Synchronizing Finite Automata

Lecture V. Expansion Method

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- Q the state set
- Σ the input alphabet
- $\delta : Q \times \Sigma \to Q$ the transition function

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1. Recap

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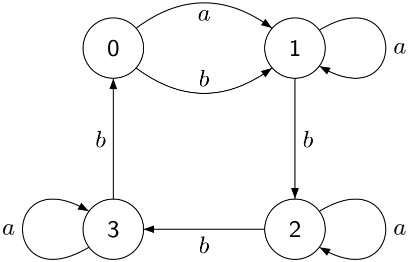
Any w with this property is a reset word for $\mathscr{A}$.



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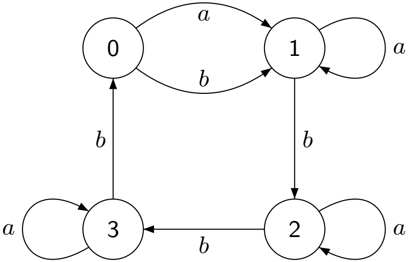
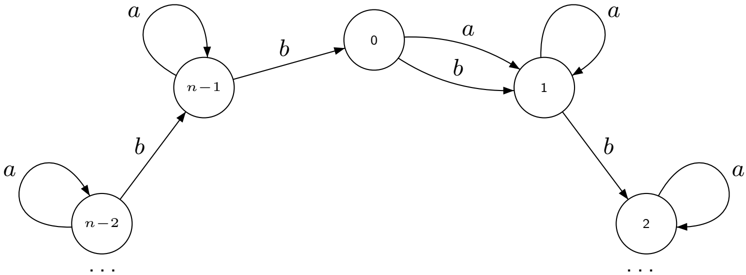


A reset word is $abbbabbba$. In fact, we have verified that this is the shortest reset word for this automaton.

In his 1964 paper Jan Černý constructed a series $\mathscr{C}_n$, $n = 2, 3, \ldots$, of synchronizing automata over 2 letters. Here is a generic automaton from the Černý series:



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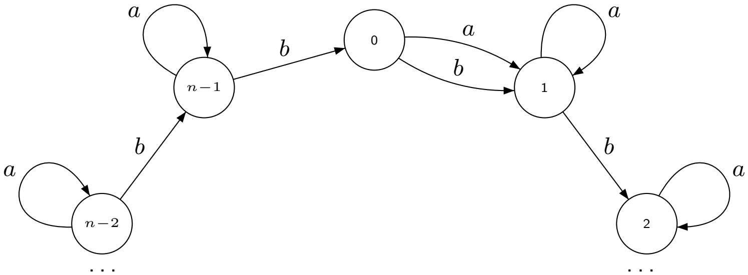

Černý has proved that the shortest reset word for $\mathscr{C}_n$ is $(ab^{n-1})^{n-2}a$ of length $(n-1)^2$.

Define the Černý function $C(n)$ as the maximum reset threshold of all synchronizing automata with n states. The above property of the series $\{\mathscr{C}_n\}$, $n = 2, 3, \dots$, yields the inequality $C(n) \geq (n-1)^2$.

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The Černý conjecture is the claim that in fact the equality $C(n) = (n-1)^2$ holds true.
Everything we know about the conjecture in general can be summarized in one line:

$$(n-1)^2 \leq C(n) \leq \frac{\min\{\frac{85059n^3+90024n^2+196504n-10648}{85184}, n^3-n\}}{6}.$$

Beyond the Černý series, the largest automaton that reaches the Černý bound is the 6-state automaton $\mathscr{K}_6$ found by Jarkko Kari (A counter example to a conjecture concerning synchronizing words in finite automata, EATCS Bull., 73, 146 (2001)).

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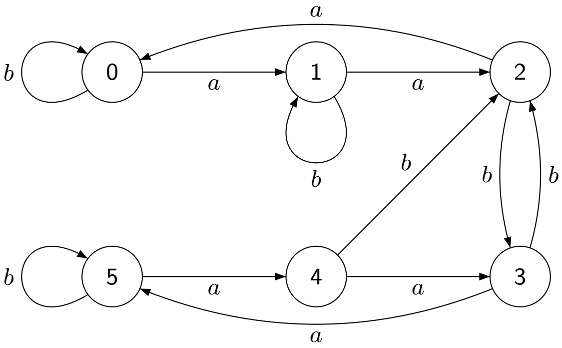

