Synchronizing Finite Automata

Lecture II: Algorithmic Issues

Mikhail Volkov

Ural Federal University

Spring of 2021

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- Σ the input alphabet
- $\delta : Q \times \Sigma \to Q$ the transition function

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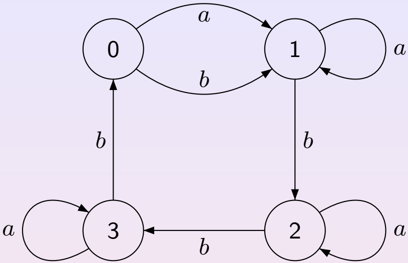
Any w with this property is a reset word for $\mathscr{A}$.

2. Example



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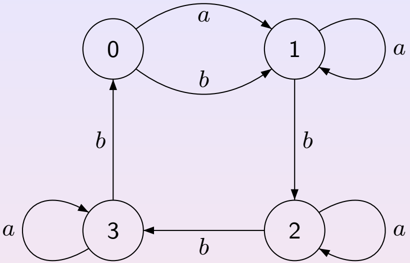

A reset word is $abbbabbba$. In fact, we will soon see that this is the shortest reset word for this automaton.

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Exercise: Write down a proof of this claim!

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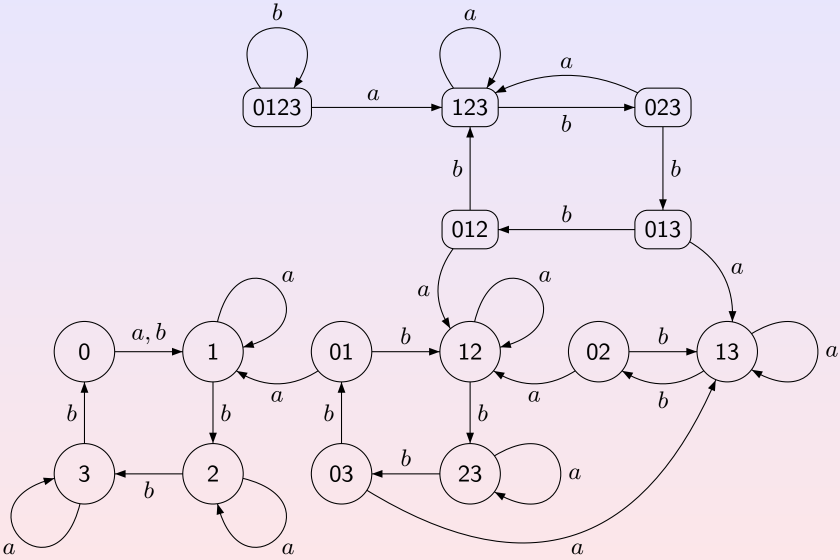

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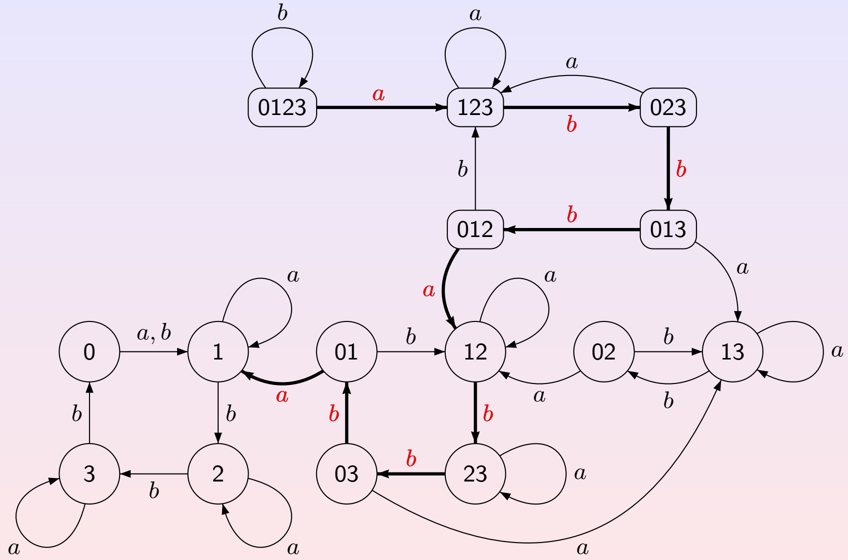

Thus, the question of whether or not a given DFA $\mathscr{A}$ is synchronizing reduces to the following reachability question in the underlying digraph of the power automaton $\mathcal{P}(\mathscr{A})$: is there a path from Q to a singleton? The latter question can be easily answered by BFS.

