Upper-modular elements of the lattice of semigroup varieties. II*

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Abstract

A semigroup variety is called a *variety of degree* ≤ 2 if all its nilsemigroups are semigroups with zero multiplication, and a *variety of degree* > 2 otherwise. We completely determine all semigroup varieties of degree > 2 that are upper-modular elements of the lattice of all semigroup varieties and find quite a strong necessary condition for semigroup varieties of degree ≤ 2 to have the same property.

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As is well known, the lattice **SEM** of all semigroup varieties is not modular. Semigroup varieties with modular subvariety lattice have been completely determined [14]. Speaking informally, this result indicates the zones of "global modularity" in the lattice **SEM**. In order to investigate the phenomenon of modularity in **SEM**, the next natural step is to consider varieties that guarantee a sort of local modularity in their environs. Saying so, we take in mind an examination of modular elements of the lattice **SEM** and other types of its elements whose definition is based on the modular law. Recall that an element x of a lattice $\langle L;\vee,\wedge\rangle$ is called *modular* if

$$\forall\, y,z\in L\colon\ y\leq z\longrightarrow (x\vee y)\wedge z=(x\wedge z)\vee y,$$

and *upper-modular* if

$$\forall\, y,z\in L\colon\ y\leq x\longrightarrow (z\vee y)\wedge x=(z\wedge x)\vee y.$$

Lower-modular elements are defined dually to upper-modular ones. A semigroup variety is called *modular* [*upper-modular, lower-modular*] if it is a modular [upper-modular, lower-modular] element of the lattice **SEM**. First results concerning modular and lower-modular varieties have appeared in the articles [3,11] where they have played an auxiliary role. The recent articles [7–10, 12, 16] are devoted to a systematic examination of modular, upper-modular and lower-modular varieties. A brief overview of results of these papers can be found in

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the survey article [6]. In particular, upper-modular nil-varieties have been described in [12], while a necessary condition for a semigroup variety to be upper-modular has been obtained and commutative upper-modular varieties have been classified in [10] (see Propositions 1 and 2 below). This note is a direct continuation of the article [10]. We completely determine here upper-modular varieties containing at least one nilsemigroup that is not a semigroup with zero multiplication (Theorem 1). We obtain also quite a strong necessary condition for varieties all whose nilsemigroups are semigroups with zero multiplication to be upper-modular (Theorem 2). These results imply affirmative answers to two questions posed in [10] (see Corollaries 2 and 3).

We need some definitions and notation. As is well known, any periodic semigroup variety $\mathcal{V}$ contains the greatest nilsubvariety that will be denoted by $\mathrm{Nil}(\mathcal{V})$. It is clear that a semigroup S satisfies an identity system of the form $wx = xw = w$, where w is a word and x is a letter that does not occur in w, if and only if S contains the zero element 0 and all values of the word w in S equal 0. As usual, we will write this identity system in the brief form $w = 0$ and refer to the equality $w = 0$ as to a usual identity. We denote by $\mathcal{T}$ the trivial variety and by $\mathcal{SEM}$ the variety of all semigroups. The notation $\mathrm{var}\,\Sigma$ stands for the semigroup variety given by the identity system Σ. Put $\mathcal{SL} = \mathrm{var}\{x^2 = x,\ xy = yx\}$ and $\mathcal{C} = \mathrm{var}\{x^2 = x^3,\ xy = yx\}$. We will use the following two results.

Proposition 1 ([10, Theorem 1.1]). *If a semigroup variety $\mathcal{V}$ is upper-modular then either $\mathcal{V} = \mathcal{SEM}$ or $\mathcal{V}$ is a periodic variety and the variety $\mathrm{Nil}(\mathcal{V})$ satisfies the identities $x^2y = xy^2$ and $xy = yx$.* □

Proposition 2 ([10, Theorem 1.2]). *A commutative semigroup variety $\mathcal{V}$ is upper-modular if and only if one of the following holds*:

(i) $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ *where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$, while $\mathcal{N}$ is a nil-variety satisfying the identities $x^2y = xy^2$ and $xy = yx$*;

