Synchronizing Finite Automata

Lecture IV: The Černý Conjecture

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- Σ the input alphabet
- $\delta : Q \times \Sigma \to Q$ the transition function

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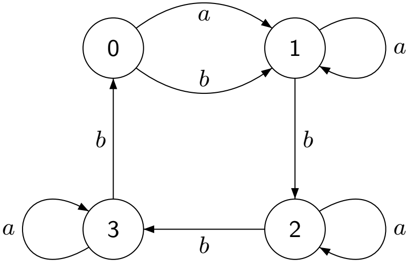
Any w with this property is a reset word for $\mathscr{A}$.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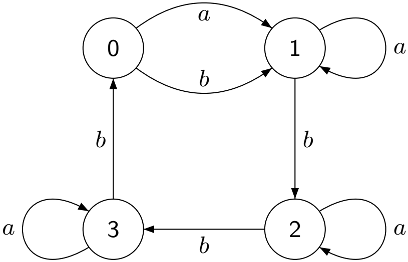

A reset word is $abbbabbba$. In fact, we have verified that this is the shortest reset word for this automaton.

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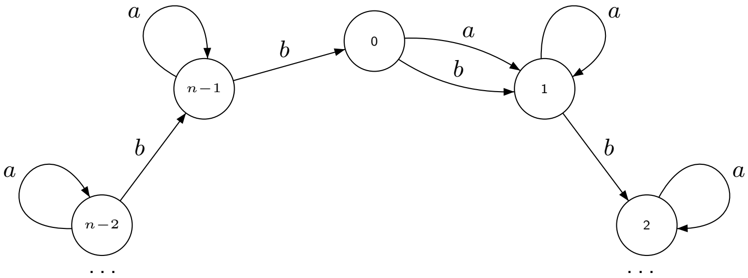
The automaton in the previous slide is $\mathscr{C}_4$.

Here is a generic automaton from the Černý series:



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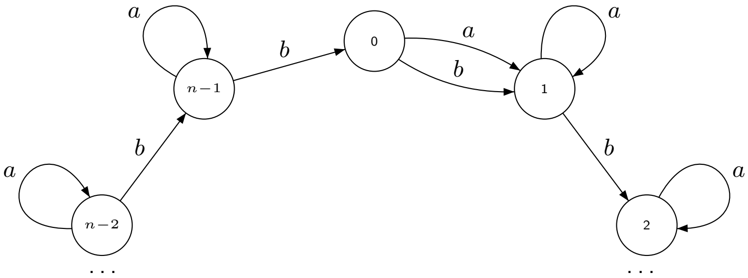

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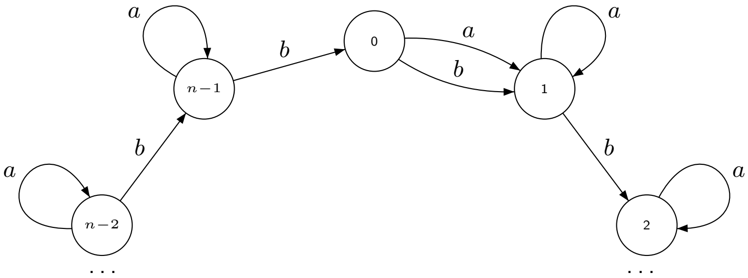

Černý has proved that the shortest reset word for $\mathscr{C}_n$ is $(ab^{n-1})^{n-2}a$ of length $(n-1)^2$. As other results from Černý's paper of 1964, this nice series of automata has been rediscovered many times.

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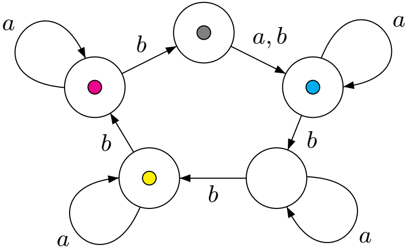
- The goal — to free all but one states.
- The only coin that remains at the end of the game is the golden coin G.

