Synchronizing Finite Automata

Lecture VII. Aperiodic Automata

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Ural Federal University

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- Q the state set
- Σ the input alphabet
- $\delta : Q \times \Sigma \to Q$ the transition function

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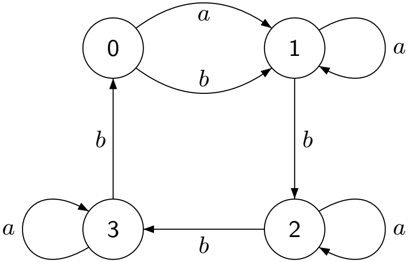
Any w with this property is a reset word for $\mathscr{A}$.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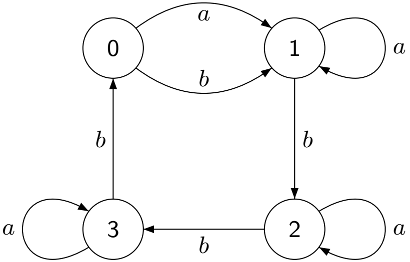

A reset word is $abbbabbba$. In fact, we have verified that this is the shortest reset word for this automaton; that is, its reset threshold is 9.

3. Černý Conjecture

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The Černý conjecture thus claims that in fact $C(n) = (n-1)^2$.

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In this lecture, we encounter a restriction of a different nature: aperiodicity.

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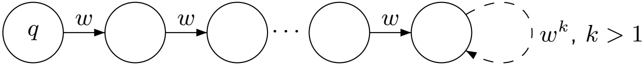

5. Aperiodic Automata

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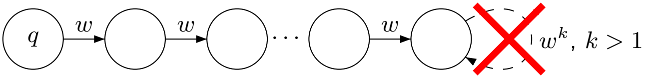

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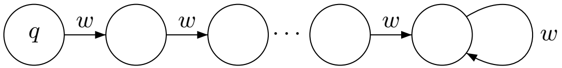
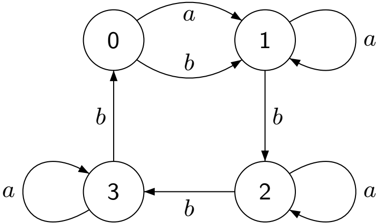
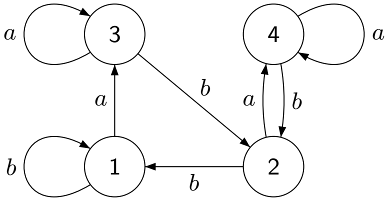


The Černý automaton $\mathscr{C}_4$ is not aperiodic since the letter b acts as a cyclic permutation of the states and thus generates a 4-element subgroup in the transition monoid of $\mathscr{C}_4$.



The following automaton is aperiodic:



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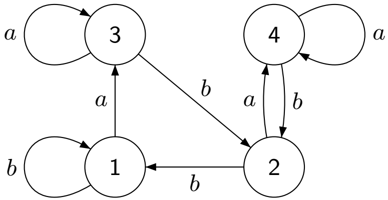

	1	2	3	4
a	3	4	3	4
b	1	1	2	2

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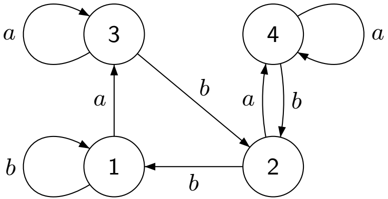

	1	2	3	4
a	3	4	3	4
b	1	1	2	2
ab	2	2	2	2
ba	3	3	4	4
b^2	1	1	1	1

$a^2 = a$

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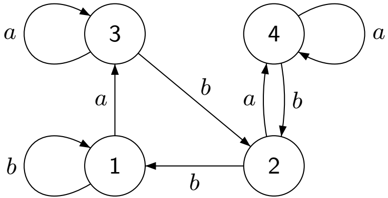

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a	3	4	3	4
b	1	1	2	2
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ba	3	3	4	4
b^2	1	1	1	1
aba	4	4	4	4
b^2a	3	3	3	3

$a^2 = a$, $ab^2 = b^2$, $bab = ab$, $b^3 = b^2$

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Also, the synchronization issues remain difficult when restricted to the class of aperiodic automata. Indeed, inspecting the reduction from SAT to $\textsc{Short-Reset-Word}$ shown in Lecture III, one can see that the construction gives an aperiodic automaton, and therefore, the question of whether or not a given aperiodic automaton admits a reset word whose length does not exceed a given positive integer is NP-complete.

