Homework №2

Predicates

1) A) Are the formulas equivalent

   $F_1 = (\forall x)F(x) \to (\exists x)G(x)$ and

   $F_2 = (\exists x)\big(F(x) \to G(x)\big)$?

   B) Are the formulas equivalent

   $F_1 = (\forall x)F(x) \to (\forall x)G(x)$ and

   $F_2 = (\forall x)\big(F(x) \to G(x)\big)$?

   C) Are the formulas equivalent

   $F_1 = (\forall x)(\exists y)(F(x,y) \wedge G(x,y))$ and

   $F_2 = (\forall x)(\exists y)F(x,y) \wedge (\forall x)(\exists y)G(x,y)$?

2) Reduce to the Skolem normal form

   $\neg[(\forall x)(\exists y)[P(x,y) \to Q(y)]]$.

3) Show that the reasoning is wrong:

   Some students like their teachers. No one likes ignorant people.

   Therefore, there are ignorant teachers.

4) Write the predicate "*There exist at least two integers*" as a logical formula of the signature $< \boldsymbol{R}, P(x), Q(x,y) >$, where $P(x)$ – "x is Integer", $Q(x,y)$ – "x is equal to y".

5) Using the resolution method prove that the formula *G* is a logical consequence of formulas F_i:

   $$F_1 = (\forall x)\big[P(x) \to (\exists y)[Q(y) \wedge S(x,y)]\big],$$

   $$F_2 = (\exists x)\big[R(x) \vee (\forall y)\neg[\neg Q(y) \to S(x,y)]\big],$$

   $$F_3 = (\exists x)P(x),$$

   $G = (\exists x)[\neg P(x) \vee R(x)]$.

6) Prove that the reasoning is right.

   (Sorit L. Carroll).

   (1) Of all birds, only ostriches reach a height of 9 feet.

   (2) In this aviary, there are no birds that belong to anyone except me.

   (3) No ostriches eat pies with filling.

   (4) I do not have any birds that do not reach a height of 9 feet. Therefore, no bird in this birdhouse eats pies with filling.

   Take the set of birds as the main set.

7) Is the formula F satisfiable? Is the formula F true identically? Is the formula F false identically?

A) $F = (\forall x)(P(x) \rightarrow (\forall y)P(y))$
B) $F = P(x) \rightarrow (\forall y)P(y)$
C) $T = (\forall x)(P(x) \rightarrow (\exists y)P(y))$
D) $R(x) = P(x) \rightarrow (\exists y)P(y)$

<u>Some laws of predicate logic</u>

22) $(\forall x)(F(x) \wedge G(x))$ is equal to $(\forall x)F(x) \wedge (\forall x)G(x)$,
23) $(\exists x)(F(x) \vee G(x))$ is equal to $(\exists x)F(x) \vee (\exists x)G(x)$,
24) $(\forall x)(\forall y)F(x,y)$ is equal to $(\forall y)(\forall x)F(x,y)$,
25) $(\exists x)(\exists y)F(x,y)$ is equal to $(\exists \mathrm{y})(\exists \mathrm{x})F(x,y)$,
26) $\neg(\forall x)F(x)$ is equal to $(\exists x)\neg F(x)$,
27) $\neg(\exists x)F(x)$ is equal to $(\forall x)\neg F(x)$,
28) $(\forall x)(F(x) \vee G)$ is equal to $(\forall x)\ F(x) \vee G$,
29) $(\exists x)(F(x) \wedge G)$ is equal to $(\exists x)\ F(x) \wedge G$,
30) $(\forall x)\ F(x)$ is equal to $(\forall z)\ F(z)$,
31) $(\exists x)\ F(x)$ is equal to $(\exists z)\ F(z)$.

Solutions

№1. A) Are the formulas equivalent?

$F_1 = (\forall x)F(x) \rightarrow (\exists x)G(x)$ and
$F_2 = (\exists x)(F(x) \rightarrow G(x))$?