6. Example



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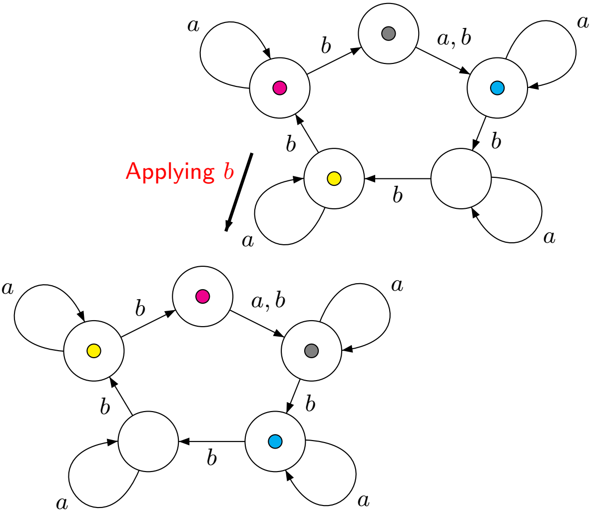

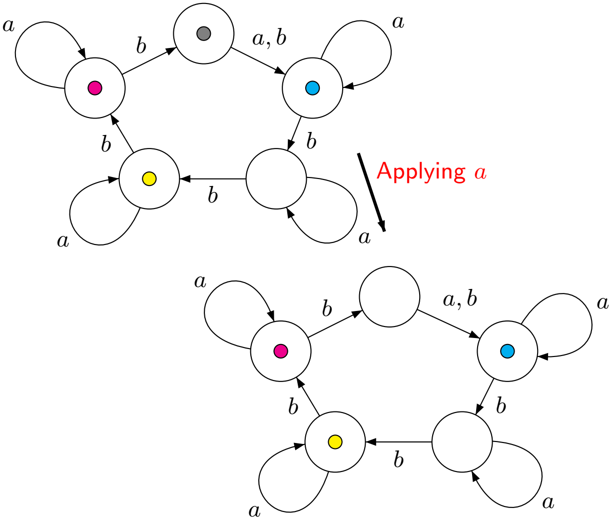

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Applying a
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(ii) for each $i = 1, \ldots, |w|$, the action of the i^{th} letter of w decreases the weight by 1 at most, that is, $1 \geq \mathrm{wg}(P_{i-1}) - \mathrm{wg}(P_i)$.

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Then $$|w| = \sum_{i=1}^{|w|} 1 \geq \sum_{i=1}^{|w|} \bigl(\mathrm{wg}(P_{i-1}) - \mathrm{wg}(P_i)\bigr) =$$
$$\mathrm{wg}(P_0) - \mathrm{wg}(P_{|w|}) \geq n(n-1) - (n-1) = (n-1)^2.$$

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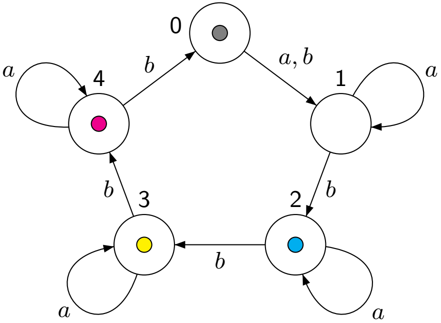
The weight of P_i is the maximum weight of the coins present in this position.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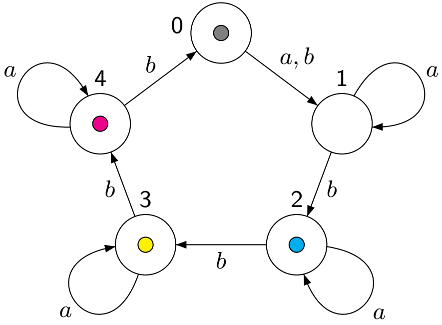

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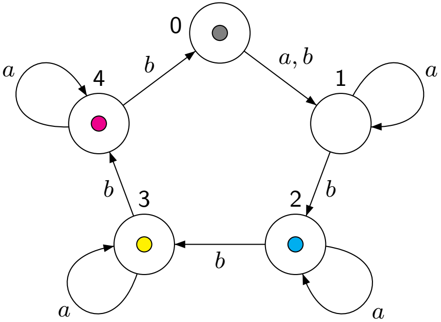

