

# Synchronizing Finite Automata

## Lecture II: Algorithmic Issues

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# 1. Recap

Deterministic finite automata (DFA):  $\mathcal{A} = \langle Q, \Sigma, \delta \rangle$ .

- $Q$  the state set
- $\Sigma$  the input alphabet
- $\delta : Q \times \Sigma \rightarrow Q$  the transition function

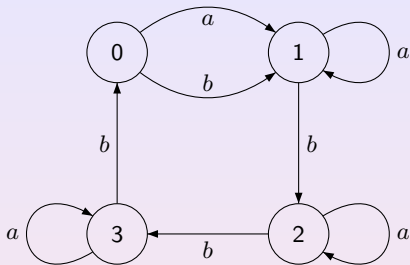
$\mathcal{A}$  is called **synchronizing** if there exists a word  $w \in \Sigma^*$  whose action resets  $\mathcal{A}$ , that is, leaves the automaton in one particular state no matter which state in

$Q$  it started at:  $\delta(q, w) = \delta(q', w)$  for all  $q, q' \in Q$ .

$|Q \cdot w| = 1$ . Here  $Q \cdot v = \{\delta(q, v) \mid q \in Q\}$ .

Any  $w$  with this property is a **reset word** for  $\mathcal{A}$ .

## 2. Example



A reset word is *abbbabba*. In fact, we will soon see that this is the **shortest** reset word for this automaton.

### 3. Power Automaton

Not every DFA is synchronizing. Therefore, the very first question is the following one: *given an automaton, how to determine whether or not it is synchronizing?* This question is easy, and a straightforward solution comes from the classic powerset construction by Rabin and Scott.

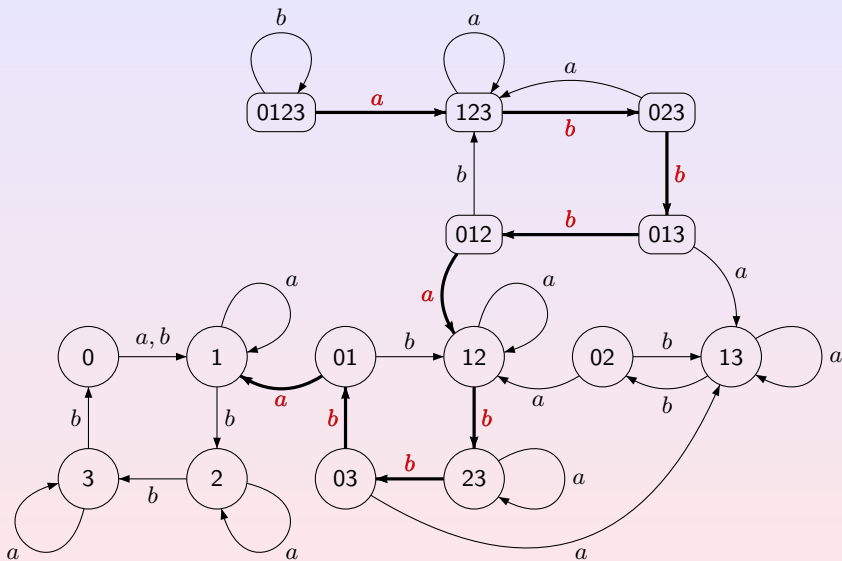
The *power automaton*  $\mathcal{P}(\mathcal{A})$  of a given DFA  $\mathcal{A} = \langle Q, \Sigma, \delta \rangle$ :

- states are the non-empty subsets of  $Q$ ,
- $\delta(P, a) = P \cdot a = \{\delta(p, a) \mid p \in P\}$

A  $w \in \Sigma^*$  is a reset word for the DFA  $\mathcal{A}$  iff  $w$  labels a path in  $\mathcal{P}(\mathcal{A})$  starting at  $Q$  and ending at a singleton.

**Exercise:** Write down a proof of this claim!

## 4. Example



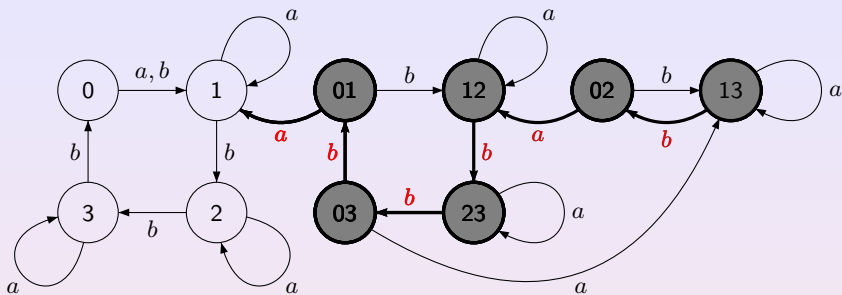
## 5. Polynomial Algorithm

Thus, the question of whether or not a given DFA  $\mathcal{A}$  is synchronizing reduces to the following reachability question in the underlying digraph of the power automaton  $\mathcal{P}(\mathcal{A})$ : is there a path from  $Q$  to a singleton? The latter question can be easily answered by BFS. This algorithm is however exponential w.r.t. the size of  $\mathcal{A}$ .