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Proposition. *A DFA* $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$ *is synchronizing iff for every* $q, q' \in Q$ *there exists a word* $w \in \Sigma^*$ *such that* $\delta(q, w) = \delta(q', w)$.

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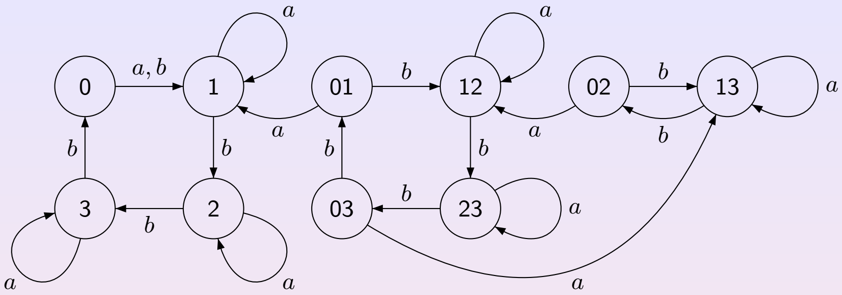

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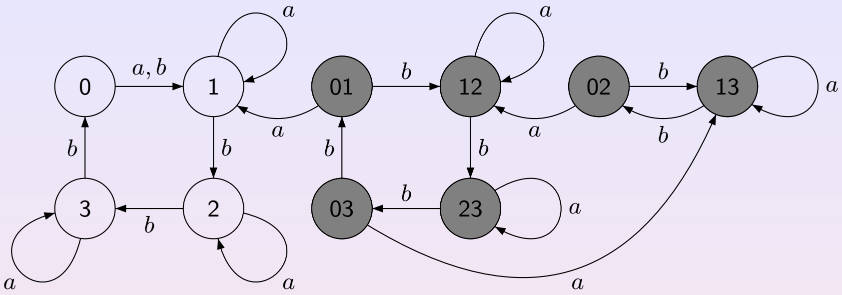

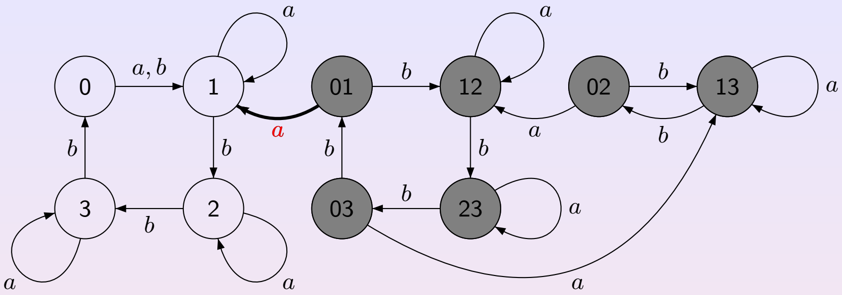

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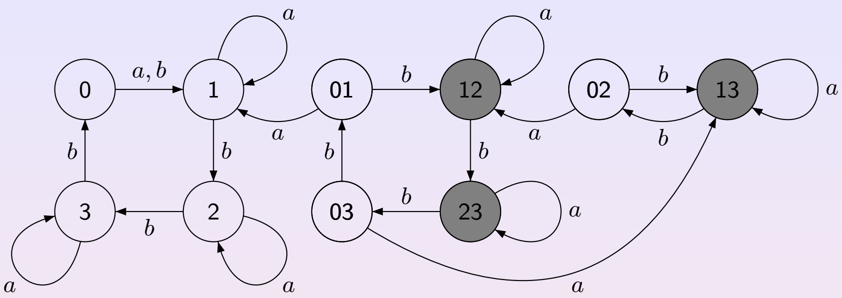

