SPECIAL ELEMENTS IN LATTICES OF SEMIGROUP VARIETIES

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Abstract. We survey results concerning special elements of nine types (modular, lower-modular, upper-modular, cancellable, distributive, codistributive, standard, costandard and neutral elements) in the lattice of all semigroup varieties and certain its sublattices, mainly in the lattices of all commutative varieties and of all overcommutative ones.

Contents

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1. Introduction

This work is an extended version of the survey [55]. The work is regularly updated and modified as new results and/or articles appear.

The collection of all semigroup varieties forms a lattice with respect to class-theoretical inclusion. This lattice will be denoted by $\mathbb{SEM}$. The lattice $\mathbb{SEM}$ has been intensively studied since the beginning of 1960s. A systematic overview of the material accumulated here is given in the survey [38].

The lattice $\mathbb{SEM}$ has an extremely complicated structure. In particular, it contains an anti-isomorphic copy of the partition lattice over a countably infinite set [2, 10], and therefore, does not satisfy any non-trivial lattice identity. Identities in subvariety lattices of semigroup varieties were intensively examined in many articles. These articles contain a number of interesting and deep results (see [38, Section 11]). The next natural step is to consider varieties that guarantee, so to speak, 'nice lattice behavior' in their neighborhood. Specifically, our attention is to study special elements of different types in the lattice $\mathbb{SEM}$.

We will consider nine types of special elements: modular, lower-modular, upper-modular, cancellable, distributive, codistributive, standard, costandard and neutral elements. Recall the corresponding definitions. An element x of a lattice $\langle L; \vee, \wedge\rangle$ is called

modular if $\forall\, y, z \in L\colon\quad y \le z \longrightarrow (x \vee y) \wedge z = (x \wedge z) \vee y;$

lower-modular if $\forall\, y, z \in L\colon\quad x \le y \longrightarrow x \vee (y \wedge z) = y \wedge (x \vee z);$

cancellable if $\forall\, y, z \in L\colon\quad x \vee y = x \vee z \;\&\; x \wedge y = x \wedge z \longrightarrow y = z;$

distributive if $\forall\, y, z \in L\colon\quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z);$

standard if $\forall\, y, z \in L\colon\quad (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z);$

neutral if, for all $y, z \in L$, the sublattice of L generated by x, y and z is distributive. It is well known (see [5, Theorem 254 on p. 226], for instance) that an element $x \in L$ is neutral if and only if

$$\forall\, y, z \in L\colon\quad (x \vee y) \wedge (y \vee z) \wedge (z \vee x) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x).$$

Upper-modular, codistributive and *costandard* elements are defined dually to lower-modular, distributive and standard ones respectively.

Special elements play an important role in the general lattice theory (see [5, Section III.2], for instance). In particular, it is well known that if a is a neutral element in a lattice L then L is decomposable into a subdirect product of the principal ideal and the principal filter of L generated by a (see [5, Theorem 254 on p. 226], for instance). Thus, the knowledge of which elements of a lattice are neutral gives essential information on the structure of the lattice as a whole. A valuable information about special elements of types mentioned above may be found in [5, 25, 43].

There is a number of interrelations between types of elements we consider. It is evident that a neutral element is both standard and costandard; a standard or costandard element is cancellable; a cancellable element is modular; a [co]distributive element is lower-modular [upper-modular]. It is well known also that a [co]standard element is [co]distributive (see [5, Theorem 253 on p. 224], for instance). These interrelations between types of elements in abstract lattices are shown in Fig. 1.

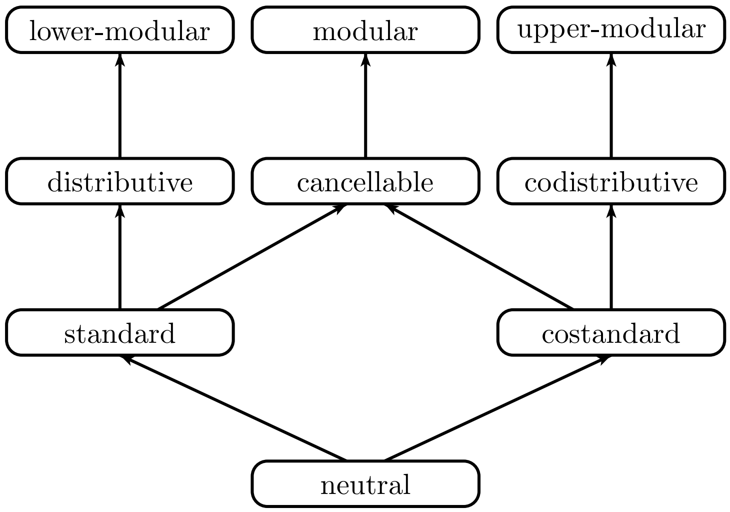

FIGURE 1. Interrelations between types of elements in abstract lattices

In fact, first information about special elements in the lattice $\mathbb{SEM}$ were obtained (in the implicit form) in the articles [1, 14, 15, 17, 23]. There were no any explicit references to special elements in these articles. But it immediately follows from the results obtained there that particular concrete varieties are special elements of some or another type in $\mathbb{SEM}$. Translated into the language of special elements, the results of these works can be formulated as follows. It is proved independently in [14] and [23] that the variety of semilattices **SL** is a modular element of $\mathbb{SEM}$. It is verified in [15] that **SL** is distributive in $\mathbb{SEM}$. The fact that the variety **ZM** of semigroups with zero multiplication is modular and distributive in $\mathbb{SEM}$ is checked in [17]. Finally, it is verified in [1] that the lattice $\mathbb{SEM}$ is 0-*disributive*, i.e., satisfies the implication

$$x \wedge y = x \wedge z = 0 \longrightarrow x \wedge (y \vee z) = 0.$$

This immediately implies that all atoms of the lattice $\mathbb{SEM}$ (in particular, the varieties **SL** and **ZM**) are codistributive in $\mathbb{SEM}$.

First explicit results about special elements in $\mathbb{SEM}$ were obtained in the articles [12, 59]. In [59], certain sufficient condition for a semigroup variety to be modular or lower-modular element of $\mathbb{SEM}$ is found. In [12], Ježek and McKenzie examined modular elements in $\mathbb{SEM}$[1]. They rediscover the sufficient

[1]Note that the paper [12] deals with the lattice of equational theories of semigroups, i.e., the dual of $\mathbb{SEM}$ rather than the lattice $\mathbb{SEM}$ itself. When reproducing results from [12], we adapt them to the terminology of the present article.

condition for modular elements of $\mathbb{SEM}$ mentioned in [59] and find a strong necessary condition for a semigroup variety to be modular element of $\mathbb{SEM}$. Note that the mentioned results of [12,59] play an auxiliary role in these works.

A systematic examination of special elements in $\mathbb{SEM}$ is the objective of the articles [9,27,28,32,33,42,48–53,57,60,64]; see also [38, Section 14]. In short, the mentioned articles contain complete descriptions of lower-modular, cancellable, distributive, standard, costandard and neutral elements of the lattice $\mathbb{SEM}$[2] and essential information (such as strong necessary conditions and descriptions in wide and important partial cases) about modular, upper-modular and codistributive elements of this lattice. In particular, it turns out that there are some interrelations between special elements of different types in $\mathbb{SEM}$ that do not hold in abstract lattices. Namely, an element of $\mathbb{SEM}$ is standard if and only if it is distributive; is costandard if and only if it is neutral; is modular whenever it is lower-modular. Interrelations between types of elements in the lattice $\mathbb{SEM}$ are shown in Fig. 2. All other possible interrelations between types of elements under consideration are not the case. Corresponding examples may be easily extracted from results formulated below. In Figures 2 and 3 we color the oval corresponding to the type of elements in green color if elements of this type in the corresponding lattice are completely determined, and in yellow color, if an essential information about such elements is known.

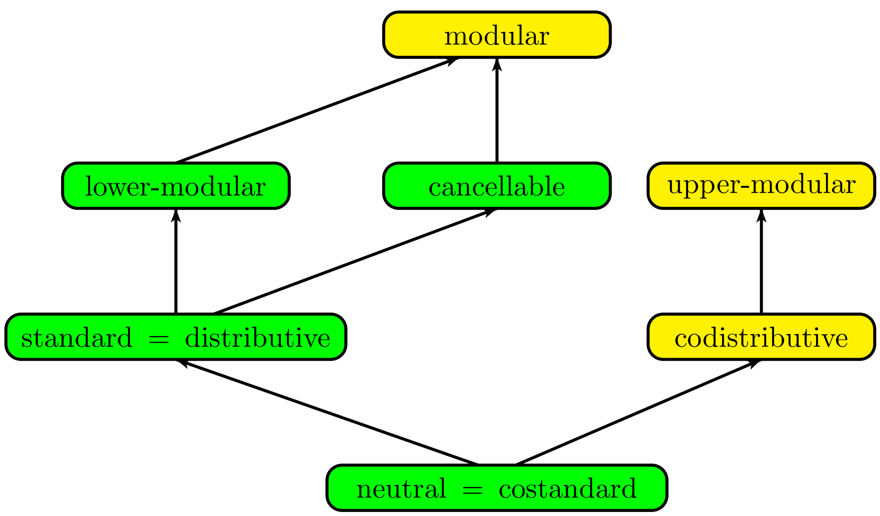

Figure 2. Interrelations between types of elements in $\mathbb{SEM}$

The lattice $\mathbb{SEM}$ contains a number of wide and important sublattices (see [38, Section 1 and Chapter 2]). It is natural to examine special elements in these sublattices. One of the most important sublattices of $\mathbb{SEM}$ is the lattice $\mathbb{COM}$ of all commutative semigroup varieties. It follows from results of [3] that this lattice contains an isomorphic copy of any finite lattice, and therefore,

[2]To prevent a possible confusion, we note that the description of standard elements of $\mathbb{SEM}$ is not formulated explicitly anywhere but readily follows from results of [57], see a comment to Theorem 3.3 below.

does not satisfy any non-trivial lattice identity. On the other hand, the lattice $\mathbb{COM}$ is known to be countably infinite [19] and can be characterized [13] (see also [38, Section 8]). Special elements in the lattice $\mathbb{COM}$ are examined in [26,27,56] where lower-modular, upper-modular, distributive, codistributive, standard, costandard and neutral elements of $\mathbb{COM}$ are completely determined and an essential information about modular elements of this lattice is obtained. As in the case of the lattice $\mathbb{SEM}$, it turns out that an element of $\mathbb{COM}$ is standard if and only if it is distributive, and is modular whenever it is lower-modular. This deep analogy between properties of special elements in the lattices $\mathbb{SEM}$ and $\mathbb{COM}$ is not accidental. As we will seen below, these similar results easily follow from some general fact concerning properties of special elements in lattices of subvarieties of overcommutative varieties (evidently, both the lattices $\mathbb{SEM}$ and $\mathbb{COM}$ are subvariety lattices of two 'extremal' overcommutative varieties, namely the variety of all semigroups and the variety of all commutative semigroups respectively). But this analogy does not extend to all types of special elements. In contrast with the case of $\mathbb{SEM}$, it turns out that the properties of being neutral and costandard elements of the lattice $\mathbb{COM}$ are not equivalent, whereas the properties of being codistributive and upper-modular elements of this lattice are, on the contrary, equivalent. Interrelations between types of elements in the lattice $\mathbb{COM}$ are shown in Fig. 3. We do not mention cancellable elements in this figure because there no any information about these elements in the lattice $\mathbb{COM}$ so far (except necessary conditions for modular elements of the lattice $\mathbb{COM}$ that evidently are also necessary conditions for cancellable elements of this lattice). No interrelations between types of elements in $\mathbb{COM}$ not specified in Fig. 3 hold. As in the case of the lattice $\mathbb{SEM}$, corresponding examples may be easily extracted from results formulated below.

Recall that a semigroup variety is called *permutative* if it satisfies a *permutational* identity, i.e., an identity of the type

$$x_1x_2\cdots x_n \approx x_{1\pi}x_{2\pi}\cdots x_{n\pi}$$

where π is a non-trivial permutation on the set $\{1,2,\dots,n\}$. This identity will be denoted by $p_n[\pi]$. The number n is called the *length* of this identity. The collection of all permutative varieties forms a sublattice $\mathbb{PERM}$ of the lattice $\mathbb{SEM}$. This lattice is located between $\mathbb{SEM}$ and $\mathbb{COM}$. It seems quite natural to examine special elements in $\mathbb{PERM}$. There are no published results here so far but some information about modular and lower-modular elements in the lattice $\mathbb{PERM}$ is found in PhD thesis by Shaprynskiĭ [30].

