Synchronizing Finite Automata

Lecture III: Complexity Issues

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Spring of 2021

1. Recap

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- Q the state set
- Σ the input alphabet
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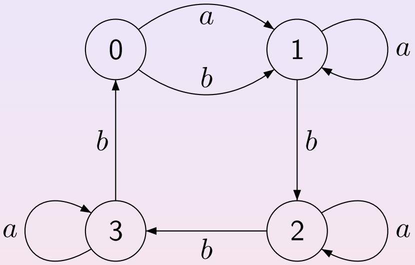
Any w with this property is a reset word for $\mathscr{A}$.

2. Example



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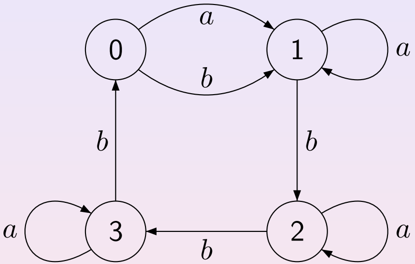

A reset word is $abbbabbba$. In fact, this is the shortest reset word for this automaton.

There is a algorithm that uses a natural greedy strategy and, when given a synchronizing automaton $\mathscr{A}$ with n states, finds a reset word of length at most $\frac{n^3-n}{6}$ for $\mathscr{A}$ spending polynomial time as a function of n. (In fact, time is $O(n^3)$).

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GreedyCompression($\mathscr{A}$)

```
w ← ε                                          ▷ Initializing the current word
P ← Q                                          ▷ Initializing the current set
while |P| > 1 do
    if |P . u| = |P| for all u ∈ Σ* then
        return Failure
    else
        take a word v ∈ Σ* of minimum length with |P . v| < |P|
    w ← wv                                     ▷ Updating the current word
    P ← P . v                                  ▷ Updating the current set
return w
```

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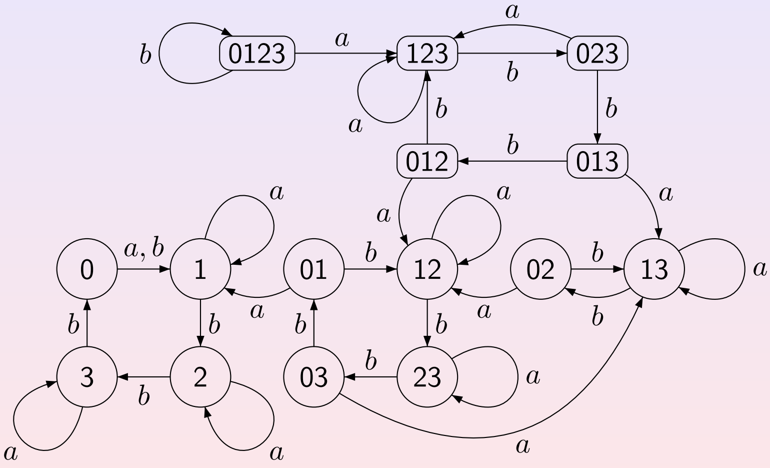
We have already seen that the greedy algorithm fails to find a reset word of minimum length.



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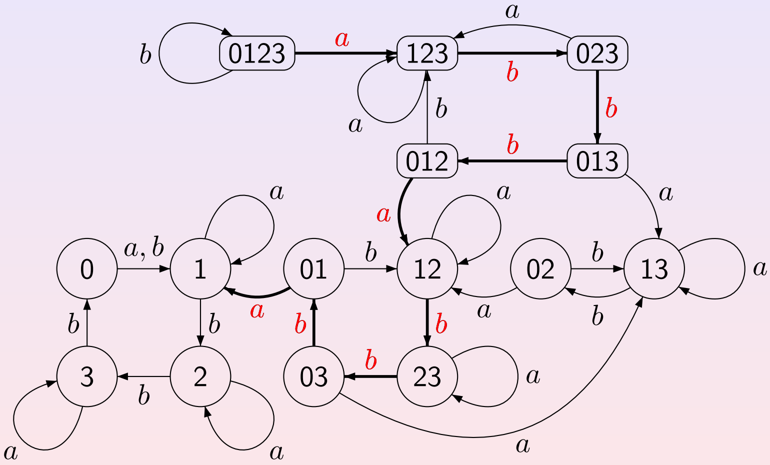