The following result found independently by Chung Laung Liu and Černý gives a polynomial algorithm:

**Proposition.** *A DFA  $\mathcal{A} = \langle Q, \Sigma, \delta \rangle$  is synchronizing iff for every  $q, q' \in Q$  there exists a word  $w \in \Sigma^*$  such that  $\delta(q, w) = \delta(q', w)$ .*

## 6. Example



$$a, Q \cdot a = \{1, 2, 3\}; \quad a \cdot bba, Q \cdot abba = \{1, 3\}$$

$$abba \cdot babbba, Q \cdot abbababbba = \{1\}$$

Observe that the reset word constructed this way is of length 10 while we know a reset word of length 9.

Thus, recognizing synchronizability reduces to a reachability problem in the automaton whose states are the 2-subsets and the 1-subsets of  $Q$ . The latter can be solved by BFS in  $O(n^2 \cdot |\Sigma|)$  time where  $n = |Q|$ .

Can one do better? It is an **open problem**.

Mikhail Berlinkov has developed a (non-trivial) algorithm that checks whether or not an automaton with  $n$  states is synchronizing and spends time  $O(n)$  **on average**. The worst case complexity of Berlinkov's algorithm is still quadratic.

The algorithm has been implemented by Pavel Ageev (Implementation of the algorithm for testing an automaton for synchronization in linear expected time.

J. Autom. Lang. Comb., 24(2-4):139–152 (2019)); the implementation outperforms the standard algorithm for random automata with  $> 30$  states.

The implementation can be found here:

<https://github.com/birneAgeev/AutomataSynchronizationChecker>



In fact, the basic algorithm not only recognizes synchronizability but also returns a reset word provided that such exists.

If one also wants to produce a reset word, one need  $O(n^3 + n^2 \cdot |\Sigma|)$  time. Why? One needs time to write down the word!

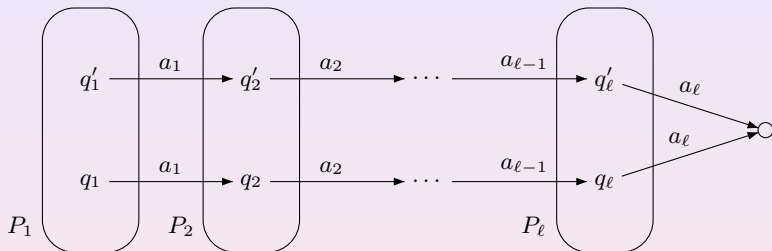
Clearly, the resulting reset word has length  $O(n^3)$ : the algorithm makes at most  $n - 1$  steps and the length of the segment added in the step when  $k$  states are still to be compressed ( $n \geq k \geq 2$ ) is at most  $1 + \#$  of blank 2-subsets, i.e.,  $1 + \binom{n}{2} - \binom{k}{2}$ . This gives the upper bound close to  $\frac{n^3 - n}{3}$ .

Can we do better? What is the exact bound?



## 10. Studying Generic Step

The sets  $P_1 = P$ ,  $P_2 = P_1 \cdot a_1$ ,  $\dots$ ,  $P_\ell = P_{\ell-1} \cdot a_{\ell-1}$  are  $k$ -subsets of  $Q$ . Since  $|P_\ell \cdot a_\ell| < |P_\ell|$ , there exist two states  $q_\ell, q'_\ell \in P_\ell$  such that  $\delta(q_\ell, a_\ell) = \delta(q'_\ell, a_\ell)$ . Now define 2-subsets  $R_i = \{q_i, q'_i\} \subseteq P_i$ ,  $i = 1, \dots, \ell$ , such that  $\delta(q_i, a_i) = q_{i+1}$ ,  $\delta(q'_i, a_i) = q'_{i+1}$  for  $i = 1, \dots, \ell - 1$ .



The condition that  $v$  is a word of minimum length with  $|P \cdot v| < |P|$  implies  $R_i \not\subseteq P_j$  for  $1 \leq j < i \leq \ell$ .

