Synchronizing Finite Automata

Lecture I. History and Motivation

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Ural Federal University

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1. Finite Automata

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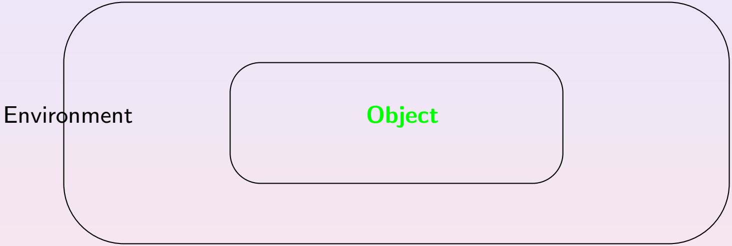

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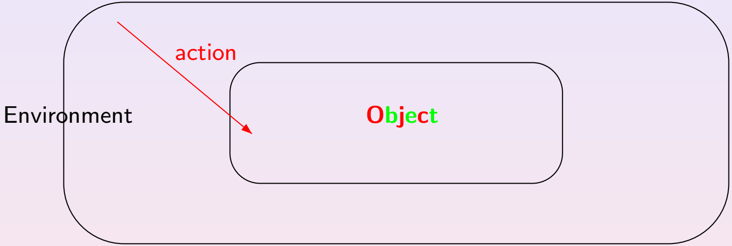

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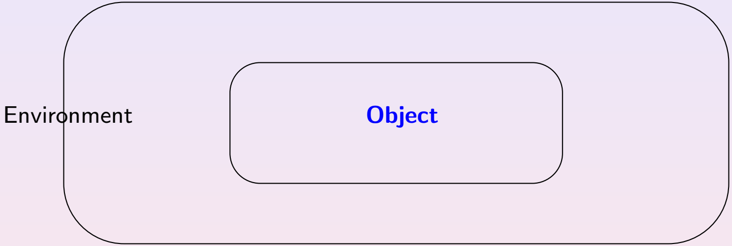

This notion originates in the seminal work by Alan Turing ("On Computable Numbers, With an Application to the Entscheidungsproblem", Proc. London Math. Soc., Ser. 2, 42 (1936), 230–265).

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Another important source is the work by neurobiologists Warren McCulloch and Walter Pitts ("A Logical Calculus of the Ideas Immanent in Nervous Activity", Bull. Math. Biophys. 5 (1943), 115–133).

3. Visualization

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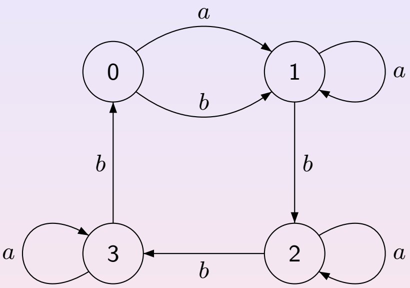

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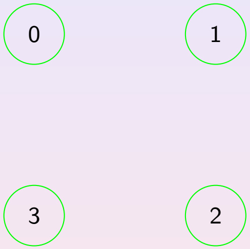

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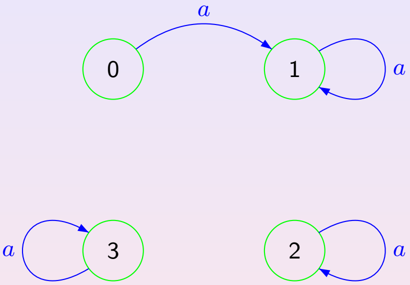

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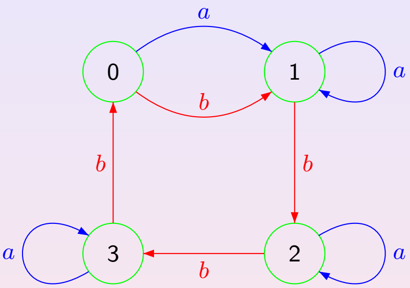

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and $P\,.\,w$ for $\{\delta(q, w) \mid q \in P\}$.