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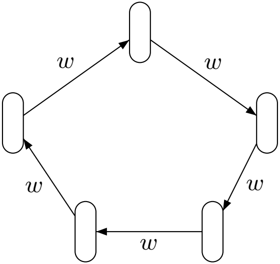

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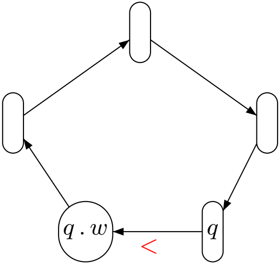

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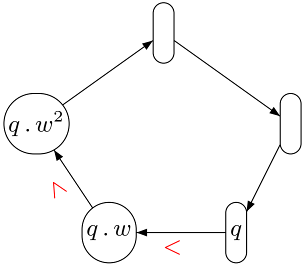

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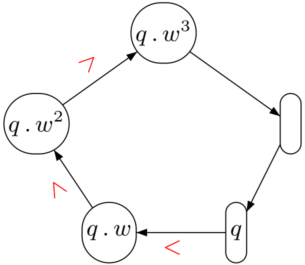

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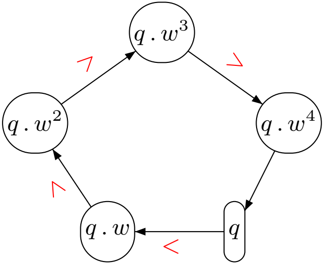

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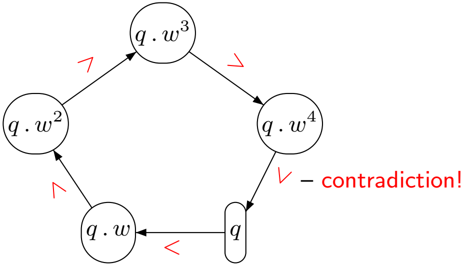

A binary relation ρ on the state set of a DFA $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ is stable if $(p,q) \in \rho$ implies $\big(\delta(p,a), \delta(q,a)\big) \in \rho$ for all $p, q \in Q$ and $a \in \Sigma$.

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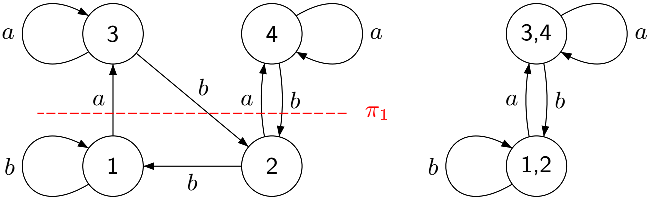

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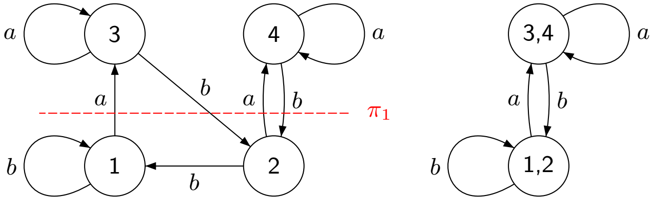

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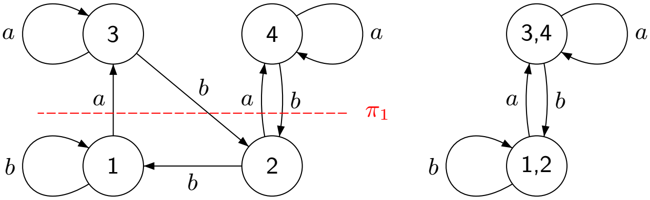

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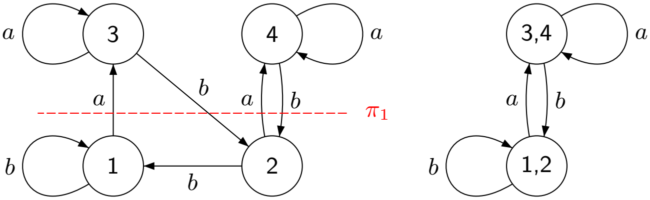

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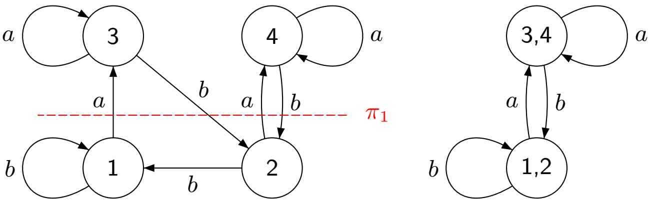