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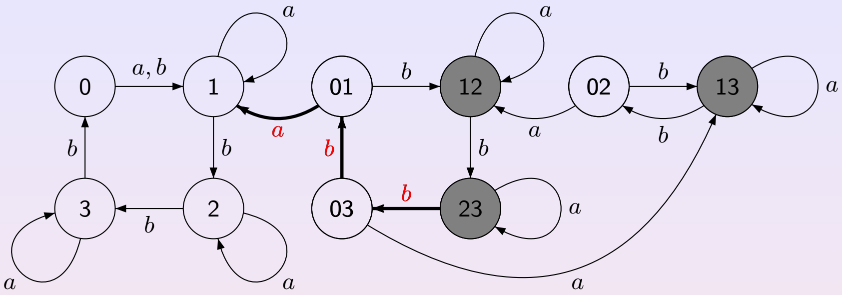

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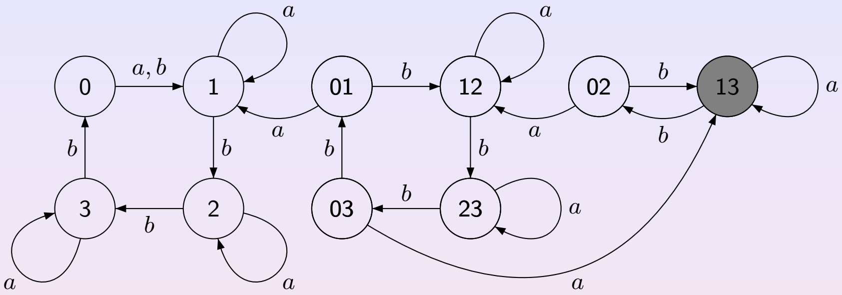

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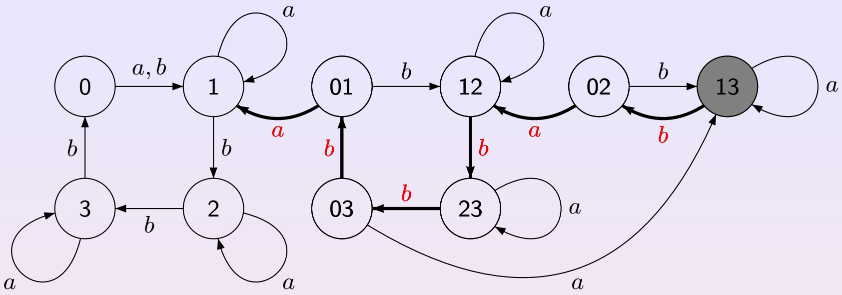

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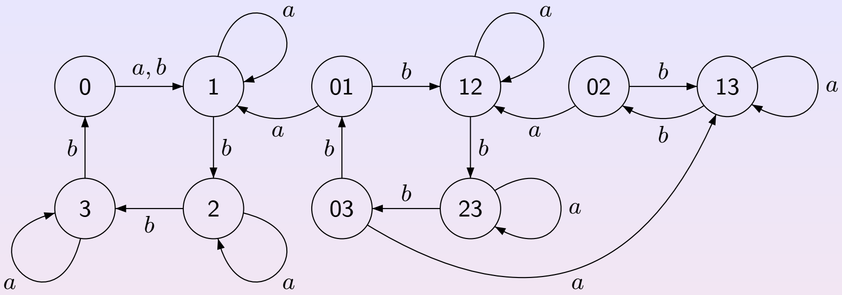

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Observe that the reset word constructed this way is of length 10 while we know a reset word of length 9.

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`https://github.com/birneAgeev/AutomataSynchronizationChecker`

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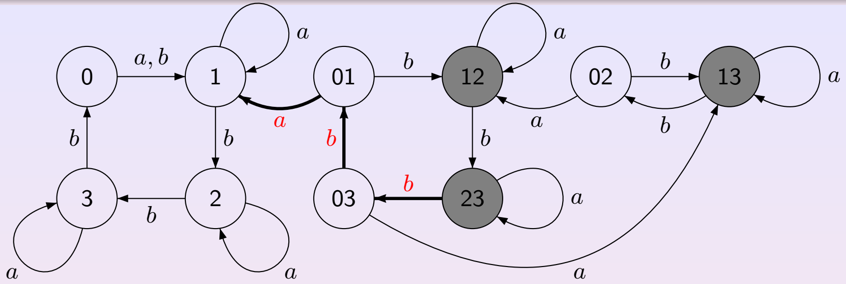
Can we do better? What is the exact bound?