## 11. Combinatorial Configuration

Our question reduces to the following problem in combinatorics of finite sets:

Let  $Q$  be an  $n$ -set,  $P_1, \dots, P_\ell$  a sequence of its  $k$ -subsets ( $k > 1$ ) such that each  $P_i$ ,  $1 < i \leq \ell$ , includes a “fresh” 2-subset that does not occur in any previous  $P_j$  ( $1 \leq j < i$ ). How long can such **renewing** sequences be?

A construction: fix a  $(k - 2)$ -subset  $W$  of  $Q$ , list all  $\binom{n-k+2}{2}$  2-subsets of  $Q \setminus W$  and let  $T_i$  be the union of  $W$  with the  $i^{\text{th}}$  2-subset in the list. This gives the renewing sequence  $T_1, \dots, T_s$  of length  $s = \binom{n-k+2}{2}$ . Is this the maximum?

## 12. Combinatorial Configuration

The question turned out to be very difficult and was solved (in the affirmative) by Peter Frankl (An extremal problem for two families of sets, Eur. J. Comb. 3:125–127 (1982)).

The proof uses linearization techniques which are quite common in combinatorics of finite sets. One reformulates the problem in linear algebra terms and then uses the corresponding machinery.

We identify  $Q$  with  $\{1, 2, \dots, n\}$  and assign to each  $k$ -subset  $I = \{i_1, \dots, i_k\}$  the following polynomial  $D(I)$  in variables  $x_{i_1}, \dots, x_{i_k}$  over the field of reals.

## 13. Linearization

$$I = \{i_1, \dots, i_k\} \mapsto D(I) = \begin{vmatrix} 1 & i_1 & i_1^2 & \cdots & i_1^{k-3} & x_{i_1} & x_{i_1}^2 \\ 1 & i_2 & i_2^2 & \cdots & i_2^{k-3} & x_{i_2} & x_{i_2}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & i_k & i_k^2 & \cdots & i_k^{k-3} & x_{i_k} & x_{i_k}^2 \end{vmatrix}_{k \times k}$$

Then one proves that:

- the polynomials  $D(P_1), \dots, D(P_\ell)$  are linearly independent whenever the  $k$ -subsets  $P_1, \dots, P_\ell$  form a renewing sequence;
- the polynomials  $D(T_1), \dots, D(T_s)$  (derived from the “standard” sequence  $T_1, \dots, T_s$  of length  $s = \binom{n-k+2}{2}$ ) generate the linear space spanned by all polynomials of the form  $D(I)$ .

## 14. Linearization, Step 1

$$D(I) = \begin{vmatrix} 1 & i_1 & i_1^2 & \cdots & i_1^{k-3} & x_{i_1} & x_{i_1}^2 \\ 1 & i_2 & i_2^2 & \cdots & i_2^{k-3} & x_{i_2} & x_{i_2}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & i_k & i_k^2 & \cdots & i_k^{k-3} & x_{i_k} & x_{i_k}^2 \end{vmatrix}_{k \times k}$$

Suppose that  $k$ -subsets  $P_1, \dots, P_\ell$  form a renewing sequence but  $D(P_1), \dots, D(P_\ell)$  are linearly dependent.

Then some polynomial  $D(P_j)$  should be expressible as a linear combination of the preceding polynomials  $D(P_1), \dots, D(P_{j-1})$ .

By the definition of a renewing sequence,  $P_j$  contains a couple  $\{p, p'\}$  such that  $\{p, p'\} \not\subseteq P_i$  for all  $i < j$ .

If we substitute  $x_p = p$ ,  $x_{p'} = p'$ , and  $x_t = 0$  for  $t \neq p, p'$  in each of the polynomials  $D(P_1), \dots, D(P_j)$ , then the polynomials  $D(P_1), \dots, D(P_{j-1})$  vanish (since the two last columns in each of the resulting determinants become proportional) and so does any linear combination of the polynomials.

## 15. Linearization, Step 1, completed

$$D(P_j) \left( \begin{array}{l} x_p = p, x_{p'} = p', \\ x_t = 0, t \neq p, p' \end{array} \right) = \begin{vmatrix} 1 & i_1 & i_1^2 & \cdots & i_1^{k-3} & p & p^2 \\ 1 & i_2 & i_2^2 & \cdots & i_2^{k-3} & p' & (p')^2 \\ 1 & i_3 & i_3^2 & \cdots & i_3^{k-3} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & i_k & i_k^2 & \cdots & i_k^{k-3} & 0 & 0 \end{vmatrix}_{k \times k}$$

(For simplicity, here we assume that  $i_1 = p, i_2 = p'$ .)