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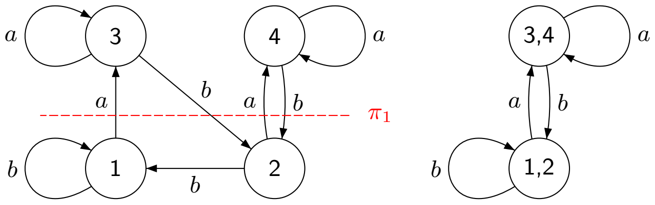

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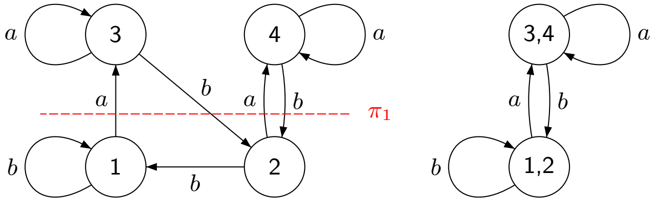
In fact, the hierarchy of generalized monotonic automata is strict: there are automata of each level $\ell = 1, 2, \ldots$, and every generalized monotonic automaton is aperiodic.



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Jump to slide 22

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12. Why Aperiodic Automata?

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It is remarkable that each of these directions has led to a major open problem, and the 3 problems play nowadays a central role in the theory of finite automata.

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The complement of a finite language is infinite. Can one get rid of the Kleene star in this setting?

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$$(ab)^* = \varepsilon + a(a + a^C)b \setminus \Big((a + a^C)aa(a + a^C) + (a + a^C)bb(a + a^C)\Big).$$

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How can one distinguish between regular languages that need star and 'star-free' languages?

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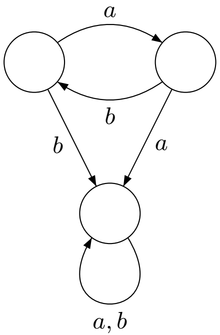
For instance, for $(ab)^*$ and $(a^2)^*$ the minimal automata are



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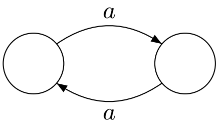

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Extended Star Height Problem

Is there a regular language of extended star height > 1?
Is the class of languages of extended star height 1 decidable?

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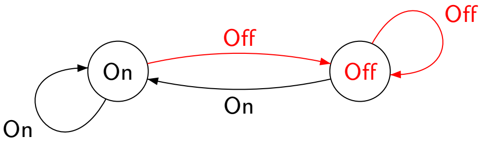

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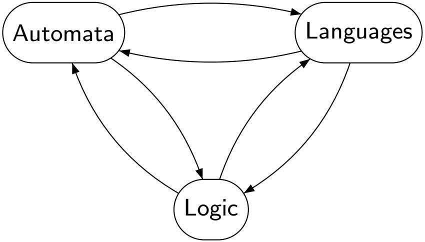
Group Complexity Problem

Given a finite automaton $\mathscr{A}$, can one decide the group complexity of $\mathscr{A}$? In particular, can we decide if the group complexity of $\mathscr{A}$ is 1?

There is a magic triangle



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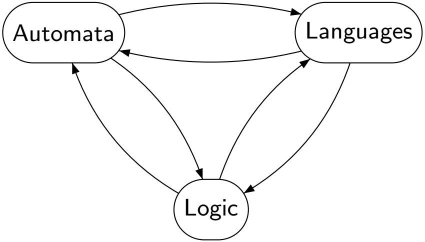

Logic for words has first order variables (positions) that take values in $\{1, 2, \dots\}$, second order variables (sets of positions) whose values are subsets of $\{1, 2, \dots\}$, the usual connectives and quantifiers, the predicate symbol $<$ with the usual meaning (and maybe some additional numerical predicates), and a special predicate Q_a for each letter a with the meaning: $Q_a x$ is true iff the position x holds the letter a.

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Monadic second order formulas define precisely regular languages (Büchi, 1960), but we would not be able to construct a first order formula defining $(a^2)^*$.

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Dot-depth 1 and dot-depth 2 are known to be decidable (Knast, 1980, for 1 and Place-Zeitoun, 2014, for 2)

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Indeed, take any $q, q' \in Q$. Since $\mathscr{A}$ is strongly connected, there exists $w \in \Sigma^*$ such that $q\,.\,w = q'$.

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Indeed, take any $q, q' \in Q$. Since $\mathscr{A}$ is strongly connected, there exists $w \in \Sigma^*$ such that $q \,.\, w = q'$. On the other hand, $\mathscr{A}$ is aperiodic whence there exists a positive integer m such that $q \,.\, w^m = q \,.\, w^{m+1}$.

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The key observation by Trahtman is that every strongly connected aperiodic automaton admits a non-trivial stable partial order.