The 'antipode' of the lattice $\mathbb{COM}$ is the lattice $\mathbb{OC}$ of all *overcommutative* semigroup varieties (i.e., varieties containing the variety of all commutative semigroups). It is well known that the lattice $\mathbb{SEM}$ is the disjoint union of $\mathbb{OC}$ and the lattice of all *periodic* semigroup varieties (i.e., varieties consisting of periodic semigroups). Results of the papers [12,49,51] imply that if a semigroup variety $\mathbf{V}$ is an element of one of the nine types mentioned above in the lattice $\mathbb{SEM}$ and $\mathbf{V}$ is different from the variety of all semigroups then $\mathbf{V}$ is a periodic variety (more general fact is proved in [28], see Proposition 3.1 below). Thus, an examination of special elements of all mentioned types in $\mathbb{SEM}$ a priori can not give any essential information about the lattice $\mathbb{OC}$. Note that the

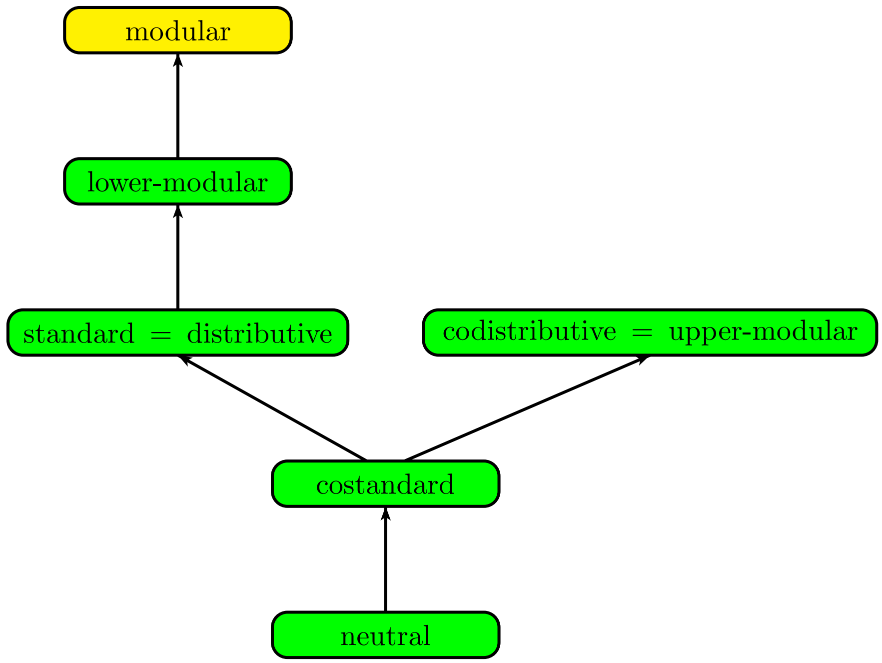

FIGURE 3. Interrelations between types of elements in $\mathbb{COM}$

lattice $\mathbb{OC}$ contains an isomorphic copy of any finite lattice [63], whence it does not satisfy any non-trivial lattice identity. Overcommutative varieties whose lattice of overcommutative subvarieties satisfies a particular lattice identity were intensively studied (see [38, Subsection 5.2] and the article [29]). All these arguments make the examination of special elements of $\mathbb{OC}$ very natural. Such an examination has been started in the article [47]. It is proved there that the properties of being a distributive, a codistributive, a standard, a costandard and a neutral element of the lattice $\mathbb{OC}$ are equivalent, and a certain characterization of corresponding overcommutative varieties is proposed. But this description turns out to be incorrect (while the result that the five mentioned conditions are equivalent is true). The correct description of distributive, codistributive, standard, costandard and neutral elements of the lattice $\mathbb{OC}$ is contained in the article [34]. Cancellable elements of the lattice $\mathbb{OC}$ are classified in [35]. More precisely, it is verified there that a property to be a cancellable element of $\mathbb{OC}$ is equivalent to the property to be a [co]distributive, [co]standard or neural element of this lattice. There are no any information about modular, lower-modular or upper-modular elements of $\mathbb{OC}$ so far.

Note that there is an interesting information about special elements in lattices of varieties of semigroups with different additional unary or nullary operations. So, a number of examples of neutral elements in the lattice of all completely regular semigroup varieties is given in [46]; here completely regular semigroups are considered as *unary semigroups*, that is, semigroups with a naturally defined additional unary operation (see [20] or [38, Section 6], for instance). Another type of unary semigroups which includes completely regular semigroups as a partial case is epigroups (see [36, 37] or [38, Section 2], for instance). Special elements of all mentioned above types in the lattice of epigroup varieties are examined in [31, 32, 39–41]. The works [7, 8] are devoted to special elements of several types in the lattice of monoid varieties. But all these results are beyond the scope of this survey.

The survey consists of six sections. In Section 2, we provide some preliminary results about special elements in abstract lattices, lattices of equivalence relations, subgroup lattices of finite symmetric groups, congruence lattices of G-sets and the lattices $\mathbb{SEM}$ and $\mathbb{COM}$. These preliminary results play an important role in the proofs of the results that we survey in Sections 3–6. In Sections 3 and 4, we overview results about special elements in the lattices $\mathbb{SEM}$ and $\mathbb{COM}$ respectively. Section 5 contains results about modular and lower-modular elements in lattices located between $\mathbb{SEM}$ and $\mathbb{COM}$, namely in subvariety lattices of overcommutative varieties and in the lattice $\mathbb{PERM}$. Finally, Section 6 is devoted to special elements in the lattice $\mathbb{OC}$. Sections 3 and 5 contain also several open questions.

2. Preliminary results

2.1. *I*-elements and *Q*-elements of lattices. Almost all types of special elements introduced above (namely, all of them except cancellable elements) are defined by the following general scheme. We take a particular lattice identity and consider it as an open formula. Then, one of the variables is left free while all the others are subjected to a universal quantifier. As a result, we obtain a first order formula $\Phi(x)$ with one free variable x in the language of lattice operations. An element w of a lattice L is said to be a special element of corresponding type of L if the sentence $\Phi(w)$ is true. It is evident that neutral, [co]standard and [co]distributive elements are defined just by this scheme. The definitions of modular, lower-modular and upper-modular elements may be easily reformulated in the framework of this approach as well because the modular law may be written as an identity.

One can formalize the approach discussed in the previous paragraph. We will write lattice or semigroup terms (rather than letters) in bold and connect two sides of a lattice or semigroup identity by the symbol $\approx$. The symbol $=$ will denote, among other things, the equality relation on a lattice or semigroup. Let ε be a lattice identity of the form $\mathbf{s} \approx \mathbf{t}$ where terms $\mathbf{s}$ and $\mathbf{t}$ depend on variables $x_1, \dots, x_n$, and $1 \le i \le n$. To write [quasi-]identities more compact, we put $X_n^i = \{x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n\}$. An element x of a lattice L is called an (ε, i)-*element* of L if

$$\forall\, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n \in L\colon \quad \mathbf{s}(X_n^i) = \mathbf{t}(X_n^i).$$

An element of a lattice L is called an *I-element* of L if it is an (ε, i)-element of L for some non-trivial identity ε depending on variables $x_1, \dots, x_n$ and some $1 \le i \le n$.

For an element a of a lattice L, we put $(a] = \{x \in L \mid x \le a\}$. If $a \in L$ and the lattice $(a]$ satisfies the identity $\mathbf{p}(x_1, \dots, x_n) \approx \mathbf{q}(x_1, \dots, x_n)$ then

$$\mathbf{p}(a \wedge x_1, \dots, a \wedge x_n) = \mathbf{q}(a \wedge x_1, \dots, a \wedge x_n)$$

for all $x_1, \ldots, x_n \in L$ because $a \wedge x_1, \ldots, a \wedge x_n \in (a]$. Therefore, in this situation, a is an $(\varepsilon, n+1)$-element of L with the following ε:

$$\mathbf{p}(x_{n+1} \wedge x_1, \ldots, x_{n+1} \wedge x_n) \approx \mathbf{q}(x_{n+1} \wedge x_1, \ldots, x_{n+1} \wedge x_n).$$

So, we have the following

Observation 2.1. *If w is an element of a lattice L and the ideal $(w]$ of L satisfies some non-trivial lattice identity then w is an I-element of L.*

The subvariety lattice of a variety $\mathbf{V}$ is denoted by $L(\mathbf{V})$. A semigroup variety $\mathbf{V}$ is called an *I-variety* if it is an I-element of the lattice $\mathbb{SEM}$. The following assertion is a specialization of Observation 2.1 for the lattice $\mathbb{SEM}$.

Corollary 2.2. *If $\mathbf{V}$ is a semigroup variety and the lattice $L(\mathbf{V})$ satisfies some non-trivial lattice identity then $\mathbf{V}$ is an I-variety.*

We will denote by $\operatorname{var}\Sigma$ the semigroup variety given by the identity system Σ. The converse statement to Corollary 2.2 is not true. Indeed, the variety $\operatorname{var}\{x^2 \approx 0\}$ is a modular and a lower-modular element of $\mathbb{SEM}$ (see Theorems 3.2 and 3.9 below) but its subvariety lattice does not satisfy any non-trivial identity [10].

The following fact turns out to be very helpful.

Lemma 2.3 ([26, Corollary 2.1])**.** *Let w be an atom and a neutral element of a lattice L, ε a lattice identity that holds in the 2-element lattice and depends on variables $x_1, \ldots, x_n$, and $1 \le i \le n$. An element $x \in L$ is an (ε, i)-element of L if and only if the element $x \vee w$ has the same property.*

It is well known that the variety $\mathbf{SL}$ is an atom of the lattice $\mathbb{SEM}$ (see [38, Section 1], for instance) and a neutral element of this lattice (see [64, Proposition 4.1] or Theorem 3.5 below). In particular, $\mathbf{SL}$ is a neutral atom of $\mathbb{COM}$. Thus, Lemma 2.3 implies the following

Corollary 2.4. *Let ε be a lattice identity that holds in the 2-element lattice and depends on variables $x_1, \ldots, x_n$, and $1 \le i \le n$. A [commutative] semigroup variety $\mathbf{V}$ is an (ε, i)-element of the lattice $\mathbb{SEM}$ [respectively $\mathbb{COM}$] if and only if the variety $\mathbf{V} \vee \mathbf{SL}$ has the same property.*

Note that a number of partial cases of Lemma 2.3 and Corollary 2.4 for special elements of different concrete types were proved earlier in [49, 53, 57, 60, 64].

The notion of I-elements does not cover the notion of cancellable elements. It seems natural to generalize the former notion in the following way. Let ξ be a lattice quasi-identity of the form

$$\mathop{\&}_{k=1}^{m} \mathbf{s}_k \approx \mathbf{t}_k \longrightarrow \mathbf{s} \approx \mathbf{t}$$

where terms $\mathbf{s}_1, \ldots, \mathbf{s}_m, \mathbf{t}_1, \ldots, \mathbf{t}_m, \mathbf{s}$ and $\mathbf{t}$ depend on variables $x_1, \ldots, x_n$, and $1 \le i \le n$. An element x of a lattice L is called a *(ξ, i)-element* of L if

$$\forall\, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in L\colon \quad \mathop{\&}_{k=1}^{m} \mathbf{s}_k(X_n^i) = \mathbf{t}_k(X_n^i) \longrightarrow \mathbf{s}(X_n^i) = \mathbf{t}(X_n^i).$$

An element of a lattice L is called a *Q-element* of L if it is a (ξ, i)-element of L for some non-trivial quasi-identity ξ depending on variables $x_1, \dots, x_n$ and some $1 \le i \le n$. Clearly, cancellable elements are Q-elements.

Unfortunately, the analogue of Lemma 2.3 for Q-elements is not the case. More precisely, the join of a (ξ, i)-element of a lattice L and a neutral atom of L need not to be a (ξ, i)-element of L. Indeed, let ξ be the following quasi-identity:

$$x_1 \approx x_2 \wedge x_3 \longrightarrow (x_2 \wedge x_3) \vee x_4 \approx (x_2 \vee x_4) \wedge (x_3 \vee x_4).$$

Further, let L be the 5-element modular non-distributive lattice M_3 with the new least element adjoined (see Fig. 4). We denote this new least element by 0 and a unique atom of L by a. Then it is easy to see that the atom a is neutral, 0 is a $(\xi, 1)$-element of L but the element $a = 0 \vee a$ does not have the last property (to verify the latter claim, it suffices to take for x_2, x_3 and x_4 three pairwise incomparable elements of L). This example is communicated to the author by Shaprynskiĭ.

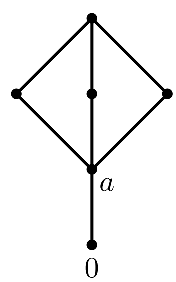

FIGURE 4. The lattice M_3 with the new least element adjoined

Nevertheless, analogues of Lemma 2.3 and Corollary 2.4 for cancellable elements are true. One can formulate these results explicitly.

Lemma 2.5 ([9, Lemma 2.1])**.** *Let w be an atom and a neutral element of a lattice L. An element $x \in L$ is a cancellable element of L if and only if the element $x \vee w$ has the same property.*

This lemma and the properties of the variety **SL** mentioned above immediately imply the following

Corollary 2.6. *A [commutative] semigroup variety* **V** *is a cancellable element of the lattice* $\mathbb{SEM}$ *[respectively* $\mathbb{COM}$*] if and only if the variety* $\mathbf{V} \vee \mathbf{SL}$ *has the same property.*

2.2. **Modular and upper-modularl elements in lattices of equivalence relations.** If S is a set then $\mathrm{Eq}(S)$ stands for the lattice of equivalence relations on S.