The value of  $D(P_j)$  under the substitution  $x_p = p, x_{p'} = p'$ , and  $x_t = 0$  for  $t \neq p, p'$  is the determinant being the product of a Vandermonde  $(k-2) \times (k-2)$ -determinant with the  $2 \times 2$ -determinant

$$\begin{vmatrix} p & p^2 \\ p' & (p')^2 \end{vmatrix} = pp'(p' - p), \text{ whence this value is not 0.}$$

Hence  $D(P_j)$  cannot be equal to any linear combination of  $D(P_1), \dots, D(P_{j-1})$ .



## 16. Linearization, Step 2

Now we aim to prove that the polynomials  $D(T_1), \dots, D(T_s)$  (derived from the “standard” sequence  $T_1, \dots, T_s$  of length  $s = \binom{n-k+2}{2}$ ) generate the linear space spanned by all polynomials of the form  $D(I)$ . Take an arbitrary  $k$ -element subset  $I = \{i_1, \dots, i_k\}$  of  $Q$ . We claim that the polynomial  $D(I)$  is a linear combination of  $D(T_1), \dots, D(T_s)$ .

Recall that each of the sets  $T_1, \dots, T_s$  is the union of some fixed  $(k-2)$ -subset  $W$  of  $Q$  with a couple of states from  $Q \setminus W$ . We prove the above claim by induction on the cardinality of the set  $I \setminus W$ . If  $|I \setminus W| = 2$ , then  $I$  is the union of  $W$  with some couple from  $Q \setminus W$ , whence  $I = T_i$  for some  $i = 1, \dots, s$ . Thus,  $D(I) = D(T_i)$  and our claim holds true.

## 17. Linearization, Step 2, continued

If  $|I \setminus W| > 2$ , there is  $i_0 \in W \setminus I$ . Let  $I' = I \cup \{i_0\}$ . There exists a polynomial  $p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 \cdots + \alpha_{k-3} x^{k-3}$  over  $\mathbb{R}$  such that  $p(i_0) = 1$  and  $p(i) = 0$  for all  $i \in W \setminus \{i_0\}$ . Consider the determinant

$$\Delta = \begin{vmatrix} p(i_0) & 1 & i_0 & i_0^2 & \cdots & i_0^{k-3} & x_{i_0} & x_{i_0}^2 \\ p(i_1) & 1 & i_1 & i_1^2 & \cdots & i_1^{k-3} & x_{i_1} & x_{i_1}^2 \\ p(i_2) & 1 & i_2 & i_2^2 & \cdots & i_2^{k-3} & x_{i_2} & x_{i_2}^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ p(i_k) & 1 & i_k & i_k^2 & \cdots & i_k^{k-3} & x_{i_k} & x_{i_k}^2 \end{vmatrix}_{(k+1) \times (k+1)}.$$

Clearly,  $\Delta = 0$  as the first column is the sum of the next  $k - 2$  columns with the coefficients  $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_{k-3}$ .

## 18. Linearization, Step 2, completed

Expanding  $\Delta$  by the first column gives  $\sum_{j=0}^k (-1)^j p(i_j) D(I' \setminus \{i_j\}) = 0$ .

$$\begin{pmatrix} p(i_0) & 1 & i_0 & i_0^2 & \cdots & i_0^{k-3} & x_{i_0} & x_{i_0}^2 \\ p(i_1) & 1 & i_1 & i_1^2 & \cdots & i_1^{k-3} & x_{i_1} & x_{i_1}^2 \\ p(i_2) & 1 & i_2 & i_2^2 & \cdots & i_2^{k-3} & x_{i_2} & x_{i_2}^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ p(i_k) & 1 & i_k & i_k^2 & \cdots & i_k^{k-3} & x_{i_k} & x_{i_k}^2 \end{pmatrix}_{(k+1) \times (k+1)}$$

Since  $p(i_0) = 1$  and  $I' \setminus \{i_0\} = I$ , the above equality rewrites as

$$D(I) = \sum_{j=1}^k (-1)^{j+1} p(i_j) D(I' \setminus \{i_j\}),$$

and since  $p(i) = 0$  for all  $i \in W \setminus \{i_0\}$ , all the non-zero summands in the right-hand side are such that  $i_j \notin W$ . For each such  $i_j$ , we have

$$(I' \setminus \{i_j\}) \setminus W = I' \setminus (W \cup \{i_j\}) = (I \cup \{i_0\}) \setminus (W \cup \{i_j\}) = (I \setminus W) \setminus \{i_j\},$$

whence  $|(I' \setminus \{i_j\}) \setminus W| = |I \setminus W| - 1$  and by the inductive assumption, the polynomials  $D(I' \setminus \{i_j\})$  are linear combinations of the polynomials  $D(T_1), \dots, D(T_s)$ .

