 1$. Therefore there is a deduction of this identity from the identity $x_1 \cdots x_{m-1} = v'$. Multiplying all words that appear in this deduction by x_m on the right, we obtain a deduction of the identity

$$x_1 \cdots x_m = x_1 \cdots x_{i-1}(x_i \cdots x_j)^t x_{j+1} \cdots x_m \tag{2.3}$$

from the identity $x_1 \cdots x_m = v$. Therefore the variety $\mathbf{V}$ satisfies (2.3). Since the identity (2.3) evidently implies (2.2), we are done. $\square$

Note that a partial case of Proposition 2.11 dealing with varieties of index ≤ 2 was proved in [3, Lemma 3]. Note also that a weaker version of Proposition 2.11 was proved in [12]. Namely, it readily follows from the proof of [12, Theorem 2] that a variety of index $\le n$ satisfies an identity of the form

$$x_1 \cdots x_n y_1 \cdots y_n z_1 \cdots z_n = x_1 \cdots x_n (y_1 \cdots y_n)^t z_1 \cdots z_n$$

for some $t > 1$.

It is evident that $\mathrm{ind}(\mathbf{X} \vee \mathbf{Y}) \geq \max\bigl\{\mathrm{ind}(\mathbf{X}), \mathrm{ind}(\mathbf{Y})\bigr\}$ for arbitrary varieties $\mathbf{X}$ and $\mathbf{Y}$. The equality $\mathrm{ind}(\mathbf{X} \vee \mathbf{Y}) = \max\bigl\{\mathrm{ind}(\mathbf{X}), \mathrm{ind}(\mathbf{Y})\bigr\}$ is wrong in the general case[1]. In the following two lemmas we find two partial cases when this equality holds. Note that if a variety $\mathbf{K}$ is completely regular then $\max\bigl\{\mathrm{ind}(\mathbf{V}), \mathrm{ind}(\mathbf{K})\bigr\} = \mathrm{ind}(\mathbf{V})$ for arbitrary variety $\mathbf{V}$ (see Lemma 2.8).

Lemma 2.12. *If* $\mathbf{V}$ *is an arbitrary semigroup variety and* $\mathbf{K}$ *is a completely regular variety then* $\mathrm{ind}(\mathbf{V} \vee \mathbf{K}) = \mathrm{ind}(\mathbf{V})$.

Proof. If $\mathrm{ind}(\mathbf{V}) = \infty$ then $\mathrm{ind}(\mathbf{V} \vee \mathbf{K}) = \infty = \mathrm{ind}(\mathbf{V})$, and we are done. So, we may assume that $\mathbf{V}$ is a variety of finite index. Let $\mathrm{ind}(\mathbf{V}) = n$. By Proposition 2.11 the variety $\mathbf{V}$ satisfies an identity of the form (2.2) for some $t > 1$ and $1 \leq i \leq j \leq n$. Suppose that $i > 1$. Substituting $x_{i-1}(x_i \cdots x_j)^{t-1}$ for x_{i-1} in the identity (2.2), we have

$$x_1 \cdots x_n = x_1 \cdots x_{i-1}(x_i \cdots x_j)^{t+(t-1)} x_{j+1} \cdots x_n.$$

Repeating k times the substitution $x_{i-1} \longmapsto x_{i-1}(x_i \cdots x_j)^{t-1}$, we obtain that $\mathbf{V}$ satisfies the identity

$$x_1 \cdots x_n = x_1 \cdots x_{i-1}(x_i \cdots x_j)^{t+k(t-1)} x_{j+1} \cdots x_n \tag{2.4}$$

for an arbitrary positive integer k. If $i = 1$ and $j < n$ then we get the same conclusion if we substitute $(x_1 \cdots x_j)^{t-1} x_{j+1}$ for x_{j+1} in the identity (2.2) k times. Finally, if $i = 1$ and $j = n$ then to deduce (2.4) from (2.2) it suffices to apply the latter identity to itself k times. Thus, $\mathbf{V}$ satisfies the identity (2.4) for every $k \geq 0$ (if $k = 0$, then (2.4) coincides with (2.2)). Let r be a positive integer such that the variety $\mathbf{K}$ satisfies the identity (2.1) (and therefore the identity $x = x^{qr+1}$ for every positive integer q). Taking into account that

$$t + (r-1)(t-1) = (t-1)r + 1,$$

we obtain that both the varieties $\mathbf{V}$ and $\mathbf{K}$ (and therefore the variety $\mathbf{V} \vee \mathbf{K}$) satisfy the identity

$$x_1 \cdots x_n = x_1 \cdots x_{i-1}(x_i \cdots x_j)^{(t-1)r+1} x_{j+1} \cdots x_n.$$

Now Lemma 2.9 applies with the conclusion that $\mathrm{ind}(\mathbf{V} \vee \mathbf{K}) \leq n = \mathrm{ind}(\mathbf{V})$. The inequality $\mathrm{ind}(\mathbf{V}) \leq \mathrm{ind}(\mathbf{V} \vee \mathbf{K})$ is evident. □

Lemma 2.13. *If* $\mathbf{V}$ *is an arbitrary semigroup variety and* $\mathbf{N}$ *is a nil-variety then* $\mathrm{ind}(\mathbf{V} \vee \mathbf{N}) = \max\bigl\{\mathrm{ind}(\mathbf{V}), \mathrm{ind}(\mathbf{N})\bigr\}$.

Proof. We may assume that $\mathbf{V}$ and $\mathbf{N}$ are varieties of finite index because

$$\mathrm{ind}(\mathbf{V} \vee \mathbf{N}) = \infty = \max\bigl\{\mathrm{ind}(\mathbf{V}), \mathrm{ind}(\mathbf{N})\bigr\}$$

in the contrary case. In particular, $\mathbf{N}$ is a nilpotent variety. Let $\mathrm{ind}(\mathbf{V}) = n$ and $\mathrm{ind}(\mathbf{N}) = m$. By Proposition 2.11 the variety $\mathbf{V}$ satisfies an identity of the form (2.2) for some $t > 1$ and $1 \leq i \leq j \leq n$. The variety $\mathbf{N}$ satisfies the identity $x_1 x_2 \cdots x_m = 0$. It is fairly easy to check that the variety $\mathbf{V} \vee \mathbf{N}$ satisfies the identity

$$x_1 \cdots x_{\max\{n,m\}} = x_1 \cdots x_{i-1}(x_i \cdots x_j)^t x_{j+1} \cdots x_{\max\{n,m\}}.$$

Now Lemma 2.9 applies with the conclusion that $\mathrm{ind}(\mathbf{V} \vee \mathbf{N}) \leq \max\{n, m\}$. The opposite inequality is evident. □

[1] For instance, $\mathrm{ind}(\mathbf{P}) = \mathrm{ind}(\overleftarrow{\mathbf{P}}) = 2$ (this follows from Lemmas 2.9 and 2.8). But $\mathrm{ind}(\mathbf{P} \vee \overleftarrow{\mathbf{P}}) = 3$. Indeed, Lemma 2.9 and the evident fact that $\mathbf{P} \vee \overleftarrow{\mathbf{P}}$ satisfies the identity $xyz = xy^2z$ imply that $\mathrm{ind}(\mathbf{P} \vee \overleftarrow{\mathbf{P}}) \leq 3$. On the other hand, Lemma 2.2(v) and its dual imply that $\mathbf{P} \vee \overleftarrow{\mathbf{P}}$ does not satisfy a non-trivial identity of the form $xy = u$. According to Proposition 2.11 this means that $\mathrm{ind}(\mathbf{P} \vee \overleftarrow{\mathbf{P}}) > 2$.

For a semigroup variety $\mathbf{V}$, we denote by $L(\mathbf{V})$ the subvariety lattice of $\mathbf{V}$. The following lemma is a part of the semigroup folklore. It easily follows from results of [8], for instance.

Lemma 2.14. *If $\mathbf{V}$ is a semigroup variety with $\mathbf{V} \not\supseteq \mathbf{SL}$ then $L(\mathbf{V} \vee \mathbf{SL}) \cong L(\mathbf{V}) \times L(\mathbf{SL})$.* $\square$

We need the following

Lemma 2.15. *Let $\mathbf{V}$ be a semigroup variety with $\mathbf{V} \not\supseteq \mathbf{C}$, $\mathbf{G}$ an abelian periodic group variety, and $\mathbf{N}$ a nil-variety. Then $\mathbf{V} \vee \mathbf{G} \vee \mathbf{SL} \vee \mathbf{N} \not\supseteq \mathbf{C}$.*

Proof. Let $\mathbf{V}_1 = \mathbf{V} \vee \mathbf{SL}$. If $\mathbf{SL} \subseteq \mathbf{V}$ then $\mathbf{V}_1 = \mathbf{V} \not\supseteq \mathbf{C}$. Suppose now that $\mathbf{SL} \nsubseteq \mathbf{V}$. By Lemma 2.14 $L(\mathbf{V}_1) \cong L(\mathbf{V}) \times L(\mathbf{SL})$. Since the lattice $L(\mathbf{SL})$ consists of two elements, every subvariety of $\mathbf{V}_1$ either is contained in $\mathbf{V}$ or has the form $\mathbf{V}' \vee \mathbf{SL}$ for some variety $\mathbf{V}' \subseteq \mathbf{V}$. Suppose that $\mathbf{V}_1 \supseteq \mathbf{C}$. Since $\mathbf{V} \not\supseteq \mathbf{C}$, we have that $\mathbf{C} = \mathbf{V}' \vee \mathbf{SL}$ for some $\mathbf{V}' \subseteq \mathbf{V}$. But this is not the case because the variety $\mathbf{C}$ is well known to be join-indecomposable (the lattice $L(\mathbf{C})$ has the form shown in Fig. 1 where $\mathbf{F}_k$ has the same sense as in the proof of Proposition 2.11 and $\mathbf{F} = \mathrm{var}\{x^2 = 0,\, xy = yx\}$). Thus $\mathbf{V}_1 \not\supseteq \mathbf{C}$.