(ii) $\mathcal{V} = \mathcal{G} \vee \mathcal{M} \vee \mathcal{N}$ *where $\mathcal{G}$ is an abelian periodic group variety, $\mathcal{M}$ is one of the varieties $\mathcal{T}$, $\mathcal{SL}$ or $\mathcal{C}$, while $\mathcal{N}$ is a variety satisfying the identities $x^2y = 0$ and $xy = yx$.* □

Let n be a natural number. A semigroup variety is said to be a *variety of degree n* if all its nilsemigroups are nilpotent of degree $\le n$ and n is the least number with such a property. Varieties that are not varieties of degree $\le n$ will be called *varieties of degree $> n$* (in particular, a variety containing a non-nilpotent nilsemigroup is a variety of degree $> n$ for any n). A semigroup variety is called *proper* if it differs from the variety $\mathcal{SEM}$. The first of two main results of this note is the following

Theorem 1. *A semigroup variety $\mathcal{V}$ of degree > 2 is upper-modular if and only if either $\mathcal{V} = \mathcal{SEM}$ or one of the conditions* (i) *or* (ii) *of Proposition* 2 *holds.*

Proof. Let $\mathcal{V}$ be a proper upper-modular semigroup variety of degree > 2. In view of Proposition 2 it suffices to verify that the variety $\mathcal{V}$ is commutative.

As is well known, an arbitrary variety of degree > 2 contains the variety $\mathcal{N}_3 = \mathrm{var}\{xyz = x^2 = 0,\ xy = yx\}$. Further, Proposition 1 implies that $\mathcal{V}$ is a periodic variety. Therefore $\mathcal{V}$ contains the greatest group subvariety that will be denoted by $\mathrm{Gr}(\mathcal{V})$. Let $\mathcal{G}$ be an arbitrary non-abelian periodic group variety whose exponent is co-prime with the exponent of the variety $\mathrm{Gr}(\mathcal{V})$. Since the variety $\mathcal{V}$ is upper-modular and $\mathcal{N}_3 \subseteq \mathcal{V}$, we have

$$(\mathcal{G} \vee \mathcal{N}_3) \wedge \mathcal{V} = (\mathcal{G} \wedge \mathcal{V}) \vee \mathcal{N}_3 = \big(\mathcal{G} \wedge \mathrm{Gr}(\mathcal{V})\big) \vee \mathcal{N}_3 = \mathcal{T} \vee \mathcal{N}_3 = \mathcal{N}_3.$$

In particular, the variety $(\mathcal{G} \vee \mathcal{N}_3) \wedge \mathcal{V}$ is commutative. Hence there exists a deduction of the identity $xy = yx$ from the identities of the varieties $\mathcal{G} \vee \mathcal{N}_3$ and $\mathcal{V}$. In particular, one of these varieties satisfies a non-trivial identity of the form $xy = w$. It is easy to check that if some variety $\mathcal{X}$ satisfies such an identity then $\mathcal{X}$ is either commutative or a variety of degree ≤ 2 (see [10, Lemma 2.10], for instance). But it is evident that both the varieties $\mathcal{G} \vee \mathcal{N}_3$ and $\mathcal{V}$ are varieties of degree > 2 and the former variety is non-commutative. Therefore, the variety $\mathcal{V}$ is commutative. $\square$

As usual, we denote by $L(\mathcal{V})$ the subvariety lattice of a variety $\mathcal{V}$. Theorem 1 and results of the article [13] imply

Corollary 1. *If $\mathcal{V}$ is a proper upper-modular semigroup variety of degree > 2 then the lattice $L(\mathcal{V})$ is distributive.* $\square$

Theorem 1 reduces the problem of description of upper-modular varieties to a consideration of varieties of degree ≤ 2. Note that, in contrast with the case of proper varieties of degree > 2, there exist non-commutative upper-modular varieties of degree ≤ 2. Simplest examples of such varieties provide the variety $\mathcal{LZ}$ of all left zero semigroups and the variety $\mathcal{RZ}$ of all right zero semigroups. Indeed, these two varieties are well known to be atoms of the lattice **SEM**, whence they are upper-modular.