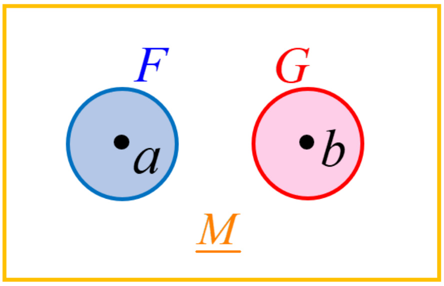
Solution:
$F_1 = (\forall x)F(x) \rightarrow (\exists x)G(x)$ |=| expanding the implication |=|
|=| $\neg(\forall x)F(x) \vee (\exists x)G(x)$ |=| Law 26 |=|
|=| $(\exists x)\neg F(x) \vee (\exists x)G(x)$ |=| Law 23 |=|
|=| $(\exists x)(\neg F(x) \vee G(x))$ |=| converse implication |=|
|=| $(\exists x)(F(x) \rightarrow G(x)) = F_2$

Answer: Formulas F_1 and F_2 are equivalent.

B) Are the formulas equivalent?

$F_1 = (\forall x)F(x) \rightarrow (\forall x)G(x)$ and
$F_2 = (\forall x)(F(x) \rightarrow G(x))$?

Solution:
Let's build an interpretation (model) $\underline{M} = \langle M; \sigma \rangle$, $M = \{a, b\}$, $\sigma = \langle F, G \rangle$, such on that on this model $F_1 = 1, F_2 = 0$.



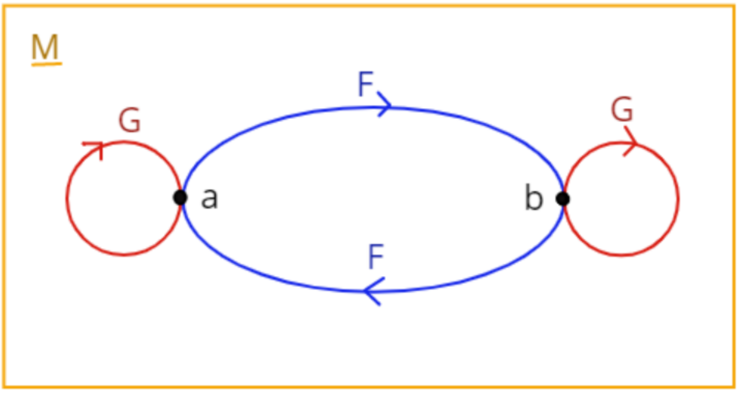
$F(a) = 1, F(b) = 0, G(a) = 0, G(b) = 1.$

Answer: Formulas F_1 and F_2 are not equivalent.

C) Are the formulas equivalent?

$F_1 = (\forall x)(\exists y)(F(x,y) \wedge G(x,y))$ и
$F_2 = (\forall x)(\exists y)F(x,y) \wedge (\forall x)(\exists y)G(x,y)$?

Solution:
Let's build an interpretation (model) $\underline{M} = \langle M; \sigma \rangle$, $M = \{a, b\}$, $\sigma = \langle F, G \rangle$, such on that on this model $F_1 = 0, F_2 = 1$.



$F(a,b) = F(b,a) = 1, F(a,a) = F(b,b) = 0,$

$G(a,b) = G(b,a) = 0, G(a,a) = G(b,b) = 1.$

Answer: Formulas F_1 and F_2 are not equivalent.

