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Semicomplements in lattices of varieties

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Let $\mathfrak{X}$ be a (quasi)variety of universal algebras and $L(\mathfrak{X})$ be the lattice of its sub(quasi)varieties. The class of all lattices of the kind $L(\mathfrak{X})$ for all (quasi)varieties $\mathfrak{X}$ is denoted by $\mathbb{V}$ (resp., $\mathbb{Q}$).

In the paper [6] complemented lattices from $\mathbb{V} \cup \mathbb{Q}$ were investigated. In particular it was proved there that for varieties and quasivarieties $\mathfrak{X}$ of all "classical" algebras $L(\mathfrak{X})$ is Boolean whenever it is complemented (see Corollary 3 below).

A natural generalization of complementedness is upper semicomplementedness. Recall that a lattice L is called *upper semicomplemented* if for each non-zero element $x \in L$, the set $U_x = \{y \in L \mid y \neq 1, x \vee y = 1\}$ is non-empty; an arbitrary element of U_x is called an upper semicomplement to x. *Lower semicomplemented* lattices and lower semicomplements to their elements are defined dually.

Upper semicomplements in $\mathbb{V}$ were studied by Ježek. In [2] he discovered that upper semicomplemented elements exist in the lattice of all varieties of a given type τ and described all such elements. Some further details for the case $\tau = \langle 2 \rangle$ can be seen in [3]. In the paper [6] was proved that a lattice from $\mathbb{Q}$ is complemented whenever it is upper semicomplemented. This fact has incited myself and M. V. Volkov to formulate the following question in [6]: is the same true for varieties, i.e. is a lattice from $\mathbb{V}$ complemented whenever it is upper semicomplemented?

The question is still open in the general case. The aim of the present paper is to give an affirmative answer in several wide partial cases including all "classical" varieties (see Theorems 1 and 2 below).

Let us fix some definitions and notations. The set of all atoms (coatoms) of a lattice L is denoted by $A(L)$ (resp., $C(L)$). Recall that a lattice with zero is called *$\mathcal{O}$-distributive* (*$\mathcal{O}$-modular*) if it satisfies the implication $x \wedge y = 0 \;\&\; x \wedge z = 0 \rightarrow x \wedge (y \vee z) = 0$ (resp., $x \leq y \;\&\; x \wedge z = 0 \rightarrow x \wedge (y \vee z) = x$). We shall say that a lattice L with zero is *$\mathcal{O}$-semimodular* if $a \wedge x = 0$ implies that $a \vee x$ covers x for all $x \in L$ and $a \in A(L)$. The crucial basic fact for our considerations is the Lampe's

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discovery [4] that a lattice L from $\mathbb{V}$ satisfies the following condition (he named it "Zipper condition"):

$$\left.\begin{array}{l}\text{if } y, z \in L,\ S \subseteq L,\ \bigwedge S = 0 \text{ and } x \vee y = z\\ \text{for all } x \in S \text{ then } y = z.\end{array}\right\} \qquad (*)$$

Define $\mathbb{Z}$ to the class of all complete atomic lattices with cocompact 0 satisfying Zipper condition. Thus $\mathbb{V} \subseteq \mathbb{Z}$. We use this fact below without references.

THEOREM 1. *If $L \in \mathbb{Z}$ is upper semicomplemented then it is lower semicomplemented and each of the following conditions implies that L is even complemented*:

(a) *for all $x \in L\backslash\{0, 1\}$ and $y \in U_x$ there exists an element $y^* \in U_x$ with $y \leq y^*$ and either $a \leq x$ or $a \leq y^*$ for all $a \in A(L)$;*

(b) *for all $A \subseteq A(L)$ there exists a finite $F \subseteq A$ with $\bigvee A = \bigvee F$;*

(c) $\bigwedge C(L) = 0$;

(d) *L is coatomic:*

(e) *L is $\mathcal{O}$-semimodular;*

(f) *L is $\mathcal{O}$-distributive.*

Proof. Let $x \in L\backslash\{0, 1\}$ and $x' = \bigwedge U_x$. Then $x' \neq 0$ by $(*)$. Suppose that $x \wedge x' \neq 0$. Then $(x \wedge x') \vee y = 1$ for some $y \in L, y \neq 1$. We see that $x' \vee y = 1$. On the other hand $y \in U_x$, hence $y \geq \bigwedge U_x = x'$ and $x' \vee y = y \neq 1$. Thus $x \wedge x' = 0$. We prove that x' is a lower semicomplement to x and hence L is lower semicomplemented.