Proposition 2.7. *Let S be a non-empty set. For an equivalence relation α on S, the following are equivalent:*

a) *α is a modular element of the lattice* $\mathrm{Eq}(S)$*;*

b) *α is an upper-modular element of the lattice* $\mathrm{Eq}(S)$*;*
c) *α has at most one non-singleton class.*

The equivalences a) $\Longleftrightarrow$ c) and b) $\Longleftrightarrow$ c) of this proposition were proved in [11, Proposition 2.2] and [59, Proposition 3] respectively.

Proposition 2.7 turns out to be very helpful for the examination of modular and lower-modular elements in varietal lattices. In order to explain how this proposition can be applied, we need some new definitions and notation. Note that a semigroup S satisfies the identity system $\mathbf{w}x \approx x\mathbf{w} \approx \mathbf{w}$ where the letter x does not occur in the word $\mathbf{w}$ if and only if S contains a zero element 0 and all values of $\mathbf{w}$ in S equal to 0. We adopt the usual convention of writing $\mathbf{w} \approx 0$ as a short form of such a system and referring to the expression $\mathbf{w} \approx 0$ as to a single identity. Identities of the form $\mathbf{w} \approx 0$ are called 0-*reduced*. Further, let $\mathbf{X}$ be a semigroup variety and $\mathbf{V} \subseteq \mathbf{X}$, F be the $\mathbf{X}$-free object and ν be the fully invariant congruence on F corresponding to $\mathbf{V}$. It is clear that if $\mathbf{V}$ may be given within $\mathbf{X}$ by the family of 0-reduced identities $\{\mathbf{w}_i \approx 0 \mid i \in I\}$ only then ν has just one non-singleton class (namely, the collection of all equivalence classes modulo $\mathbf{X}$ that contain the words $\mathbf{w}_i$ where i runs over I). Now recall the generally known fact that the lattice $L(\mathbf{X})$ is anti-isomorphic to the lattice of all fully invariant congruences on F. Therefore, the lattice $\mathrm{Eq}(F)$ contains an anti-isomorphic copy of $L(\mathbf{X})$. Finally, we note that the notion of a modular element is self-dual. Combining all these observations with Proposition 2.7, we have the following

Corollary 2.8. *Let $\mathbf{X}$ be a semigroup variety and $\mathbf{V} \subseteq \mathbf{X}$. If $\mathbf{V}$ is defined within $\mathbf{X}$ by* 0*-reduced identities only then $\mathbf{V}$ is a modular and lower-modular element of the lattice $L(\mathbf{X})$.*

2.3. **Special elements in subgroup lattices of finite symmetric groups.** The subgroup lattice of a group G is denoted by $\mathrm{Sub}(G)$. We denote by S_n the full symmetric group on the set $\{1, 2, \ldots, n\}$. If $\mathbf{V}$ is a semigroup variety and n is a natural number then we put

$$\mathrm{Perm}_n(\mathbf{V}) = \{\pi \in S_n \mid \mathbf{V} \text{ satisfies the identity } p_n[\pi]\}.$$

Clearly, $\mathrm{Perm}_n(\mathbf{V})$ is a subgroup in S_n. The following assertion explains our interest to modular and cancellable elements in lattices of the form $\mathrm{Sub}(S_n)$.

Lemma 2.9. *Let $\mathbf{V}$ be a semigroup variety and n be a natural number. If $\mathbf{V}$ is a modular* [*cancellable*] *element of the lattice* $\mathbb{SEM}$ *then the group* $\mathrm{Perm}_n(\mathbf{V})$ *is a modular* [*cancellable*] *element of the lattice* $\mathrm{Sub}(S_n)$*.*

This lemma is proved in [48, Corollary 4.3] for modular elements and in [32, Proposition 3.2] for cancellable ones.

We denote by A_n the alternative subgroup of S_n, by V_4 the Klein four-group and by T the singleton group.

Proposition 2.10 ([11, Propositions 3.1, 3.7 and 3.8])**.** *Let n be a natural number. A subgroup G of a group S_n is a modular element of the lattice* $\mathrm{Sub}(S_n)$ *if and only if one of the following holds:*

(i) *$n \leq 3$;*

(ii) $n = 4$ *and either* $G = T$ *or* $G \supseteq V_4$*;*
(iii) $n \geq 5$ *and* G *coincides with either* T *or* A_n *or* S_n*.*

This proposition is applied in the examination of modular elements in $\mathbb{SEM}$.

Proposition 2.11. *Let* n *be a natural number. For a subgroup* G *of the group* S_n*, the following are equivalent:*

a) G *is a neutral element of the lattice* $\mathrm{Sub}(S_n)$*;*
b) G *is a standard element of the lattice* $\mathrm{Sub}(S_n)$*;*
c) G *is a costandard element of the lattice* $\mathrm{Sub}(S_n)$*;*
d) G *is a cancellable element of the lattice* $\mathrm{Sub}(S_n)$*;*
e) G *is a distributive element of the lattice* $\mathrm{Sub}(S_n)$*;*
f) G *is a codistributive element of the lattice* $\mathrm{Sub}(S_n)$*;*
g) *either* $G = T$ *or* $G = S_n$*.*

The equivalence of the claims a)–c) and e)–g) of this proposition is checked in the proof of [47, Theorem 2], while the equivalence d) $\Longleftrightarrow$ g) is verified in [32, Lemma 2.3]. Note that the key role in the proof of the equivalence d) $\Longleftrightarrow$ g) plays Proposition 2.10. Proposition 2.11 is applied in the proof of Theorems 3.13 and 6.1.

2.4. **Special elements in congruence lattices of G-sets.** A unary algebra with the carrier A and the set of (unary) operations G is called a *G-set* if G is equipped by a structure of a group and this group structure on G is compatible with the unary structure on A (this means that if $g, h \in G$, $x \in A$ and e is the unit element of G then $g\big(h(x)\big) = (gh)(x)$ and $e(x) = x$). Our interest to G-sets is explained by the fact that the lattice $\mathbb{OC}$ admits a concise and transparent description in terms of congruence lattices of G-sets. More precisely, $\mathbb{OC}$ is anti-isomorphic to a subdirect product of congruence lattices of countably infinite series of certain G-sets (see [63] or [38, Subsection 5.1]). To apply this result for examination of special elements in $\mathbb{OC}$, some information about special elements in congruence lattices of G-sets is required.

A G-set A is said to be *transitive* if, for all $a, b \in A$, there exists $g \in G$ such that $g(a) = b$. If A is a G-set and $a \in A$ then we put

$$\mathrm{Stab}_A(a) = \big\{g \in G \mid g(a) = a\big\}.$$

Clearly, $\mathrm{Stab}_A(a)$ is a subgroup in G. This subgroup is called a *stabilizer* of an element a in A. The congruence lattice of a G-set A is denoted by $\mathrm{Con}(A)$.

Proposition 2.12. *Let* A *be a non-transitive* G*-set such that* $\mathrm{Stab}_A(x) = \mathrm{Stab}_A(y)$ *for all elements* $x, y \in A$*. For a congruence* α *on* A*, the following are equivalent:*

a) α *is a neutral element of the lattice* $\mathrm{Con}(A)$*;*
b) α *is a standard element of the lattice* $\mathrm{Con}(A)$*;*
c) α *is a costandard element of the lattice* $\mathrm{Con}(A)$*;*
d) α *is a cancellable element of the lattice* $\mathrm{Con}(A)$*;*
e) α *is a distributive element of the lattice* $\mathrm{Con}(A)$*;*
f) α *is a codistributive element of the lattice* $\mathrm{Con}(A)$*;*
g) α *is either the universal relation or the equality relation on* A*.*

The equivalence of the claims a)–c) and e)–g) of this proposition was proved in [47, Theorem 1]. The equivalence d) $\Longleftrightarrow$ g) is verified in [35, Proposition 3.4].

G-sets that appear in [63] in the description of the lattice $\mathbb{OC}$ have the property that the stabilizer of any element in these G-sets is the trivial group. Thus, the application of Proposition 2.12 is not hindered by the hypothesis that stabilizers of all elements in A coincide. It is presently unknown, whether the proposition holds without this hypothesis.

2.5. **Upper-modular and codistributive elements: interrelations between lattice identities and a hereditary property.** The following easy observation turns out to be helpful.

Observation 2.13. *Let L be a lattice. If an element $a \in L$ is upper-modular [codistributive] in L and the lattice $(a]$ is modular [distributive] then every element of $(a]$ is upper-modular [codistributive] in L.*

This claim was noted in [51, Lemma 2.1] for upper-modular elements and in [53, Lemma 2.2] for codistributive ones.

Observation 2.13 immediately implies the following

Corollary 2.14. *Let $\mathbf{V}$ be a [commutative] semigroup variety.*

(i) *If $\mathbf{V}$ is an upper-modular element of the lattice $\mathbb{SEM}$ [respectively $\mathbb{COM}$] and the lattice $L(\mathbf{V})$ is modular then every subvariety of $\mathbf{V}$ is an upper-modular element of $\mathbb{SEM}$ [respectively $\mathbb{COM}$].*

(ii) *If $\mathbf{V}$ is a codisributive element of the lattice $\mathbb{SEM}$ [respectively $\mathbb{COM}$] and the lattice $L(\mathbf{V})$ is disributive then every subvariety of $\mathbf{V}$ is a codisributive element of $\mathbb{SEM}$ [respectively $\mathbb{COM}$].*

3. The lattice $\mathbb{SEM}$

For convenience, we call a semigroup variety *modular* if it is a modular element of the lattice $\mathbb{SEM}$ and adopt similar agreement for all other types of special elements. The main results of this section provide:

- a complete classification of lower-modular, cancellable, distributive, standard, costandard or neutral varieties (Theorems 3.2, 3.3, 3.5 and 3.13),
- a classification of modular, upper-modular or codistributive varieties in some wide partial cases (Theorems 3.12, 3.16, 3.24 and 3.32),
- strong necessary conditions for a semigroup variety to be modular, upper-modular or codistributive (Theorems 3.7, 3.8, 3.18 and 3.31),
- a sufficient condition for a semigroup variety to be modular (Theorem 3.9).

One can mention also Proposition 3.1 that gives an important information about I-varieties and Proposition 3.11 containing an interesting partial information about modular varieties.

3.1. ***I*-varieties.** We denote by $\mathbf{SEM}$ the variety of all semigroups. A semigroup variety $\mathbf{V}$ is called *proper* if $\mathbf{V} \neq \mathbf{SEM}$.

The class of I-varieties includes all varieties with non-trivial identities in subvariety lattices (see Corollary 2.2). It follows from results of [3] that a semigroup variety $\mathbf{V}$ is periodic whenever the lattice $L(\mathbf{V})$ satisfies some non-trivial identity. As we have already mentioned in Section 1, results of the articles [12, 49, 51] imply that if a proper variety $\mathbf{V}$ is a special element of one of the nine types we consider in $\mathbb{SEM}$ then it is periodic too. All these statements are generalized by the following

Proposition 3.1 ([28, Theorem 1]). *A proper I-variety of semigroups is periodic.*

Formally speaking, this proposition is not applicable to cancellable varieties because cancellable elements of a lattice are not I-elements. But, in actual fact, Proposition 3.1 implies that any proper cancellable variety is periodic because a cancellable variety is a modular one and modular elements are I-elements.

On the other hand, it is verified in [28, Theorem 2] that there are periodic varieties (moreover, nil-varieties) of semigroups that are not I-varieties. However, as far as we know, an explicit example of a periodic semigroup variety that is not an I-variety is unknown so far.

3.2. **Lower-modular varieties.** Varieties that may be given by 0-reduced identities only are called 0-*reduced*. We denote by $\mathbf{T}$ the trivial semigroup variety. A number of partial results concerning lower-modular varieties were obtained in [49, 50, 59]. All of them are covered by the following

Theorem 3.2. *A semigroup variety $\mathbf{V}$ is lower-modular if and only if either $\mathbf{V} = \mathbf{SEM}$ or $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ where $\mathbf{M}$ is one of the varieties $\mathbf{T}$ or $\mathbf{SL}$, while $\mathbf{N}$ is a 0-reduced variety.*

This theorem was verified for the first time in [33, Theorem 1.1] and was reproved in a simpler and shorter way in [27]. The proof of Theorem 3.2 given in [27] is based on Theorem 5.1 below. Note that the 'if' part of Theorem 3.2 immediately follows from Corollaries 2.8 (with $\mathbf{X} = \mathbf{SEM}$) and 2.4.

Neutral, standard and distributive varieties are lower-modular. In view of Theorem 3.2, a description of varieties of these three types should look as follows:

A semigroup variety $\mathbf{V}$ is distributive [*standard, neutral*] *if and only if either $\mathbf{V} = \mathbf{SEM}$ or $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ where $\mathbf{M}$ is one of the varieties $\mathbf{T}$ or $\mathbf{SL}$, while $\mathbf{N}$ is a 0-reduced variety such that ...* (with some additional restriction to $\mathbf{N}$ depending on the type of element we consider).