It is known that a semigroup variety is a variety of degree ≤ 2 if and only if it satisfies one of the identities

$$xy = (xy)^{r+1}, \tag{1}$$

$$xy = x^{r+1}y, \tag{2}$$

$$xy = xy^{r+1} \tag{3}$$

for some natural r (see [2, Lemma 3] or [10, Proposition 2.11]). If a variety $\mathcal{V}$ satisfies the identity (1) then the square of any semigroup in $\mathcal{V}$ satisfies the identity $x = x^{r+1}$. As is known, this identity holds in a semigroup S if and only if S is *completely regular* (i. e. a union of groups). By this reason varieties satisfying the identity (1) are called *varieties of semigroups with completely regular square*. Put $\mathcal{P} = \mathrm{var}\{xy = x^2y,\ x^2y^2 = y^2x^2\}$. The variety dual to $\mathcal{P}$ is denoted by $\overleftarrow{\mathcal{P}}$. Note that the varieties $\mathcal{P}$ and $\overleftarrow{\mathcal{P}}$ satisfy the identities $xyz = yxz$ and $xyz = xzy$ respectively. The second main result of this article is

Theorem 2. *If $\mathcal{V}$ is an upper-modular semigroup variety of degree ≤ 2 then one of the following holds*:

(i) $\mathcal{V}$ *is a variety of semigroups with completely regular square*;

(ii) $\mathcal{V} = \mathcal{K} \vee \mathcal{P}$ *where* $\mathcal{K}$ *is a completely regular variety with* $\mathcal{RZ} \nsubseteq \mathcal{K}$;

(iii) $\mathcal{V} = \mathcal{K} \vee \overleftarrow{\mathcal{P}}$ *where* $\mathcal{K}$ *is a completely regular variety with* $\mathcal{LZ} \nsubseteq \mathcal{K}$.

Proof. We need some notation. For any prime number p we denote by $\mathcal{A}_p$ the variety of all abelian groups of exponent dividing p, and by $\mathcal{CSA}_p$ the variety of all completely simple semigroups over groups from $\mathcal{A}_p$. Put

$$\begin{aligned}
\mathcal{LSNB} &= \mathrm{var}\{x^2 = x,\, xyz = xyzxz\},\\
\mathcal{Q} &= \mathrm{var}\{xy = xy^2,\, xyz^2 = yxz^2,\, xyx = yx^2\},\\
\mathcal{RRB} &= \mathrm{var}\{x^2 = x,\, xy = xyx\},\\
\mathcal{RZM} &= \{xyz = yz\}.
\end{aligned}$$

Note that $\mathcal{P} \subseteq \mathcal{Q}$. We formulate several auxiliary statements now. Lemmas 2 and 3 of the paper [15] imply

Lemma 1. *If* $\mathcal{X}$ *is one of the varieties* $\mathcal{CSA}_p$, $\mathcal{LSNB}$, $\mathcal{RRB}$ *or* $\mathcal{RZM}$ *then* $\mathcal{X} \vee \mathcal{P} \supseteq \mathcal{Q}$. □

Lemma 2 ([15, Lemma 7])**.** *If a semigroup variety satisfies the identity* (2) *but does not satisfy the identity* (1) *then it contains the variety* $\mathcal{P}$. □

Any periodic semigroup variety $\mathcal{X}$ contains the greatest completely regular subvariety. We will denote this subvariety by $\mathrm{CR}(\mathcal{X})$. Put $\mathcal{ZM} = \mathrm{var}\{xy = 0\}$. The following lemma is implied by [2, Lemma 4] and [15, Lemma 14].

Lemma 3. *If a semigroup variety* $\mathcal{X}$ *satisfies the identity* (2) *and does not contain the variety* $\mathcal{RZM}$ *then* $\mathcal{X} = \mathrm{CR}(\mathcal{X}) \vee \mathcal{M}$ *where* $\mathcal{M}$ *is one of the varieties* $\mathcal{T}$, $\mathcal{ZM}$, $\mathcal{P}$ *or* $\mathcal{Q}$. □

For a word u and a letter x, we denote by $c(u)$ the set of all letters occurring in u, by $\ell(u)$ the length of u, by $\ell_x(u)$ the number of occurrences of x in u, and by $t(u)$ the last letter of u. The claims (i)–(iii) of the following lemma are well known and can be easily verified. The claim (iv) was proved in [2, Lemma 7].