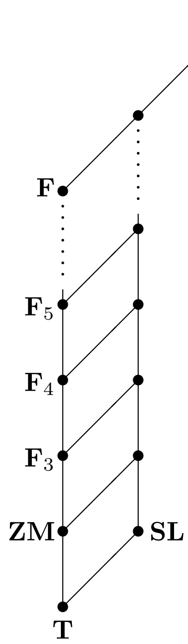

FIGURE 1. The lattice $L(\mathbf{C})$

Let now $\mathbf{V}_2 = \mathbf{V}_1 \vee \mathbf{N}$. Since $\mathbf{V}_1 \supseteq \mathbf{SL}$ and $\mathbf{V}_1 \not\supseteq \mathbf{C}$, the claims (ii) and (iii) of Lemma 2.2 imply that $\mathbf{V}_1$ satisfies some identity $u = v$ such that $c(u) = c(v)$, $\ell_x(u) = 1$, and $\ell_x(v) > 1$ for some letter $x \in c(u)$. The variety $\mathbf{N}$ satisfies an identity of the form $x^n = 0$ for some positive integer n. Let y be a letter with $y \not\equiv x$. Then the variety $\mathbf{V}_2$ satisfies the identity $uy^n = vy^n$, $c(uy^n) = c(vy^n)$, $\ell_x(uy^n) = 1$, and $\ell_x(vy^n) > 1$. Therefore $\mathbf{V}_2 \not\supseteq \mathbf{C}$ by Lemma 2.2(iii).

Finally, let $\mathbf{V}_3 = \mathbf{V}_2 \vee \mathbf{G}$. Clearly, $\mathbf{V}_3 = \mathbf{V} \vee \mathbf{G} \vee \mathbf{SL} \vee \mathbf{N}$. We have to verify that $\mathbf{V}_3 \not\supseteq \mathbf{C}$. If $\mathbf{G} = \mathbf{T}$ then $\mathbf{V}_3 = \mathbf{V}_2 \not\supseteq \mathbf{C}$. Whence we may assume that $\mathbf{G} \neq \mathbf{T}$, that is $\mathbf{G} = \mathbf{A}_k$ for some $k > 1$. Since $\mathbf{V}_2 \supseteq \mathbf{SL}$ and $\mathbf{V}_2 \not\supseteq \mathbf{C}$, the claims (ii) and (iii) of Lemma 2.2 imply that $\mathbf{V}_2$ satisfies some identity $u = v$ such that $c(u) = c(v)$, $\ell_x(u) = 1$, and $\ell_x(v) > 1$ for some letter $x \in c(u)$. If $c(u) = \{x\}$ then $u = v$ is an identity of the form (2.1). By Lemma 2.8 this means that $\mathbf{V}_2$ is a completely regular variety. Then the variety $\mathbf{V}_3$ is completely regular as well, and therefore $\mathbf{V}_3 \not\supseteq \mathbf{C}$. Thus, we may assume that $c(u) \neq \{x\}$.

Let y be a letter with $y \not\equiv x$. Substituting y for all letters except x in the identity $u = v$, we deduce from $u = v$ an identity of the form

$$y^{m_1}xy^{n_1} = w_1 \tag{2.5}$$

where $m_1, n_1 \geq 0$, $m_1 + n_1 > 0$, $c(w_1) = \{x, y\}$, and $\ell_x(w_1) > 1$. If $m_1 = 0$ then multiplying the identity (2.5) by y on the left, we get an identity of the form (2.5) with $m_1 > 0$. Thus we may assume that $m_1 > 0$. Analogous arguments permit to assume that $n_1 > 0$. Put $\ell = \ell_x(w_1)$.

Now let us substitute $y^{m_1}xy^{n_1}$ for x in the identity (2.5). We obtain an identity of the form

$$y^{m_2}xy^{n_2} = w_2 \tag{2.6}$$

where $m_2, n_2 > 0$, $c(w_2) = \{x, y\}$, $\ell_x(w_2) = \ell > 1$, and $y^{m_1}xy^{n_1}$ is a subword of w_2. Let us fix some subword w' of the word w_2 such that $w' \equiv y^{m_1}xy^{n_1}$. In view of the identity (2.5), we may substitute w_1 for w' in the identity (2.6). Since $\ell_x(w_1) = \ell_x(w_2) = \ell$, after this substitution the number of occurrences of the letter x in the right-hand side of the identity increases on $\ell - 1$. Then we obtain an identity of the form (2.6) with $\ell_x(w_2) = \ell + \ell - 1 = 2(\ell - 1) + 1$.

Repeating the procedure described in the previous paragraph $k - 1$ times, we get an identity of the form

$$y^{m_k}xy^{n_k} = w_k \tag{2.7}$$

where $c(w_k) = \{x, y\}$ and

$$\ell_x(w_k) = \ell + (k - 1)(\ell - 1) = (\ell - 1)k + 1.$$

In particular, $\ell_x(w_k) - \ell_x(y^{m_k}xy^{n_k}) = (\ell - 1)k$. Substituting y^k for y in the identity (2.7), we get an identity $u' = v'$ such that $u' = v'$ follows from $u = v$, $c(u') = c(v') = \{x, y\}$, $\ell_x(v') - \ell_x(u')$ and $\ell_y(v') - \ell_y(u')$ are divisible by k, $\ell_x(u') = 1$, and $\ell_x(v') > 1$. The identity $u' = v'$ holds in both the varieties $\mathbf{V}_2$ and $\mathbf{A}_k$ (the latter follows from Lemma 2.2(i)). Therefore it holds in the variety $\mathbf{V}_3$. On the other hand, Lemma 2.2(iii) implies that $u' = v'$ fails in $\mathbf{C}$. This means that $\mathbf{V} \vee \mathbf{G} \vee \mathbf{SL} \vee \mathbf{N} = \mathbf{V}_3 \not\supseteq \mathbf{C}$. □

Let p and q be different prime numbers. We denote by $\mathbf{A}_p\mathbf{A}_q$ the product of the varieties $\mathbf{A}_p$ and $\mathbf{A}_q$ in the sense of the theory of group varieties, that is the class of all groups G possessing a normal subgroup $H \in \mathbf{A}_p$ such that the quotient group G/H belongs to $\mathbf{A}_q$. As is well known, the lattice $L(\mathbf{A}_p\mathbf{A}_q)$ has the form shown in Fig. 2 (this readily follows from [9, Theorem 54.41], for instance). In particular, $\mathbf{A}_p\mathbf{A}_q$ is a minimal non-abelian periodic group variety.

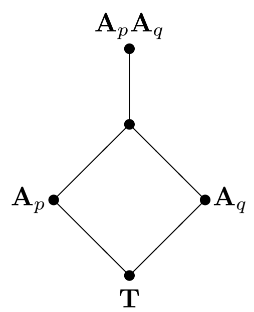

FIGURE 2. The lattice $L(\mathbf{A}_p\mathbf{A}_q)$

We conclude this section with the following lemma[2].

[2]The proof of this lemma was communicated to the author by M. V. Volkov.

Lemma 2.16. *The join of all varieties of the kind* $\mathbf{A}_p\mathbf{A}_q$ *where* p *and* q *are different prime numbers* (*and therefore the join of all minimal non-abelian periodic group varieties*) *coincides with the variety* **SEM**.

Proof. It suffices to show that a free non-cyclic semigroup embeds in a direct product of some groups from $\mathbf{A}_p\mathbf{A}_q$. For this, we shall employ a representation of the semigroup by triangular matrices (inspired by the classical Magnus representation of the free metabelian group, see [7]).

Let $\mathbb{Z}[x]$ stand for the ring of polynomials in x with integer coefficients. Consider the set S of 2×2-matrices over $\mathbb{Z}[x]$ of the form

$$\begin{pmatrix} x^n & 0 \\ f(x) & 1 \end{pmatrix}$$

where n is a non-negative integer and $f(x)\in\mathbb{Z}[x]$. Obviously, S is a semigroup under the usual matrix multiplication. Let

$$Y=\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix},\quad Z=\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

and let F be the subsemigroup in S generated by the matrices Y and YZ. Our first aim is to verify that F is free with Y and YZ being free generators. This amounts to showing that every matrix in F has a unique representation as a word in the generators.