0
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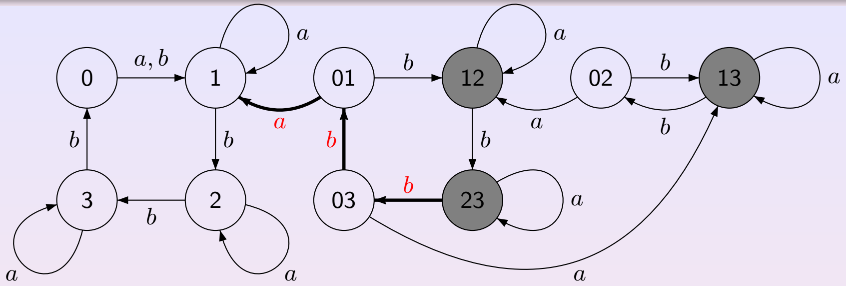

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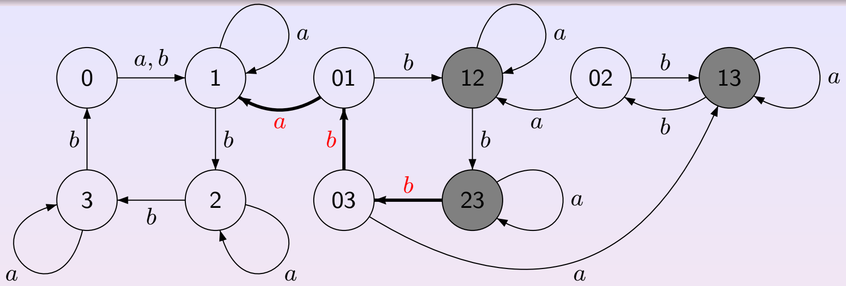

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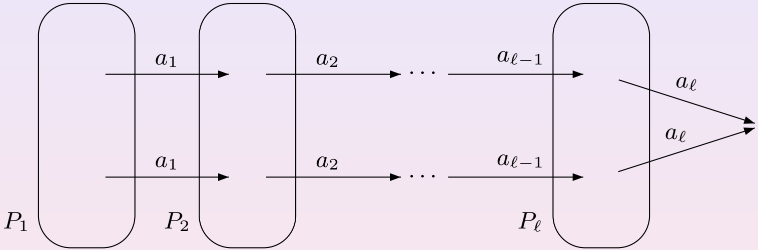
Consider a generic step of the algorithm at which states to be compressed form a set P with $|P| = k > 1$ and let $v = a_1 \cdots a_\ell$ with $a_i \in \Sigma$, $i = 1, \ldots, \ell$, be a word of minimum length such that $|P \,.\, v| < k$.

The sets $P_1 = P,\ P_2 = P_1 \,.\, a_1,\ \dots,\ P_\ell = P_{\ell-1} \,.\, a_{\ell-1}$ are k-subsets of Q.



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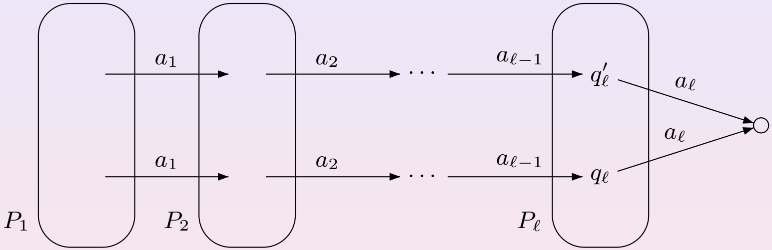