Exact formulations of corresponding results are given in the following two subsections.

3.3. **Distributive and standard varieties.** Put

$$\mathbf{Q} = \mathrm{var}\{x^2y \approx xyx \approx yx^2 \approx 0\},$$
$$\mathbf{Q}_n = \mathrm{var}\{x^2y \approx xyx \approx yx^2 \approx x_1x_2\cdots x_n \approx 0\},$$
$$\mathbf{R} = \mathrm{var}\{x^2 \approx xyx \approx 0\},$$
$$\mathbf{R}_n = \mathrm{var}\{x^2 \approx xyx \approx x_1x_2\cdots x_n \approx 0\}$$

where $n \geq 2$. It is easy to see that varieties of these four types are precisely all non-trivial 0-reduced varieties satisfying the identities $x^2y \approx xyx \approx yx^2 \approx 0$.

Theorem 3.3. *For a semigroup variety* **V***, the following are equivalent:*

a) **V** *is distributive;*
b) **V** *is standard;*
c) *either* **V** = **SEM** *or* **V** = **M** ∨ **N** *where* **M** *is one of the varieties* **T** *or* **SL***, while* **N** *is one of the varieties* **T***,* **Q***,* $\mathbf{Q}_n$*,* **R** *or* $\mathbf{R}_n$*.*

The equivalence a) ⟺ c) is proved in [57, Theorem 1.1]. Note that the proof of the implication a) ⟹ c) given in [57] may be essentially simplified by using Theorem 3.2. The implication b) ⟹ a) is evident. To verify the implication a) ⟹ b), we need two ingredients. First, it is verified in [57, Corollary 1.2] that a distributive semigroup variety is a modular one[3]. Second, it is well known that an element of a lattice is standard whenever it is simultaneously distributive and modular (see [6, Lemma II.1.1], for instance). Note that the former statement is strengthened by Corollary 3.10 below.

It is well known that the set of all standard elements of an arbitrary lattice L forms a sublattice in L [5, Theorem 259 on p. 230]. Theorem 3.3 shows that the lattice of all distributive (equivalently, standard) varieties has the form shown in Fig. 5.

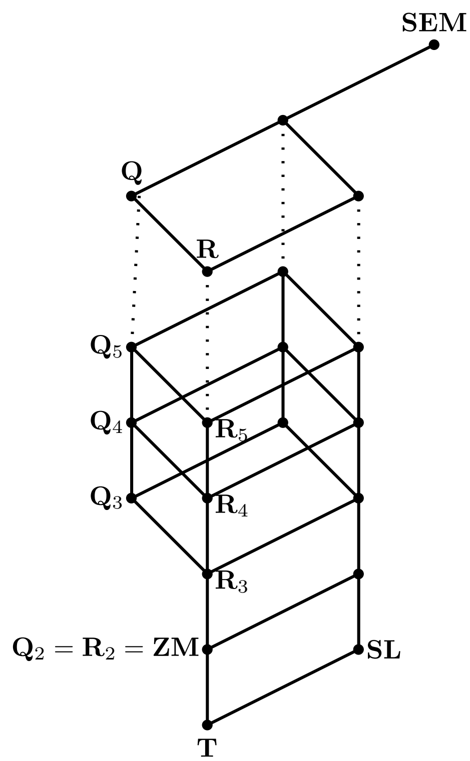

FIGURE 5. The lattice of distributive varieties

Thus, we have

[3]In afctually fact, this claim immediately follows from the implication a) ⟹ c) of Theorem 3.3 and Corollaries 2.8 (with **X** = **SEM**) and 2.4.

Corollary 3.4. *The class of all distributive* [*standard*] *semigroup varieties forms a countably infinite distributive sublattice of the lattice* $\mathbb{SEM}$.

3.4. **Costandard and neutral varieties.** Results of the earlier works [1, 14, 15, 17, 23] mentioned in the Introduction easily imply that the varieties $\mathbf{SL}$ and $\mathbf{ZM}$ are neutral. Since the set of all neutral elements of a lattice L forms a sublattice of L (see [5, Theorem 259(iii) on p. 230]), the variety $\mathbf{SL} \vee \mathbf{ZM}$ is neutral too. The neutrality of the varieties $\mathbf{T}$ and $\mathbf{SEM}$ is evident. It turns out that there are no neutral varieties except the five mentioned ones. More exactly, the following statement is true.

Theorem 3.5. *For a semigroup variety* $\mathbf{V}$*, the following are equivalent:*

a) $\mathbf{V}$ *is both lower-modular and upper-modular;*
b) $\mathbf{V}$ *is both distributive and codistributive;*
c) $\mathbf{V}$ *is costandard;*
d) $\mathbf{V}$ *is neutral;*
e) $\mathbf{V}$ *is one of the varieties* $\mathbf{T}$*,* $\mathbf{SL}$*,* $\mathbf{ZM}$*,* $\mathbf{SL} \vee \mathbf{ZM}$ *or* $\mathbf{SEM}$.

Clearly, the claim e) of this theorem may be reformulated in the manner specified in Subsection 3.2: either $\mathbf{V} = \mathbf{SEM}$ or $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ where $\mathbf{M}$ is one of the varieties $\mathbf{T}$ or $\mathbf{SL}$, while $\mathbf{N}$ is a 0-reduced variety such that $xy \approx 0$ in $\mathbf{N}$.

We do not include in Theorem 3.5 the claim that $\mathbf{V}$ is both standard and costandard because it is well known that an element of arbitrary lattice is both standard and costandard if and only if it is neutral (see [5, Theorem 255 on p. 228], for instance). The equivalences a) $\Longleftrightarrow$ e), c) $\Longleftrightarrow$ e) and d) $\Longleftrightarrow$ e) of Theorem 3.5 were verified in [50, Corollary 3.5], [53, Theorem 1.3] and [64, Proposition 4.1] respectively, while the implications d) $\Longrightarrow$ b) $\Longrightarrow$ a) are evident.

Since a neutral element of a lattice is standard, Theorem 3.5 implies the following

Corollary 3.6 ([53, Corollary 1.1])**.** *Every costandard semigroup variety is standard.*

3.5. **An application to definable varieties.** Here we discuss an interesting application of results overviewed above. We need some new definitions. A subset A of a lattice $\langle L; \vee, \wedge\rangle$ is called *definable in* L if there exists a first-order formula $\Phi(x)$ with one free variable x in the language of lattice operations $\vee$ and $\wedge$ which *defines* A *in* L. This means that, for an element $a \in L$, the sentence $\Phi(a)$ is true if and only if $a \in A$. If A consists of a single element, then we talk about definability of this element. A set $\mathcal{X}$ of semigroup varieties (or a single semigroup variety $\mathbf{X}$) is said to be *definable* if it is definable in $\mathbb{SEM}$. In this situation we will say that the corresponding first-order formula *defines* the set $\mathcal{X}$ or the variety $\mathbf{X}$.

A number of deep results about definable varieties and sets of varieties of semigroups have been obtained in [12] by Ježek and McKenzie. It has been conjectured there that every finitely based semigroup variety is definable up to duality. The conjecture is confirmed in [12] for locally finite finitely based varieties. On their way to obtaining this fundamental result, Ježek and McKenzie proved the definability of several important sets of semigroup varieties such as the sets of all finitely based, all locally finite, all finitely generated and all 0-reduced semigroup varieties. But the article [12] contains no explicit first-order formulas that define any of these sets of varieties. The task of writing an explicit formula that defines the set of all finitely based or the set of all locally finite or the set of all finitely generated varieties seems to be extremely difficult. On the other hand, the set of all 0-reduced varieties can be defined by a quite simple first-order formula based on descriptions of lower-modular and neutral varieties.

Indeed, Theorem 3.2 shows that a semigroup variety is 0-reduced if and only if it is lower-modular and does not contain the variety **SL**. It remains to define the variety **SL**. Theorem 3.5 together with the well-known description of atoms of the lattice $\mathbb{SEM}$ (see [38, Section 1], for instance) imply that this lattice contains exactly two neutral atoms, namely the varieties **SL** and **ZM**. Recall that a semigroup variety **V** is called *chain* if the lattice $L(\mathbf{V})$ is a chain. It is well known that the variety **ZM** is properly contained in some chain variety, while the variety **SL** is not (see [44] or [54] for more details). Combining the mentioned observations, we see that the set of all 0-reduced varieties may be defined as the set $\mathcal{K}$ of semigroup varieties with the following properties:

(i) every member of $\mathcal{K}$ is a lower-modular variety;
(ii) if $\mathbf{V} \in \mathcal{K}$ and **V** contains some neutral atom **A** then **A** is properly contained in some chain variety.

It is evident that properties (i) and (ii) may be written by simple first-order formulas with one free variable.

An explicit formula that defines the set of all 0-reduced varieties is written in [54, Section 3]. Note that the description of distributive semigroup varieties given by Theorem 3.3 may also be applied to define some interesting varieties (see [54, Section 6]).

3.6. **Modular varieties.** The problem of description of modular semigroup varieties is open so far. Here we provide some partial results concerning this problem.

Recall that a semigroup variety is called a *nil-variety* if it consists of nilsemigroups or, equivalently, satisfies an identity of the form $x^n \approx 0$ for some natural n. Clearly, every 0-reduced variety is a nil-variety. The following theorem gives a strong necessary condition for a semigroup variety to be modular.

Theorem 3.7. *If* **V** *is a modular semigroup variety then either* $\mathbf{V} = \mathbf{SEM}$ *or* $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* **M** *is one of the varieties* **T** *or* **SL**, *while* **N** *is a nil-variety.*

This theorem readily follows from [12, Proposition 1.6]. A deduction of Theorem 3.7 from [12, Proposition 1.6] is given explicitly in [48, Proposition 2.1]. A direct and transparent proof of Theorem 3.7 not depending on a technique from [12] is given in [27]. This proof is based on Theorem 5.1 below.

Theorem 3.7 and Corollary 2.4 completely reduce the examination of modular varieties to nil-varieties. There is a strong necessary condition for a nil-variety to be modular. To formulate this result, we need some additional definitions.

We call an identity $\mathbf{u} \approx \mathbf{v}$ *substitutive* if the words **u** and **v** depend on the same letters and **v** may be obtained from **u** by renaming of letters. In [11], Ježek describes modular elements of the lattice of all varieties (more precisely, all equational theories) of any given type. In particular, it follows from [11, Lemma 6.3] that if a nil-variety of semigroups $\mathbf{V}$ is a modular element of the lattice of all groupoid varieties then $\mathbf{V}$ may be given by 0-reduced and substitutive identities only. This does not imply directly the same conclusion for modular nil-varieties because a modular element of $\mathbb{SEM}$ need not be a modular element of the lattice of all groupoid varieties. Nevertheless, the following assertion shows that the 'semigroup analogue' of the mentioned result of Ježek holds true.

Theorem 3.8 ([48, Proposition 2.2])**.** *A modular nil-variety of semigroups may be given by* 0*-reduced and substitutive identities only.*

Corollary 2.8 with $\mathbf{X} = \mathbf{SEM}$ immediately implies the following

Theorem 3.9. *Every* 0*-reduced semigroup variety is modular.*

This fact was noted for the first time in [59, Corollary 3] and rediscovered (in other terminology) in [12, Proposition 1.1][4].

Theorems 3.8 and 3.9 provide a necessary and a sufficient condition for a nil-variety to be modular respectively. The gap between these conditions seems to be not very large. But the necessary condition is not a sufficient one, while the sufficient condition is not a necessary one (this follows from Corollary 3.15 below).

Theorems 3.2 and 3.9 and Corollary 2.4 immediately imply the following

Corollary 3.10 ([33, Corollary 1.2])**.** *Every lower-modular semigroup variety is modular.*

Theorems 3.8 and 3.9 show that in order to describe modular nil-varieties (and therefore, all modular varieties) we need to examine nil-varieties satisfying substitutive identities. A natural partial case of substitutive identities are permutational ones. Some interesting information concerning modular varieties satisfying a permutational identity give the following assertion.

Proposition 3.11 ([48, Theorem 4.5])**.** *Let* $\mathbf{V}$ *be a modular variety of semigroups satisfying a permutational identity* $p_n[\pi]$*. Then* $\mathbf{V}$ *satisfies also:*

(i) *all permutational identities of length* $n+1$*;*
(ii) *all permutational identities of length* n *whenever* $n \geq 5$ *and the permutation* π *is odd;*
(iii) *an arbitrary identity of the form* $\mathbf{u} \approx 0$*, where* $\mathbf{u}$ *is a word of length* n *depending on* $n-1$ *letters, whenever* $n \geq 4$ *and* $\mathbf{V}$ *is a nil-variety.*

The proof of this proposition is based on Theorem 3.8, Proposition 2.10, Lemma 2.9 and results of the article [21].

The following statement gives a complete classification of modular varieties satisfying a permutational identity of length 3.