5. Kari's Automaton

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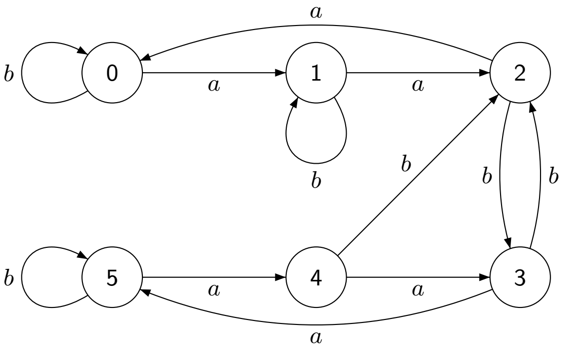

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Let $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ be a DFA. For $P \subseteq Q$ and $w \in \Sigma^*$,

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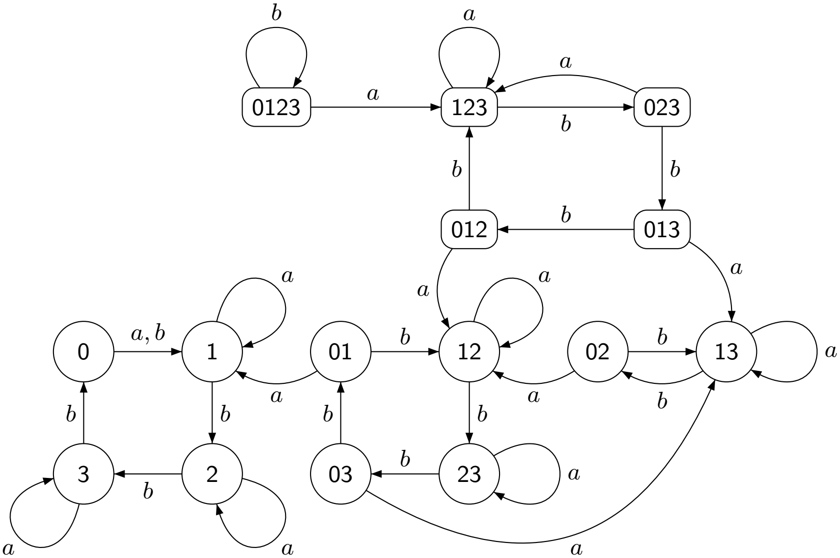
It was conjectured that in synchronizing automata every proper non-singleton subset is extensible.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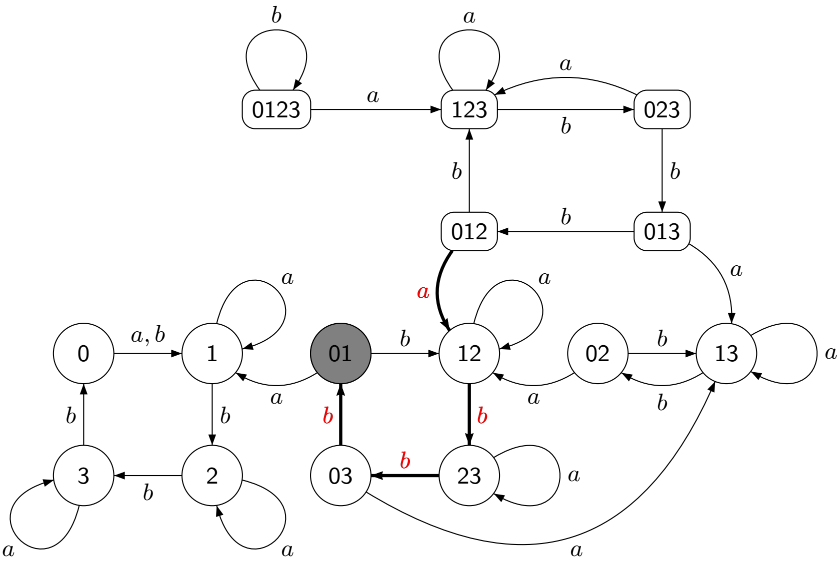
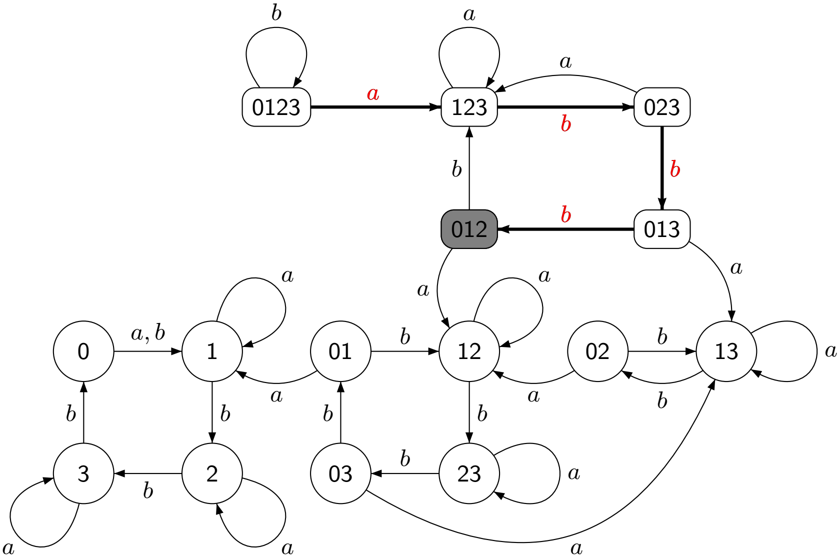
7. Example