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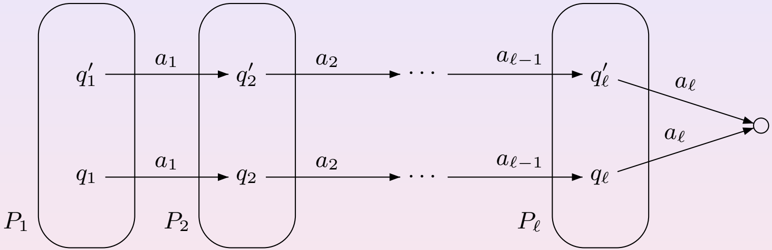

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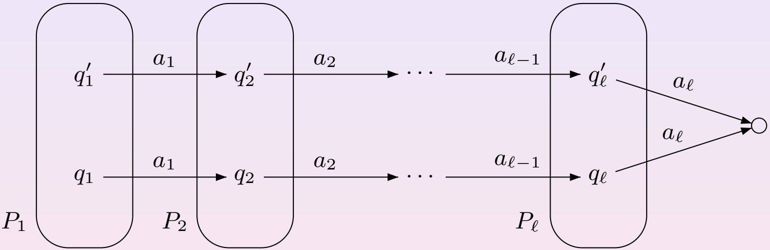

The condition that v is a word of minimum length with $|P \,.\, v| < |P|$ implies $R_i \nsubseteq P_j$ for $1 \leq j < i \leq \ell$.

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A construction: fix a $(k-2)$-subset W of Q, list all $\binom{n-k+2}{2}$ 2-subsets of $Q \setminus W$ and let T_i be the union of W with the i^{th} 2-subset in the list.

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We identify Q with $\{1, 2, \ldots, n\}$ and assign to each k-subset $I = \{i_1, \ldots, i_k\}$ the following polynomial $D(I)$ in variables $x_{i_1}, \ldots, x_{i_k}$ over the field of reals.

$$I = \{i_1, \dots, i_k\} \mapsto D(I) = \begin{vmatrix} 1 & i_1 & i_1^2 & \cdots & i_1^{k-3} & x_{i_1} & x_{i_1}^2 \\ 1 & i_2 & i_2^2 & \cdots & i_2^{k-3} & x_{i_2} & x_{i_2}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & i_k & i_k^2 & \cdots & i_k^{k-3} & x_{i_k} & x_{i_k}^2 \end{vmatrix}_{k \times k}$$

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Then one proves that:
• the polynomials $D(P_1), \ldots, D(P_\ell)$ are linearly independent whenever the k-subsets $P_1, \ldots, P_\ell$ form a renewing sequence;

$$I = \{i_1, \dots, i_k\} \mapsto D(I) = \begin{vmatrix} 1 & i_1 & i_1^2 & \cdots & i_1^{k-3} & x_{i_1} & x_{i_1}^2 \\ 1 & i_2 & i_2^2 & \cdots & i_2^{k-3} & x_{i_2} & x_{i_2}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & i_k & i_k^2 & \cdots & i_k^{k-3} & x_{i_k} & x_{i_k}^2 \end{vmatrix}_{k \times k}$$

Then one proves that:

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Suppose that k-subsets $P_1, \ldots, P_\ell$ form a renewing sequence but $D(P_1), \ldots, D(P_\ell)$ are linearly dependent.

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Then some polynomial $D(P_j)$ should be expressible as a linear combination of the preceding polynomials $D(P_1), \ldots, D(P_{j-1})$.

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Then some polynomial $D(P_j)$ should be expressible as a linear combination of the preceding polynomials $D(P_1),\dots,D(P_{j-1})$.
By the definition of a renewing sequence, P_j contains a couple $\{p,p'\}$ such that $\{p,p'\}\nsubseteq P_i$ for all $i<j$.

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By the definition of a renewing sequence, P_j contains a couple $\{p,p'\}$ such that $\{p,p'\}\nsubseteq P_i$ for all $i<j$.