[4] In fact, Theorem 3.9 is an immediate consequence of the equivalence of the claims a) and c) of Proposition 2.7. This equivalence is proved by Ježek in [11, Proposition 2.2]. In [59], Theorem 3.9 is justified just by reference to this result by Ježek. It is rather unexpected that Ježek and McKenzie give much more complex proof of Theorem 3.9 in [12].

Theorem 3.12 ([42, Theorem 1.1])**.** *A semigroup variety* $\mathbf{V}$ *satisfying a permutational identity of length* 3 *is modular if and only if* $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$ *or* $\mathbf{SL}$*, while the variety* $\mathbf{N}$ *satisfies one of the following identity systems:*

$$xyz \approx zyx,\ x^2y \approx 0;$$
$$xyz \approx yzx,\ x^2y \approx 0;$$
$$xyz \approx yxz,\ xyzt \approx xzty,\ xy^2 \approx 0;$$
$$xyz \approx xzy,\ xyzt \approx yzxt,\ x^2y \approx 0.$$

3.7. **Cancellable varieties.** Some partial results about cancellable semigroup varieties are proved in the articles [9, 42]. All these results are covered by the complete description of cancellable varieties obtained in [32, Theorem 1.1]. To formulate this result, we need notation for varieties introduced in Subsection 3.3 and also some new notation. If $\mathbf{X}$ is one of the varieties $\mathbf{Q}$, $\mathbf{Q}_n$, $\mathbf{R}$ or $\mathbf{R}_n$, and $m \ge 2$ then we denote by $\mathbf{X}[m]$ the subvariety of $\mathbf{X}$ given within $\mathbf{X}$ by all permutational identities of length m. In particular, $\mathbf{Q}_n[m] = \mathbf{Q}_n$ and $\mathbf{R}_n[m] = \mathbf{R}_n$ whenever $m \ge n$.

Theorem 3.13 ([32, Theorem 1.1])**.** *A semigroup variety* $\mathbf{V}$ *is cancellable if and only if either* $\mathbf{V} = \mathbf{SEM}$ *or* $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$ *or* $\mathbf{SL}$*, while* $\mathbf{N}$ *is one of the varieties* $\mathbf{T}$*,* $\mathbf{Q}$*,* $\mathbf{R}$*,* $\mathbf{Q}[m]$ *with* $m \ge 2$*,* $\mathbf{R}[m]$ *with* $m \ge 2$*,* $\mathbf{Q}_n[m]$ *with* $2 \le m \le n$ *or* $\mathbf{R}_n[m]$ *with* $2 \le m \le n$*.*

This theorem shows that if a cancellable semigroup variety satisfies a permutational identity of length m then it satisfies all permutational identities of length m (see [32, Proposition 3.2]). In fact, this claim follows from Lemma 2.9 and Proposition 2.11. Theorem 3.13 shows also that cancellable nil-varieties satisfies the following claim which stronger than Theorem 3.8: such varieties can be given by 0-reduced and permutational identities only.

It is known that the set of all cancellable elements of a lattice L may not be a sublattice of L. However, Theorem 3.3 readily implies the following analogue of Corollary 3.4.

Corollary 3.14. *The class of all cancellable semigroup varieties forms a countably infinite distributive sublattice of the lattice* $\mathbb{SEM}$*.*

In Fig. 6 we show the 'main part' of this lattice, namely, the lattice of cancellable nil-varieties. To get the whole lattice of cancellable varieties, one need to adjoin the new greatest element (the variety $\mathbf{SEM}$) to the direct product of the lattice shown in Fig. 6 and the 2-element chain (consisting of the varieties $\mathbf{T}$ and $\mathbf{SL}$).

Theorems 3.12 and 3.13 readily imply the following

Corollary 3.15. *For a commutative semigroup variety* $\mathbf{V}$*, the following are equivalent:*

a) $\mathbf{V}$ *is modular;*
b) $\mathbf{V}$ *is cancellable;*

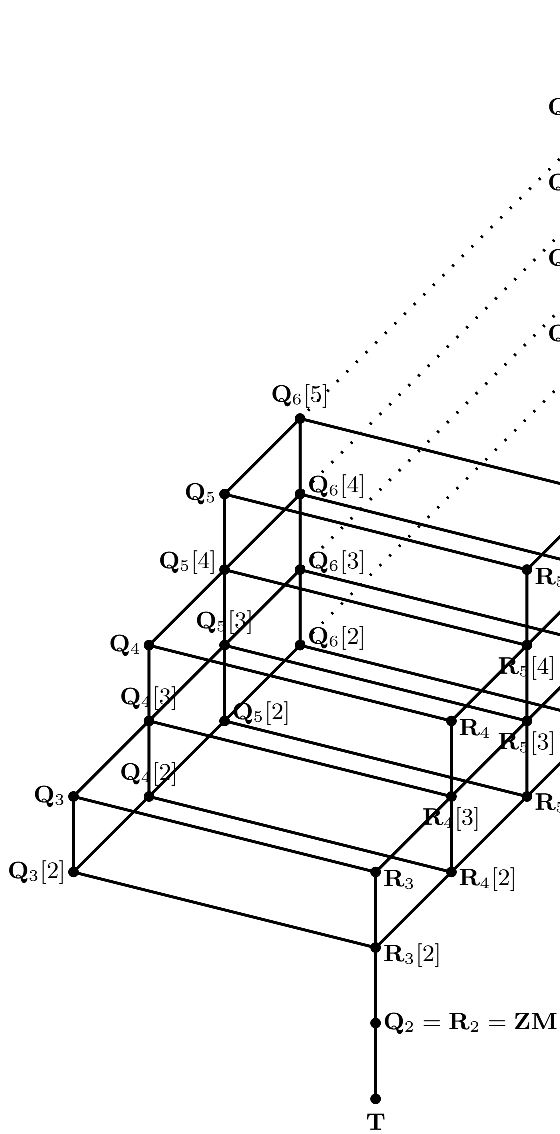

FIGURE 6. The lattice of cancellable nil-varieties

c) $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$ *or* $\mathbf{SL}$*, while* $\mathbf{N}$ *is a commutative variety satisfying the identity*

$$x^2y \approx 0. \tag{1}$$

The equivalences a) $\Longleftrightarrow$ c) and b) $\Longleftrightarrow$ c) in this corollary were first proved in [48, Theorem 3.1] and [9, Theorem 1.1] respectively. Note that the part of Corollary 3.15 concerning modular varieties is reproduced in [55] with inaccuracy[5].

[5]Namely, it is written in [55, Theorem 3.10] that a commutative semigroup variety is modular if and only if it satisfies the identity (1).

3.8. **Upper-modular varieties.** The problem of description of upper-modular semigroup varieties is open so far. Here we provide some partial results concerning this problem. The first result classifies upper-modular varieties in some wide class of varieties. To formulate this statement we need some additional definitions and notation.

A semigroup variety $\mathbf{V}$ is called a variety of *finite degree* [a variety of *degree* n] if all nilsemigroups in $\mathbf{V}$ are nilpotent [if nilpotency degrees of nilsemigroups in $\mathbf{V}$ are bounded by the number n and n is the least number with this property]. We say that a semigroup variety is a *variety of degree* $> n$ if it is either a variety of a finite degree m with $m > n$ or not a variety of finite degree. Put

$$\mathbf{A}_n = \mathrm{var}\{x^n y \approx y,\, xy \approx yx\} \text{ where } n \geq 1,$$
$$\mathbf{C} = \mathrm{var}\{x^2 \approx x^3,\, xy \approx yx\}.$$

In particular, $\mathbf{A}_1 = \mathbf{T}$. Note that $\mathbf{A}_n$ is the variety of all Abelian groups of exponent n.

Theorem 3.16 ([52, Theorem 1])**.** *A semigroup variety* $\mathbf{V}$ *of degree* > 2 *is upper-modular if and only if one of the following holds:*

(i) $\mathbf{V} = \mathbf{SEM}$*;*
(ii) $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$ *or* $\mathbf{SL}$*, while* $\mathbf{N}$ *is a nil-variety satisfying the identities*

$$x^2 y \approx xy^2,\ xy \approx yx; \tag{2}$$

(iii) $\mathbf{V} = \mathbf{A}_n \vee \mathbf{M} \vee \mathbf{N}$ *where* $n \geq 1$*,* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$*,* $\mathbf{SL}$ *or* $\mathbf{C}$*, while* $\mathbf{N}$ *is a commutative variety satisfying the identity* (1)*.*

Theorem 3.16 readily implies a necessary condition for a semigroup variety to be upper-modular given by [51, Theorem 1.1] and the following

Corollary 3.17 ([60, Theorem 2])**.** *A nil-variety of semigroups is upper-modular if and only if it satisfies the identities* (2)*.*

It is easy to list explicitly all upper-modular varieties of degree > 2. Indeed, a simple arguments show that if a nil-variety satisfies the identity system (2) then it satisfies also the identity $x^2yz \approx 0$. Thus, it suffices to describe the subvariety lattice of the variety $\mathbf{U} = \mathrm{var}\{x^2y \approx xy^2,\, xy \approx yx,\, x^2yz \approx 0\}$. Roughtine calculations show that this lattice has the form shown in Fig. 7. The fact that some element of the lattice is marked in Fig. 7 by an identity ε means that the corresponding variety is given within $\mathbf{W}$ by ε .

Theorem 3.16 reduces the examination of upper-modular varieties to varieties of degree ≤ 2. To formulate a result concerning this case, we need some new definitions and notation. Recall that a semigroup variety is called *completely regular* if it consists of *completely regular* semigroups — unions of groups. A semigroup variety $\mathbf{V}$ is called a *variety of semigroups with completely regular*

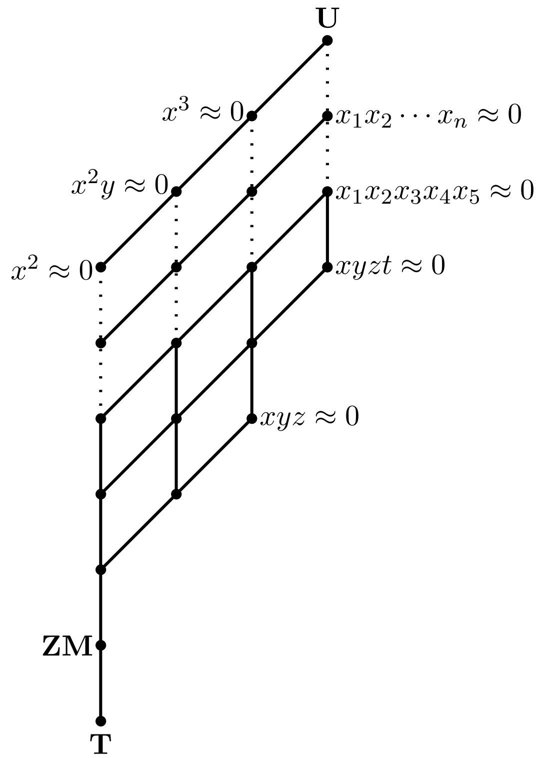

FIGURE 7. The lattice $L(\mathbf{U})$

square if, for any member S of $\mathbf{V}$, the semigroup S^2 is completely regular. Put

$$\mathbf{LZ} = \mathrm{var}\{xy \approx x\},$$
$$\mathbf{RZ} = \mathrm{var}\{xy \approx y\},$$
$$\mathbf{P} = \mathrm{var}\{xy \approx x^2y,\ x^2y^2 \approx y^2x^2\},$$
$$\overleftarrow{\mathbf{P}} = \mathrm{var}\{xy \approx xy^2,\ x^2y^2 \approx y^2x^2\}.$$

All we know about upper-modular varieties of degree ≤ 2 is the following

Theorem 3.18 ([52, Theorem 2])**.** *If* $\mathbf{V}$ *is an upper-modular semigroup variety of degree* ≤ 2 *then one of the following holds:*

(i) $\mathbf{V}$ *is a variety of semigroups with completely regular square;*
(ii) $\mathbf{V} = \mathbf{K} \vee \mathbf{P}$ *where* $\mathbf{K}$ *is a completely regular semigroup variety such that* $\mathbf{RZ} \nsubseteq \mathbf{K}$*;*
(iii) $\mathbf{V} = \mathbf{K} \vee \overleftarrow{\mathbf{P}}$ *where* $\mathbf{K}$ *is a completely regular semigroup variety such that* $\mathbf{LZ} \nsubseteq \mathbf{K}$.

We do not know any example of a non-upper-modular variety that satisfies one of the claims (i)–(iii) of Theorem 3.18. This inspires the following two questions.

Question 3.19. Is every variety of semigroups with completely regular square upper-modular?

Question 3.20. Is every semigroup variety satisfying one of the claims (ii) or (iii) of Theorem 3.18 upper-modular?

A natural weaker version of Question 3.19 is the following

Question 3.21. Is every completely regular semigroup variety upper-modular?

