Synchronizing Finite Automata
Lecture VI. Automata with Zero

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1. Recap

Deterministic finite automata (DFA): $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$.

- Q the state set
- Σ the input alphabet
- $\delta : Q \times \Sigma \to Q$ the transition function

2. Algebraic Perspective

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Any such DFA is said to be a subautomaton of $\mathscr{A}$.

3. Subautomata

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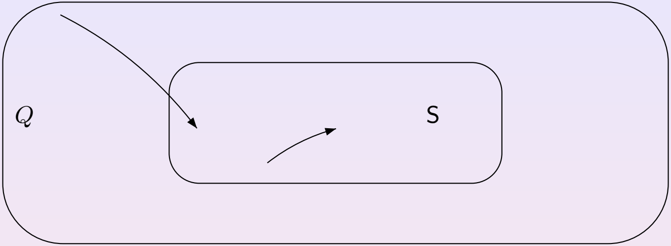

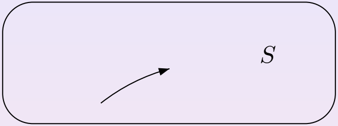

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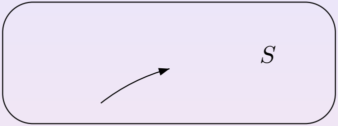

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Exercise: show that a DFA has no proper subautomata iff it is strongly connected.

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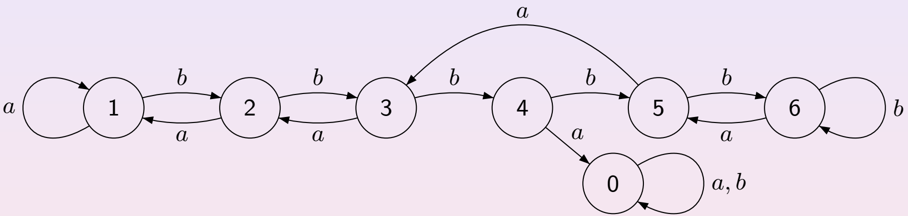

An equivalence π on the state set Q of a DFA $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$ is called a congruence if $(p, q) \in \pi$ implies $\big(\delta(p, a), \delta(q, a)\big) \in \pi$ for all $p, q \in Q$ and all $a \in \Sigma$.

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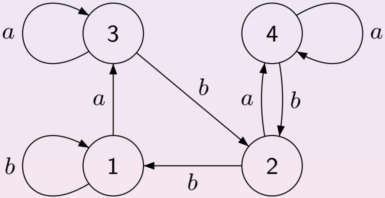

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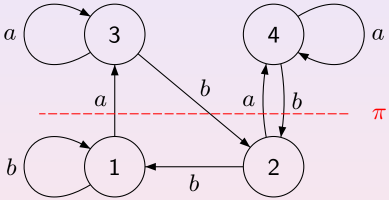

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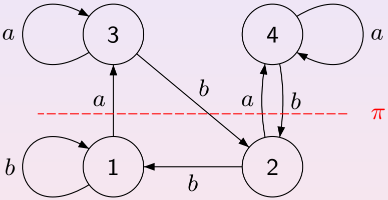

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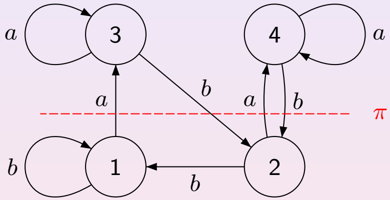

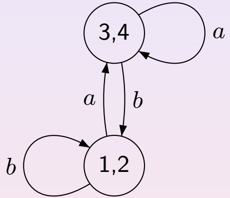

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Clearly, in the quotient automaton $\mathscr{A}/\rho_{\mathscr{S}}$ the state S is a sink.

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7. Useful Observations

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2. Any quotient of a synchronizing automaton is synchronizing, and every reset word for an automaton also serves as a reset word for any of its quotients.