Clearly, every word in Y and YZ can be uniquely rewritten as either

$$Y^\alpha \tag{2.8}$$

where $\alpha>0$ or

$$Y^{\alpha_1}ZY^{\alpha_2}Z\cdots Y^{\alpha_k}ZY^{\alpha_{k+1}} \tag{2.9}$$

where $k,\alpha_1,\dots,\alpha_k>0$ and $\alpha_{k+1}\ge 0$. Put $\beta_i=\sum\limits_{j=i}^{k+1}\alpha_j$ for all $i=1,2,\dots,k+1$. Calculating the corresponding matrix, one readily obtains that it is equal to

$$\begin{pmatrix} x^\alpha & 0 \\ 0 & 1 \end{pmatrix} \tag{2.10}$$

for the word (2.8) and

$$\begin{pmatrix} x^{\beta_1} & 0 \\ \sum\limits_{i=1}^{k+1} x^{\beta_i} & 1 \end{pmatrix} \tag{2.11}$$

for the word (2.9). It is clear that the matrix (2.10) (respectively (2.11)) provides enough information to uniquely recover the word (2.8) (respectively (2.9)).

It remains to show that our semigroup S is a subdirect product of some groups from varieties of the form $\mathbf{A}_p\mathbf{A}_q$ where p and q are different primes. Let $\xi=\cos\frac{2\pi}{q}+i\sin\frac{2\pi}{q}$ be the q^{th} primitive root of unity, $\mathbb{Z}_p$ the p-element field, and $\mathbb{Z}_p[\xi]$ the group ring of the q-element group $\mathbb{C}_q=\{1,\xi,\dots,\xi^{q-1}\}$ over $\mathbb{Z}_p$. Consider the set G_{pq} of 2×2-matrices over $\mathbb{Z}_p[\xi]$ of the form

$$\begin{pmatrix} \xi^n & 0 \\ f(\xi) & 1 \end{pmatrix}$$

where $0\le n<q$ and $f(\xi)\in\mathbb{Z}_p[\xi]$. It is easy to check that G_{pq} is a group under the usual matrix multiplication; moreover, the mapping

$$\begin{pmatrix} \xi^n & 0 \\ f(\xi) & 1 \end{pmatrix}\longmapsto \xi^n$$

is a homomorphism from G_{pq} onto $\mathbb{C}_q$ whose kernel

$$\left\{\begin{pmatrix} 1 & 0 \\ f(\xi) & 1 \end{pmatrix}\ \middle|\ f(\xi)\in\mathbb{Z}_p[\xi]\right\}$$

is isomorphic to the additive subgroup of the ring $\mathbb{Z}_p[\xi]$. We conclude that $G_{pq} \in \mathbf{A}_p\mathbf{A}_q$. (The reader acquainted with the concept of wreath product will recognize in G_{pq} a realization of the wreath product of the p-element group with the q-element group.)

Now consider the mapping $\varphi_{pq} : S \longmapsto G_{pq}$ induced by the natural mapping $\mathbb{Z}[x] \longmapsto \mathbb{Z}_p[\xi]$; for a matrix

$$A = \begin{pmatrix} x^m & 0 \\ \sum \alpha_j x^j & 1 \end{pmatrix} \in S$$

with $\alpha_j \in \mathbb{Z}$, $m, j \geq 0$, one has

$$\varphi_{pq}(A) = \begin{pmatrix} \xi^{m \,(\mathrm{mod}\, q)} & 0 \\ \sum \overline{\alpha}_j \xi^{j \,(\mathrm{mod}\, q)} & 1 \end{pmatrix}$$

where $\overline{\alpha}_j$ is the residue of α_j modulo p. Clearly, φ_{pq} is a semigroup homomorphism, and it remains to show that any two different matrices $A, B \in S$ can be separated by a suitable φ_{pq}. Let

$$A = \begin{pmatrix} x^m & 0 \\ \sum \alpha_j x^j & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} x^n & 0 \\ \sum \beta_j x^j & 1 \end{pmatrix}.$$

If we choose q to be a prime bigger than

$$\max\{m, n\} + \max\{j \mid \alpha_j \neq 0 \text{ or } \beta_j \neq 0\},$$

then $m \,(\mathrm{mod}\, q) = m$, $n \,(\mathrm{mod}\, q) = n$, and $j \,(\mathrm{mod}\, q) = j$ for all j such that $\alpha_j \neq 0$ or $\beta_j \neq 0$. Similarly, if p is chosen to be a prime bigger than $2 \max\{|\alpha_j|, |\beta_j|\}$, then $\alpha_j = \beta_j$ whenever the residues of α_j and β_j modulo p coincide. Therefore, for the chosen values of p and q, the equality $\varphi_{pq}(A) = \varphi_{pq}(B)$ is only possible if $A = B$. $\square$

3. Proof of Theorem 1.1

Throughout this section $\mathbf{V}$ *is a fixed proper upper-modular semigroup variety.* We have to verify that $\mathbf{V}$ is periodic and the variety $\mathrm{Nil}(\mathbf{V})$ is commutative and satisfies the identity (1.1).

Suppose that $\mathbf{V}$ is not periodic. Then $\mathbf{V} \supseteq \mathbf{COM}$. Further considerations are divided into two cases.

Case 1: $\mathbf{V} \supset \mathbf{COM}$. By Lemma 2.16 there exists a minimal non-abelian periodic group variety $\mathbf{G}$ with $\mathbf{G} \nsubseteq \mathbf{V}$. Clearly, $\mathbf{G} \wedge \mathbf{V}$ is an abelian periodic group variety. Let $\mathbf{N}$ be a commutative nil-variety with $\mathbf{N} \nsubseteq \mathbf{ZM}$. Then the variety $(\mathbf{G} \wedge \mathbf{V}) \vee \mathbf{N}$ satisfies the identity $xy = yx$. Since $\mathbf{V}$ is upper-modular and $\mathbf{N} \subseteq \mathbf{COM} \subseteq \mathbf{V}$, we have $(\mathbf{G} \vee \mathbf{N}) \wedge \mathbf{V} = (\mathbf{G} \wedge \mathbf{V}) \vee \mathbf{N}$, whence the variety $(\mathbf{G} \vee \mathbf{N}) \wedge \mathbf{V}$ satisfies the identity $xy = yx$ as well. Then there is a deduction of this identity from the identities of the varieties $\mathbf{G} \vee \mathbf{N}$ and $\mathbf{V}$. In particular, there is a word u such that $u \not\equiv xy$ and $xy = u$ holds in either $\mathbf{G} \vee \mathbf{N}$ or $\mathbf{V}$. Suppose first that $xy = u$ in the variety $\mathbf{V}$. By Corollary 2.10 this means that $\mathbf{V}$ is either a commutative variety or a variety of index ≤ 2. Since $\mathbf{V} \supset \mathbf{COM}$, this is a contradiction. Therefore the identity $xy = u$ holds in the variety $\mathbf{G} \vee \mathbf{N}$. Now Corollary 2.10 applies again and we conclude that the variety $\mathbf{G} \vee \mathbf{N}$ is either a commutative variety or a variety of index ≤ 2. In the former case the variety $\mathbf{G}$ is abelian while in the latter case $\mathbf{N} \subseteq \mathrm{Nil}(\mathbf{G} \vee \mathbf{N}) \subseteq \mathbf{ZM}$. We have a contradiction with the choice of the varieties $\mathbf{G}$ and $\mathbf{N}$. This completes the consideration of Case 1.

Case 2: $\mathbf{V} = \mathbf{COM}$. Put

$$W_1 = \{x^2y, xyx, yx^2\}, \ W_2 = \{y^2x, yxy, xy^2\}, \text{ and } W = W_1 \cup W_2.$$

For the sequel, we need the following

Lemma 3.1 ([19, Lemma 2.5])**.** *If a commutative nil-variety* $\mathbf{X}$ *satisfies an identity of the form* $u = v$ *with* $u \in W$ *then either* $v \in W$ *or* $\mathbf{X}$ *satisfies the identity* $u = 0$. □

Put $\mathbf{U} = \mathrm{var}\{x^2y = xy^2,\, xy = yx,\, x^2yz = 0\}$ and $\mathbf{SI} = \mathrm{var}\{xy = (xy)^2\}$. One can mention some properties of the variety $\mathbf{SI}$. Substituting 1 for y in the identity $xy = (xy)^2$, we have $x = x^2$, whence the variety $\mathbf{SI}$ is combinatorial. Further, Lemma 2.9 and the evident inclusion $\mathbf{ZM} \subseteq \mathbf{SI}$ imply that $\mathbf{SI}$ is a variety of index 2. Finally, it follows from [2] and can be easily verified directly that a non-trivial identity of the form $u = v$ with $u \in W_1$ and $v \in W_2$ fails in $\mathbf{SI}$. Put $\mathbf{G} = \mathrm{Gr}(\mathbf{SI} \wedge \mathbf{COM})$, $\mathbf{M} = \mathrm{Nil}^1(\mathbf{SI} \wedge \mathbf{COM})$, and $\mathbf{N} = \mathrm{Nil}(\mathbf{SI} \wedge \mathbf{COM})$. Clearly, $\mathbf{G} = \mathbf{T}$ and $\mathbf{N} = \mathbf{ZM}$. The latter equality, Lemma 2.4, and the evident fact that $\mathrm{Nil}(\mathbf{C}) \nsubseteq \mathbf{ZM}$ imply that $\mathbf{M} = \mathbf{SL}$. By Lemma 2.5

$$\mathbf{SI} \wedge \mathbf{COM} = \mathbf{G} \vee \mathbf{M} \vee \mathbf{N} = \mathbf{SL} \vee \mathbf{ZM}.$$