Lemma 4. *The identity* $u = v$ *holds in the variety*:

(i) $\mathcal{RZ}$ *if and only if the letters* $t(u)$ *and* $t(v)$ *coincide*;

(ii) $\mathcal{SL}$ *if and only if* $c(u) = c(v)$;

(iii) $\mathcal{C}$ *if and only if* $c(u) = c(v)$ *and, for every letter* $x \in c(u)$*, either* $\ell_x(u) = \ell_x(v) = 1$ *or* $\ell_x(u), \ell_x(v) > 1$;

(iv) $\mathcal{P}$ *if and only if* $c(u) = c(v)$ *and either* $\ell_{t(u)}(u), \ell_{t(v)}(v) > 1$ *or* $\ell_{t(u)}(u) = \ell_{t(v)}(v) = 1$ *and the letters* $t(u)$ *and* $t(v)$ *coincide.* □

Lemma 5. $\mathcal{C} \vee \mathcal{RZ} \supseteq \mathcal{P}$.

Proof. Let an identity $u = v$ hold in the variety $\mathcal{C} \vee \mathcal{RZ}$. It is evident that $u = v$ holds in the varieties $\mathcal{RZ}$ and $\mathcal{SL}$. The claims (i) and (ii) of Lemma 4 imply now that $c(u) = c(v)$ and the letters $t(u)$ and $t(v)$ coincide. Besides that, the identity $u = v$ holds in the variety $\mathcal{C}$. In view of Lemma 4(iii) this means that every letter from $c(u)$ either occurs in both the words u and v once or occurs in both these words more than once. All the saying together with Lemma 4(iv) show that the identity $u = v$ holds in the variety $\mathcal{P}$. □

For an element a of a lattice L, we denote by $(a]$ the *principal ideal* of the lattice L generated by a; in other words, $(a] = \{x \in L \mid x \leq a\}$. Simple lattice-theoretical arguments permit to verify the following

Lemma 6 ([10, Lemma 2.1])**.** *Let L be a lattice and $a \in L$ an upper-modular element. The lattice $(a]$ is modular if and only if every element of this lattice is an upper-modular element of L.* □

With all the above preliminaries in hand, we proceed with the proof of the claim of Theorem 2. Let $\mathcal{V}$ be an upper-modular semigroup variety of degree ≤ 2. Suppose that $\mathcal{V} \supseteq \mathcal{Q}$. By Proposition 1, $\mathcal{V}$ is periodic. Let p be an arbitrary prime number which does not divide the exponent of the variety $\mathrm{Gr}(\mathcal{V})$. Then $\mathcal{A}_p \nsubseteq \mathcal{V}$. As is well known, the lattice $L(\mathcal{CSA}_p)$ has the form shown in Fig. 1. Therefore $\mathcal{CSA}_p \wedge \mathcal{V} \subseteq \mathcal{LZ} \vee \mathcal{RZ}$. On the other hand, $\mathcal{CSA}_p \vee \mathcal{P} \supseteq \mathcal{Q}$ by Lemma 1. Since $\mathcal{P} \subseteq \mathcal{Q} \subseteq \mathcal{V}$ and the variety $\mathcal{V}$ is upper-modular, we have:

$$\mathcal{Q} \subseteq (\mathcal{CSA}_p \vee \mathcal{P}) \wedge \mathcal{V} = (\mathcal{CSA}_p \wedge \mathcal{V}) \vee \mathcal{P} \subseteq \mathcal{LZ} \vee \mathcal{RZ} \vee \mathcal{P}.$$

But the inclusion $\mathcal{Q} \subseteq \mathcal{LZ} \vee \mathcal{RZ} \vee \mathcal{P}$ cannot hold because the identity $xyzt = xzyt$ is true in the variety $\mathcal{LZ} \vee \mathcal{RZ} \vee \mathcal{P}$ and fails in $\mathcal{Q}$ (the last claim follows from [15, Lemma 1]). Thus $\mathcal{V} \nsupseteq \mathcal{Q}$.