Let $\mathbf{C}$ be any class of automata closed under taking subautomata and quotients, and let $\mathbf{C}_n$ stand for the class of all automata with n states in $\mathbf{C}$.

8. A Reduction

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Theorem (Folklore)

If each synchronizing automaton in $\mathbf{C}_n$ which either is strongly connected or possesses a zero has a reset word of length $f(n)$, then the same holds true for all synchronizing automata in $\mathbf{C}_n$.

9. A Reduction: Proof

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We see that it suffices to prove the Černý conjecture
1) for strongly connected automata and
2) for automata with zero.

11. Automata with Zero

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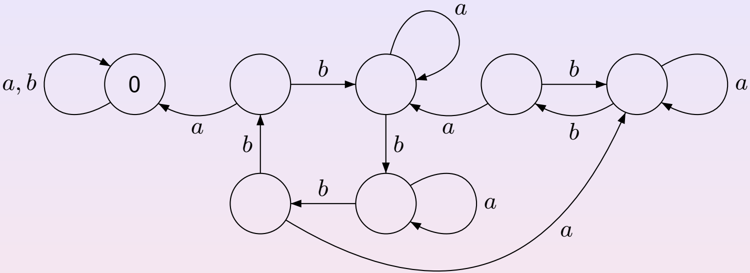

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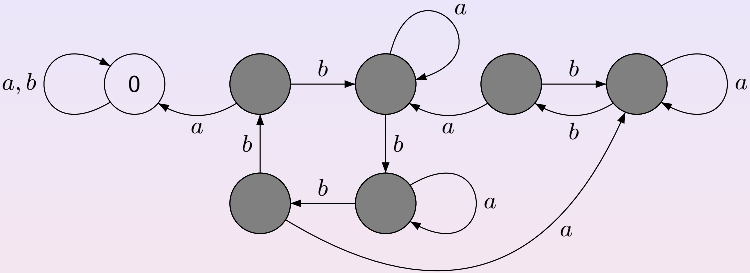

We cover all non-zero states with coins and move the closest coin to 0 until all coins disappear.

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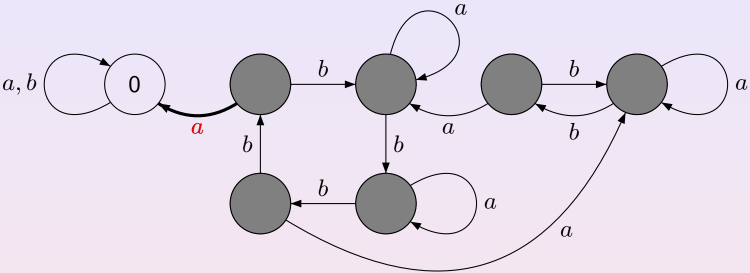

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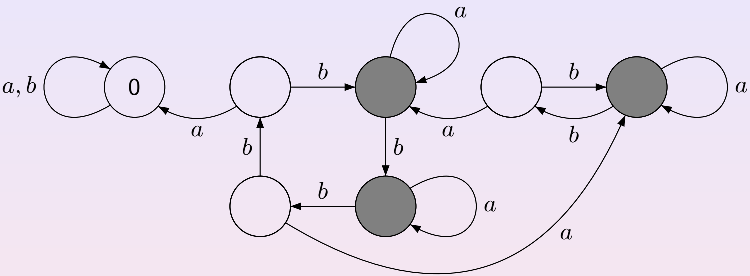

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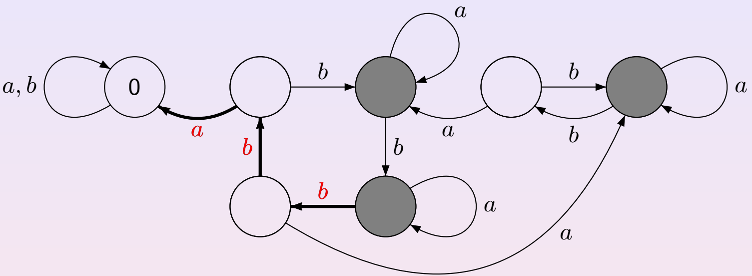