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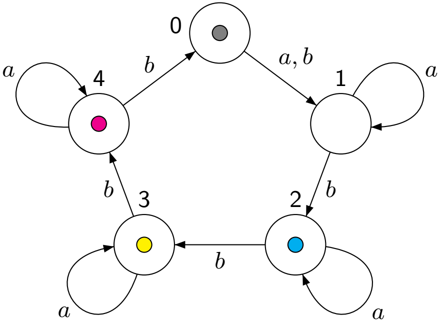

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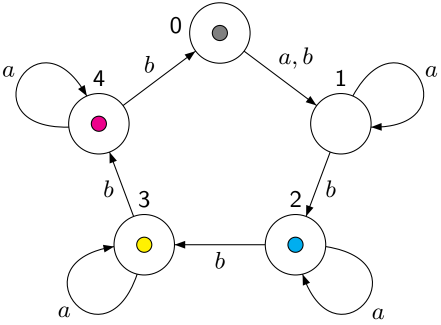

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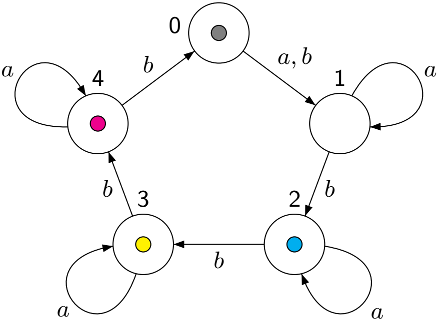

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The weight of the gray coin is $5 \cdot 3 + 0 = 15$.
The weight of the red coin is $5 \cdot 4 + 1 = 21$,
and this is the weight of the position.

We have to check that our weight function satisfies the conditions

(i) $\mathrm{wg}(P_0) \geq n(n-1)$ and $\mathrm{wg}(P_{|w|}) \leq n-1$;

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Since the weight of a position is not less than the weight of any coin in this position, we have $\mathrm{wg}(P_0) \geq n(n-1)$.

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This completes the proof.

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This simply looking conjecture is arguably the most longstanding open problem in the combinatorial theory of finite automata. Everything we know about the conjecture in general can be summarized in just one line:

$$(n-1)^2 \leq C(n) \leq \frac{\min\{\frac{85059n^3+90024n^2+196504n-10648}{85184}, n^3-n\}}{6}.$$

14. Why it is hard?

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- non-locality: prefixes of optimal solutions need not be optimal (that is why the greedy algorithm fails);
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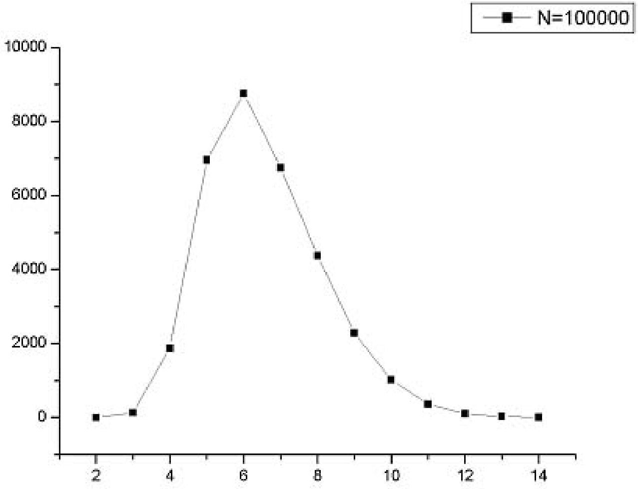
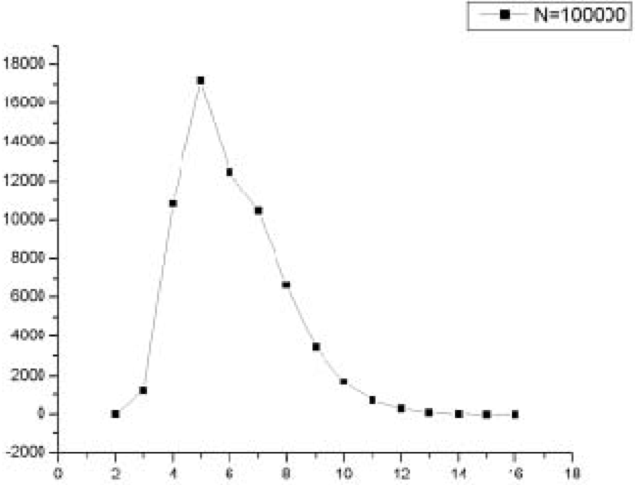
- non-locality: prefixes of optimal solutions need not be optimal (that is why the greedy algorithm fails);
- combinatorics of finite sets is encoded in the problem.