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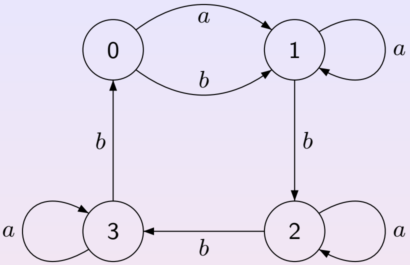
Other names:

- for automata: directable, cofinal, collapsible, etc;
- for words: directing, recurrent, synchronizing, etc.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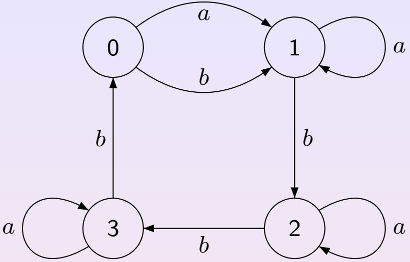

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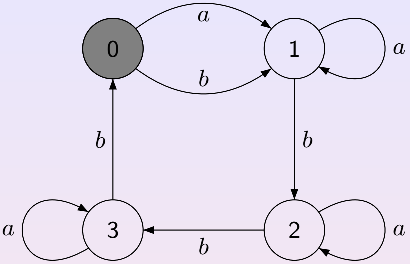

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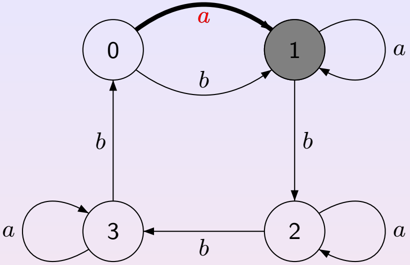

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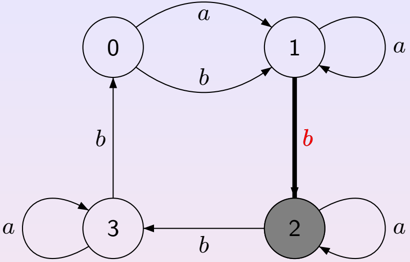

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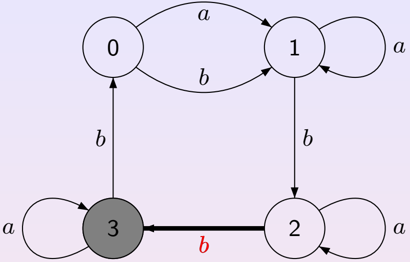

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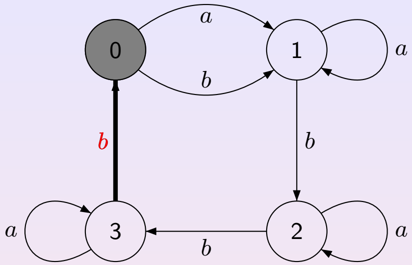

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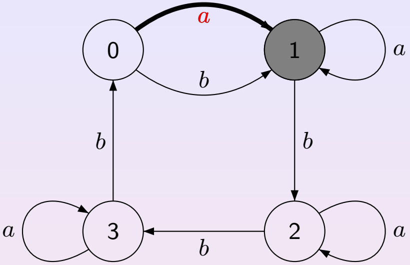

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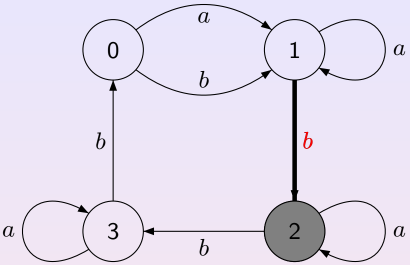

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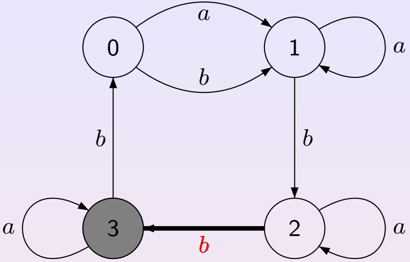

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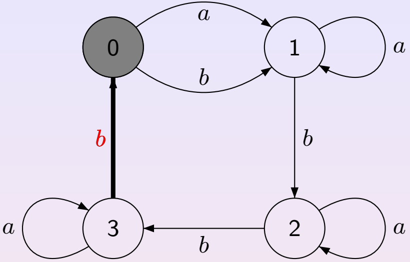

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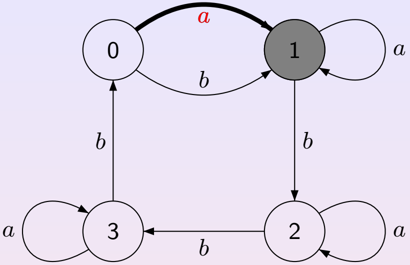