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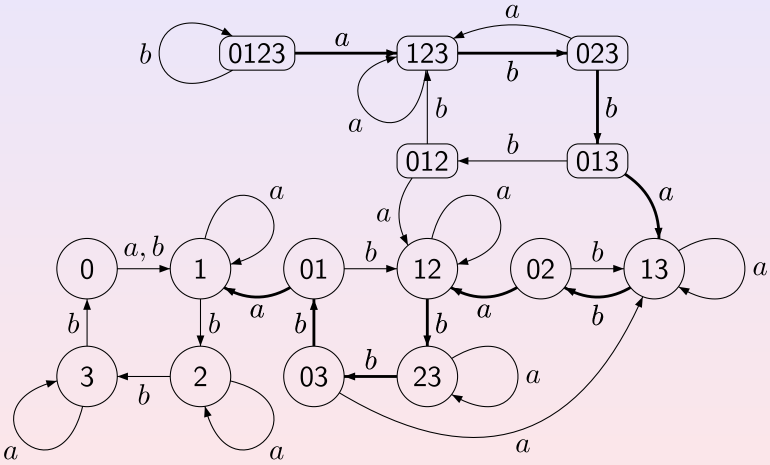

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Now we aim to prove that under standard assumptions (like NP $\neq$ coNP) no polynomial algorithm, even non-deterministic, can find the minimum length of reset words for synchronizing automata.

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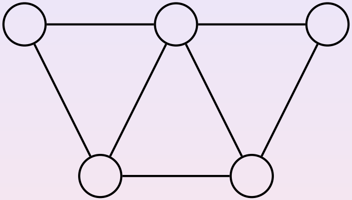

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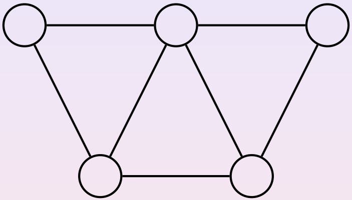

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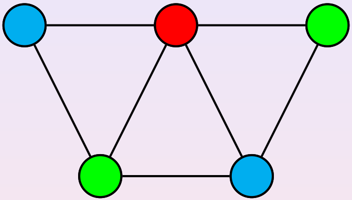

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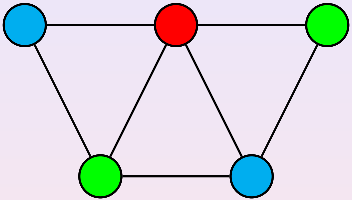

The question is whether the vertices of G can be labeled
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For the above graph, the answer to 3-COLOR is YES
while the answer to 2-COLOR is NO.

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A problem is in coNP if, whenever the answer to its instance is NO, Merlin can convince Arthur that the answer is NO in polynomial time (of the size of the input).

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How can one prove that a problem is NP-hard? Via a polynomial reduction from some problem known to be NP-complete.

Consider the following decision problem:

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Several authors have observed that SHORT-RESET-WORD is NP-hard by a transparent reduction from SAT which is a classical NP-complete problem.

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The answer to an instance C is YES if C has a satisfying assignment (i.e., a truth assignment on V that satisfies C) and NO otherwise.

Given an instance C of SAT with n variables $x_1, \dots, x_n$ and m clauses $c_1, \dots, c_m$, one constructs $\mathscr{A}(C)$ with 2 input letters a and b and the state set $\{z, q_{i,j} \mid 1 \le i \le m,\ 1 \le j \le n+1\}$.