Although Theorems 3.16 and 3.18 do not provide a classification of all upper-modular varieties, they permit to deduct some important and surprising properties of such varieties. Theorems 3.16 and 3.18 together with results of the articles [62, 65] imply the following

Corollary 3.22 ([52, Corollary 2])**.** *A proper upper-modular semigroup variety has a modular subvariety lattice.*

Corollaries 3.22 and 2.14(i) imply the following

Corollary 3.23 ([52, Corollary 3])**.** *If a proper semigroup variety is upper-modular then every its subvariety is also upper-modular.*

Now we describe upper-modular varieties in one more class of varieties. A semigroup variety is called *strongly permutative* if it satisfies an identity of the form $p_n[\pi]$ with $1\pi \neq 1$ and $n\pi \neq n$.

Theorem 3.24. *A strongly permutative semigroup variety* $\mathbf{V}$ *is upper-modular if and only if it satisfies one of the claims* (ii) *or* (iii) *of Theorem* 3.16*.*

A partial case of this statement concerning commutative varieties is proved in [51, Theorem 1.2]. Theorem 3.24 may be easily deduced from the proof of this partial case. A scheme of this deduction is provided in [50].

As we have seen above (see Corollary 3.22), the subvariety lattice of arbitrary proper upper-modular variety is modular. It turns out that such a lattice is even distributive in several wide classes of varieties. So, Theorem 3.16 together with results of the paper [61] imply the following

Corollary 3.25 ([52, Corollary 1])**.** *A proper upper-modular semigroup variety of degree* > 2 *has a distributive subvariety lattice.*

Theorem 3.24 together with results of [61] imply the following

Corollary 3.26. *A strongly permutative upper-modular semigroup variety has a distributive subvariety lattice.*

The special case of this claim dealing with commutative varieties was mentioned in [51, Corollary 4.4].

Theorem 3.18 together with results of the articles [22] and [62] readily imply the following

Corollary 3.27 ([52, Corollary 4])**.** *Let* $\mathbf{V}$ *be a proper upper-modular semigroup variety that is not a variety of semigroups with completely regular square and let* ε *be a non-trivial lattice identity. The lattice* $L(\mathbf{V})$ *satisfies the identity* ε (*in particular, is distributive*) *if and only if the subvariety lattice of any group subvariety of* $\mathbf{V}$ *has the same property.*

Further, a semigroup variety $\mathbf{V}$ is called *aperiodic* if all groups in $\mathbf{V}$ are trivial. Corollary 3.27 together with the result of the paper [4] readily imply the following

Corollary 3.28 ([52, Corollary 5])**.** *An aperiodic upper-modular semigroup variety has a distributive subvariety lattice.*

Corollaries 3.25–3.28 inspire the following open

Question 3.29. Is the subvariety lattice of every proper upper-modular semigroup variety distributive?

All proper upper-modular varieties that appeared above are varieties mentioned in Theorem 3.16. These varieties are commutative. Based on this observation, one can conjecture that any proper upper-modular variety is commutative. But this is not the case. Evident counter-examples are the varieties $\mathbf{LZ}$ and $\mathbf{RZ}$. The claim that these two varieties are upper-modular immediately follows from the fact that they are atoms of the lattice $\mathbb{SEM}$. Two more examples of proper non-commutative upper-modular varieties are the varieties $\mathbf{P}$ and $\overleftarrow{\mathbf{P}}$. Indeed, it is well known that if a variety $\mathbf{V}$ is properly contained in one of these two varieties then $\mathbf{V} \subseteq \mathbf{SL} \vee \mathbf{ZM}$, whence $\mathbf{V}$ is lower-modular by Theorem 3.2. This readily implies that $\mathbf{P}$ and $\overleftarrow{\mathbf{P}}$ are upper-modular.

3.9. **Varieties that are both modular and upper-modular.** It is interesting to examine varieties that satisfy different combinations of the properties we consider. Corollary 3.10 implies that a variety is both modular and lower-modular if and only if it is lower-modular. So, Theorem 3.2 gives, in fact, a complete description of varieties that are both modular and lower-modular (this result was obtained for the first time in [64, Theorem 3.1]). A description of varieties that both are lower-modular and upper-modular as well as varieties that are both distributive and codistributive is given in Theorem 3.5. The following assertion classifies varieties that are both modular and upper-modular.

Proposition 3.30 ([60, Theorem 1])**.** *A semigroup variety* $\mathbf{V}$ *is both modular and upper-modular if and only if either* $\mathbf{V} = \mathbf{SEM}$ *or* $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$ *or* $\mathbf{SL}$*, while* $\mathbf{N}$ *is a commutative variety satisfying the identity* (1)*.*

3.10. **Codistributive varieties.** The problem of description of codistributive semigroup varieties is open so far. Here we provide some partial results concerning this problem. The following theorem gives a strong necessary condition for a semigroup variety to be codistributive.

Theorem 3.31 ([53, Theorem 1.1])**.** *If a semigroup variety* $\mathbf{V}$ *is codistributive then either* $\mathbf{V} = \mathbf{SEM}$ *or* $\mathbf{V}$ *is a variety of semigroups with completely regular square.*

Note that Theorems 3.16 and 3.18 are crucial in the proof of Theorem 3.31.

It is easy to see that a variety of semigroups with completely regular square is a variety of degree ≤ 2 (this readily follows from [24, Lemma 1] or [51, Proposition 2.11]). Therefore, Theorem 3.31 implies that a proper codistributive variety has degree ≤ 2. The following assertion shows that, for strongly permutative varieties, the converse statement holds as well.

Theorem 3.32 ([53, Theorem 1.2])**.** *For a strongly permutative semigroup variety* $\mathbf{V}$*, the following are equivalent:*

a) $\mathbf{V}$ *is codistributive;*
b) $\mathbf{V}$ *has a degree* ≤ 2*;*
c) $\mathbf{V} = \mathbf{A}_n \vee \mathbf{X}$ *where* $n \geq 1$ *and* $\mathbf{X}$ *is one of the varieties* $\mathbf{T}$*,* $\mathbf{SL}$*,* $\mathbf{ZM}$ *or* $\mathbf{SL} \vee \mathbf{ZM}$.

It is easy to see that there exist non-codistributive varieties of semigroups with completely regular square and moreover, non-codistributive periodic group varieties. Indeed, the lattice of periodic group varieties is modular but not distributive. Therefore, it contains the 5-element modular non-distributive sublattice. It is evident that all three pairwise non-comparable elements of this sublattice are non-codistributive periodic group varieties. We see that the problem of description of codistributive varieties is closely related to the problem of description of periodic group varieties with distributive subvariety lattice. The latter problem seems to be extremely difficult (see [38, Subsection 11.2] for more detailed comments), whence the former problem is extremely difficult too.

However, we do not know any examples of non-codistributive varieties of semigroups with completely regular square except ones mentioned in the previous paragraph. This inspires us to eliminate an examination of codistributive varieties with non-trivial groups. In other words, it seems natural to consider aperiodic codistributive varieties only. It is easy to see that if $\mathbf{V}$ is an aperiodic variety of semigroups with completely regular square then, for every $S \in \mathbf{V}$, the semigroups S^2 is a band. A variety with the last property is called a *variety of semigroups with idempotent square*. In view of Theorem 3.31, every aperiodic codistributive variety is a variety of semigroups with idempotent square. Thus, the following question seems to be natural.

Question 3.33. Is every variety of semigroups with idempotent square codistributive?

A natural weaker version of this question is the following

Question 3.34. Is every variety of bands codistributive?

Clearly, every variety of semigroups with idempotent square satisfies the identity $xy \approx (xy)^2$. Put

$$\mathbf{IS} = \mathrm{var}\{xy \approx (xy)^2\},$$
$$\mathbf{BAND} = \mathrm{var}\{x \approx x^2\}.$$

It is verified in [4] that the lattice $L(\mathbf{IS})$ is distributive. Then Corollary 2.14(ii) shows that Question 3.33 is equivalent to the following: is the variety $\mathbf{IS}$ codistributive? Similarly, Question 3.34 is equivalent to asking, whether the variety $\mathbf{BAND}$ is codistributive, i.e., whether the equality

$$\mathbf{BAND} \wedge (\mathbf{X} \vee \mathbf{Y}) = (\mathbf{BAND} \wedge \mathbf{X}) \vee (\mathbf{BAND} \wedge \mathbf{Y})$$

holds for arbitrary varieties $\mathbf{X}$ and $\mathbf{Y}$ or not. It is verified in [18, Corollary 5.9] that this is the case whenever the varieties $\mathbf{X}$ and $\mathbf{Y}$ are locally finite.

A strongly permutative codistributive variety has a distributive subvariety lattice (this follows from Corollary 3.26 and may be easily deduced from Theorem 3.32). Aperiodic codistributive varieties also have a distributive subvariety lattice (here it suffices to refer to either Corollary 3.28 or Theorem 3.31 and the mentioned result of [4]). We do not know any example of proper codistributive variety with non-distributive subvariety lattice. This inspires the following

Question 3.35. Is the subvariety lattice of every proper codistributive semigroup variety distributive?

This question is closely related to the following

Question 3.36. Is every subvariety of an arbitrary proper codistributive semigroup variety codistributive?

Corollary 2.14(ii) shows that the affirmative answer to Question 3.35 would imply the affirmative answer to Question 3.36.

As we have mentioned in Introduction, results of the article [1] imply that all atoms of the lattice $\mathbb{SEM}$ are codistributive. This fact can be generalized by the following way.

Remark 3.37 ([53, Remark 4.1])**.** *If* $\mathbf{V}_1$, $\mathbf{V}_2$, $\ldots$, $\mathbf{V}_k$ *are atoms of the lattice* $\mathbb{SEM}$ *then the variety* $\bigvee_{i=1}^{k} \mathbf{V}_i$ *is codistributive.*

In connection with Questions 3.35 and 3.36, we note that if $\mathbf{V}_1$, $\mathbf{V}_2$, $\ldots$, $\mathbf{V}_k$ are atoms of the lattice $\mathbb{SEM}$ and $\mathbf{V} = \bigvee_{i=1}^{k} \mathbf{V}_i$ then:

(i) the lattice $L(\mathbf{V})$ is distributive (in fact, $L(\mathbf{V})$ is a direct product of k copies of the 2-element chain),

(ii) if $\mathbf{X} \subseteq \mathbf{V}$ then $\mathbf{X}$ is the join of those of the atoms $\mathbf{V}_1$, $\mathbf{V}_2$, $\ldots$, $\mathbf{V}_k$ that are contained in $\mathbf{X}$, and therefore, $\mathbf{X}$ is codistributive by Remark 3.37.

The claim (i) is a part of [58, Proposition 1], while the statement (ii) follows from (i).

4. The lattice $\mathbb{COM}$

For convenience, we call a commutative semigroup variety $\mathbb{COM}$-*modular* if it is a modular element of the lattice $\mathbb{COM}$ and adopt similar agreement for all other types of special elements. The main results of this section provide:

- a complete classification of $\mathbb{COM}$-lower-modular, $\mathbb{COM}$-upper-modular, $\mathbb{COM}$-distributive, $\mathbb{COM}$-codistributive, $\mathbb{COM}$-standard, $\mathbb{COM}$-costandard or $\mathbb{COM}$-neutral varieties (Theorems 4.1, 4.2, 4.3, 4.9 and 4.13),
- necessary conditions for a commutative semigroup variety to be $\mathbb{COM}$-modular (Theorems 4.5 and 4.6),
- a sufficient condition for a commutative semigroup variety to be $\mathbb{COM}$-modular (Theorem 4.7).

Note that there are no any essential information about $\mathbb{COM}$-cancellable varieties so far.

4.1. **$\mathbb{COM}$-lower-modular varieties.** We denote by **COM** the variety of all commutative semigroups. A commutative semigroup variety is called $\mathbb{COM}$-0-*reduced* if it may be given by the commutative law and some non-empty set of 0-reduced identities only. Some partial information about $\mathbb{COM}$-lower-modular varieties was obtained in [26]. It is covered by the following 'commutative analogue' of Theorem 3.2.

Theorem 4.1 ([27, Theorem 1.6]). *A commutative semigroup variety* $\mathbf{V}$ *is* $\mathbb{COM}$*-lower-modular if and only if either* $\mathbf{V} = \mathbf{COM}$ *or* $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$ *or* $\mathbf{SL}$*, while* $\mathbf{N}$ *is a* $\mathbb{COM}$*-0-reduced variety.*

Note that the 'if' part of Theorem 4.1 immediately follows from Corollaries 2.8 (with $\mathbf{X} = \mathbf{COM}$) and 2.4. The proof of the 'only if' part given in [27] is based on Theorem 5.1 below.

As in the case of the lattice $\mathbb{SEM}$ (see Subsection 3.2), Theorem 4.1 implies that a description of $\mathbb{COM}$-distributive, $\mathbb{COM}$-standard and $\mathbb{COM}$-neutral varieties should look as follows:

A commutative semigroup variety $\mathbf{V}$ *is* $\mathbb{COM}$*-distributive* [$\mathbb{COM}$*-standard,* $\mathbb{COM}$*-neutral*] *if and only if either* $\mathbf{V} = \mathbf{COM}$ *or* $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$ *or* $\mathbf{SL}$*, while* $\mathbf{N}$ *is a* $\mathbb{COM}$*-0-reduced variety such that ...* (with some additional restriction to $\mathbf{N}$ depending on the type of element we consider).