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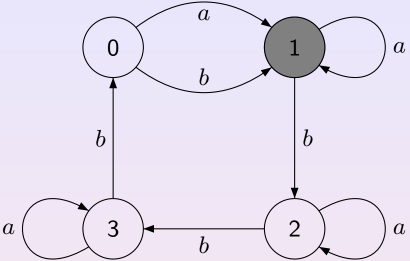

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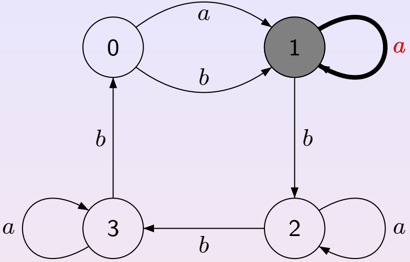

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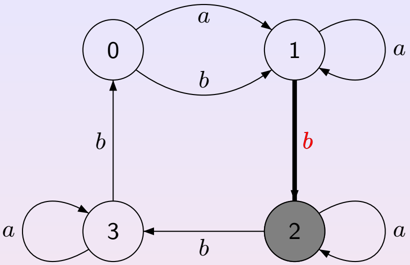

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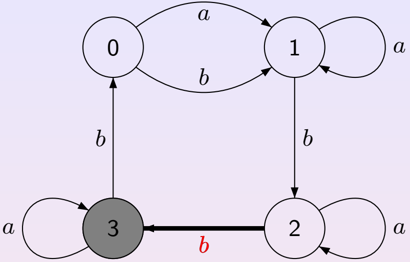

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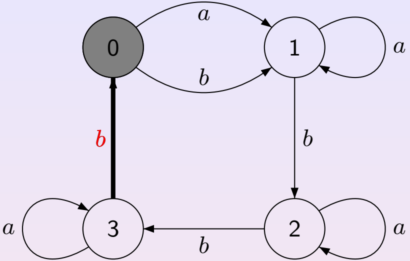

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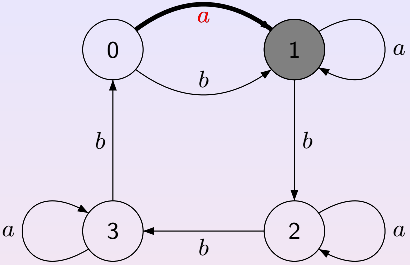

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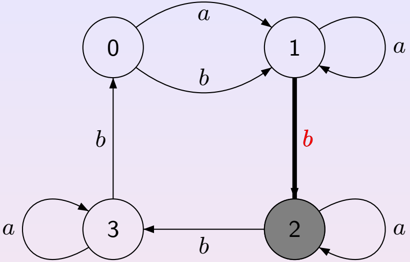

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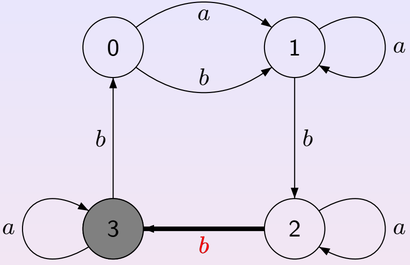

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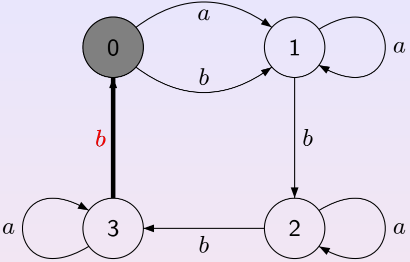

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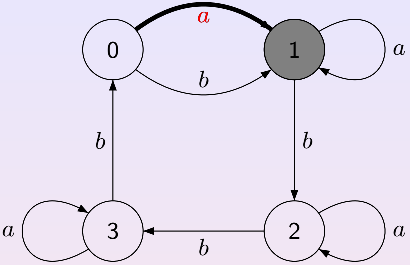

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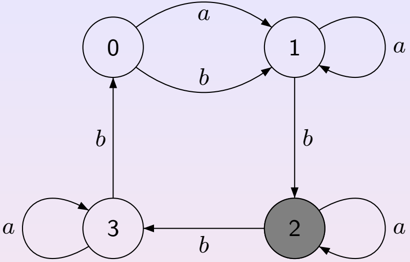

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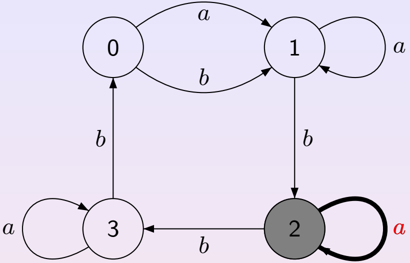