Since $\mathbf{U} \subseteq \mathbf{COM}$, the variety $\mathbf{V} = \mathbf{COM}$ is upper-modular, by the hypothesis, and $\mathbf{ZM} \subseteq \mathbf{U}$, we have

$$(\mathbf{SI} \vee \mathbf{U}) \wedge \mathbf{COM} = (\mathbf{SI} \wedge \mathbf{COM}) \vee \mathbf{U} = (\mathbf{SL} \vee \mathbf{ZM}) \vee \mathbf{U} = \mathbf{SL} \vee \mathbf{U}.$$

In particular, the variety $(\mathbf{SI} \vee \mathbf{U}) \wedge \mathbf{COM}$ satisfies the identity (1.1). Then there is a sequence of words $u_0, u_1, \dots, u_n$ such that $u_0 \equiv x^2y$, $u_n \equiv xy^2$, and, for each $i = 0, 1, \dots, n-1$, the identity $u_i = u_{i+1}$ holds in one of the varieties $\mathbf{SI} \vee \mathbf{U}$ or $\mathbf{COM}$. Of course, we may assume that the words $u_0, u_1, \dots, u_n$ are pairwise different. Since $u_0 \in W_1$ and $u_n \notin W_1$, there is an index $i > 0$ such that $u_{i-1} \in W_1$ while $u_i \notin W_1$. The identity $u_{i-1} = u_i$ holds in one of the varieties $\mathbf{SI} \vee \mathbf{U}$ or $\mathbf{COM}$. Lemma 2.2(iv) implies that if the variety $\mathbf{COM}$ satisfies an identity of the form $u = v$ with $u \in W_1$ then $v \in W_1$ as well. Therefore the identity $u_{i-1} = u_i$ is false in $\mathbf{COM}$, whence it holds in the variety $\mathbf{SI} \vee \mathbf{U}$. In particular $u_{i-1} = u_i$ holds in $\mathbf{SI}$. Then $u_i \notin W_2$, and therefore $u_i \notin W$. Since the identity $u_{i-1} = u_i$ holds in the variety $\mathbf{U}$, Lemma 3.1 applies with the conclusion that $\mathbf{U}$ satisfies the identity (1.2). But this is not the case.

This completes Case 2.

We have proved that the variety $\mathbf{V}$ is periodic. Put $\mathbf{N} = \mathrm{Nil}(\mathbf{V})$. One can verify that the variety $\mathbf{N}$ is commutative. Arguing by contradiction, suppose that the commutative law fails in $\mathbf{N}$. In particular, this means that the variety $\mathbf{V}$ is non-commutative. Put $\mathbf{V}' = \mathbf{V} \wedge \mathbf{COM}$ and $\mathbf{N}' = \mathrm{Nil}(\mathbf{V}')$. The exponent of a periodic group variety $\mathbf{X}$ will be denoted by $\exp(\mathbf{X})$. Let $\mathbf{G}$ be a non-abelian periodic group variety such that $\exp(\mathbf{G})$ is co-prime with $\exp\bigl(\mathrm{Gr}(\mathbf{V})\bigr)$. Clearly, $\mathbf{G} \wedge \mathbf{V} = \mathbf{G} \wedge \mathrm{Gr}(\mathbf{V}) = \mathbf{T}$. Since $\mathbf{N}' \subseteq \mathbf{V}$ and the variety $\mathbf{V}$ is upper-modular, we have

$$(\mathbf{G} \vee \mathbf{N}') \wedge \mathbf{V} = (\mathbf{G} \wedge \mathbf{V}) \vee \mathbf{N}' = \mathbf{T} \vee \mathbf{N}' = \mathbf{N}' \subseteq \mathbf{V}' \subseteq \mathbf{COM}.$$

We see that the variety $(\mathbf{G} \vee \mathbf{N}') \wedge \mathbf{V}$ is commutative. Therefore there is a deduction of the identity $xy = yx$ from the identities of the varieties $\mathbf{G} \vee \mathbf{N}'$ and $\mathbf{V}$. In particular there is a word u such that $u \not\equiv xy$ and the identity $xy = u$ holds in either $\mathbf{G} \vee \mathbf{N}'$ or $\mathbf{V}$. Suppose at first that $xy = u$ holds in $\mathbf{G} \vee \mathbf{N}'$. Then Corollary 2.10 applies and we conclude that either $\mathbf{G} \vee \mathbf{N}'$ is commutative or $\mathrm{ind}(\mathbf{G} \vee \mathbf{N}') \le 2$. The former is impossible because the variety $\mathbf{G}$ is non-abelian, and therefore $\mathrm{ind}(\mathbf{G} \vee \mathbf{N}') \le 2$. Since $\mathrm{ind}(\mathbf{COM}) = \infty$, we have

$$\begin{aligned}\mathrm{ind}(\mathbf{V}) &= \min\bigl\{\mathrm{ind}(\mathbf{V}), \mathrm{ind}(\mathbf{COM})\bigr\} = \mathrm{ind}(\mathbf{V} \wedge \mathbf{COM})\\ &= \mathrm{ind}(\mathbf{V}') = \mathrm{ind}\bigl(\mathrm{Nil}(\mathbf{V}')\bigr) = \mathrm{ind}(\mathbf{N}') \le \mathrm{ind}(\mathbf{G} \vee \mathbf{N}') \le 2.\end{aligned}$$

We have proved that $\mathrm{ind}(\mathbf{V}) \le 2$ whenever the identity $xy = u$ holds in the variety $\mathbf{G} \vee \mathbf{N}'$. If $xy = u$ holds in $\mathbf{V}$ then Corollary 2.10 and the fact that the variety $\mathbf{V}$ is non-commutative imply that $\mathrm{ind}(\mathbf{V}) \le 2$ as well. We see that this inequality holds in all cases. Therefore $\mathbf{N} = \mathrm{Nil}(\mathbf{V}) \subseteq \mathbf{ZM}$, contradicting the hypothesis that $\mathbf{N}$ is non-commutative.

We have proved that the variety $\mathbf{N}$ is commutative. It remains to verify that this variety satisfies the identity (1.1). Recall that we denote by $\mathbf{U}$ the variety $\mathrm{var}\{x^2y = xy^2,\ xy = yx,\ x^2yz = 0\}$. Put $\mathbf{N}^* = \mathbf{N} \wedge \mathbf{U}$. Let $\mathbf{G}$ be a non-trivial periodic group variety such that $\exp(\mathbf{G})$ is co-prime with $\exp\bigl(\mathrm{Gr}(\mathbf{V})\bigr)$. Since $\mathbf{V}$ is upper-modular and $\mathbf{N}^* \subseteq \mathbf{V}$, we have

$$(\mathbf{G} \vee \mathbf{N}^*) \wedge \mathbf{V} = (\mathbf{G} \wedge \mathbf{V}) \vee \mathbf{N}^* = \mathbf{T} \vee \mathbf{N}^* = \mathbf{N}^* \subseteq \mathbf{U}.$$