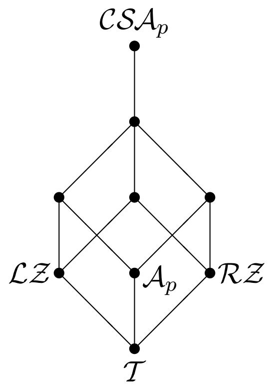

Figure 1: the lattice $L(\mathcal{CSA}_p)$

Suppose that $\mathcal{V}$ is not a variety of semigroups with completely regular square, i. e. it does not satisfy the identity (1). Then one of the identities (2) or (3) holds in $\mathcal{V}$. By symmetry, we may assume that $\mathcal{V}$ satisfies the identity (2). We are going to show the the condition (ii) of Theorem 2 is true in this case.

According to Lemma 2, $\mathcal{V} \supseteq \mathcal{P}$. Now Lemma 1 applies with the conclusion that $\mathcal{V}$ does not contain the varieties $\mathcal{CSA}_p$ (for any p), $\mathcal{LSNB}$, $\mathcal{RRB}$ and $\mathcal{RZM}$.

Lemma 3 and the fact that $\mathcal{Q} \nsubseteq \mathcal{V}$ imply that $\mathcal{V} = \mathcal{K} \vee \mathcal{U}$ where $\mathcal{K} = \mathrm{CR}(\mathcal{V})$ and $\mathcal{U}$ is one of the varieties $\mathcal{T}$, $\mathcal{ZM}$ or $\mathcal{P}$. If $\mathcal{U} = \mathcal{T}$ or $\mathcal{U} = \mathcal{ZM}$ then the variety $\mathcal{V} = \mathcal{K} \vee \mathcal{U}$ satisfies the identity (1). Therefore $\mathcal{U} = \mathcal{P}$ and $\mathcal{V} = \mathcal{K} \vee \mathcal{P}$.

Thus $\mathcal{V} = \mathcal{K} \vee \mathcal{P}$ where $\mathcal{K}$ is a completely regular variety that does not contain the varieties $\mathcal{CSA}_p$, $\mathcal{RRB}$ and $\mathcal{LSNB}$. Together with the results of the article [14], this implies that the lattice $L(\mathcal{V})$ is modular. Now Lemma 6 applies and we conclude that any subvariety of the variety $\mathcal{V}$ is upper-modular.

Suppose that $\mathcal{RZ} \subseteq \mathcal{K}$. Then $\mathcal{P} \vee \mathcal{RZ} \subseteq \mathcal{V}$, whence the variety $\mathcal{P} \vee \mathcal{RZ}$ is upper-modular. Applying Lemma 5, the claim that the variety $\mathcal{P} \vee \mathcal{RZ}$ is upper-modular and the inclusion $\mathcal{RZ} \subseteq \mathcal{P} \vee \mathcal{RZ}$, we have

$$\big(\mathcal{C} \wedge (\mathcal{P} \vee \mathcal{RZ})\big) \vee \mathcal{RZ} = (\mathcal{C} \vee \mathcal{RZ}) \wedge (\mathcal{P} \vee \mathcal{RZ}) = \mathcal{P} \vee \mathcal{RZ}.$$

On the other hand, the variety $\mathcal{C} \wedge (\mathcal{P} \vee \mathcal{RZ})$ satisfies the identities $xy = x^2y$ and $xy = yx$. It is well known and easily verified that these two identities determine the variety $\mathcal{SL} \vee \mathcal{ZM}$. Hence this variety contains $\mathcal{C} \wedge (\mathcal{P} \vee \mathcal{RZ})$. The opposite inclusion is evident, and therefore

$$\mathcal{P} \vee \mathcal{RZ} = \big(\mathcal{C} \wedge (\mathcal{P} \vee \mathcal{RZ})\big) \vee \mathcal{RZ} = \mathcal{SL} \vee \mathcal{ZM} \vee \mathcal{RZ}.$$

But the varieties $\mathcal{SL} \vee \mathcal{ZM} \vee \mathcal{RZ}$ and $\mathcal{P} \vee \mathcal{RZ}$ are distinct because the former variety satisfies the identity $xy = xy^2$ which is false in the latter variety. We have a contradiction, whence $\mathcal{RZ} \nsubseteq \mathcal{K}$. □

Theorems 1 and 2 and results of the article [14] imply

Corollary 2. *If $\mathcal{V}$ is a proper upper-modular semigroup variety then the lattice $L(\mathcal{V})$ is modular.* □

Corollary 2 and Lemma 6 imply

Corollary 3. *Any subvariety of a proper upper-modular semigroup variety is upper-modular.* □

Corollaries 2 and 3 give the affirmative answers to Questions 5.4b) and 5.4′ of the article [10] respectively.