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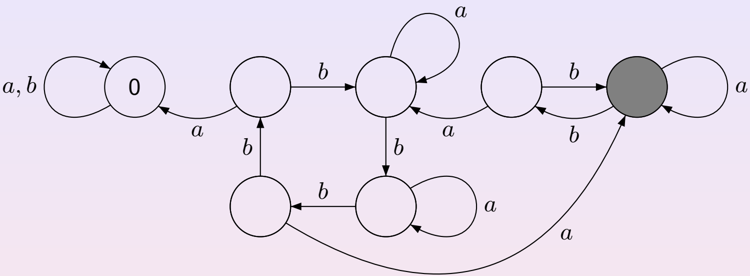

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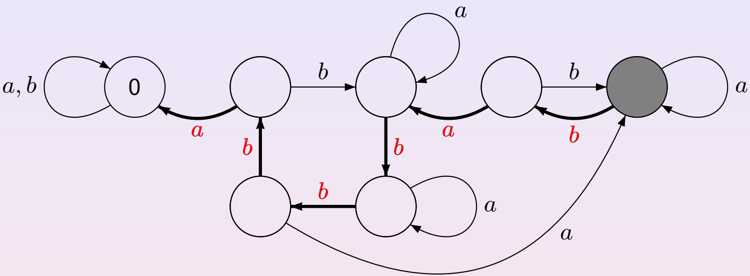

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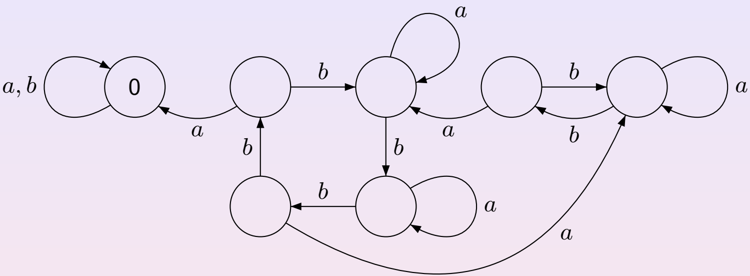

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The algorithm makes at most $n-1$ steps and the length of the segment added in the step when t states still hold coins $(n-1 \geq t \geq 1)$ is at most $n-t$. The total length is $\leq 1+2+\cdots+(n-1) = \frac{n(n-1)}{2}$.

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For each $n \geq 2$, Igor Rystsov (Reset words for commutative and solvable automata, Theoret. Comput. Sci. 172, 273–279 (1997)) constructed an n-state and $(n-1)$-letter synchronizing automaton $\mathscr{R}_n = \langle Q, \Sigma, \delta \rangle$ with zero and reset threshold equal to $\frac{n(n-1)}{2}$.

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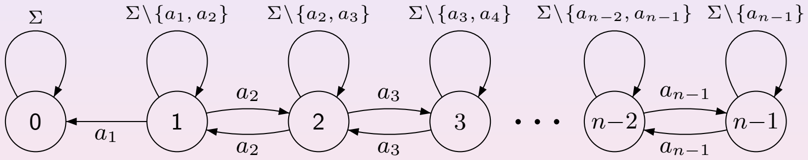
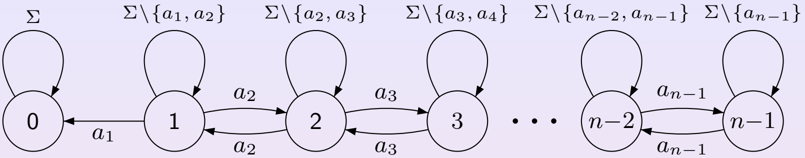
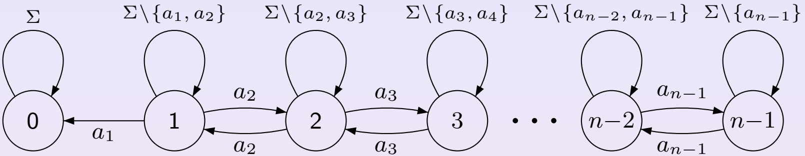