If we substitute $x_p=p$, $x_{p'}=p'$, and $x_t=0$ for $t\neq p,p'$ in each of the polynomials $D(P_1),\dots,D(P_j)$, then the polynomials $D(P_1),\dots,D(P_{j-1})$ vanish (since the two last columns in each of the resulting determinants become proportional) and so does any linear combination of the polynomials.

$$D(P_j)\binom{x_p = p,\ x_{p'} = p',}{x_t = 0,\ t \neq p, p'} = \begin{vmatrix} 1 & i_1 & i_1^2 & \cdots & i_1^{k-3} & p & p^2 \\ 1 & i_2 & i_2^2 & \cdots & i_2^{k-3} & p' & (p')^2 \\ 1 & i_3 & i_3^2 & \cdots & i_3^{k-3} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & i_k & i_k^2 & \cdots & i_k^{k-3} & 0 & 0 \end{vmatrix}_{k\times k}$$

(For simplicity, here we assume that $i_1 = p$, $i_2 = p'$.)

The value of $D(P_j)$ under the substitution $x_p = p$, $x_{p'} = p'$, and $x_t = 0$ for $t \neq p, p'$ is the determinant being the product of a Vandermonde $(k-2) \times (k-2)$-determinant with the 2×2-determinant $\begin{vmatrix} p & p^2 \\ p' & (p')^2 \end{vmatrix} = pp'(p'-p)$, whence this value is not 0.

$$D(P_j)\binom{x_p = p,\, x_{p'} = p',}{x_t = 0,\, t \neq p, p'} = \begin{vmatrix} 1 & i_1 & i_1^2 & \cdots & i_1^{k-3} & p & p^2 \\ 1 & i_2 & i_2^2 & \cdots & i_2^{k-3} & p' & (p')^2 \\ 1 & i_3 & i_3^2 & \cdots & i_3^{k-3} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & i_k & i_k^2 & \cdots & i_k^{k-3} & 0 & 0 \end{vmatrix}_{k\times k}$$

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The value of $D(P_j)$ under the substitution $x_p = p$, $x_{p'} = p'$, and $x_t = 0$ for $t \neq p, p'$ is the determinant being the product of a Vandermonde $(k-2)\times(k-2)$-determinant with the 2×2-determinant $\begin{vmatrix} p & p^2 \\ p' & (p')^2 \end{vmatrix} = pp'(p'-p)$, whence this value is not 0.

Hence $D(P_j)$ cannot be equal to any linear combination of $D(P_1),\dots,D(P_{j-1})$.

Now we aim to prove that the polynomials $D(T_1), \dots, D(T_s)$ (derived from the "standard" sequence $T_1, \dots, T_s$ of length $s = \binom{n-k+2}{2}$) generate the linear space spanned by all polynomials of the form $D(I)$.

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Now we aim to prove that the polynomials $D(T_1), \ldots, D(T_s)$ (derived from the "standard" sequence $T_1, \ldots, T_s$ of length $s = \left(\binom{n-k+2}{2}\right)$ generate the linear space spanned by all polynomials of the form $D(I)$. Take an arbitrary k-element subset $I = \{i_1, \ldots, i_k\}$ of Q. We claim that the polynomial $D(I)$ is a linear combination of $D(T_1), \ldots, D(T_s)$.

Recall that each of the sets $T_1, \ldots, T_s$ is the union of some fixed $(k-2)$-subset W of Q with a couple of states from $Q \setminus W$.

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Recall that each of the sets $T_1, \dots, T_s$ is the union of some fixed $(k-2)$-subset W of Q with a couple of states from $Q \setminus W$. We prove the above claim by induction on the cardinality of the set $I \setminus W$. If $|I \setminus W| = 2$, then I is the union of W with some couple from $Q \setminus W$, whence $I = T_i$ for some $i = 1, \dots, s$. Thus, $D(I) = D(T_i)$ and our claim holds true.