Yet another reason: "slowly" synchronizing automata turn out to be extremely rare. The only known infinite series of n-state synchronizing automata with reset threshold $(n-1)^2$ is the Černý series $\mathscr{C}_n$, $n = 2, 3, \ldots$, with a few sporadic examples for $n \leq 6$.

15. 20-State Experiment




N=100000
18000
16000
14000
12000
10000
8000
6000
4000
2000
0
-2000
0
2
4
6
8
10
12
14
16
18

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Thus, Černý conjecture holds true almost surely.

Moreover, even “slowly” synchronizing automata cannot be discovered via a random sampling.

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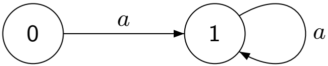
For the sake of completeness, we start with $n = 2$:



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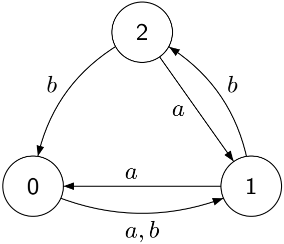

19. Sporadic Examples: $n = 3$

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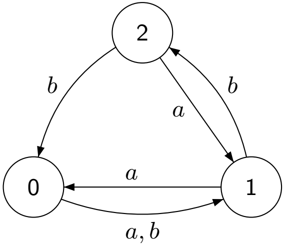

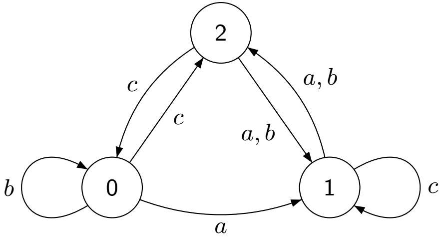

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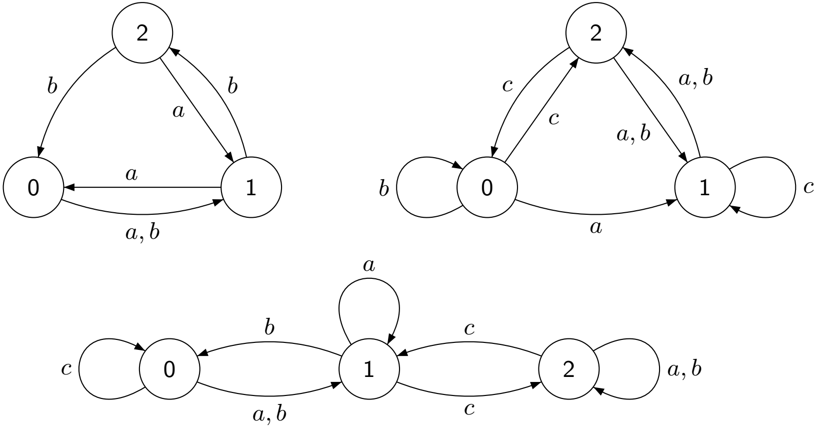

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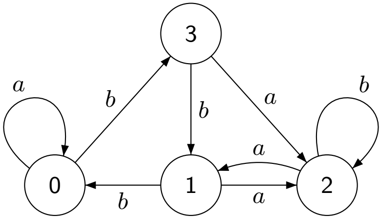

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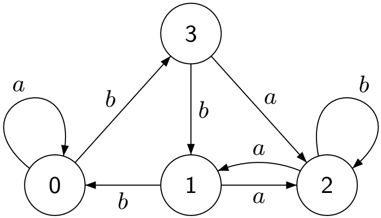