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Then in at most $n-2$ steps the sequence $P_0, P_1, P_2, \ldots$ reaches Q and

$$Q \,.\, w_{n-2}w_{n-3}\cdots w_1 a = \{p\},$$

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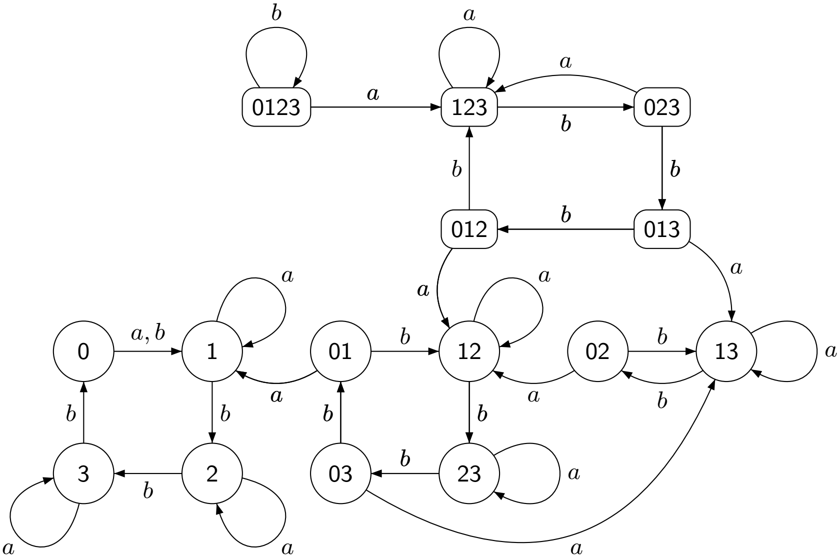
The length of this reset word is at most $n(n-2) + 1 = (n-1)^2$.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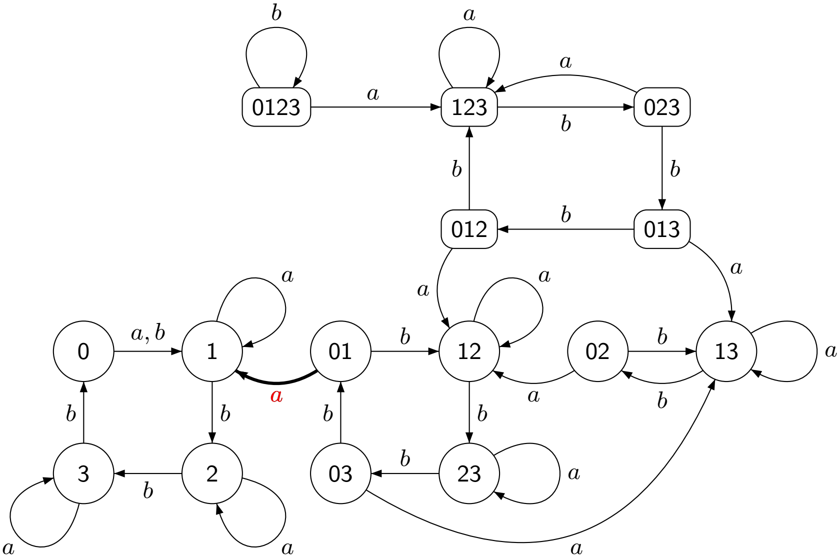