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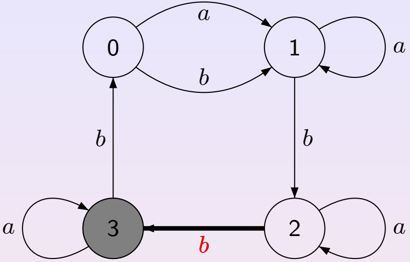

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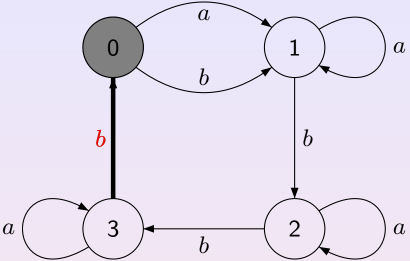

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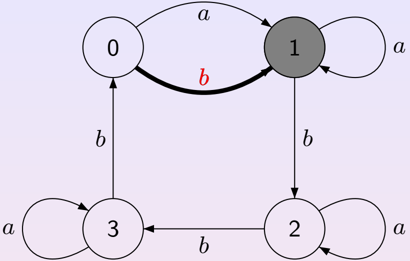

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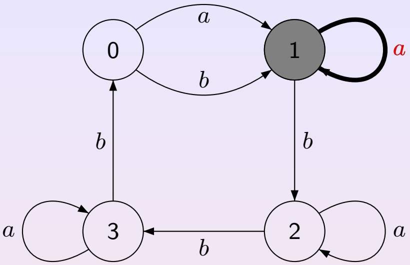

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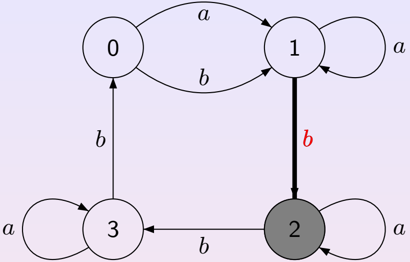

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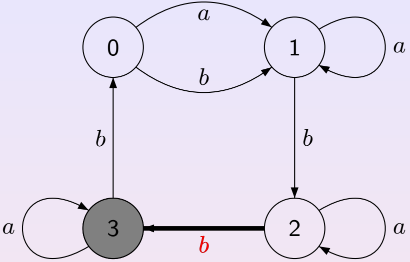

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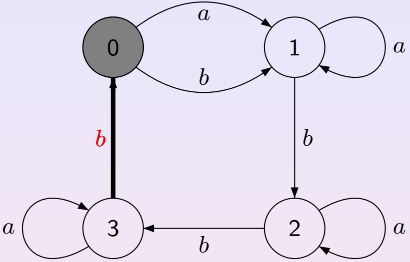

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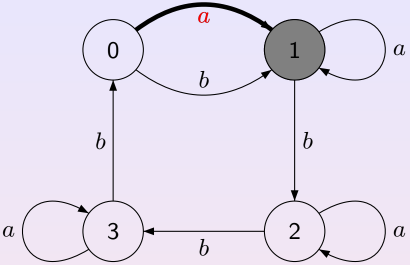

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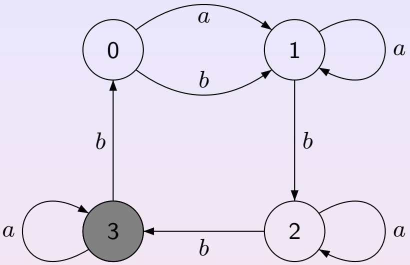

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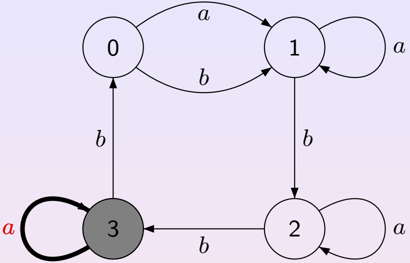

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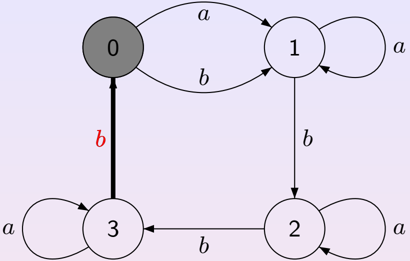