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$$\Delta = \begin{vmatrix} p(i_0) & 1 & i_0 & i_0^2 & \cdots & i_0^{k-3} & x_{i_0} & x_{i_0}^2 \\ p(i_1) & 1 & i_1 & i_1^2 & \cdots & i_1^{k-3} & x_{i_1} & x_{i_1}^2 \\ p(i_2) & 1 & i_2 & i_2^2 & \cdots & i_2^{k-3} & x_{i_2} & x_{i_2}^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ p(i_k) & 1 & i_k & i_k^2 & \cdots & i_k^{k-3} & x_{i_k} & x_{i_k}^2 \end{vmatrix}_{(k+1)\times(k+1)}.$$

17. Linearization, Step 2, continued

If $|I \setminus W| > 2$, there is $i_0 \in W \setminus I$. Let $I' = I \cup \{i_0\}$. There exists a polynomial $p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 \cdots + \alpha_{k-3} x^{k-3}$ over $\mathbb{R}$ such that $p(i_0) = 1$ and $p(i) = 0$ for all $i \in W \setminus \{i_0\}$. Consider the determinant

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Clearly, $\Delta = 0$ as the first column is the sum of the next $k-2$ columns with the coefficients $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{k-3}$.

18. Linearization, Step 2, completed

Expanding Δ by the first column gives the identity

$$\sum_{j=0}^{k}(-1)^j p(i_j)D(I' \setminus \{i_j\}) = 0.$$

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Since $p(i_0) = 1$ and $I' \setminus \{i_0\} = I$, the identity rewrites as

$$D(I) = \sum_{j=1}^{k}(-1)^{j+1} p(i_j) D(I' \setminus \{i_j\}),$$

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\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
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$$D(I)=\sum_{j=1}^{k}(-1)^{j+1}p(i_j)D(I'\setminus\{i_j\}),$$

and since $p(i)=0$ for all $i\in W\setminus\{i_0\}$, all the non-zero summands in the right-hand side are such that $i_j\notin W$.

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and since $p(i) = 0$ for all $i \in W \setminus \{i_0\}$, all the non-zero summands in the right-hand side are such that $i_j \notin W$. For each such i_j, we have

$$(I' \setminus \{i_j\}) \setminus W = I' \setminus (W \cup \{i_j\}) = (I \cup \{i_0\}) \setminus (W \cup \{i_j\}) = (I \setminus W) \setminus \{i_j\},$$

whence $|(I' \setminus \{i_j\}) \setminus W| = |I \setminus W| - 1$ and by the inductive assumption, the polynomials $D(I' \setminus \{i_j\})$ are linear combinations of the polynomials $D(T_1), \dots, D(T_s)$.

Thus, in the step when k states are still to be compressed, the compression can always be achieved by applying a suitable word of length $\leq \binom{n-k+2}{2}$.

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Summing up over $k = n, \dots, 2$, we see that the greedy algorithm always returns a reset word of length $\le \frac{n^3-n}{6}$:

Thus, in the step when k states are still to be compressed, the compression can always be achieved by applying a suitable word of length $\leq \binom{n-k+2}{2}$. Summing up over $k = n, \ldots, 2$, we see that the greedy algorithm always returns a reset word of length $\leq \frac{n^3-n}{6}$:

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n-1}{2} + \binom{n}{2} =$$

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20. Szykuła's Improvement

Up to recently, the bound $\frac{n^3-n}{6}$ (obtained almost 40 years ago) remained the best upper bound for the length of the shortest reset words for synchronizing automata with n states.

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An improvement on this bound has been found by Marek Szykuła (Improving the upper bound on the length of the shortest reset word. In STACS 2018, volume 96 of LIPIcs, pages 56:1–56:13 (2018)): the new bound is still cubic in n but improves the coefficient $\frac{1}{6} = 0.1666\ldots$ at n^3 by $\frac{125}{511104} \approx 0.000245$ so that it becomes ≈ 0.1664.