In particular, the variety $(\mathbf{G} \vee \mathbf{N}^*) \wedge \mathbf{V}$ satisfies the identity (1.1). Hence there is a sequence of words $u_0, u_1, \ldots, u_n$ such that $u_0 \equiv x^2y$, $u_n \equiv xy^2$, and, for all $i = 0, 1, \ldots, n-1$, the identity $u_i = u_{i+1}$ holds in one of the varieties $\mathbf{G} \vee \mathbf{N}^*$ or $\mathbf{V}$. Since $u_0 \in W_1$ and $u_n \notin W_1$, there is an index $i > 0$ such that $u_{i-1} \in W_1$ while $u_i \notin W_1$. The identity $u_{i-1} = u_i$ holds in one of the varieties $\mathbf{G} \vee \mathbf{N}^*$ or $\mathbf{V}$. Suppose that $u_{i-1} = u_i$ in $\mathbf{V}$. Then, in particular, the identity $u_{i-1} = u_i$ holds in the variety $\mathbf{N}$. The variety $\mathbf{N}$ is commutative. Therefore it satisfies all identities of the form $w_1 = x^2y$ with $w_1 \in W_1$ and $w_2 = xy^2$ with $w_2 \in W_2$. So, if $u_i \in W_2$ then $\mathbf{N}$ satisfies the identity (1.1). Furthermore, if $u_i \notin W_2$ then $u_i \notin W$. According to Lemma 3.1 we conclude that $\mathbf{N}$ satisfies the identity (1.2). Therefore $xy^2 = 0$ in $\mathbf{N}$, whence $\mathbf{N}$ satisfies the identity (1.1) as well. We have shown that if $u_{i-1} = u_i$ holds in $\mathbf{V}$ then $\mathbf{N}$ satisfies the identity (1.1). Suppose now that $u_{i-1} = u_i$ holds in $\mathbf{G} \vee \mathbf{N}^*$. If $u_i \in W_2$, then the variety $\mathbf{G} \vee \mathbf{N}^*$ satisfies the identity $u_{i-1} = u_i$ where $u_{i-1} \in W_1$ and $u_i \in W_2$. In particular, this identity holds in the variety $\mathbf{G}$. Substituting 1 for y in the identity $u_{i-1} = u_i$, we obtain that $x^2 = x$ in $\mathbf{G}$. But this is not the case because the variety $\mathbf{G}$ is non-trivial. Thus $u_i \notin W_2$, whence $u_i \notin W$. The identity $u_{i-1} = u_i$ holds in the variety $\mathbf{N}^*$. By Lemma 3.1 $u_{i-1} = 0$ in $\mathbf{N}^*$. Since the variety $\mathbf{N}^*$ is commutative, we have that it satisfies the identity (1.2). Recall that $\mathbf{N}^* = \mathbf{N} \wedge \mathbf{U}$. Hence there is a sequence of words $v_0, v_1, \ldots, v_m$ such that $v_0 \equiv x^2y$ and each of the identities $v_0 = v_1$, $v_1 = v_2$, $\ldots$, $v_{m-1} = v_m$, and $v_m = 0$ holds in one of the varieties $\mathbf{N}$ or $\mathbf{U}$. Let $v_0, v_1, \ldots, v_m$ be the shortest sequence with these properties. Then neither of the varieties $\mathbf{N}$ and $\mathbf{U}$ satisfies the identity $v_i = 0$ for each $i = 0, 1, \ldots, m-1$. Since $v_0 \in W$, Lemma 3.1 implies that $v_1, \ldots, v_m \in W$. Thus, one of the varieties $\mathbf{N}$ or $\mathbf{U}$ satisfies the identity $v_m = 0$ and $v_m \in W$. If $v_m = 0$ in $\mathbf{N}$ then $\mathbf{N}$ satisfies the identity (1.1). Finally, if $v_m = 0$ in $\mathbf{U}$ then $\mathbf{U}$ satisfies the identity (1.2) but this is not the case.

Theorem 1.1 is proved. □

4. Proof of Theorem 1.2

Necessity. Let $\mathbf{V}$ be a commutative upper-modular semigroup variety. We have to verify that $\mathbf{V}$ satisfies one of the claims (i) or (ii) of Theorem 1.2. By Theorem 1.1, $\mathbf{V}$ is periodic whence, by Lemma 2.5, $\mathbf{V} = \mathbf{G} \vee \mathbf{M} \vee \mathbf{N}$ where $\mathbf{G} = \mathrm{Gr}(\mathbf{V})$, $\mathbf{M} = \mathrm{Nil}^1(\mathbf{V})$, and $\mathbf{N} = \mathrm{Nil}(\mathbf{V})$. Clearly, $\mathbf{G}$ is an abelian periodic group variety and $\mathbf{N}$ is a commutative nil-variety. By Theorem 1.1, $\mathbf{N}$ satisfies the identity (1.1). Now Lemma 2.7 applies with the conclusion that $\mathbf{M} \in \{\mathbf{T}, \mathbf{SL}, \mathbf{C}\}$. If the identity (1.2) holds in $\mathbf{N}$, then the variety $\mathbf{V}$ satisfies the claim (ii) of Theorem 1.2. Suppose now that the identity (1.2) fails in $\mathbf{N}$. It follows from [20, Lemma 7] that $\mathbf{N}$ contains the variety $\mathbf{D} = \mathrm{var}\{xyzt = x^3 = 0,\ x^2y = xy^2,\ xy = yx\}$ in this case. It suffices to verify that $\mathbf{G} = \mathbf{T}$ and $\mathbf{M}$ is one of the varieties $\mathbf{T}$ or $\mathbf{SL}$ because $\mathbf{V}$ satisfies the claim (i) of Theorem 1.2 in this case. Arguing by contradiction, suppose that either $\mathbf{G} \neq \mathbf{T}$ or $\mathbf{M} \notin \{\mathbf{T}, \mathbf{SL}\}$. In the former case $\mathbf{G} = \mathbf{A}_k$ for some $k > 1$, while in the latter case $\mathbf{M} \supseteq \mathbf{C}$ (see Lemma 2.4). Thus $\mathbf{V}$ contains a variety of the kind $\mathbf{X} \vee \mathbf{D}$ where $\mathbf{X}$ is one of the varieties $\mathbf{A}_k$ or $\mathbf{C}$. The variety $\mathbf{X}$ either consists of groups or is generated by a semigroup with unit. Suppose that $\mathbf{X}$ satisfies the identity (1.1).

Substituting 1 for y in this identity, we obtain that $x^2 = x$ holds in $\mathbf{X}$. Since none of the varieties $\mathbf{A}_k$ and $\mathbf{C}$ satisfies $x^2 = x$, the identity (1.1) fails in $\mathbf{X}$. This fact and the proof of [20, Lemma 8] imply that (1.1) fails in $\mathrm{Nil}(\mathbf{X} \vee \mathbf{D})$ as well. But this is impossible because $\mathrm{Nil}(\mathbf{X} \vee \mathbf{D}) \subseteq \mathrm{Nil}(\mathbf{V}) = \mathbf{N}$ and (1.1) holds in $\mathbf{N}$. Necessity is proved.

Sufficiency. We aim to verify that varieties satisfying one of the claims (i) or (ii) of Theorem 1.2 are upper-modular. For varieties satisfying the claim (i), the desired conclusion immediately follows from the following two facts:

- a commutative nil-variety satisfying the identity (1.1) is upper-modular (the 'if' part of [19, Theorem 2][3]);
- a semigroup variety $\mathbf{X}$ is upper-modular if and only if the variety $\mathbf{SL} \vee \mathbf{X}$ is also [19, Corollary 1.5(ii)].

It remains to consider varieties satisfying the claim (ii) of Theorem 1.2. *Throughout the rest of this section* $\mathbf{V} = \mathbf{G} \vee \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{G}$ *is an abelian periodic group variety,* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$, $\mathbf{SL}$ *or* $\mathbf{C}$, *and* $\mathbf{N}$ *satisfies the identities* $xy = yx$ *and* (1.2); *besides that, let* $\mathbf{Y} \subseteq \mathbf{V}$ *and* $\mathbf{Z}$ *be an arbitrary semigroup variety.* We aim to verify that $(\mathbf{Z} \vee \mathbf{Y}) \wedge \mathbf{V} = (\mathbf{Z} \wedge \mathbf{V}) \vee \mathbf{Y}$.

We may assume that the variety $\mathbf{Z}$ is periodic since otherwise

$$\mathbf{Z} \supseteq \mathbf{COM} \supseteq \mathbf{V} \supseteq \mathbf{Y} \ \text{ and } \ (\mathbf{Z} \vee \mathbf{Y}) \wedge \mathbf{V} = \mathbf{V} = (\mathbf{Z} \wedge \mathbf{V}) \vee \mathbf{Y}.$$

Both the varieties $(\mathbf{Z} \vee \mathbf{Y}) \wedge \mathbf{V}$ and $(\mathbf{Z} \wedge \mathbf{V}) \vee \mathbf{Y}$ are commutative and periodic. In view of Lemma 2.5, it suffices to verify the following three equalities:

$$\mathrm{Gr}\big((\mathbf{Z} \vee \mathbf{Y}) \wedge \mathbf{V}\big) = \mathrm{Gr}\big((\mathbf{Z} \wedge \mathbf{V}) \vee \mathbf{Y}\big), \tag{4.1}$$

$$\mathrm{Nil}^1\big((\mathbf{Z} \vee \mathbf{Y}) \wedge \mathbf{V}\big) = \mathrm{Nil}^1\big((\mathbf{Z} \wedge \mathbf{V}) \vee \mathbf{Y}\big), \tag{4.2}$$

$$\mathrm{Nil}\big((\mathbf{Z} \vee \mathbf{Y}) \wedge \mathbf{V}\big) = \mathrm{Nil}\big((\mathbf{Z} \wedge \mathbf{V}) \vee \mathbf{Y}\big). \tag{4.3}$$

The equality (4.1). Lemma 2.6 and the fact that the variety $\mathbf{M} \vee \mathbf{N}$ is combinatorial imply that $\mathbf{G} = \mathrm{Gr}(\mathbf{V})$. The varieties $\mathrm{Gr}\big((\mathbf{Z} \vee \mathbf{Y}) \wedge \mathbf{V}\big)$ and $\mathrm{Gr}\big((\mathbf{Z} \wedge \mathbf{V}) \vee \mathbf{Y}\big)$ are contained in $\mathrm{Gr}(\mathbf{V}) = \mathbf{G}$, whence they are abelian periodic group varieties. Therefore to prove that these two varieties coincide, it suffices to check that they have the same exponent. As usual, we denote by $\mathrm{lcm}\{m, n\}$ (respectively $\gcd\{m, n\}$) the least common multiple (the greatest common divisor) of positive integers m and n. It is easy to see that if $\mathbf{X}_1$ and $\mathbf{X}_2$ are arbitrary periodic semigroup varieties then

$$\exp\big(\mathrm{Gr}(\mathbf{X}_1 \wedge \mathbf{X}_2)\big) = \gcd\big\{\exp\big(\mathrm{Gr}(\mathbf{X}_1)\big), \exp\big(\mathrm{Gr}(\mathbf{X}_2)\big)\big\}, \tag{4.4}$$

$$\exp\big(\mathrm{Gr}(\mathbf{X}_1 \vee \mathbf{X}_2)\big) = \mathrm{lcm}\big\{\exp\big(\mathrm{Gr}(\mathbf{X}_1)\big), \exp\big(\mathrm{Gr}(\mathbf{X}_2)\big)\big\}. \tag{4.5}$$