Given a DFA $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$, its square $\mathscr{A}^{[2]} = \langle Q \times Q, \Sigma, \delta^{[2]} \rangle$ is defined by $\delta^{[2]}\big((q,p),a\big) = \big(\delta(q,a), \delta(p,a)\big)$.

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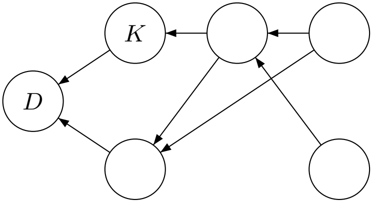

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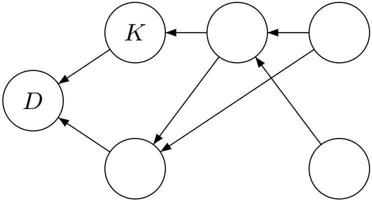

Then $K \cup D$ is a non-trivial stable reflexive relation on Q.

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25. $\mathscr{A}$ is aperiodic

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$$\begin{tikzcd} p \arrow[r, bend left, "w"] & q \arrow[l, bend left, "w"] \end{tikzcd}$$

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- for each $i = 1, \ldots, \ell$, the congruence π_{i-1} generated by ρ_{i-1} is contained in ρ_i and the relation ρ_i/π_{i-1} is a partial order on Q/π_{i-1};

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This differs from the notion of a generalized monotonic automaton by just dropping the restriction that the order ρ_i/π_{i-1} is linear on each π_i/π_{i-1}-class.

30. Examples

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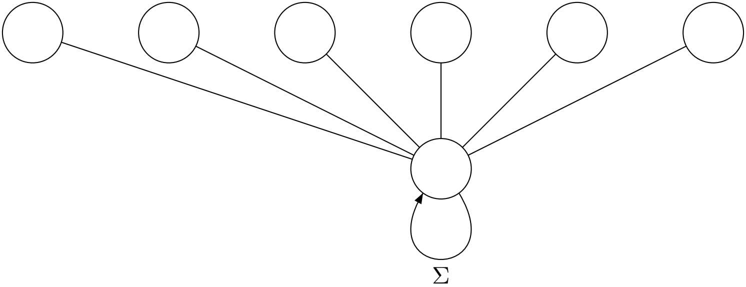

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$$n - 1 \leq C_{SCA}(n) \leq \left\lfloor \frac{n(n+1)}{6} \right\rfloor \quad \text{(Volkov, 2009).}$$

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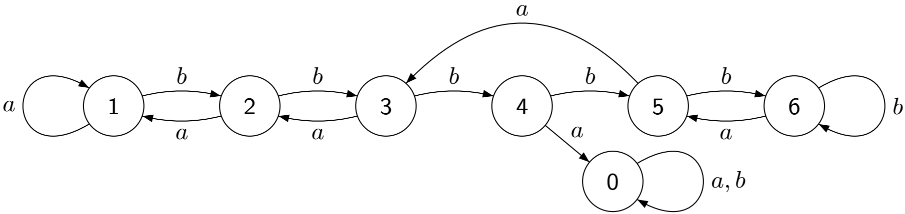

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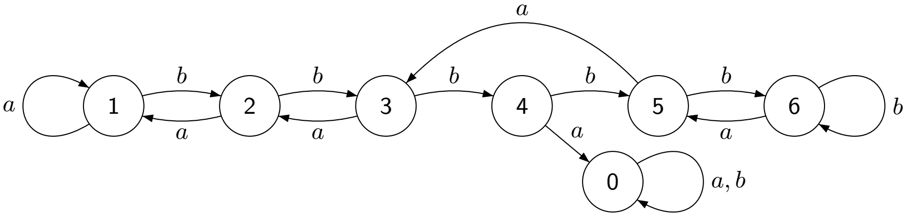

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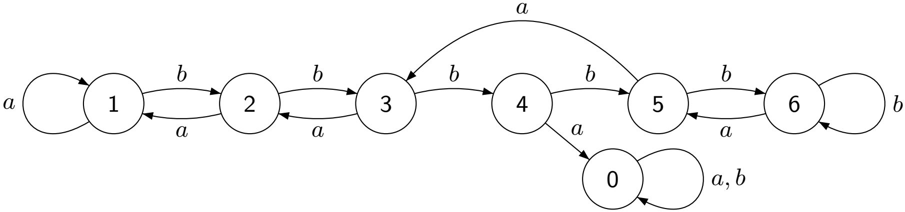

This is the first automaton in Ananichev's series that yields the best up to now lower bound for $C_A(n)$. It has 7 states and its shortest reset word is a^4b^3a of length $7 + \left\lfloor \frac{7}{2} \right\rfloor - 2 = 8$.