№2. Reduce to the Skolem normal form
$\neg[(\forall x)(\exists y)[P(x,y) \rightarrow Q(y)]]$

Solution:
$\neg[(\forall x)(\exists y)[P(x,y) \rightarrow Q(y)]]$ |=| expanding the implication |=|

$|=| \neg(\forall x)\left[(\exists y)[\neg P(x,y) \vee Q(y)]\right] |=|$ Law 26 $|=|$
$|=| (\exists x)\neg(\exists y)[\neg P(x,y) \vee Q(y)]$ $|=|$ Law 27 $|=|$
$|=| (\exists x)(\forall y)\neg[\neg P(x,y) \vee Q(y)] |=|$ we apply the negation $|=|$
$|=| (\exists x)(\forall y)[P(x,y) \wedge \neg Q(y)] |=|$ We remove $\exists$: $substitute\ x = c \sim$
$\sim(\forall y)[P(c,y) \wedge \neg Q(y)]$

Answer: $(\forall y)[P(c,y) \wedge \neg Q(y)]$

№3. Show that the reasoning is wrong:
Some students like their teachers. No one likes ignorant people. Therefore, there are ignorant teachers.

Solution:

Let's take the set of people as the main set M.

Let

$P(x) = 1$: "$x -$ is student", $D(x) = 1$: "$x -$ is teacher",

$Q(x) = 1$: "$x -$ ignorant", $L(x,y) = 1$: "x likes y".

Then
(1) $F_1: (\exists x)[P(x) \wedge (\forall y)(D(y) \rightarrow L(x,y))]$
(2) $F_2: (\forall x)(\forall y)[Q(y) \rightarrow \neg L(x,y)]$
(3) $\mathrm{G}: (\exists x)[D(x) \wedge Q(x)]$

Let's take the negation of G:

$$\neg \mathrm{G} = \neg(\exists x)[D(x) \wedge Q(x)] |=| (\forall x)[\neg D(x) \vee \neg Q(x)]$$

Let's build an interpretation (model) $\underline{M} = \langle M; \sigma \rangle$, $M = \{a, b, c\}$, $\sigma = \langle Q, P, D, L \rangle$, such on that on this model $F_1 = F_2 = 1, \mathrm{G} = 0$.

Let

$P(a) = 1,\ P(b) = 0,\ P(c) = 0,$
$D(a) = 0,\ D(b) = 1,\ D(c) = 0,$
$Q(a) = 0,\ Q(b) = 0,\ Q(c) = 1,$
$L(a,b) = 1,\ L(x,y) = 0,$ if $x \neq a$ or $y \neq b$

Then as it is easy to understand, $F_1 = F_2 = 1, \mathrm{G} = 0$.

Answer: This reasoning is illogical.

№4. Write the predicate "*There exist at least two integers*" as a logical formula of the signature $< \boldsymbol{R}, P(x), Q(x,y) >$, where $P(x)$ – "x is Integer", $Q(x,y)$ – "x is equal to y".

Solution:

F - "*There exist at least two integers*",

$P(x) = 1$: " x – is Integer",

$P(y) = 1$: " y – is Integer",

$Q(x,y) = 1$: " x is equal to y ".

" There exist at least two unequal integers":

$F = (\exists x)(\exists y)[P(x) \wedge P(y) \ \wedge \neg Q(x,y)]$

Answer: $F = (\exists x)(\exists y)[P(x) \wedge P(y) \ \wedge \neg Q(x,y)]$

№5. Using the resolution method prove that the formula *G* is a logical consequence of formulas F_i:

$F_1 = (\forall x)[P(x) \rightarrow \ (\exists y)(Q(y) \wedge S(x,y))]$,

$F_2 = (\exists x)\big[R(x) \vee \ (\forall y)\neg\big(Q(y) \wedge S(x,y)\big)\big]$,

$F_3 = (\exists x)P(x)$,

$G = (\exists x)[\neg P(x) \vee R(x)]$.