13. Rystsov's Series: Proof





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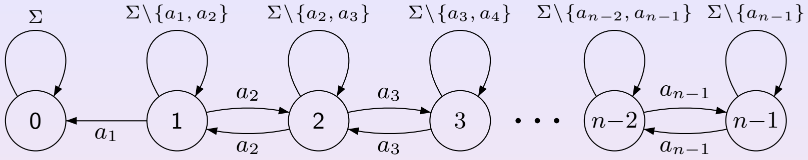

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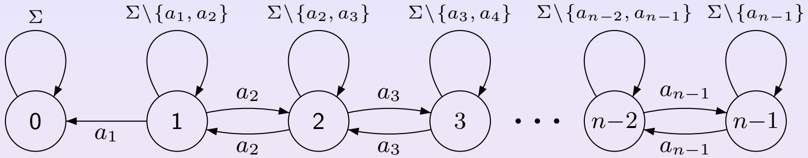

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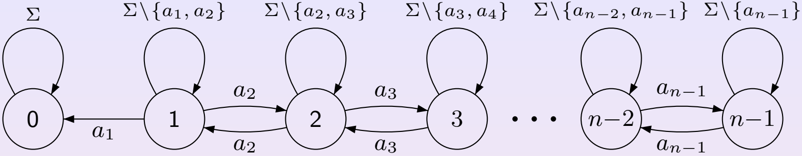
For $S = \{s_1, \ldots, s_t\} \subseteq Q$, let $f(S) := \sum_{i=1}^{t} s_i$. Then $f(\{0\}) = 0$ and $f(Q) = \frac{n(n-1)}{2}$.

13. Rystsov's Series: Proof



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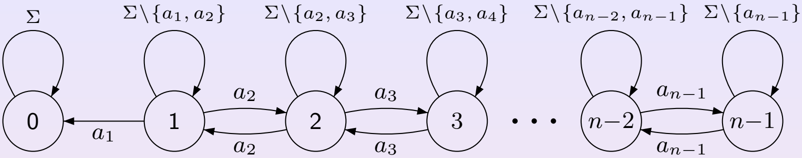

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$$0 = f(\{0\}) = f(Q \,.\, w) \geq f(Q) - |w| = \frac{n(n-1)}{2} - |w|$$

whence $|w| \geq \frac{n(n-1)}{2}$.

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14. Binary Case

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In the binary case (2 input letters), for quite a long time, the lower bound for the reset threshold of n-state synchronizing automata with 0 found by Pavel Martyugin (A series of slowly synchronizing automata with zero state over a small alphabet, Inf. Comput. 19, 517–536 (2009)) remained the best.

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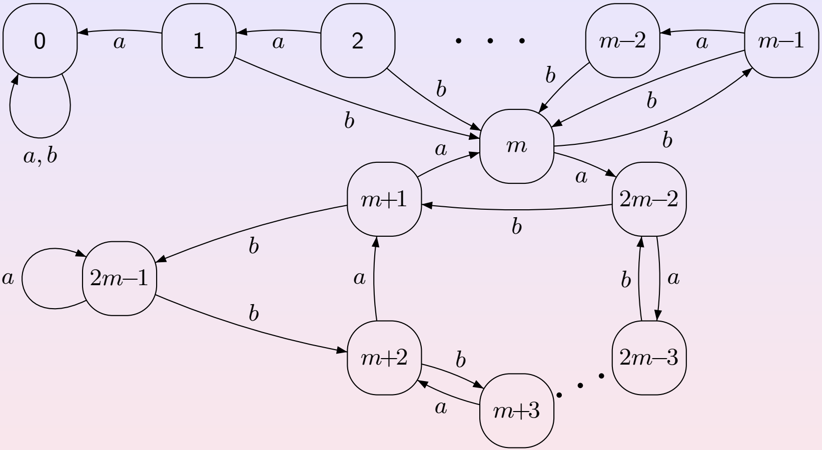
Here is the automaton $\mathscr{M}_n$ from Martyugin's series for $n = 2m$.