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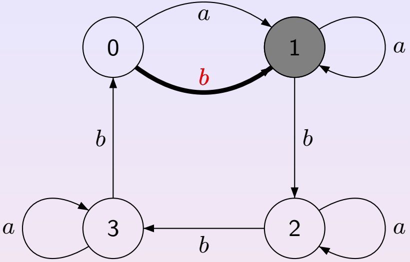

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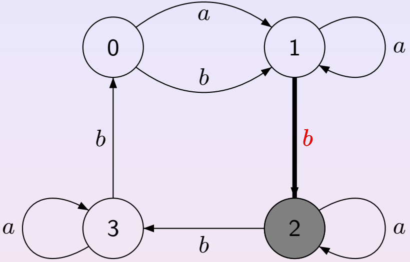

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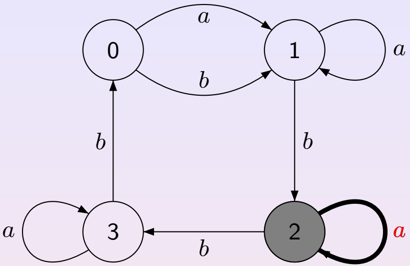

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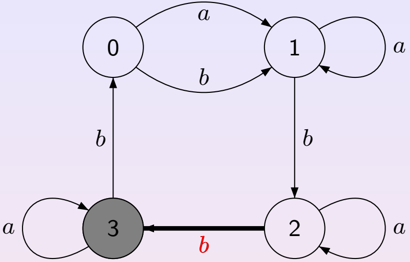

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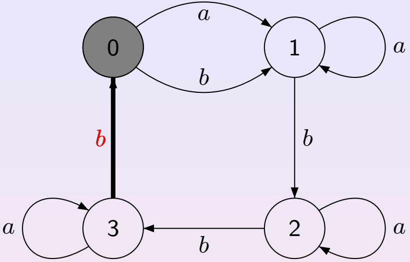

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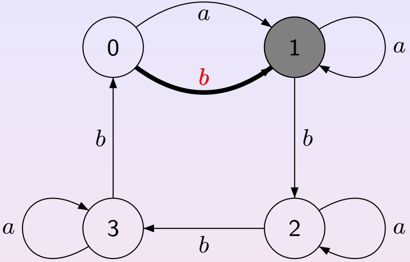

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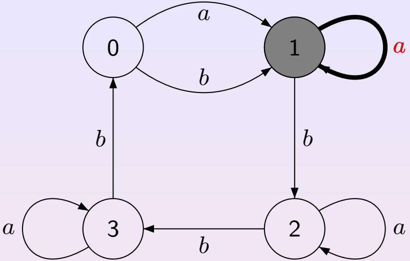

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The notion was formalized in 1964 in a paper by Jan Černý (Poznámka k homogénnym eksperimentom s konečnými automatami, Matematicko-fyzikalny Časopis Slovensk. Akad. Vied, 14, no.3, 208–216 [in Slovak]) though implicitly it had been around since at least 1956.

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7. Cerný's Paper

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The idea of synchronization is pretty natural and of obvious importance: we aim to restore control over a device whose current state is not known.

Think of a satellite which loops around the Moon and cannot be controlled from the Earth while "behind" the Moon (Černý's original motivation).

8. Ashby's Ghost Taming Automaton

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'**4/15. Materiality**. *The reader may now like to test the methods of this chapter as an aid to solving the problem set by the following letter. It justifies the statement made in S.1/2 that cybernetics is not bound to the properties found in terrestrial matter, nor does it draw its laws from them. What is important in cybernetics is the extent to which the observed behaviour is regular and reproducible.*'

The earliest synchronizing automaton that I was able to trace back in the literature appeared in Ross Ashby's 'An Introduction to Cybernetics' (1956), pp. 60–61.

The letter presents a puzzle about two ghostly noises, Singing and Laughter, in a haunted mansion. Each of the noises can be either on or off, and their behaviour depends on combinations of two possible actions, playing the organ or burning incense.