Corollary 1 provides a wide class of semigroup varieties where the property of being upper-modular implies the distributive law in their subvariety lattices. Two more classes of varieties with such a property are given by the following two claims.

Corollary 4. *Let $\mathcal{V}$ be a proper upper-modular semigroup variety that is not a variety of semigroups with completely regular square. The lattice $L(\mathcal{V})$ is distributive [satisfies a non-trivial lattice identity] if and only if the subvariety lattice of any group subvariety of the variety $\mathcal{V}$ is distributive [satisfies this identity].*

Proof. The necessity is evident. Let us prove the sufficiency. Let $\mathcal{V}$ be a proper upper-modular semigroup variety that is not a variety of semigroups with completely regular square, and ε a non-trivial lattice identity (may be, in particular, the distributive law). If $\mathcal{V}$ is a variety of degree > 2 then the lattice $L(\mathcal{V})$ is distributive (and therefore satisfies the identity ε) by Corollary 1. Let now $\mathcal{V}$ be a variety of degree ≤ 2. Applying Theorem 2, we may assume by symmetry that $\mathcal{V} = \mathcal{K} \vee \mathcal{P}$ where $\mathcal{K}$ is a completely regular variety that does not contain $\mathcal{RZ}$. Recall that a completely regular variety is called *orthodox* if, in every its semigroup, the set of all idempotents forms a subsemigroup. As is well known (see [4], for instance), a completely regular variety is orthodox if and only if it does not contain the variety $\mathcal{CSA}_p$ for all prime p. Since $\mathcal{CSA}_p \supseteq \mathcal{RZ}$ for every prime p (see Fig. 1) and $\mathcal{K} \not\supseteq \mathcal{RZ}$, we obtain that the variety $\mathcal{K}$ is orthodox. Now it follows from [5, Corollary 5] that the lattice $L(\mathcal{K})$ satisfies the identity ε whenever the subvariety lattice of every group subvariety of $\mathcal{K}$ satisfies ε. It remains to take into account that the lattice $L(\mathcal{V})$ is isomorphic to a subdirect product of the lattice $L(\mathcal{K})$ and the 3-element chain by [15, Lemma 15]. □

Recall that a semigroup variety is called *combinatorial* if all its groups are singleton.

Corollary 5. *If $\mathcal{V}$ is a combinatorial upper-modular semigroup variety then the lattice $L(\mathcal{V})$ is distributive.*

Proof. Corollary 4 permits to assume that $\mathcal{V}$ is a variety of semigroups with completely regular square. Let $S \in \mathcal{V}$ and $x, y \in S$. Since $\mathcal{V}$ satisfies the identity (1), xy is a group element in S. But the variety $\mathcal{V}$ is combinatorial, whence all subgroups in S are singleton. Therefore xy is an idempotent in S. Thus $\mathcal{V}$ satisfies the identity $xy = (xy)^2$. It remains to refer to the result from [1] that the variety given by the last identity has a distributive subvariety lattice. □

We do not know any example of a proper upper-modular variety with non-distributive subvariety lattice. We do not know also any example of a non-upper-modular variety that satisfies one of the conditions (i)–(iii) of Theorem 2. Thus, the following three questions arise naturally. The first of them have been already mentioned in [10], while the second have been mentioned in [6].

Question 1. *Does there exist a proper upper-modular semigroup variety with non-distributive subvariety lattice*?

Question 2. *Does there exist a non-upper-modular variety of semigroups with completely regular square*?

Question 3. *Does there exist a non-upper-modular semigroup variety satisfying one of the conditions* (ii) *or* (iii) *of Theorem* 2?

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