Put $\mathbf{G}' = \mathrm{Gr}(\mathbf{Y})$, $\mathbf{M}' = \mathrm{Nil}^1(\mathbf{Y})$, and $\mathbf{N}' = \mathrm{Nil}(\mathbf{Y})$. The variety $\mathbf{V}$ is periodic and commutative, whence so is $\mathbf{Y}$. By Lemma 2.5, $\mathbf{Y} = \mathbf{G}' \vee \mathbf{M}' \vee \mathbf{N}'$. Put $\mathbf{Z}' = \mathbf{Z} \vee \mathbf{M}' \vee \mathbf{N}'$. Then $\mathbf{Z} \vee \mathbf{Y} = \mathbf{Z}' \vee \mathbf{G}'$. Since the variety $\mathbf{M}' \vee \mathbf{N}'$ is combinatorial, we may apply Lemma 2.6 and conclude that $\mathrm{Gr}(\mathbf{Z}') = \mathrm{Gr}(\mathbf{Z})$. Put

$$r = \exp\big(\mathrm{Gr}(\mathbf{V})\big),\ s = \exp\big(\mathrm{Gr}(\mathbf{Y})\big) = \exp(\mathbf{G}'),\ t = \exp\big(\mathrm{Gr}(\mathbf{Z})\big) = \exp\big(\mathrm{Gr}(\mathbf{Z}')\big).$$

Applying the equalities (4.4) and (4.5), we have

$$\exp\big(\mathrm{Gr}\big((\mathbf{Z} \vee \mathbf{Y}) \wedge \mathbf{V}\big)\big) = \exp\big(\mathrm{Gr}\big((\mathbf{Z}' \vee \mathbf{G}') \wedge \mathbf{V}\big)\big) = \gcd\big\{\mathrm{lcm}\{t, s\}, r\big\},$$

[3] To prevent a possible confusion, one should note that there is some inaccuracy in the proof of this theorem in [19]. Namely, the claim was made in this proof that $\mathrm{ind}(\mathbf{X}_1 \vee \mathbf{X}_2) = \max\{\mathrm{ind}(\mathbf{X}_1), \mathrm{ind}(\mathbf{X}_2)\}$ for arbitrary semigroup varieties $\mathbf{X}_1$ and $\mathbf{X}_2$. As we have already mentioned in Section 2, the claim is wrong in the general case. Fortunately, it was used in [19] in the case when one of the varieties $\mathbf{X}_1$ and $\mathbf{X}_2$ is a nil-variety only, and the claim is true in this partial case (see Lemma 2.13).

while

$$\exp\bigl(\mathrm{Gr}\bigl((\mathbf{Z}\wedge\mathbf{V})\vee\mathbf{Y}\bigr)\bigr)=\mathrm{lcm}\bigl\{\gcd\{t,r\},s\bigr\}.$$

Note that s divides r because $\mathrm{Gr}(\mathbf{Y})\subseteq\mathrm{Gr}(\mathbf{V})$. It is easy to see that in this situation $\gcd\bigl\{\mathrm{lcm}\{t,s\},r\bigr\}=\mathrm{lcm}\bigl\{\gcd\{t,r\},s\bigr\}$. This completes the proof of the equality (4.1).

To verify the equalities (4.2) and (4.3), we need the following

Lemma 4.1. *The variety* $\mathrm{Nil}(\mathbf{V})$ *satisfies the identities* $xy=yx$ *and* (1.2).

Proof. Since $\mathbf{V}=\mathbf{G}\vee\mathbf{M}\vee\mathbf{N}$, the variety $\mathbf{V}$ satisfies the identity $x^{r+2}y=x^2y$ where $r=\exp(\mathbf{G})$. Now Lemma 2.3(ii) applies with the conclusion that the variety $\mathrm{Nil}(\mathbf{V})$ satisfies the identity (1.2). It remains to take into account that the variety $\mathrm{Nil}(\mathbf{V})$ is commutative because $\mathbf{V}$ is. □

The equality (4.2). Put

$$\mathbf{M}_1=\mathrm{Nil}^1\bigl((\mathbf{Z}\vee\mathbf{Y})\wedge\mathbf{V}\bigr)\ \text{ and }\ \mathbf{M}_2=\mathrm{Nil}^1\bigl((\mathbf{Z}\wedge\mathbf{V})\vee\mathbf{Y}\bigr).$$

It suffices to verify that $\mathbf{M}_1\subseteq\mathbf{M}_2$ because the opposite inclusion is evident. Clearly, $\mathrm{Nil}^1(\mathbf{Y})\subseteq\mathrm{Nil}^1(\mathbf{V})$. Lemmas 2.4, 2.7, and 4.1 imply that each of the varieties $\mathrm{Nil}^1(\mathbf{V})$ and $\mathrm{Nil}^1(\mathbf{Y})$ coincides with one of the varieties $\mathbf{T}$, $\mathbf{SL}$ or $\mathbf{C}$, while $\mathrm{Nil}^1(\mathbf{Z})$ either coincides with one of the varieties $\mathbf{T}$, $\mathbf{SL}$ or contains $\mathbf{C}$. If $\mathrm{Nil}^1(\mathbf{Y})=\mathrm{Nil}^1(\mathbf{V})$ then $\mathbf{M}_1=\mathbf{M}_2$ because $\mathrm{Nil}^1(\mathbf{Y})\subseteq\mathbf{M}_1\subseteq\mathrm{Nil}^1(\mathbf{V})$ and $\mathrm{Nil}^1(\mathbf{Y})\subseteq\mathbf{M}_2\subseteq\mathrm{Nil}^1(\mathbf{V})$. Therefore we may assume that $\mathrm{Nil}^1(\mathbf{Y})\subset\mathrm{Nil}^1(\mathbf{V})$. Hence it remains to consider the following nine cases:

1) $\mathrm{Nil}^1(\mathbf{V})=\mathbf{SL}$, $\mathrm{Nil}^1(\mathbf{Y})=\mathbf{T}$, and $\mathrm{Nil}^1(\mathbf{Z})=\mathbf{T}$;
2) $\mathrm{Nil}^1(\mathbf{V})=\mathbf{SL}$, $\mathrm{Nil}^1(\mathbf{Y})=\mathbf{T}$, and $\mathrm{Nil}^1(\mathbf{Z})=\mathbf{SL}$;
3) $\mathrm{Nil}^1(\mathbf{V})=\mathbf{SL}$, $\mathrm{Nil}^1(\mathbf{Y})=\mathbf{T}$, and $\mathrm{Nil}^1(\mathbf{Z})\supseteq\mathbf{C}$;
4) $\mathrm{Nil}^1(\mathbf{V})=\mathbf{C}$, $\mathrm{Nil}^1(\mathbf{Y})=\mathbf{T}$, and $\mathrm{Nil}^1(\mathbf{Z})=\mathbf{T}$;
5) $\mathrm{Nil}^1(\mathbf{V})=\mathbf{C}$, $\mathrm{Nil}^1(\mathbf{Y})=\mathbf{T}$, and $\mathrm{Nil}^1(\mathbf{Z})=\mathbf{SL}$;
6) $\mathrm{Nil}^1(\mathbf{V})=\mathbf{C}$, $\mathrm{Nil}^1(\mathbf{Y})=\mathbf{T}$, and $\mathrm{Nil}^1(\mathbf{Z})\supseteq\mathbf{C}$;
7) $\mathrm{Nil}^1(\mathbf{V})=\mathbf{C}$, $\mathrm{Nil}^1(\mathbf{Y})=\mathbf{SL}$, and $\mathrm{Nil}^1(\mathbf{Z})=\mathbf{T}$;
8) $\mathrm{Nil}^1(\mathbf{V})=\mathbf{C}$, $\mathrm{Nil}^1(\mathbf{Y})=\mathbf{SL}$, and $\mathrm{Nil}^1(\mathbf{Z})=\mathbf{SL}$;
9) $\mathrm{Nil}^1(\mathbf{V})=\mathbf{C}$, $\mathrm{Nil}^1(\mathbf{Y})=\mathbf{SL}$, and $\mathrm{Nil}^1(\mathbf{Z})\supseteq\mathbf{C}$.