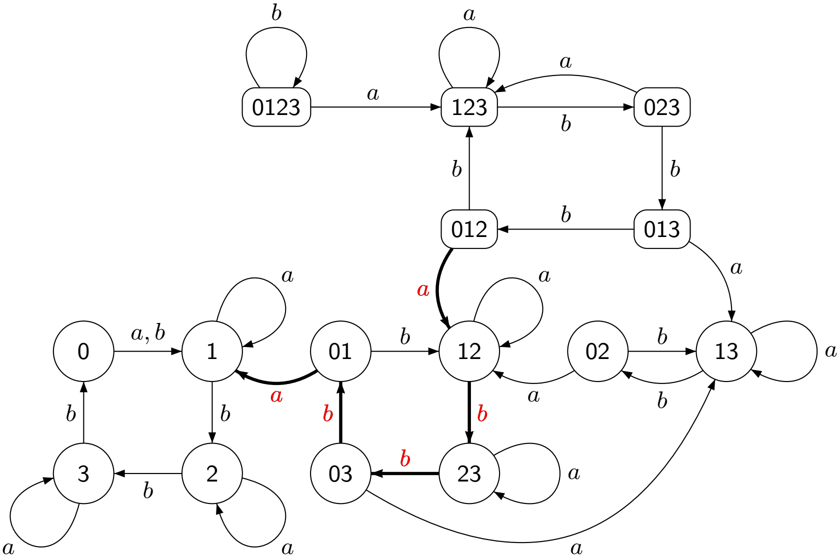
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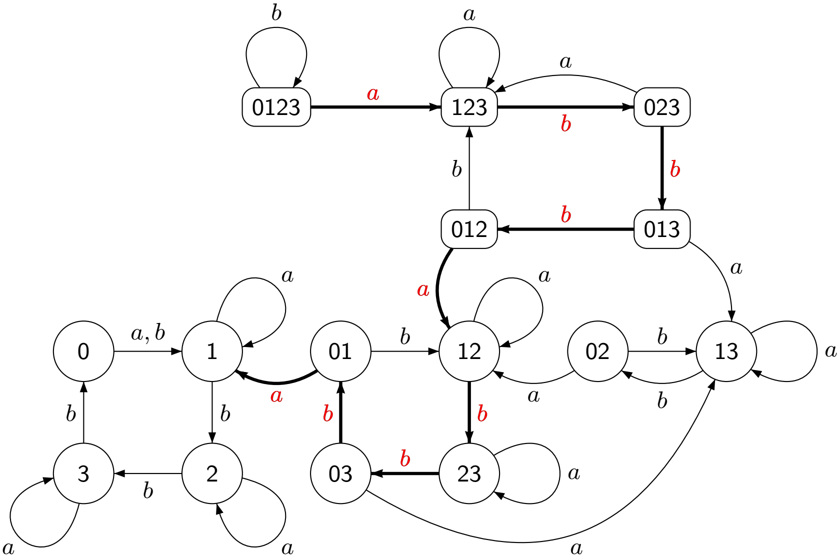

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• Jarkko Kari's result for automata with Eulerian digraphs (Synchronizing finite automata on Eulerian digraphs, Theoret. Comput. Sci., 295, 223–232 (2003).)
• Benjamin Steinberg's result for automata in which a letter labels only one cycle (one-cluster automata) and this cycle is of prime length (The Černý conjecture for one-cluster automata with prime length cycle. Theoret. Comput. Sci., 412, 5487–5491 (2011)).

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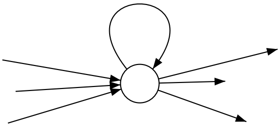

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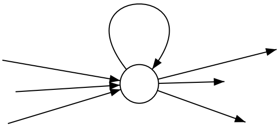

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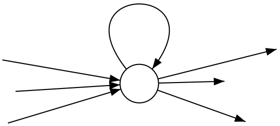

A DFA is said to be Eulerian if so is its underlying graph.
Since in any DFA the number of edges starting at a given state is the same (the cardinality of the input alphabet), in an Eulerian DFA the number of edges ending at any state is the same.

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$$\sum_{a \in \Sigma} |Pa^{-1}| = k|P| \qquad (*)$$

holds true since the left-hand side is the number of edges in the underlying graph of $\mathscr{A}$ with ends in P.