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The new bound is

$$\frac{85059n^3 + 90024n^2 + 196504n - 10648}{511104}.$$

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The new bound is

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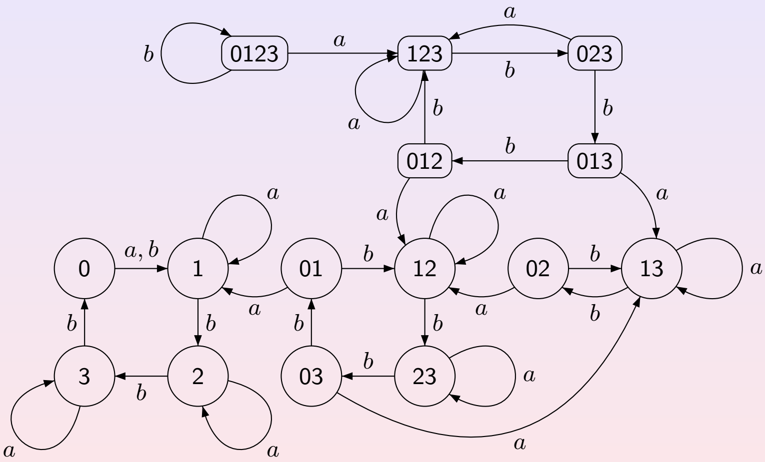
Yaroslav Shitov (An improvement to a recent upper bound for synchronizing words of finite automata. J. Autom. Lang. Comb., 24(2-4):367–373 (2019)) found a further improvement to ≈ 0.1654.

21. Greedy Algorithm

GreedyCompression($\mathscr{A}$)

```
w ← ε                                   ▷ Initializing the current word
P ← Q                                   ▷ Initializing the current set
while |P| > 1 do
    if |P . u| = |P| for all u ∈ Σ* then
        return Failure
    else
        take a word v ∈ Σ* of minimum length with |P . v| < |P|
    w ← wv                              ▷ Updating the current word
    P ← P . v                           ▷ Updating the current set
return w
```

We have already seen that the greedy algorithm fails to find a reset word of minimum length.



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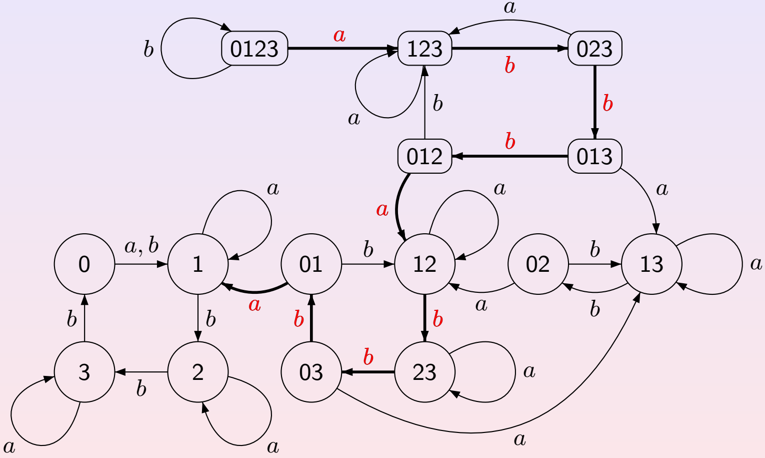

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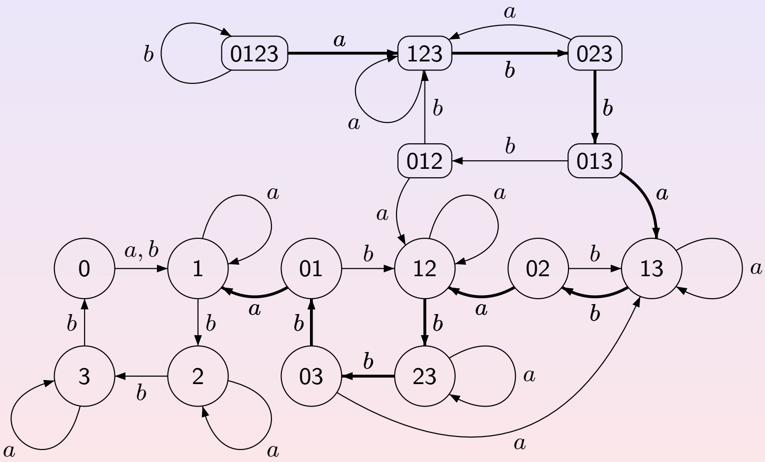

Actually, the gap between the minimum length of a reset word and the length of the word produced by the greedy algorithm may be arbitrarily large:

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23. Greedy Algorithm: Conclusion

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The behaviour of the greedy algorithm on average is not yet well understood; practically it behaves rather satisfactory.