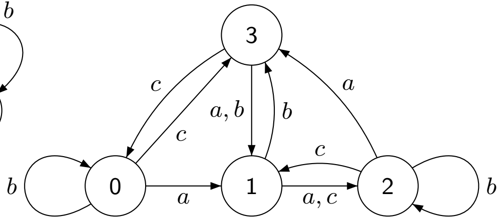

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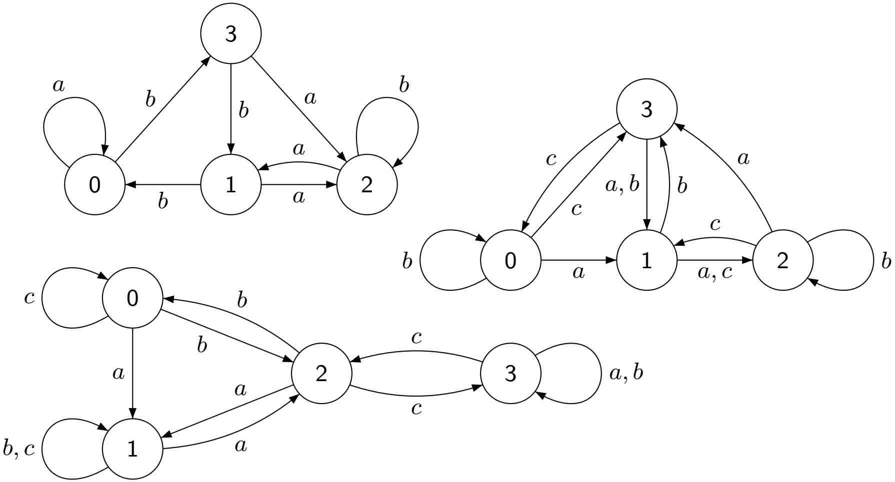

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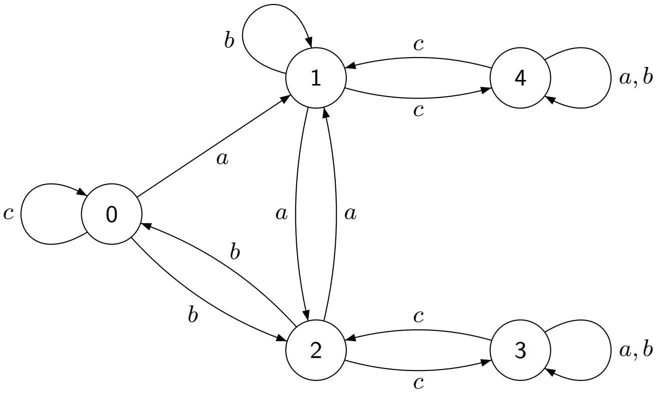

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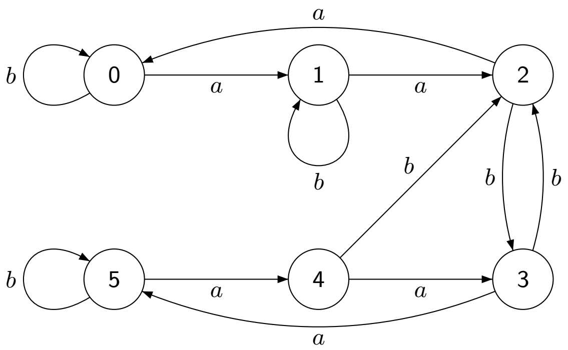

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Recent exhaustive search experiments (Andrzej Kisielewicz, Jakub Kowalski and Marek Szykuła, Experiments with synchronizing automata, CIAA 2016, LNCS 9705, 176–188, 2016) have indicated that likely $\mathscr{K}_6$ is the only 'proper' counter example to Pin's conjecture.

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Again, the Černý conjecture corresponds to the case $k = 1$.

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However, 4 is not the maximum number of tokens that can be removed! One can show that 5 states can be freed by a sequence of 25 moves — in full accordance with the rank conjecture.