Solution:

Let's build the set $\{F_1,\ F_2,\ F_3,\ \neg\text{G}\}$. We will convert each of the formulas into Skolem normal form, resulting in the following formulas:

F_1: $(\forall x)[P(x) \rightarrow \ (\exists y)(Q(y) \wedge S(x,y))] | {=} |$

$| {=} |\ (\forall x)[\neg P(x) \vee \ (\exists y)(Q(y) \wedge S(x,y))] | {=} |$

$| {=} |\ (\forall x)[(\exists y)(Q(y) \wedge S(x,y)) \vee \neg P(x)] | {=} |$ Law 29 $| {=} |$

$| {=} |\ (\forall x)(\exists y)[(Q(y) \wedge S(x,y)) \vee \neg P(x)] | {=} |$

$| {=} |\ (\forall x)[(Q(y) \wedge S(x,y)) \vee \neg P(x)] \sim$

$\sim (\forall x)[(\neg P(x) \vee Q(a)) \wedge (\neg P(x) \vee S(x,a))]$

F_2: $(\exists x)[R(x) \vee (\forall y)\neg(Q(y) \wedge S(x,y))]|{=}|$

$|{=}|\ (\exists x)[R(x) \vee (\forall y)\neg(Q(y) \wedge S(x,y))]|{=}|$ Law 28 $|{=}|$

$|{=}|\ (\exists x)(\forall y)[R(x) \vee (Q(y) \wedge S(x,y))]|{=}|$

$|{=}|\ (\exists x)(\forall y)[[R(x) \vee \neg Q(y)] \wedge [R(x) \vee \neg S(x,y)]]\sim$

$\sim (\forall y)[[R(b) \vee \neg Q(y)] \wedge [R(b) \vee \neg S(b,y)]]$

F_3: $(\exists x)P(x) \sim P(c)$

$\neg G$: $\neg(\exists x)[\neg P(x) \vee R(x)]\ |{=}|$ Law 27 $|{=}|(\forall x)\neg[\neg P(x) \vee R(x)]\ |{=}|$
$|{=}|\ (\forall x)[P(x) \wedge \neg R(x)]$

The set S will consist of seven disjunctions:

$S = \{\neg P(x) \vee Q(a), \neg P(u) \vee S(u,a), R(b) \vee \neg Q(y),$
$R(b) \vee \neg S(b,z), P(c), P(v), \neg R(v)\}$

Let's build a resolutive conclusion:

1. $\neg P(x) \vee Q(a)$
2. $\neg P(u) \vee S(u,a)$
3. $R(b) \vee \neg Q(y)$
4. $R(b) \vee \neg S(b,z)$
5. $P(c)$
6. $P(v)$
7. $\neg R(v)$
8. $Q(a)$ $\{x = v\}$ from 1, 6
9. $\neg Q(y)$ $\{v = b\}$ from 3, 7
10. $\blacksquare\{y = a\}$ from 8, 9

Answer: G is a logical consequence of the formulas F_i.

№6. Prove that the reasoning is right.

(1) Of all birds, only ostriches reach a height of 9 feet.

(2) In this aviary, there are no birds that belong to anyone except me.

(3) No ostriches eat pies with filling.

(4) I do not have any birds that do not reach a height of 9 feet. Therefore, no bird in this birdhouse eats pies with filling.

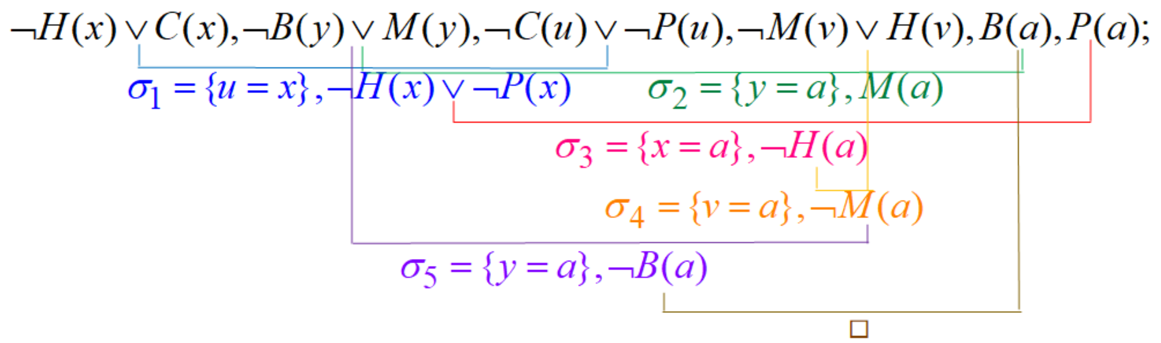
Решение:

Пусть $M = \{\text{The set of birds}\}$
$\mathrm{C}(x) = 1 \Leftrightarrow x - \text{Ostrich}$
$\mathrm{H}(x) = 1 \Leftrightarrow x - Reach\text{ a height of 9 feet}$
$\mathrm{B}(x) = 1 \Leftrightarrow x - \text{Bird in this birdhouse}$
$M(x) = 1 \Leftrightarrow x - \text{A bird belonging to me}$
$P(x) = 1 \Leftrightarrow x\ \text{Eats pies with filling}$

$F_1 = \forall x(H(x) \to C(x)) \equiv \forall x(\neg H(x) \vee C(x))$. Disjunct: $\neg H(x) \vee C(x)$.
$F_2 = \neg\exists x(\mathrm{B}(x) \wedge \neg M(x) \equiv \forall x(\neg B(x) \vee M(x))$. Disjunct: $\neg B(y) \vee M(y)$.
$F_3 = \neg\exists x(\mathrm{C}(x) \wedge P(x)) \equiv \forall x(\neg \mathrm{C}(x) \vee \neg P(x))$. Disjunct: $\neg \mathrm{C}(u) \vee \neg P(u)$.
$F_4 = \neg\exists x(M(x) \wedge \neg \mathrm{H}(x) \equiv \forall x(\neg M(x) \vee \mathrm{H}(x))$. Disjunct: $\neg M(v) \vee \mathrm{H}(v)$
$G = \neg\exists x(B(x) \wedge P(x)) \equiv \forall x(\neg B(x) \vee \neg P(x))$
$\neg G = \exists x\big(B(x) \wedge P(x)\big) \sim B(a) \wedge P(a)$. Disjunctions: $B(\mathrm{a}), P(a)$.

Let's derive the empty disjunct from a set of disjunctions.

$S = \{\neg H(x) \vee C(x), \neg B(y) \vee M(y), \neg \mathrm{C}(u) \vee \neg P(u),$
$\neg M(v) \vee \mathrm{H}(v), B(a), P(a)\}$



$\neg H(x) \vee \neg P(x)$, $M(a)$, $\neg H(a)$, $\neg M(a)$, $\neg B(a)$, $\square$

№7. Is the formula F satisfiable? Is the formula F true identically? Is the formula F false identically?

A) $F = (\forall x)(P(x) \to (\forall y)P(y))$

B) $F = P(x) \to (\forall y)P(y)$

C) $T = (\forall x)(P(x) \to (\exists y)P(y))$
D) $R(x) = P(x) \to (\exists y)P(y)$

Let us first recall the definitions from the lectures.

The formula F of signature σ is called *satisfiable* [*true*] on the model $\underline{M} = \langle M; \sigma \rangle$, if it is true for some [respectively, for any] interpretation into this model. The formula F is simply *satisfiable* if it is satisfiable on some model. Note that for closed formulas the concepts of satisfiability and on the model and truth on the model coincide.

The formula F is called *logically valid* if it is true on any model of signature σ. Finally, the formula F is called logically *contradictory* if the formula $\neg F$ is logically valid.