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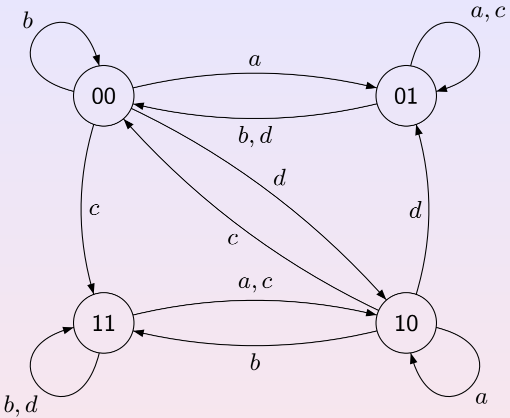
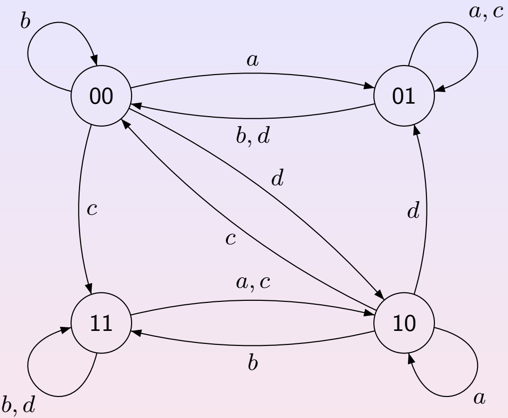
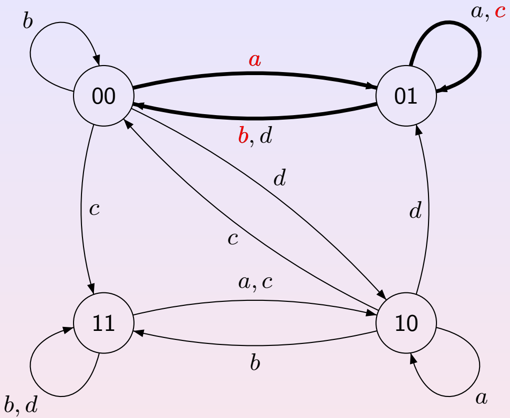
Under a suitable encoding, this leads to an automaton with 4 states and 4 input letters shown in the next slide.



00
01
10
11
b
a, c
a
b, d
d
d
c
c
a, c
b
b, d
a



It is easy to see that this is a synchronizing automaton and acb is its shortest reset word.



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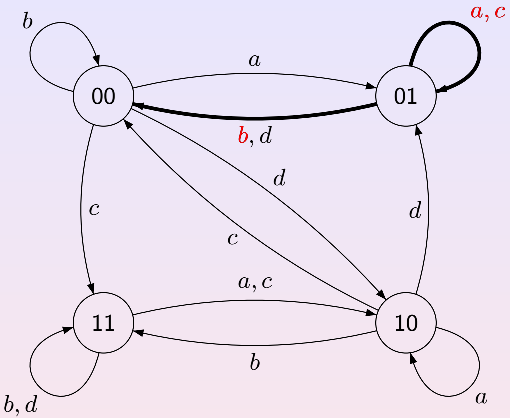

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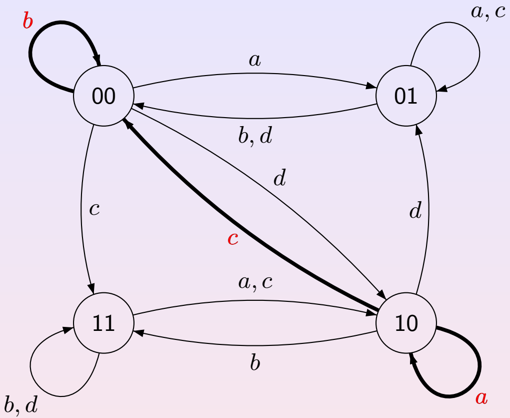

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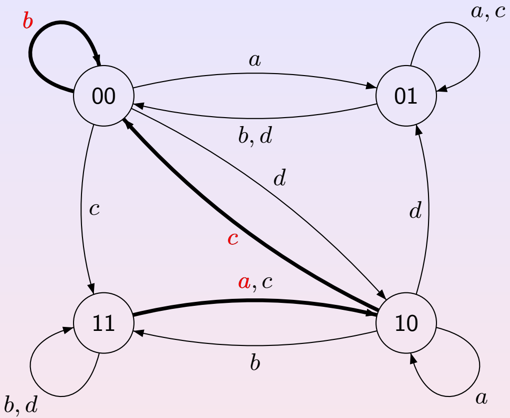

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10. Other Sources

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Example: A. E. Laemmel, B. Rudner, Study of the application of coding theory, Report PIBEP-69-034, Polytechnic Inst. Brooklyn, Dept. Electrophysics, Farmingdale, N.Y., 94 pp.

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This simple idea was already used in Morse code of the 19th century: 'e', the most common letter in English has the shortest Morse code, a single dot.