Exact formulations of corresponding results are given in the following two subsections.

4.2. **$\mathbb{COM}$-distributive and $\mathbb{COM}$-standard varieties.** The following statement is the 'commutative analogue' of Theorem 3.3.

Theorem 4.2 ([26, Theorem 1.1]). *For a commutative semigroup variety* $\mathbf{V}$*, the following are equivalent:*

a) $\mathbf{V}$ *is* $\mathbb{COM}$*-distributive;*
b) $\mathbf{V}$ *is* $\mathbb{COM}$*-standard;*
c) *either* $\mathbf{V} = \mathbf{COM}$ *or* $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$ *or* $\mathbf{SL}$*, while* $\mathbf{N}$ *is a* $\mathbb{COM}$*-0-reduced variety that satisfies the identities* $x^3yz \approx x^2y^2z \approx 0$ *and either satisfies both the identities* $x^3y \approx 0$ *and* $x^2y^2 \approx 0$ *or does not satisfy any of them.*

Note that the item c) of Theorem 4.2 is reproduced in [55] with inaccuracy[6].

It is verified in [26, Corollary 1.1] that a $\mathbb{COM}$-distributive variety is $\mathbb{COM}$-modular. This statement is generalized by Corollary 4.8 below.

4.3. **$\mathbb{COM}$-neutral varieties.** A complete description of $\mathbb{COM}$-neutral varieties is given by the following partial analogue of Theorem 3.5.

Theorem 4.3. *For a commutative semigroup variety* $\mathbf{V}$*, the following are equivalent:*

[6]Namely, the identity $x^3 \approx 0$ rather than $x^3y \approx 0$ is written in the item c) of [55, Theorem 4.2].

a) **V** *is both* $\mathbb{COM}$*-upper-modular and* $\mathbb{COM}$*-lower-modular;*
b) **V** *is both* $\mathbb{COM}$*-distributive and* $\mathbb{COM}$*-codistributive;*
c) **V** *is* $\mathbb{COM}$*-neutral;*
d) *either* $\mathbf{V} = \mathbf{COM}$ *or* $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* **M** *is one of the varieties* **T** *or* **SL** *and the variety* **N** *satisfies the identity* (1).

The equivalence of the claims b)–d) of this theorem is verified in [26, Theorem 1.2], while the equivalence a) $\Longleftrightarrow$ c) is proved in [27, Corollary 4.2].

Theorem 4.3 and Corollary 3.15 immediately implies the following

Corollary 4.4. *For a commutative semigroup variety* **V** *with* $\mathbf{V} \neq \mathbf{COM}$*, the following are equivalent:*

a) **V** *is* $\mathbb{COM}$*-neutral;*
b) **V** *is modular;*
c) **V** *is cancellable.*

4.4. $\mathbb{COM}$**-modular varieties.** The problem of description of $\mathbb{COM}$-modular commutative semigroup varieties is open so far. Here we provide some partial results concerning this problem. Note that these results are 'commutative analogues' of Theorems 3.7, 3.8 and 3.9.

First of all, the following necessary condition for a commutative semigroup variety to be $\mathbb{COM}$-modular is true.

Theorem 4.5 ([27, Theorem 1.4])**.** *If* **V** *is a* $\mathbb{COM}$*-modular commutative semigroup variety then either* $\mathbf{V} = \mathbf{COM}$ *or* $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* **M** *is one of the varieties* **T** *or* **SL***, while* **N** *is a nil-variety.*

In fact, this theorem readily follows from Theorem 5.1 below. Theorem 4.5 and Corollary 2.4 completely reduce an examination of $\mathbb{COM}$-modular varieties to the nil-case. The following theorem is yet another analogue of the result of Ježek [11] (see Theorem 3.8 and the paragraph before this theorem).

Theorem 4.6 ([27, Theorem 1.5])**.** *A* $\mathbb{COM}$*-modular commutative nil-variety of semigroups may be given within the variety* **COM** *by* 0*-reduced and substitutive identities only.*

Corollary 2.8 with $\mathbf{X} = \mathbf{COM}$ immediately implies the following

Theorem 4.7 ([26, Proposition 2.1])**.** *Every* $\mathbb{COM}$*-*0*-reduced commutative semigroup variety is* $\mathbb{COM}$*-modular.*

Theorems 4.6 and 4.7 provide, respectively, a necessary and a sufficient condition for a commutative nil-variety to be $\mathbb{COM}$-modular. The gap between these conditions does not seem to be very large. But the necessary condition is not a sufficient one, while the sufficient condition is not a necessary one. Indeed:

- the variety $\mathrm{var}\{xyzt \approx x^3 \approx 0,\ x^2y \approx y^2x,\ xy \approx yx\}$ is $\mathbb{COM}$-modular although it is not $\mathbb{COM}$-0-reduced [30, Proposition 4.1],
- the variety
  $\mathrm{var}\{x^4y^3z^2t \approx y^4x^3z^2t,\ x_1x_2\cdots x_{11} \approx \mathbf{w}_i \approx 0,\ xy \approx yx \mid \mathbf{w}_i \in W_{10,3}\}$
  where $W_{10,3}$ is the set of all words of length 10 depending on ≤ 3 letters is not $\mathbb{COM}$-modular although it is given within $\mathbb{COM}$ by 0-reduced and substitutive identities only [30, Proposition 4.2].

Theorems 4.1 and 4.7 and Corollary 2.4 imply the following 'commutative analogue' of Corollary 3.10.

Corollary 4.8 ([27, Corollary 4.1]). *Every* $\mathbb{COM}$*-lower-modular commutative semigroup variety is* $\mathbb{COM}$*-modular.*

4.5. $\mathbb{COM}$**-cancellable varieties.** There are no any significant information about $\mathbb{COM}$-cancellable commutative semigroup varieties so far. We mention only a few observations here.

First, it is clear that the conclusions of Theorems 4.5 and 4.6 remain true whenever $\mathbf{V}$ is a $\mathbb{COM}$-cancellable commutative semigroup variety.

Second, the 'cancellable analogue' of Theorem 4.7 is false. This immediately follows from the complete description of the subvariety lattice of the variety

$$\mathbf{N}_5^c = \mathrm{var}\{x_1x_2x_3x_4x_5 \approx 0,\, xy \approx yx\}$$

obtained in [16, Fig. 3]. We reproduce the diagram of this lattice in Fig. 8 below. This figure shows that the $\mathbb{COM}$-0-reduced varieties $\mathrm{var}\{x_1x_2x_3x_4x_5 \approx x^3 \approx 0,\, xy \approx yx\}$ and $\mathrm{var}\{x_1x_2x_3x_4x_5 \approx x^3y \approx 0,\, xy \approx yx\}$ are not $\mathbb{COM}$-cancellable.

4.6. $\mathbb{COM}$**-upper-modular and** $\mathbb{COM}$**-codistributive varieties.** A complete description of $\mathbb{COM}$-upper-modular and $\mathbb{COM}$-codistributive varieties is given by the following

Theorem 4.9 ([56, Theorem 1.1]). *For a commutative semigroup variety* $\mathbf{V}$*, the following are equivalent:*

a) $\mathbf{V}$ *is* $\mathbb{COM}$*-upper-modular;*
b) $\mathbf{V}$ *is* $\mathbb{COM}$*-codistributive;*
c) *either* $\mathbf{V} = \mathbf{COM}$ *or* $\mathbf{V}$ *satisfies one of the claims* (ii) *or* (iii) *of Theorem* 3.16*.*

Note that, in contrast with the equivalence of the claims a) and b) of this theorem, the properties of being upper-modular and codistributive varieties are not equivalent. This follows from comparison of Theorems 3.24 and 3.32.

Theorems 3.24 and 4.9 implies immediately

Corollary 4.10 ([56, Corollary 4.1]). *If* $\mathbf{V}$ *is a commutative semigroup variety and* $\mathbf{V} \neq \mathbf{COM}$ *then the following are equivalent:*

a) $\mathbf{V}$ *is* $\mathbb{COM}$*-upper-modular;*
b) $\mathbf{V}$ *is* $\mathbb{COM}$*-codistributive;*
c) $\mathbf{V}$ *is upper-modular.*

Theorem 4.3 implies the following 'commutative analogue' of Corollary 3.23.

Corollary 4.11 ([56, Corollary 4.3]). *If* $\mathbf{V}$ *is a* $\mathbb{COM}$*-upper-modular commutative semigroup variety and* $\mathbf{V} \neq \mathbf{COM}$ *then every subvariety of* $\mathbf{V}$ *is* $\mathbb{COM}$*-upper-modular.*

Theorem 4.9 together with results of [61] imply the following

Corollary 4.12 ([56, Corollary 4.4]). *If* $\mathbf{V}$ *is a* $\mathbb{COM}$*-upper-modular commutative semigroup variety and* $\mathbf{V} \neq \mathbf{COM}$ *then the lattice* $L(\mathbf{V})$ *is distributive.*

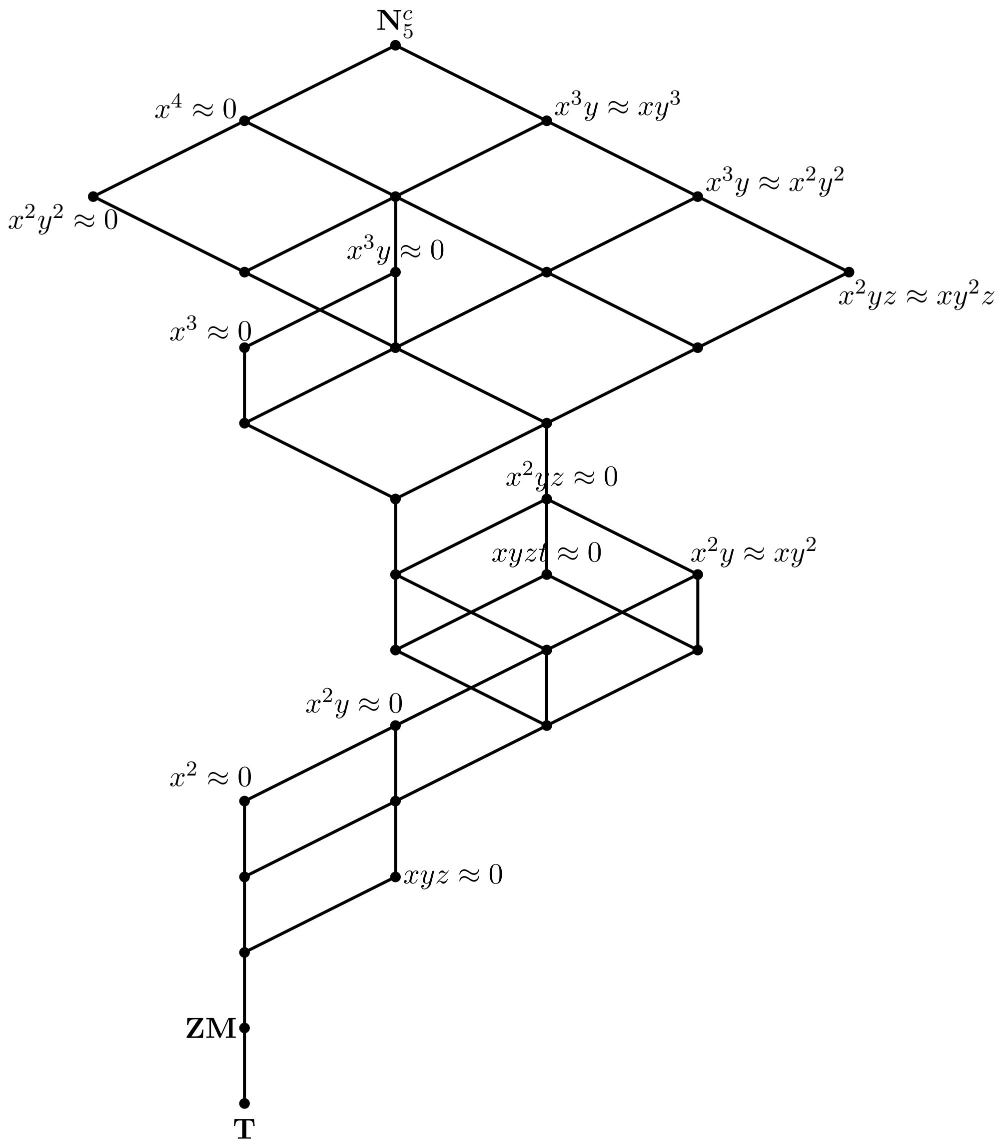

FIGURE 8. The lattice $L(\mathbf{N}_5^c)$

4.7. $\mathbb{COM}$**-costandard varieties.** A complete description of $\mathbb{COM}$-costandard varieties is given by the following

Theorem 4.13 ([56, Theorem 1.2])**.** *For a commutative semigroup variety* $\mathbf{V}$*, the following are equivalent:*

a) $\mathbf{V}$ *is both* $\mathbb{COM}$*-modular and* $\mathbb{COM}$*-upper-modular;*
b) $\mathbf{V}$ *is* $\mathbb{COM}$*-costandard;*
c) *either* $\mathbf{V} = \mathbf{COM}$ *or* $\mathbf{V}$ *satisfies the claim* (ii) *of Theorem* 3.16*.*

A comparison of Theorems 4.3 and 4.13 shows that, in contrast with Theorem 3.5, the properties of being $\mathbb{COM}$-neutral and $\mathbb{COM}$-costandard varieties are not equivalent.