Thus, in the step when  $k$  states are still to be compressed, the compression can always be achieved by applying a suitable word of length  $\leq \binom{n-k+2}{2}$ . Summing up over  $k = n, \dots, 2$ , we see that the greedy algorithm always returns a reset word of length  $\leq \frac{n^3-n}{6}$ :

$$\begin{aligned} & \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \dots + \binom{n-1}{2} + \binom{n}{2} = \\ & \binom{3}{3} + \binom{3}{2} + \binom{4}{2} + \dots + \binom{n-1}{2} + \binom{n}{2} = \\ & \binom{4}{3} + \binom{4}{2} + \dots + \binom{n-1}{2} + \binom{n}{2} = \dots = \binom{n+1}{3} = \frac{n^3-n}{6}. \end{aligned}$$

Up to recently, the bound  $\frac{n^3-n}{6}$  (obtained almost 40 years ago) remained the best upper bound for the length of the shortest reset words for synchronizing automata with  $n$  states.

An improvement on this bound has been found by Marek Szykuła (Improving the upper bound on the length of the shortest reset word. In STACS 2018, volume 96 of LIPIcs, pages 56:1–56:13 (2018)): the new bound is still cubic in  $n$  but improves the coefficient  $\frac{1}{6} = 0.1666\dots$  at  $n^3$  by  $\frac{125}{511104} \approx 0.000245$  so that it becomes  $\approx 0.1664$ .

The new bound is

$$\frac{85059n^3 + 90024n^2 + 196504n - 10648}{511104}.$$

Yaroslav Shitov (An improvement to a recent upper bound for synchronizing words of finite automata. J. Autom. Lang. Comb., 24(2-4):367–373 (2019)) found a further improvement to  $\approx 0.1654n^3 + o(n^3)$ .

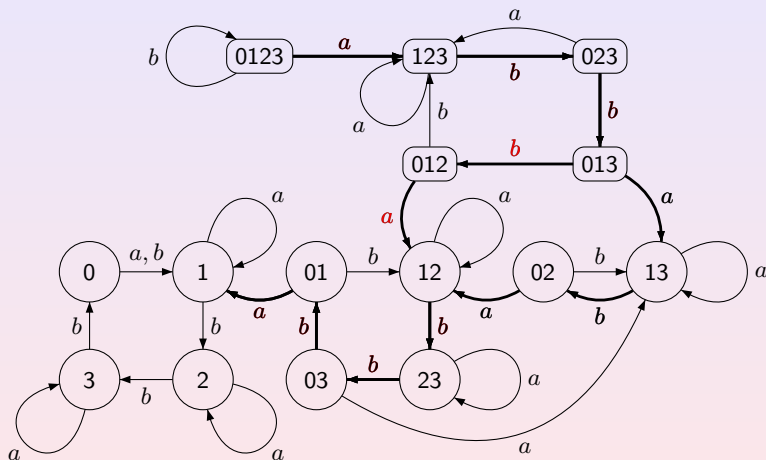
## 21. Greedy Algorithm

GREEDYCOMPRESSION( $\mathcal{A}$ )

- 1:  $w \leftarrow \varepsilon$  ▷ Initializing the current word
- 2:  $P \leftarrow Q$  ▷ Initializing the current set
- 3: **while**  $|P| > 1$  **do**
- 4:   **if**  $|P \cdot u| = |P|$  for all  $u \in \Sigma^*$  **then**
- 5:     **return** Failure
- 6:   **else**
- 7:     take a word  $v \in \Sigma^*$  of minimum length with  $|P \cdot v| < |P|$
- 8:      $w \leftarrow wv$  ▷ Updating the current word
- 9:      $P \leftarrow P \cdot v$  ▷ Updating the current set
- 10: **return**  $w$

## 22. Example Revisited

We have already seen that the greedy algorithm fails to find a reset word of minimum length.



## 23. Greedy Algorithm: Conclusion

Actually, the gap between the minimum length of a reset word and the length of the word produced by the greedy algorithm may be arbitrarily large: for each  $n > 1$  there exists a synchronizing automaton with  $n$  states whose shortest reset word has length  $(n - 1)^2$  while the greedy algorithm produces a reset word of length  $\Omega(n^2 \log n)$ .

Dmitry Ananichev and Vladimir Gusev (Approximation of reset thresholds with greedy algorithms, *Fundam. Inform.* 145:3, 221–227 (2016)) provided a deep analysis of the worst case behaviour of all natural variants of the greedy algorithm.

The behaviour of the greedy algorithm on average is not yet well understood; practically it behaves rather satisfactory.