In the cases 1) and 4) $\mathbf{SL}\nsubseteq\mathbf{Z}$ and $\mathbf{SL}\nsubseteq\mathbf{Y}$. It follows from results of [1] that in these cases $\mathbf{SL}\nsubseteq\mathbf{Z}\vee\mathbf{Y}$ too. By Lemma 2.4, we have $\mathbf{M}_1=\mathbf{T}\subseteq\mathbf{M}_2$. In the cases 2) and 3)

$$\mathbf{M}_1\subseteq\mathrm{Nil}^1(\mathbf{V})=\mathbf{SL}\subseteq\mathrm{Nil}^1(\mathbf{Z})\wedge\mathrm{Nil}^1(\mathbf{V})=\mathrm{Nil}^1(\mathbf{Z}\wedge\mathbf{V})\subseteq\mathbf{M}_2.$$

In the cases 5), 7), and 8) $\mathrm{Nil}^1(\mathbf{Y})\subseteq\mathbf{SL}$ and $\mathbf{Z}\nsupseteq\mathbf{C}$. Applying Lemmas 2.5 and 2.15, we conclude that

$$\mathbf{Z}\vee\mathbf{Y}=\mathbf{Z}\vee\mathrm{Gr}(\mathbf{Y})\vee\mathrm{Nil}^1(\mathbf{Y})\vee\mathrm{Nil}(\mathbf{Y})\nsupseteq\mathbf{C},$$

and therefore $\mathbf{M}_1\subseteq\mathrm{Nil}^1(\mathbf{Z}\vee\mathbf{Y})\subseteq\mathbf{SL}$ by Lemma 2.4. On the other hand $\mathbf{SL}\subseteq(\mathbf{Z}\wedge\mathbf{V})\vee\mathbf{Y}$, and therefore $\mathbf{SL}\subseteq\mathbf{M}_2$ in the cases under consideration. Whence $\mathbf{M}_1\subseteq\mathbf{SL}\subseteq\mathbf{M}_2$. Finally, in the cases 6) and 9) $\mathbf{C}\subseteq\mathbf{Z}\wedge\mathbf{V}$, whence $\mathbf{C}\subseteq\mathbf{M}_2$. Therefore $\mathbf{M}_1\subseteq\mathrm{Nil}^1(\mathbf{V})=\mathbf{C}\subseteq\mathbf{M}_2$. The equality (4.2) is proved.

The equality (4.3). Clearly, $\mathbf{V}=\mathbf{G}\vee\mathbf{M}\vee\mathrm{Nil}(\mathbf{V})$. By Lemma 4.1 the variety $\mathrm{Nil}(\mathbf{V})$ is commutative and satisfies the identity (1.2). Therefore we may assume that $\mathbf{N}=\mathrm{Nil}(\mathbf{V})$. Put $\mathbf{E}=\mathrm{var}\{x^2y=0,\,xy=yx\}$. Then both the varieties $\mathrm{Nil}\bigl((\mathbf{Z}\vee\mathbf{Y})\wedge\mathbf{V}\bigr)$ and $\mathrm{Nil}\bigl((\mathbf{Z}\wedge\mathbf{V})\vee\mathbf{Y}\bigr)$ are contained in $\mathrm{Nil}(\mathbf{V})=\mathbf{N}$ and $\mathbf{N}\subseteq\mathbf{E}$. It is evident that if the identity $u=0$ fails in the variety $\mathbf{E}$ then either $u\equiv x^2$ or $u\equiv x_1x_2\cdots x_n$ for some positive integer n. Lemma 2.3 implies now that each proper subvariety of $\mathbf{E}$ is given within $\mathbf{E}$ either by the identity

$$x^2=0 \tag{4.6}$$

or by the identity $x_1x_2\cdots x_n = 0$ for some n or by both these identities. Therefore subvarieties $\mathbf{X}_1$ and $\mathbf{X}_2$ of the variety $\mathbf{E}$ coincide if and only if $\mathrm{ind}(\mathbf{X}_1) = \mathrm{ind}(\mathbf{X}_2)$ and the variety $\mathbf{X}_1$ satisfies the identity (4.6) if and only if the variety $\mathbf{X}_2$ does so.

First, one can verify that

$$\mathrm{ind}\bigl(\mathrm{Nil}\bigl((\mathbf{Z}\vee\mathbf{Y})\wedge\mathbf{V}\bigr)\bigr) = \mathrm{ind}\bigl(\mathrm{Nil}\bigl((\mathbf{Z}\wedge\mathbf{V})\vee\mathbf{Y}\bigr)\bigr). \tag{4.7}$$

Clearly, $\mathrm{ind}\bigl(\mathrm{Nil}(\mathbf{X})\bigr) = \mathrm{ind}(\mathbf{X})$ for an arbitrary periodic variety $\mathbf{X}$. So, we have to check that $\mathrm{ind}\bigl((\mathbf{Z}\vee\mathbf{Y})\wedge\mathbf{V}\bigr) = \mathrm{ind}\bigl((\mathbf{Z}\wedge\mathbf{V})\vee\mathbf{Y}\bigr)$. Moreover, it suffices to verify that $\mathrm{ind}\bigl((\mathbf{Z}\vee\mathbf{Y})\wedge\mathbf{V}\bigr) \le \mathrm{ind}\bigl((\mathbf{Z}\wedge\mathbf{V})\vee\mathbf{Y}\bigr)$ because the opposite inequality is guaranteed by the inclusion $(\mathbf{Z}\wedge\mathbf{V})\vee\mathbf{Y} \subseteq (\mathbf{Z}\vee\mathbf{Y})\wedge\mathbf{V}$. Arguing by contradiction, suppose that

$$\mathrm{ind}\bigl((\mathbf{Z}\vee\mathbf{Y})\wedge\mathbf{V}\bigr) > \mathrm{ind}\bigl((\mathbf{Z}\wedge\mathbf{V})\vee\mathbf{Y}\bigr). \tag{4.8}$$

Clearly, $(\mathbf{Z}\wedge\mathbf{V})\vee\mathbf{Y}$ is a variety of finite index in this case.

Put $\mathbf{G}' = \mathrm{Gr}(\mathbf{Y})$, $\mathbf{M}' = \mathrm{Nil}^1(\mathbf{Y})$, and $\mathbf{N}' = \mathrm{Nil}(\mathbf{Y})$. Note that $\mathbf{Y}$ is a variety of finite index because $(\mathbf{Z}\wedge\mathbf{V})\vee\mathbf{Y}$ is. Since $\mathrm{ind}(\mathbf{C}) = \infty$, we have $\mathbf{M}' \not\supseteq \mathbf{C}$. By Lemma 2.4 this means that $\mathbf{M}' \subseteq \mathbf{SL}$. Therefore the variety $\mathbf{G}'\vee\mathbf{M}'$ is completely regular. Applying Lemma 2.5, we have

$$\mathbf{Z}\vee\mathbf{Y} = \mathbf{Z}\vee(\mathbf{G}'\vee\mathbf{M}')\vee\mathbf{N}' \ \text{ and } \ (\mathbf{Z}\wedge\mathbf{V})\vee\mathbf{Y} = (\mathbf{Z}\wedge\mathbf{V})\vee(\mathbf{G}'\vee\mathbf{M}')\vee\mathbf{N}'.$$

Put $n = \mathrm{ind}(\mathbf{Z})$, $m = \mathrm{ind}(\mathbf{N}') = \mathrm{ind}(\mathbf{Y})$, and $\ell = \mathrm{ind}(\mathbf{V})$. Since the variety $\mathbf{G}'\vee\mathbf{M}'$ is completely regular and $\mathbf{N}'$ is a nil-variety, we may apply Lemmas 2.12 and 2.13. Since $\mathrm{ind}(\mathbf{X}_1\wedge\mathbf{X}_2) = \min\bigl\{\mathrm{ind}(\mathbf{X}_1), \mathrm{ind}(\mathbf{X}_2)\bigr\}$ for arbitrary semigroup varieties $\mathbf{X}_1$ and $\mathbf{X}_2$, we have

$$\mathrm{ind}\bigl((\mathbf{Z}\vee\mathbf{Y})\wedge\mathbf{V}\bigr) = \min\bigl\{\max\{n,m\},\ell\bigr\}$$

and

$$\mathrm{ind}\bigl((\mathbf{Z}\wedge\mathbf{V})\vee\mathbf{Y}\bigr) = \max\bigl\{\min\{n,\ell\},m\bigr\}.$$

Clearly, $m \le \ell$ because $\mathbf{Y} \subseteq \mathbf{V}$. Evidently,

$$\min\bigl\{\max\{n,m\},\ell\bigr\} = \max\bigl\{\min\{n,\ell\},m\bigr\} = \begin{cases} m, & \text{whenever } n \le m \le \ell,\\ n, & \text{whenever } m < n \le \ell,\\ \ell, & \text{whenever } m \le \ell < n.\end{cases}$$

Whence $\mathrm{ind}\bigl((\mathbf{Z}\vee\mathbf{Y})\wedge\mathbf{V}\bigr) = \mathrm{ind}\bigl((\mathbf{Z}\wedge\mathbf{V})\vee\mathbf{Y}\bigr)$ that contradicts (4.8). The equality (4.7) is proved.