Solution:

A) Let's build an interpretation (model) $\underline{M} = \langle M; \sigma \rangle, M = \{a\}, \ \sigma = \langle P \rangle,$ such that the closed formula F is true on this model. To do this, it suffices to define $P(a) = 1$. Therefore, the formula F is **satisfiable.**

Let's show that the formula $\neg F$ is also satisfiable. Since the formula

$$\neg F \equiv (\exists x)\big(P(x) \wedge \neg(\forall y)P(y)\big) \equiv (\exists x)\big(P(x) \wedge (\exists y)\neg P(y)\big)$$
$$\equiv (\exists x)(\exists y)(P(x) \wedge \neg P(y))$$

Has a (Skolem normal form) $G = P(a) \wedge \neg P(b)$, which is satisfiable or unsatisfiable at the same time as the formula $\neg F$, we need to construct a model that satisfies this condition $\underline{N} = \langle N; \sigma' \rangle$, where $\sigma' = \langle P, a, b \rangle$, such that G is true on this model. To do this, it is enough to take

$$N = \{a, b\}, P(a) = 1, P(b) = 0$$

Therefore, the formula $\neg F$ is satisfiable, which means that there exists a model on which the formula F is false. Hence, the formula F is **not logically valid**. Furthermore, since there exists a model for which F is true, i.e. $\neg F$ is false, $\neg F$ is not logically valid, and therefore F is **not logically contradictory**.

B) On a model, $\underline{M} = \langle M; \sigma \rangle, M = \{a\}, \sigma = \langle P \rangle, P(a) = 1,$ the formula $H(x)$ is true at $x = a$. Therefore, the formula $H(x)$ is satisfiable on this model, i.e. simply satisfiable.

The formula $\neg H(x) \equiv P(x) \wedge \neg(\forall y)P(y) \equiv (\exists y)(P(x) \wedge \neg P(y))$ has a (Skolem normal form) $K(x) = P(x) \wedge \neg P(b)$, which is satisfiable or unsatisfiable at the same time as the formula $\neg H(x)$. Clearly, the formula $K(x)$ is true in the model

$\underline{N} = \langle N; \sigma' \rangle$ at $x = a$, т.е. is satisfiable on this model, i.e. simply satisfiable. Therefore, the formula $\neg H(x)$ is also satisfiable. This means that the formula $H(x)$ is false in some model for some interpretation of the free variable x, and therefore, it is not **logically valid**. Since $\neg H(x)$ is false in the model $\underline{M} = \langle M; \sigma \rangle$ for some interpretation of the free variable x, then the formula $\neg H(x)$ is not logically valid, and therefore the formula $H(x)$ is **not logically contradictory**.

C) Let's show that the closed formula T is logically valid. To do this, we will prove that the formula$\neg T$ is logically contradictory, i.e. false in any model. By definition (see theoretical material on the resolution method in predicate logic), this means that the formula $\neg T$ has no model. To do this, we can show that from the set S of disjuncts in the (Skolem normal form) of this formula, an empty disjunct is derived (see the same theoretical material). Let's transform the formula $\neg T$ to (ПНФ = не понятно что именно значит) and then to (СНФ = не понятно что именно значит):

$$\neg T \equiv (\exists x)\big(P(x) \wedge \neg(\exists y)P(y)\big) \equiv (\exists x)\big(P(x) \wedge (\forall y)\neg P(y)\big)$$

$$\equiv (\exists x)(\forall y)(P(x) \wedge \neg P(y)) \sim (\forall y)(P(a) \wedge \neg P(y))$$

Then, from the set $S = \{P(a), \neg P(y)\}$, it is obvious that an empty disjunct is derived (for this, it is sufficient to take the most general unifier $\sigma = \{y = a\}$).

Thus, the formula $\neg T$ has no model, i.e. logically contradictory, and therefore the formula T **is logically valid**. Consequently, the last formula is **satisfiable** and not **logically contradictory.**

D) Consider the closure of the formula $R(x)$. It is a formula T. Since the formula T is logically valid, i.e. true on any model, the formula $R(x)$. is also logically valid, which means it is satisfiable and not logically contradictory.

Answer:

A) and B) are satisfiable, but not logically valid and not logically contradictory.

C) and G) are satisfiable, logically valid, and not logically contradictory.