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The equality (*) readily implies that for each $P \subseteq Q$, exactly one of the following alternatives takes place:
either

$$|Pa^{-1}| = |P| \text{ for all letters } a \in \Sigma$$

or

$$|Pb^{-1}| > |P| \text{ for some letter } b \in \Sigma.$$

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$$(n-2)(n-1) + 1 = n^2 - 3n + 3 < (n-1)^2.$$

Assume that $Q = \{1, 2, \ldots, n\}$. Assign to each subset $P \subseteq Q$ its characteristic vector $[P]$ in the linear space $\mathbb{R}^n$ of n-dimensional row vectors over $\mathbb{R}$ as follows: i-th entry of $[P]$ is 1 if $i \in P$, otherwise it is equal to 0.

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Observe that for any vector $x \in \mathbb{R}^n$, the inner product $\langle x, [Q] \rangle$ is equal to the sum of all entries of x. In particular, for each subset $P \subseteq Q$, we have $\langle [P], [Q] \rangle = |P|$.

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Assign to each word $w \in \Sigma^*$ the linear operator φ_w on $\mathbb{R}^n$ defined by $\varphi_w([i]) := [iw^{-1}]$ for each $i \in Q$.

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For instance, $[Q]$ is the all ones row vector and the vectors $[1], \dots, [n]$ form the standard basis of $\mathbb{R}^n$.
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Assign to each word $w \in \Sigma^*$ the linear operator φ_w on $\mathbb{R}^n$ defined by $\varphi_w([i]) := [iw^{-1}]$ for each $i \in Q$. Then for each $P \subseteq Q$, we get

$$\varphi_w([P]) = \varphi_w(\sum_{i \in P}[i]) = \sum_{i \in P} \varphi_w[i] = \sum_{i \in P}[iw^{-1}] = [Pw^{-1}].$$

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Let U be the subspace of all vectors orthogonal to the vector $[Q]$. Then $x \in U$ and the inequality $\langle \varphi_u(x), [Q] \rangle \neq 0$ means that $\varphi_u(x) \notin U$. We aim to bound the minimum length of such word u but first we explain why words sending x beyond U exist.

Since the automaton $\mathscr{A}$ is synchronizing and strongly connected, there exists a word $w \in \Sigma^*$ such that $Q\,.\,w \subseteq S$—one can first synchronize $\mathscr{A}$ to a state q and then move q into S by applying a word that labels a path from q to a state in S.

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Hence

$$1 = \dim U_0 < \dim U_1 < \cdots < \dim U_{\ell-1} < \dim U_\ell$$

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$$(n-2)(n-1)+1 = n^2 - 3n + 3 < (n-1)^2.$$

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The best theoretical lower bounds for the restriction of the Černý function to the class of Eulerian synchronizing automata known so far are of magnitude $\frac{n^2}{2}$ (Pavel Martyugin, Vladimir Gusev, Marek Szykuła, Vojtěch Vorel).

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18. Open Problem

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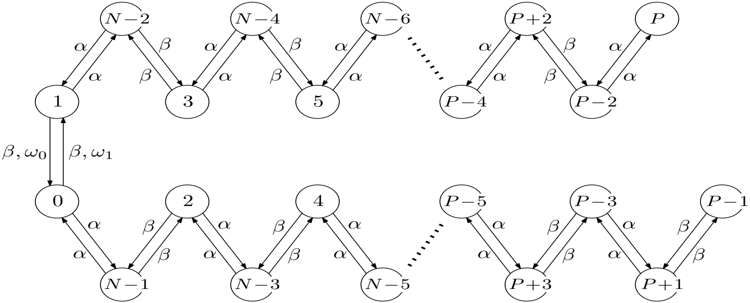

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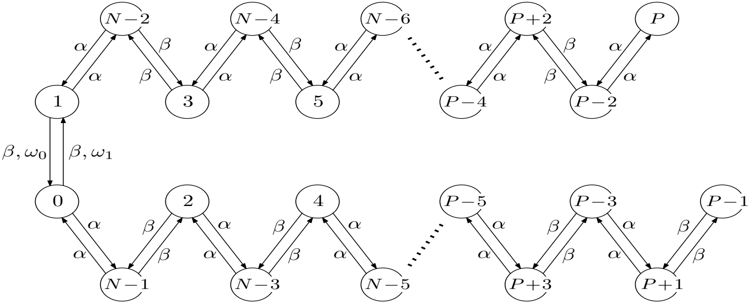

$P = \frac{N+1}{2}$, loops are not shown. The proof is quite non-trivial.