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allows one to encode the sentence more efficiently:

space	C	E	O	U	A	D	S	Y
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Sent 0 0 0

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Received	1 0 0 0 0 1 0 0 1 1 1 ...
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Sent	$0\,0\,0 \mid 0\,0\,1\,0 \mid 0\,1\,1\,1 \mid \ldots$
Received	1 0 0 0 0 1 0 0 1 1 1 . . .
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Received	$1\,0\,0\;\;0\;0\,1\,0\;\;\;0\,1\,1\,1\;\ldots$
Decoded	$1\,0 \mid 0\,0\,0 \mid 1\,0 \mid \mathbf{0\,1\,1\,1} \mid \ldots$

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The boldfaced code words indicate the position at which the decoder resynchronizes.

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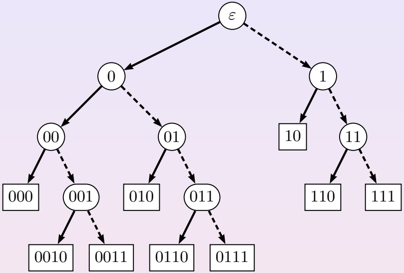

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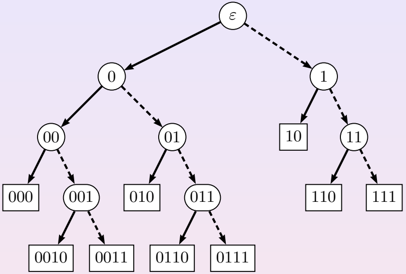

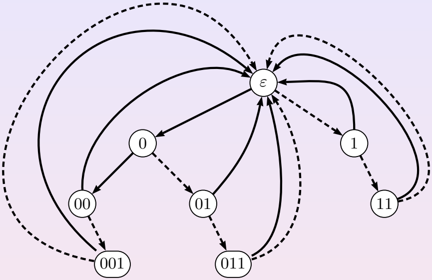

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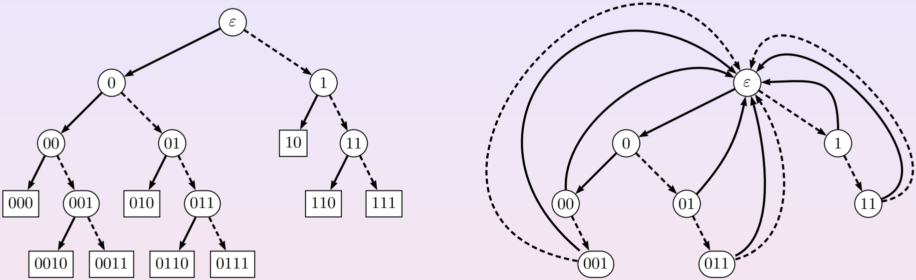

Synchronized codes precisely correspond to synchronizing automata!

Since the 60s synchronizing automata have been considered as a useful tool for testing of reactive systems (first circuits, later protocols)

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18. Re-inventing by Engineers

Since the 60s synchronizing automata have been considered as a useful tool for testing of reactive systems (first circuits, later protocols) and have been also applied in coding theory.
In the 80s, the notion was reinvented by engineers working in a branch of robotics which deals with part handling problems in industrial automation.

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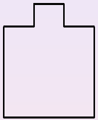
Suppose that one of the parts of a certain device has the following shape:



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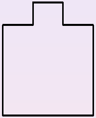

Such parts arrive at manufacturing sites in boxes and they need to be sorted and oriented before assembly.

Assume that only four initial orientations of the part shown above are possible, namely, the following ones:

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Suppose that prior the assembly the part should take the 'bump-left' orientation (the second one in the picture). Thus, one has to construct an orienter which action will put the part in the prescribed position independently of its initial orientation.

We put parts to be oriented on a conveyer belt which takes them to the assembly point and let the stream of the parts encounter a series of passive obstacles of two types (*tall* and *short*) positioned along the belt.