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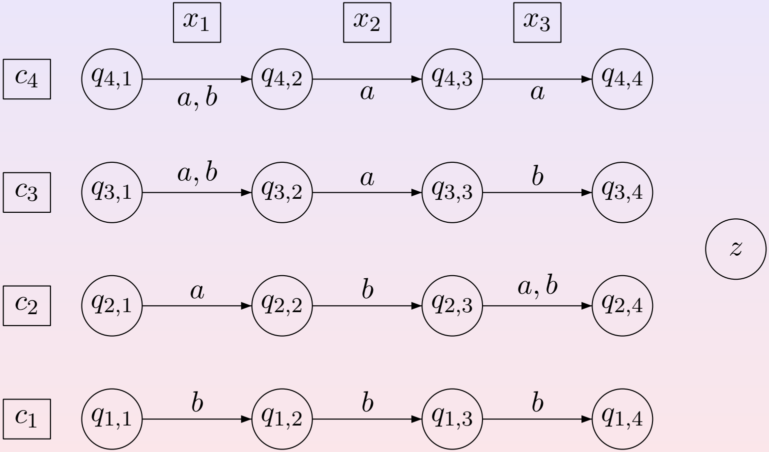

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If we change C to $C' = \{x_1 \vee x_2, \neg x_1 \vee x_2, \neg x_2 \vee x_3, \neg x_2 \vee \neg x_3\}$, it becomes unsatisfiable and $\mathscr{A}(C')$ is reset by no word of length 3.

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Thus, assigning the instance $(\mathscr{A}(C), n)$ of SHORT-RESET-WORD to an arbitrary n-variable instance C of SAT, one gets a polynomial reduction

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Thus, assigning the instance $(\mathscr{A}(C), n)$ of SHORT-RESET-WORD to an arbitrary n-variable instance C of SAT, one gets a polynomial reduction which is in fact parsimonious, i.e., there is a 1-1 correspondence between the satisfying assignments for C and reset words of length n for $\mathscr{A}(C)$.

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16. Reduction from SAT

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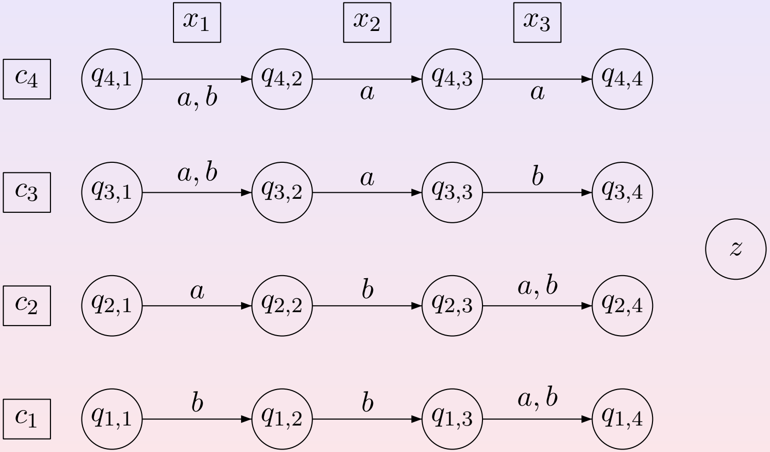

17. Shortest Reset Words are Even Harder to Decide

Now consider the following decision problem:

SHORTEST-RESET-WORD: *Given a synchronizing automaton* $\mathscr{A}$ *and a positive integer* ℓ, *is it true that the minimum length of a reset word for* $\mathcal{A}$ *is equal to* ℓ?

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Assigning the instance $(\mathcal{A}(C), n+1)$ of SHORTEST-RESET-WORD to an arbitrary system C of clauses on n variables, one sees that the answer to the instance is "Yes" if and only if C is not satisfiable.

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SHORTEST-RESET-WORD has been shown to be complete for DP (Difference Polynomial-Time) by Jörg Olschewski and Michael Ummels, The complexity of finding reset words in finite automata, MFCS 2010, LNCS 6281: 568–579, 2010.