Gusev (Lower bounds for the length of reset words in Eulerian automata, Reachability Problems, LNCS 6945, 180–190 (2011)) has constructed another series of Eulerian synchronizing automata with n states and 2 input letters whose reset threshold is $\frac{n^2-3n+4}{2}$.

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Define the automaton $\mathscr{M}_n$ (from Matricaria) on the state set $\{1, 2, \dots, n\}$, where $n \geq 5$ is odd, in which a and b act as follows:

$$k \, . \, a = \begin{cases} k & \text{if } k \text{ is odd,} \\ k+1 & \text{if } k \text{ is even;} \end{cases} \quad k \, . \, b = \begin{cases} k+1 & \text{if } k \neq n \text{ is odd,} \\ k & \text{if } k \text{ is even,} \\ 1 & \text{if } k = n. \end{cases}$$

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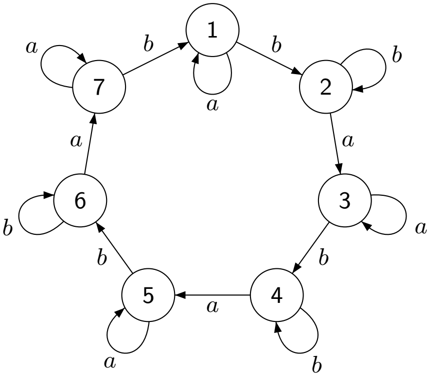

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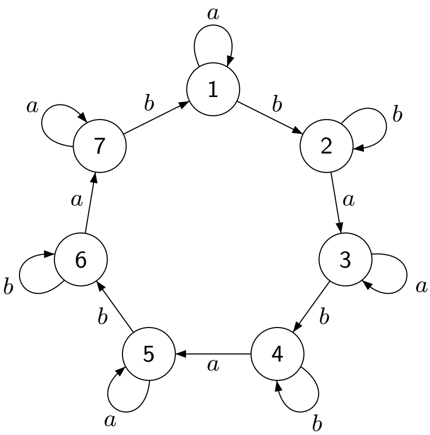

The automaton $\mathscr{M}_7$ and the automaton induced by the actions of b and $c = ab$

After applying the first letter of u it remains to synchronize the subautomaton on the set of states $S = \{1\} \cup \{2k \mid 1 \leq k \leq \frac{n-1}{2}\}$, and this subautomaton is isomorphic to $\mathscr{C}_{\frac{n+1}{2}}$.

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Thus, if $C_E(n)$ is the restriction of the Černý function to the class of Eulerian automata, then

(Szykuła and Vorel, 2016) $\dfrac{n^2-3}{2} \le C_E(n) \le n^2 - 3n + 3$ (Kari, 2003).

Back to extensibility, in $\mathscr{K}_6$ there exists a 2-subset that cannot be extended to a larger subset by any word of length 6 (and even by any word of length 7).

23. Extensibility vs Kari's Example

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However, studying the extensibility phenomenon in synchronizing automata appears to be worthwhile: if there is a linear bound on the minimum length of words extending non-singleton proper subsets of a synchronizing automaton, then there is a quadratic bound on the minimum length of reset words for the automaton.

24. Extension Algorithm

GREEDYEXTENSTION($\mathscr{A}$)

if $|qa^{-1}| = 1$ for all $q \in Q$ and $a \in \Sigma$ **then**
  return Failure
else
  $w \leftarrow a$ such that $|qa^{-1}| > 1$ ▷ Initializing the current word
  $P \leftarrow qa^{-1}$ such that $|qa^{-1}| > 1$ ▷ Initializing the current set
while $|P| < |Q|$ **do**
  if $|Pu^{-1}| \leq |P|$ for all $u \in \Sigma^*$ **then**
    return Failure
  else
    take a word $v \in \Sigma^*$ of minimum length with $|Pv^{-1}| > |P|$
  $w \leftarrow vw$ ▷ Updating the current word
  $P \leftarrow Pv^{-1}$ ▷ Updating the current set
return w

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Let α be a positive real number. An automaton $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ is α-extensible if for any subset $P \subset Q$ there is $w \in \Sigma^*$ of length at most $\alpha|Q|$ such that $|Pw^{-1}| > |P|$.