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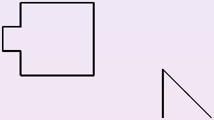

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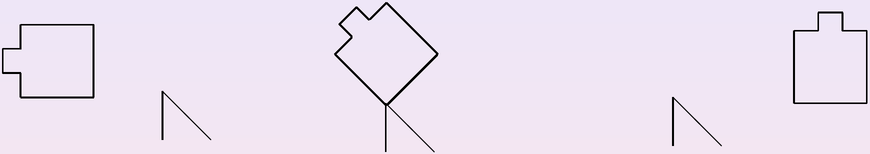

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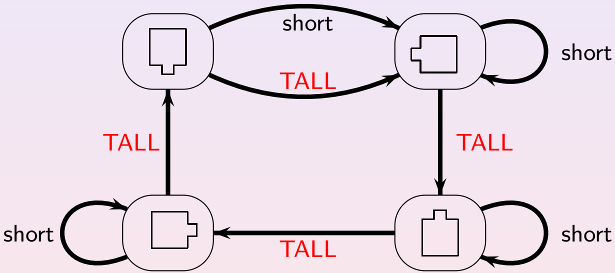
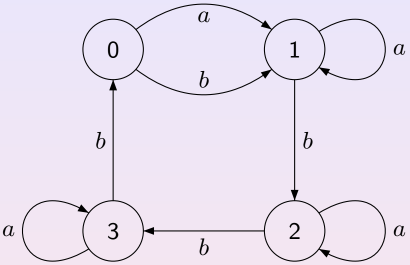
The following schema summarizes how the obstacles effect the orientation of the part in question:



We met this picture a few slides ago:



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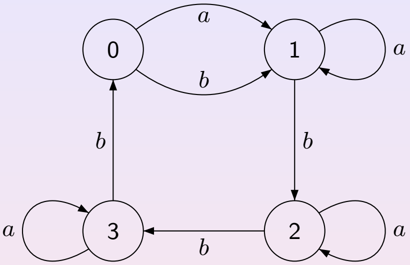

– this was our example of a synchronizing automaton, and we saw that $abbbabbba$ is a reset sequence of actions. Hence the series of obstacles

short-TALL-TALL-TALL-short-TALL-TALL-TALL-short

yields the desired sensorless orienter.

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Thus, τ satisfies the coincidence condition (with $k = 4$, $m = 7$).
The coincidence condition completely characterizes the constant length substitutions that give rise to dynamical systems measure-theoretically isomorphic to a translation on a compact Abelian group (Dekking, 1978).

There is a straightforward bijection between DFAs and constant length substitutions. Each DFA $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ with $\Sigma = \{a_1, \dots, a_\ell\}$ defines a length ℓ substitution on Q that maps every $q \in Q$ to the word $(q \,.\, a_1) \dots (q \,.\, a_\ell) \in Q^+$.

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$$\begin{tikzcd} 0 \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & 1 \arrow[loop right, "a"] \arrow[d, "b"] \\ 3 \arrow[u, "b"] \arrow[loop left, "a"] & 2 \arrow[l, "b"] \arrow[loop right, "a"] \end{tikzcd}$$

induces the substitution $0 \mapsto 11,\ 1 \mapsto 12,\ 2 \mapsto 23,\ 3 \mapsto 30.$

Conversely, each substitution $\sigma : X \to X^+$ such that all words $\sigma(x)$, $x \in X$, have the same length ℓ gives rise to a DFA for which X is the state set and which has ℓ input letters $a_1, \dots, a_\ell$ acting on X as follows: $x \, . \, a_i$ is the symbol in the i-th position of the word $\sigma(x)$.

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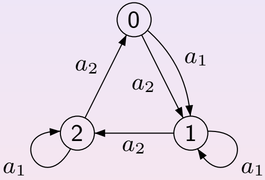

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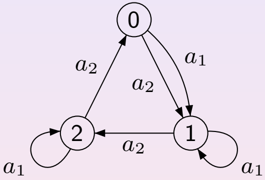

Under this bijection substitutions satisfying the coincidence condition correspond precisely to synchronizing automata, and moreover, given a substitution, the number of iterations at which the coincidence first occurs is equal to the minimum length of reset word for the corresponding automaton.

26. An Algebraic Framework

One may treat DFAs as unary algebras since each letter of the input alphabet defines a unary operation on the state set.

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In DNA-computing, there is fast progressing work by Ehud Shapiro's group on "*soup of automata*" (Programmable and autonomous computing machine made of biomolecules, Nature 414, no.1 (November 22, 2001) 430–434; DNA molecule provides a computing machine with both data and fuel, Proc. National Acad. Sci. USA 100 (2003) 2191–2196, etc).

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They have produced a solution containing 3×10^{12} identical DNA-based automata per μl. These automata can work in parallel on different inputs (DNA strands), thus ending up in different and unpredictable states. One has to feed the automata with an reset sequence (again encoded by a DNA-strand) in order to get them ready for a new use.

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