18. Computing is Harder than Deciding

$P^{NP[\log]}$ is the class of all problems that can be solved by a deterministic polynomial-time Turing machine that has an access to an oracle for an NP-complete problem, with the number of queries being logarithmic in the size of the input.

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Finding the shortest reset words may be even harder than computing their length but the exact complexity is not yet known.

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Mikhail Berlinkov has shown that under NP $\neq$ P, for no k, there may exist a polynomial algorithm that, given a synchronizing automaton with two input letters, produces a reset word whose length is less than $k\times$minimum possible length of a reset word (Approximating the minimum length of synchronizing words is hard, Theory Comput. Syst. 54:2, 211–223, 2014).

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The next question was: is approximating within a logarithmic factor possible?

20. Non-approximability: Logarithmic Factor

Michael Gerbush and Brent Heeringa (Approximating minimum reset sequence, CIAA 2010, LNCS 6482: 154–162, 2010) have observed that SET COVER admits a transparent reduction to the problem of finding a reset word of minimum length for a given synchronizing automaton.

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Using a difficult result on SET COVER by Alon, Moshkovitz and Safra, Gerbush and Heeringa have deduced that the minimum length of reset words for synchronizing automata with n states and unbounded alphabet cannot be approximated within the factor $c \log n$ for some constant $c > 0$ unless P = NP.

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Berlinkov has obtained a similar result for synchronizing automata with only 2 input letters (On two algorithmic problems about synchronizing automata, DLT 2014, LNCS 8633: 61–67, 2014).

21. Non-approximability: Sublinear Factor

Finally, Pawel Gawrychowski and Damian Straszak have shown that for every $\varepsilon > 0$ it is not possible to approximate the length of the shortest reset word for synchronizing automata with n states within a factor of $n^{1-\varepsilon}$ in polynomial time, unless P = NP (Strong inapproximability of the shortest reset word, MFCS 2015 Part 1, LNCS 9234: 243–255, 2015).

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22. Constrained Synchronization

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In symbols, given a DFA $\mathscr{A} = \langle Q, \Sigma \rangle$, we fix a letter $a \in \Sigma$ and seek a reset word from the (regular) language $a(\Sigma \setminus \{a\})^* a$.

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In symbols, given a DFA $\mathscr{A} = \langle Q, \Sigma \rangle$, we fix a letter $a \in \Sigma$ and seek a reset word from the (regular) language $a(\Sigma \setminus \{a\})^* a$. Does $\mathscr{A}$ admit synchronization under such a constraint?

The constraint language $a(\Sigma \setminus \{a\})^*a$ is very simple and looks innocent.

23. Constrained Synchronization: An Example

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The problem of deciding whether a given $\mathscr{A} = \langle Q, \Sigma \rangle$ has a reset word from the language $a(\Sigma \setminus \{a\})^*a$, for a fixed letter $a \in \Sigma$, is PSPACE-complete if $|\Sigma| \geq 3$ and NP-complete if $|\Sigma| = 2$.

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To prove hardness we reduce from DFA-Intersection Nonemptiness. Membership in PSPACE is easy due to Savitch's Theorem. A relatively nontrivial part is membership in NP for $|\Sigma| = 2$. There are DFAs $\mathscr{A} = \langle Q, \{a, b\} \rangle$ for which the least N such that $ab^N a$ is a reset word for $\mathscr{A}$ is exponentially big with respect to $|Q|$. One can guess N in binary, but a direct verification that $ab^N a$ resets $\mathscr{A}$ may be unfeasible.

24. Regular Constraints

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Recall that $L = ab^*a$ provides NP-completeness while for $L = a(b+c)^*a$ we get PSPACE-completeness. Of course, L-Synchronization can be in P.

So far we have not found representatives for any other complexity class so that one can state a trichotomy conjecture (but I do not believe in it).