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Several important classes of synchronizing automata are known to be 2-extensible, for instance, one-cluster automata (Marie-Pierre Béal, Mikhail Berlinkov, Dominique Perrin, A quadratic upper bound on the size of a synchronizing word in one-cluster automata, Int. J. Found. Comput. Sci., 22, 277–288 (2011)).

On the other hand, for any $\alpha < 2$ Mikhail Berlinkov (On a conjecture by Carpi and D'Alessandro, Int. J. Found. Comput. Sci. 22, 1565–1576 (2011)) constructed a synchronizing one-cluster automaton that is not α-extensible.

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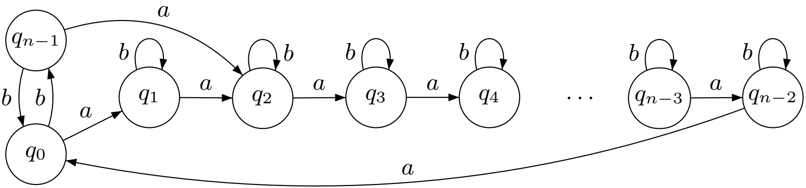

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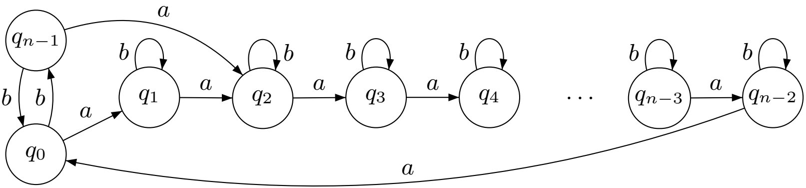

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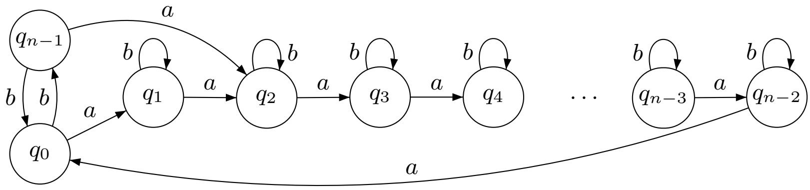

For $n > \frac{3}{2-\alpha}$, this automaton is not α-extensible. In fact, the shortest word that extends the set $\{q_0, q_{n-1}\}$ is $a^{n-2}ba^{n-2}$.

Finally, Andrzej Kisielewicz and Marek Szykuła (Synchronizing automata with extremal properties, MFCS 2015, LNCS 9234, 331–343 (2015)) constructed a series of synchronizing automata that are not α-extensible for any α.

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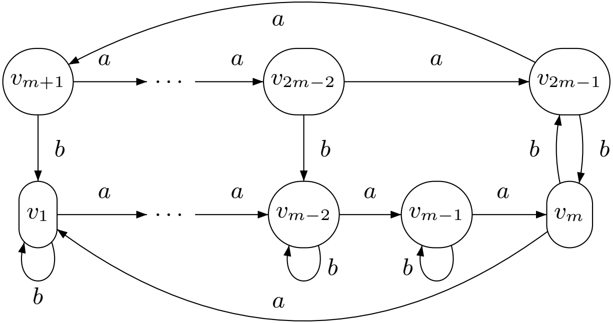

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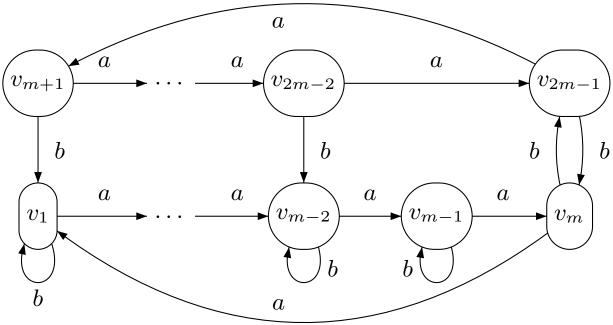

The automata in the series have subsets that require words of length as big as $m^2 + O(m)$ in order to be extended.

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Some experimental work that can be used in this direction has been done by Adam Roman and Marek Szykuła (Forward and backward synchronizing algorithms, Expert Systems with Applications, 42, 9512–9527 (2015)).