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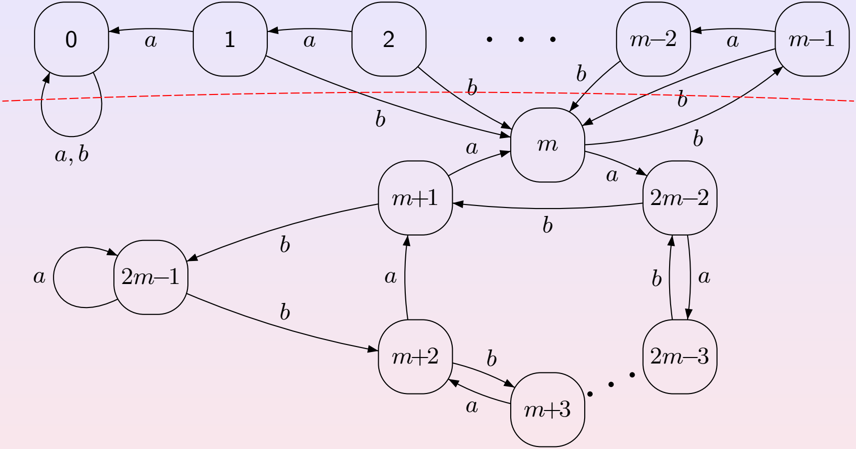

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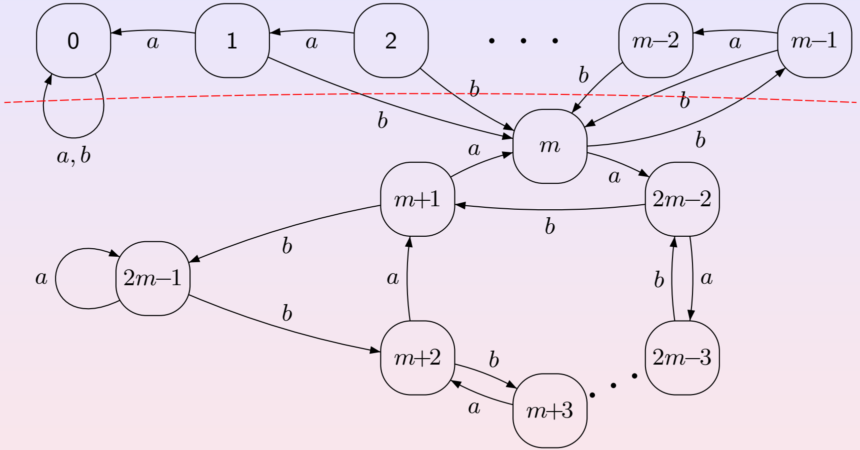

$\mathscr{M}_n$ consists of the "body" formed by $m, m+1, \ldots, 2m-1$ and the "tail" formed by $0, 1, \ldots, m-1$.

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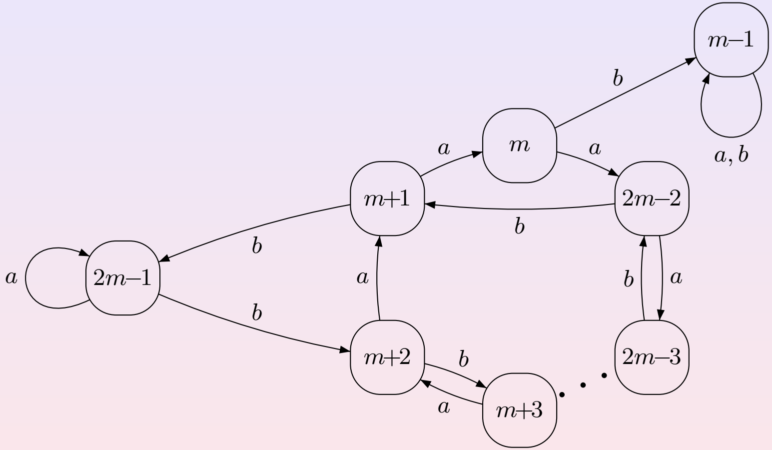
A synchronizing binary DFA $(Q, \{a, b\}, \delta)$ with a sink state $q_0 \in Q$ is an *almost permutation automaton* if it fulfils the following three conditions:

1. There is a unique state *pre-sink* $r \in Q \setminus \{q_0\}$ such that $\delta(r, b) = q_0$.
2. The letter b acts as a permutation on the set $Q \setminus \{r\}$.
3. The letter a acts as a permutation on Q.

For instance, the "body" of Martyugin's automaton $\mathscr{M}_{2m}$ is the following almost permutation automaton:



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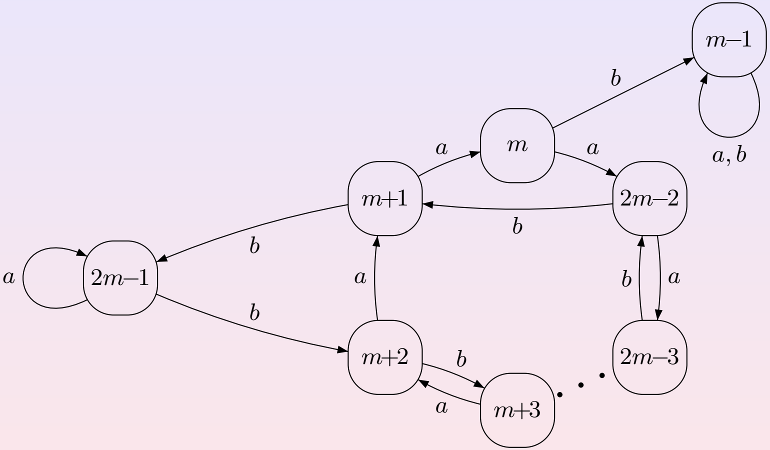

Here the state $m-1$ is a sink and m is pre-sink state.

If $\mathscr{A} = (Q, \{a, b\}, \delta)$ is an almost permutation automaton, the least k such that a^k acts as the identity permutation is called the *order* of a. Clearly, the order of a is the least common multiple of the lengths of cycles with respect to a.

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Let $\mathscr{A} = \langle Q, \{a, b\}, \delta\rangle$ be an n-state synchronizing almost permutation automaton and let k be a multiple of the order of a. Then one can add a "tail" to $\mathscr{A}$ so that the reset threshold of the resulting automaton is $\mathrm{rt}(\mathscr{A}) + nk$.

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Then one can add tails of lengths $k = k(n)$ and obtain a series of binary automata $\mathscr{B}_N$ with 0 and $N = n + k$ states. If $k(n)$ is chosen to be the order of the letter a in $\mathscr{A}_n$, then Vorel's Lemma implies that

$$\mathrm{rt}(\mathscr{B}_N) = An^2 + Bn + C + nk.$$

Suppose that $k = Dn + E$, where D and E are constants. Then,

$$\mathrm{rt}(\mathscr{B}_N) = \frac{A+D}{(1+D)^2} \cdot N^2 + O(N).$$

If $D = 1 - 2A$, then the first coefficient is maximal and is equal to $\frac{1}{4(1-A)}$. This implies that if $A < 1/2$, then $\mathrm{rt}(\mathscr{B}_N)$ grows faster than $\mathrm{rt}(\mathscr{A}_n)$.

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In Martyugin's example, the reset threshold of the "body" grows as $3n + C$, thus, $A = 0$ (and hence $D = 1$). Also $k = n - 2$. This gives

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