It remains to verify that the variety $\mathrm{Nil}\bigl((\mathbf{Z}\vee\mathbf{Y})\wedge\mathbf{V}\bigr)$ satisfies the identity (4.6) if and only if the variety $\mathrm{Nil}\bigl((\mathbf{Z}\wedge\mathbf{V})\vee\mathbf{Y}\bigr)$ does so. It suffices to check that the variety $\mathrm{Nil}\bigl((\mathbf{Z}\vee\mathbf{Y})\wedge\mathbf{V}\bigr)$ satisfies the identity (4.6) whenever $\mathrm{Nil}\bigl((\mathbf{Z}\wedge\mathbf{V})\vee\mathbf{Y}\bigr)$ does so (because the opposite claim is evident). Suppose that the identity (4.6) holds in $\mathrm{Nil}\bigl((\mathbf{Z}\wedge\mathbf{V})\vee\mathbf{Y}\bigr)$. Since the variety $(\mathbf{Z}\wedge\mathbf{V})\vee\mathbf{Y}$ is periodic, it satisfies an identity of the form $x^k = x^m$ for some positive integers k and m with $m > k$. Let k be the least number with such a property. By Lemma 2.3(ii), $\mathrm{Nil}\bigl((\mathbf{Z}\wedge\mathbf{V})\vee\mathbf{Y}\bigr)$ satisfies the identity $x^k = 0$; clearly, k is the least number with such a property. Therefore $k = 2$. Thus $(\mathbf{Z}\wedge\mathbf{V})\vee\mathbf{Y}$ satisfies an identity of the form $x^2 = x^m$ for some $m > 2$. In particular, this identity holds in both the varieties $\mathbf{Y}$ and $\mathbf{Z}\wedge\mathbf{V}$. Therefore there is a sequence of words $u_0, u_1, \dots, u_k$ such that $u_0 \equiv x^2, u_k \equiv x^m$, and, for each $i = 0, 1, \dots, k-1$, the identity $u_i = u_{i+1}$ holds in one of the varieties $\mathbf{Z}$ or $\mathbf{V}$. We may assume that $u_i \not\equiv u_{i+1}$ for each $i = 0, 1, \dots, k-1$. If $c(u_1) \ne \{x\}$ then by Lemma 2.3(i) the identity (4.6) holds in one of the varieties $\mathrm{Nil}(\mathbf{Z})$ or $\mathrm{Nil}(\mathbf{V})$. Now let $c(u_1) = \{x\}$. Then $u_1 \equiv x^s$ for some $s \ne 2$ and one of the varieties $\mathbf{Z}$ or $\mathbf{V}$ satisfies the identity $x^2 = x^s$. Clearly, we may assume that $s > 2$ (in the contrary case $s = 1$ but $x^2 = x$ implies $x^3 = x^2$). By Lemma 2.3(ii) this implies that (4.6) holds in one of the varieties $\mathrm{Nil}(\mathbf{Z})$ or $\mathrm{Nil}(\mathbf{V})$ again. If (4.6) holds in $\mathrm{Nil}(\mathbf{V})$ then the variety $\mathrm{Nil}\bigl((\mathbf{Z}\vee\mathbf{Y})\wedge\mathbf{V}\bigr)$ satisfies the identity (4.6) too. Finally, let (4.6) hold in $\mathrm{Nil}(\mathbf{Z})$. As above, this implies that $\mathbf{Z}$ satisfies an identity of the form $x^2 = x^s$ for some $s > 2$. Recall that the variety $\mathbf{Y}$ satisfies the identity $x^2 = x^m$ for some $m > 2$. Therefore the variety $\mathbf{Z} \vee \mathbf{Y}$ satisfies $x^2 = x^{2+(s-2)(m-2)}$. Now Lemma 2.3(ii) applies and we conclude that the variety $\mathrm{Nil}(\mathbf{Z} \vee \mathbf{Y})$ satisfies the identity (4.6), whence the variety $\mathrm{Nil}\big((\mathbf{Z} \vee \mathbf{Y}) \wedge \mathbf{V}\big)$ does so.

The equality (4.3) is proved. This completes the proof of sufficiency in Theorem 1.2 and of Theorem 1.2 as a whole. □

Theorems 1.1 and 1.2 imply the following two corollaries.

Corollary 4.2 ([19, Theorem 2])**.** *A nil-variety is an upper-modular element of the lattice* $\mathbb{SEM}$ *if and only if it satisfies the identities* $xy = yx$ *and* (1.1). □

Corollary 4.3. *If a proper semigroup variety is an upper-modular element of the lattice* $\mathbb{SEM}$ *then every nil-subvariety is also.* □

Theorem 1.2 and results of the article [21] imply the following

Corollary 4.4. *If a commutative semigroup variety* $\mathbf{V}$ *is an upper-modular element of the lattice* $\mathbb{SEM}$ *then the lattice* $L(\mathbf{V})$ *is distributive.* □

Theorem 1.2 evidently implies also the following

Corollary 4.5. *If a commutative semigroup variety is an upper-modular element of the lattice* $\mathbb{SEM}$ *then every subvariety is also.* □

Note that Lemma 2.1 permits one to deduce Corollary 4.5 from Corollary 4.4 without direct reference to Theorem 1.2.

5. Open questions

It follows from [6, Proposition 1.6] that there are only two completely regular modular varieties, namely the varieties $\mathbf{T}$ and $\mathbf{SL}$. It is proved in [16] that an arbitrary abelian periodic group variety is not lower-modular. In contrast with these facts, Theorem 1.2 shows that every commutative completely regular variety is upper-modular (because, as follows from Lemma 2.5, commutative completely regular varieties are exhausted by the varieties $\mathbf{T}$, $\mathbf{A}_k$, $\mathbf{SL}$, and $\mathbf{A}_k \vee \mathbf{SL}$). Varieties $\mathrm{var}\{xy = x\}$ and $\mathrm{var}\{xy = y\}$ provide examples of non-commutative completely regular upper-modular varieties. Indeed, these varieties are known to be minimal non-trivial semigroup varieties, whence they are upper-modular. Moreover, we do not know whether or not a completely regular but not upper-modular variety exists.

Question 5.1. *Is every completely regular semigroup variety an upper-modular element of the lattice* $\mathbb{SEM}$?

For a positive integer r, put $\mathbf{K}_r = \mathrm{var}\{x = x^{r+1}\}$. According to Lemma 2.8, if $\mathbf{V}$ is a completely regular semigroup variety then $\mathbf{V} \subseteq \mathbf{K}_r$ for some r. The lattice of all completely regular varieties is known to be modular [10]. In view of Lemma 2.1, Question 5.1 is equivalent to the following

Question 5.1′. *Let* r *be a positive integer. Is the variety* $\mathbf{K}_r$ *an upper-modular element of the lattice* $\mathbb{SEM}$?

The natural first step is to consider the case when $r = 1$. The variety $\mathbf{K}_1$ is nothing but the variety of all bands (idempotent semigroups). We denote this variety by $\mathbf{B}$. We go to the following

Question 5.2. *Is the variety* $\mathbf{B}$ *an upper-modular element of the lattice* $\mathbb{SEM}$?

In other words, we are interested, whether or not the implication

$$\mathbf{Y} \subseteq \mathbf{B} \longrightarrow (\mathbf{Z} \vee \mathbf{Y}) \wedge \mathbf{B} = (\mathbf{Z} \wedge \mathbf{B}) \vee \mathbf{Y} \tag{5.1}$$

holds for arbitrary semigroup varieties $\mathbf{Y}$ and $\mathbf{Z}$. Recall that a semigroup variety is called *locally finite* if all its finitely generated members are finite. It follows from [11, Corollary 5.9] that if $\mathbf{Y}$ and $\mathbf{Z}$ are locally finite semigroup varieties then

$$(\mathbf{Z} \vee \mathbf{Y}) \wedge \mathbf{B} = (\mathbf{Z} \wedge \mathbf{B}) \vee (\mathbf{Y} \wedge \mathbf{B}).$$

It is well known that an arbitrary variety of bands is locally finite. Therefore the implication (5.1) holds whenever the variety $\mathbf{Z}$ is locally finite.

Semigroup varieties that are both modular and lower-modular were completely described in [22, Theorem 3.1], while varieties that are both modular and upper-modular were completely determined in [19, Theorem 1]. This inspires the following

Problem 5.3. *Describe semigroup varieties that are both upper-modular and lower-modular elements of the lattice* $\mathbb{SEM}$.

In [16] this problem is solved within the classes of commutative varieties and nil-varieties; besides that, it is proved there that if a proper semigroup variety is both upper-modular and lower-modular then it is a variety of index ≤ 2.

Analogues of Corollaries 4.4 and 4.5 for arbitrary semigroup varieties fail: an evident example is provided by the variety **SEM**. But we do not know whether or not other examples of such a type there exist.

Question 5.4. *Let* $\mathbf{V}$ *be a proper semigroup variety and an upper-modular element of the lattice* $\mathbb{SEM}$*. Is the lattice* $L(\mathbf{V})$*:*

a) *distributive;*
b) *modular*?

Since the lattice of all completely regular varieties is not distributive, Question 5.4a) is answered in negative if the answer to Question 5.1 is affirmative.

In view of Lemma 2.1, Question 5.4b) is equivalent to the following

Question 5.4′. *Let* $\mathbf{V}$ *be a proper semigroup variety and an upper-modular element of the lattice* $\mathbb{SEM}$*. Is an arbitrary subvariety of* $\mathbf{V}$ *an upper-modular element of this lattice*?

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