As we have already mentioned in Subsection 3.8, in every nil-variety, the identities (2) imply the identity $x^2yz \approx 0$. Then comparison of Theorems 4.2 and 4.13 implies the following analog of Corollary 3.6 which was not mentioned explicitly anywhere.

Corollary 4.14. *Every $\mathbb{COM}$-costandard commutative semigroup variety is $\mathbb{COM}$-standard.*

Note that a description of varieties that are both $\mathbb{COM}$-modular and $\mathbb{COM}$-lower-modular immediately follows from Corollary 4.8 and Theorem 4.1, while a description of varieties that are both $\mathbb{COM}$-upper-modular and $\mathbb{COM}$-lower-modular is given by Theorem 4.3. We note also that the analogue of Theorem 4.13 in the lattice $\mathbb{SEM}$ is not the case. Indeed, a comparison of Theorem 3.5 and Proposition 3.30 show that there exist semigroup varieties that are simultaneously modular and upper-modular but not costandard.

Theorems 4.9 and 4.13 imply immediately

Corollary 4.15 ([56, Corollary 4.2])**.** *For a commutative nil-variety of semigroups* $\mathbf{V}$*, the following are equivalent:*

a) $\mathbf{V}$ *is $\mathbb{COM}$-upper-modular;*
b) $\mathbf{V}$ *is $\mathbb{COM}$-codistributive;*
c) $\mathbf{V}$ *is $\mathbb{COM}$-costandard;*
d) $\mathbf{V}$ *satisfies the identities* (2)*.*

Note that the equivalence of the claims a)–c) of Corollary 4.15 for $\mathbb{COM}$-0-reduced varieties immediately follows from [26, Theorem 1.2 and Proposition 2.3]. Moreover, if a variety $\mathbf{V}$ is $\mathbb{COM}$-0-reduced then the claims a)–c) are equivalent to the claim that $\mathbf{V}$ is $\mathbb{COM}$-neutral. This also follows from the mentioned results of [26].

5. Lattices located between $\mathbb{SEM}$ and $\mathbb{COM}$

In this section, we examine modular and lower-modular elements only. It turns out that properties of such elements in the lattices $\mathbb{SEM}$ and $\mathbb{COM}$ discussed in Subsections 3.2, 3.6, 4.1 and 4.4 may be partially extended to some sublattices of $\mathbb{SEM}$ that contain $\mathbb{COM}$. More precisely, we have in mind subvariety lattices of overcommutative semigroup varieties and the lattice $\mathbb{PERM}$.

5.1. **Subvariety lattices of overcommutative varieties.** As we have seen in Subsections 4.1 and 4.4, there are numerous parallels between results about modular and lower-modular elements in the lattices $\mathbb{SEM}$ and $\mathbb{COM}$. The following result partially explains these parallels and permits us to give unified proofs of several results about [lower-]modular elements in $\mathbb{SEM}$ and $\mathbb{COM}$.

Theorem 5.1 ([27, Proposition 3.3])**.** *Let* $\mathbf{X}$ *be an overcommutative semigroup variety and* $\mathbf{V}$ *a periodic subvariety of* $\mathbf{X}$*. If* $\mathbf{V}$ *is either a modular or a lower-modular element of the lattice* $L(\mathbf{X})$ *then* $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$ *or* $\mathbf{SL}$*, while* $\mathbf{N}$ *is a nil-variety.*

Applying this theorem with $\mathbf{X} = \mathbf{SEM}$ [respectively $\mathbf{X} = \mathbf{COM}$], we obtain an important information about [$\mathbb{COM}$-]modular and [$\mathbb{COM}$-]lower-modular varieties. After that, only some simple additional arguments are needed to verify Theorems 3.9 and 4.7, as well as the 'if' parts of Theorems 3.2 and 4.1. It is natural to ask if it is possible to eliminate these additional arguments altogether. To do this, we should verify an analogue of Theorem 5.1 without the assumption that the variety $\mathbf{V}$ is periodic. Unfortunately, it turns out that this is impossible. Indeed, it is verified in [45] that every proper semigroup variety is covered in $\mathbb{SEM}$ by some other variety (see also [38, Subsection 3.1]). It is evident that if an overcommutative variety $\mathbf{V}$ is covered by a variety $\mathbf{X}$ then $\mathbf{X}$ is overcommutative and $\mathbf{V}$ is a lower-modular element of the lattice $L(\mathbf{X})$. Thus, the 'lower-modular half' of Theorem 5.1 would be false if we eliminate the assumption that $\mathbf{V}$ is periodic. The same is true for the 'modular half' of this theorem. For example, the variety $\mathbf{COM}$ is a modular element in the lattice $L(\mathbf{W})$ where $\mathbf{W} = \mathrm{var}\{xyz \approx yzx \approx zyx\}$ [30, p. 29]. Note that $\mathbf{COM}$ is also a lower-modular element in $L(\mathbf{W})$ because $\mathbf{W}$ covers $\mathbf{COM}$.

5.2. **The lattice $\mathbb{PERM}$.** By analogy with the commutative case, we call a permutative semigroup variety $\mathbb{PERM}$-[*lower-*]*modular* if it is a [lower-]modular element of the lattice $\mathbb{PERM}$.

Theorem 5.2 ([30, Proposition 2.2])**.** *If a permutative semigroup variety* $\mathbf{V}$ *is either* $\mathbb{PERM}$*-modular or* $\mathbb{PERM}$*-lower-modular then* $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$ *or* $\mathbf{SL}$*, while* $\mathbf{N}$ *is a nil-variety.*

This result does not give any information about $\mathbb{PERM}$-modular or $\mathbb{PERM}$-lower-modular nil-varieties. Recall that:

(i) by Theorems 3.8 and 4.6, every [$\mathbb{COM}$-]modular nil-variety may be given [within $\mathbf{COM}$] by substitutive and 0-reduced identities only;
(ii) by Theorems 3.2 and 4.1, every [$\mathbb{COM}$-]lower-modular nil-variety is [$\mathbb{COM}$-]0-reduced;
(iii) by Corollary 2.8, every [$\mathbb{COM}$-]0-reduced variety is both [$\mathbb{COM}$-]modular and [$\mathbb{COM}$-]lower-modular.

Note that we cannot use Corollary 2.8 to obtain a 'permutative analogue' of the claim (iii) because the class of all permutative semigroups does not form a variety.

We do not know, whether a 'permutative analogue' of the claim (i) true. So, we formulate the following

Question 5.3. Is it true that every $\mathbb{PERM}$-modular permutative nil-variety of semigroups may be given by substitutive and 0-reduced identities only?

As to 'permutative analogues' of claims (ii) and (iii), they do not hold. For instance:

- the variety $\mathrm{var}\{xyzt \approx 0,\ x^2y \approx xyx\}$ is $\mathbb{PERM}$-lower-modular although it may not be given by permutational and 0-reduced identities only;
- the variety $\mathbf{N}_5^c$ is neither $\mathbb{PERM}$-modular nor $\mathbb{PERM}$-lower-modular although it is permutative and is given by permutational and 0-reduced identities only (the variety $\mathbf{N}_5^c$ was introduced in Subsection 4.5).

Both these claims are communicated to the author by Shaprynskiĭ.

6. The lattice $\mathbb{OC}$

For convenience, we call an overcommutative semigroup variety $\mathbb{OC}$-*modular* if it is a modular element of the lattice $\mathbb{OC}$ and adopt similar agreement for all other types of special elements.

The problems of description of $\mathbb{OC}$-modular, $\mathbb{OC}$-lower-modular and $\mathbb{OC}$-upper-modular varieties are open so far. Moreover, any essential information about varieties of these three types is absent. On the other hand, $\mathbb{OC}$-distributive, $\mathbb{OC}$-codistributive, $\mathbb{OC}$-cancellable, $\mathbb{OC}$-standard, $\mathbb{OC}$-costandard and $\mathbb{OC}$-neutral varieties are completely determined (see Theorem 6.1 below). To formulate this result, we need some new definitions and notation.

Let m and n be positive integers with $2 \le m \le n$. A sequence of positive integers $(\ell_1, \ell_2, \dots, \ell_m)$ is called a *partition of n into m parts* if

$$\sum_{i=1}^{m} \ell_i = n \quad \text{and} \quad \ell_1 \ge \ell_2 \ge \cdots \ge \ell_m.$$

The set of all partitions of n into m parts is denoted by $\Lambda_{n,m}$. Let $\lambda = (\ell_1, \ell_2, \dots, \ell_m) \in \Lambda_{n,m}$. We define numbers $q(\lambda)$, $r(\lambda)$ and $s(\lambda)$ as follows:

$q(\lambda)$ is the number of ℓ_i's with $\ell_i = 1$;

$r(\lambda) = n - q(\lambda)$ (in other words, $r(\lambda)$ is the sum of all ℓ_i's with $\ell_i > 1$);

$s(\lambda) = \max\bigl\{r(\lambda) - q(\lambda) - \delta, 0\bigr\}$ where

$$\delta = \begin{cases} 0 & \text{if } n = 3,\ m = 2 \text{ and } \lambda = (2,1), \\ 1 & \text{otherwise.} \end{cases}$$

If $k \ge 0$ then $\lambda^{(k)}$ stands for the following partition of $n + k$ into $m + k$ parts:

$$\lambda^{(k)} = (\ell_1, \ell_2, \dots, \ell_m, \underbrace{1, \dots, 1}_{k \text{ times}})$$

(in particular, $\lambda^{(0)} = \lambda$). If $\mu = (m_1, m_2, \dots, m_s) \in \Lambda_{r,s}$ then $W_{r,s,\mu}$ stands for the set of all words $\mathbf{u}$ such that:

- the length of $\mathbf{u}$ equals r;
- $\mathbf{u}$ depends on the letters $x_1, x_2, \dots, x_s$;
- for every $i = 1, 2, \dots, s$, the number of occurrences of x_i in $\mathbf{u}$ equals m_i.

For a partition $\lambda = (\ell_1, \ell_2, \dots, \ell_m) \in \Lambda_{n,m}$, we put

$$\mathbf{S}_\lambda = \operatorname{var}\bigl\{\mathbf{u} \approx \mathbf{v} \mid \text{there is } i \in \{0, 1, \dots, s(\lambda)\} \text{ such that } \mathbf{u}, \mathbf{v} \in W_{n+i,m+i,\lambda^{(i)}}\bigr\}.$$

We call sets of the form $W_{n,m,\lambda}$ *transversals*. We say that an overcommutative variety $\mathbf{V}$ *reduces* [*collapses*] a transversal $W_{n,m,\lambda}$ if $\mathbf{V}$ satisfies some non-trivial identity [all identities] of the form $\mathbf{u} \approx \mathbf{v}$ with $\mathbf{u}, \mathbf{v} \in W_{n,m,\lambda}$. An overcommutative variety $\mathbf{V}$ is said to be *greedy* if it collapses any transversal it reduces.

Theorem 6.1. *For an overcommutative semigroup variety $\mathbf{V}$, the following are equivalent:*

a) $\mathbf{V}$ *is $\mathbb{OC}$-neutral;*
b) $\mathbf{V}$ *is $\mathbb{OC}$-standard;*

c) $\mathbf{V}$ *is* $\mathbb{OC}$*-costandard;*
d) $\mathbf{V}$ *is* $\mathbb{OC}$*-cancellable;*
e) $\mathbf{V}$ *is* $\mathbb{OC}$*-distributive;*
f) $\mathbf{V}$ *is* $\mathbb{OC}$*-codistributive;*
g) $\mathbf{V}$ *is greedy;*
h) *either* $\mathbf{V} = \mathbf{SEM}$ *or* $\mathbf{V} = \bigwedge_{i=1}^{k} \mathbf{S}_{\lambda_i}$ *for some partitions* $\lambda_1, \lambda_2, \ldots, \lambda_k$.

The equivalence of the claims a)–c) and e)–g) of this theorem was proved in [47] (claim g) was not mentioned in [47] explicitly but the fact that this claim is equivalent to each of the claims a)–c), e) and f) readily follows from the proof of [47, Theorem 2]). The equivalences d) $\Longleftrightarrow$ g) and g) $\Longleftrightarrow$ h) are verified in [35] and [34] respectively. The results of the paper [63] and Propositions 2.11 and 2.12 play the crucial role in the parts of the proof of Theorem 6.1 given in [35] and [47].

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