The scheme of our further considerations is:

$$\begin{array}{ccc}(d) & (b) & \\ \downarrow & \downarrow & \\ (c) \rightarrow & (a) \rightarrow & L \text{ is complemented.}\\ \uparrow & \uparrow & \\ (e) & (f) & \end{array}$$

(a) $\rightarrow L$ is complemented. Let $x \in L\backslash\{0, 1\}$. Put $x' = \bigwedge_{y \in U_x} y^*$. If $a \in A(L)$ then either $a \leq x$ or $a \leq x'$ by (a). Hence $x \vee x' \geq \bigvee A(L)$. It is easy to verify that $\bigvee A(M) = 1$ for an arbitrary complete atomic and lower semicomplemented lattice M. Hence $\bigvee A(L) = 1$ and $x \vee x' = 1$. Suppose that $x \wedge x' \neq 0$. Then $(x \wedge x') \vee y = 1$ for some $y \in L, y \neq 1$. It is clear that $y \in U_x$. We have $(x \wedge x') \vee y^* = 1$ and $x' \vee y^* = 1$. On the other hand $y^* \geq \bigwedge_{z \in U_x} z^* = x'$ and $x' \vee y^* = y^* \neq 1$, a contradiction. We see that x' is a complement to x.

(b) $\rightarrow$ (a). Let $x \in L\backslash\{0, 1\}$, $y \in U_x$ and $A_{x,y} = \{a \in A(L) \mid a \nleq x, a \nleq y\}$. Put $b = \bigvee A_{x,y}$ and $y^* = y \vee b$. It is clear that $y \leq y^*$ and either $a \leq x$ or $a \leq y^*$ for all

$a \in A(L)$. It remains to prove that $y^* \in U_x$. Since $x \vee y^* = 1$ we need only verify that $y^* \neq 1$. Suppose that $y^* = 1$. By the hypothesis $b = \bigvee_{i=1}^{n} a_i$ for some $a_1, \ldots, a_n \in A_{x,y}$. Put $b' = \bigvee_{i=1}^{n-1} a_i$. Then $(y \vee b') \vee x = 1$, $(y \vee b') \vee a_n = y \vee b = y^* = 1$ and $x \wedge a_n = 0$. Hence $y \vee b' = 1$ by (∗). Analogously $y \vee b'' = 1$ where $b'' = \bigvee_{i=1}^{n-2} a_i$. Repeating these considerations we prove that $y = 1$. It is impossible because $y \in U_x$.

(c) → (a). Let $x \in L \backslash \{0, 1\}$ and $y \in U_x$. If $y \vee c = 1$ for all $c \in C(L)$ then $y = 1$ by (∗). Hence there exists an element $y^* \in C(L)$ with $y \leq y^*$. It is evident that $y^* \in U_x$. Suppose that $a \nleq x$ and $a \nleq y^*$ for some $a \in A(L)$. Then $x \vee y^* = 1$, $a \vee y^* = 1$, and $x \wedge a = 0$. By (∗) this means that $y^* = 1$, a contradiction.

(d) → (c). It is easy to see that $\bigwedge C(M) = 0$ for an arbitrary complete coatomic and upper semicomplemented lattice M. Hence $\bigwedge C(L) = 0$.

(e) → (c). Suppose that $k = \bigwedge C(L) \neq 0$. Then $k \geq a$ for some $a \in A(L)$. Let $b \in U_a$. It is clear that $a \wedge b = 0$. Since L is $\mathcal{O}$-semimodular, it would follow that $1 = a \vee b$ covers b, i.e. $b \in C(L)$. Then $b \geq \bigwedge C(L) = k \geq a$ and $a \vee b = b \neq 1$, a contradiction.

(f) → (a). Let $x \in L \backslash \{0, 1\}$ and $y \in U_x$. Suppose that $a \nleq x$ and $a \nleq y$ for some $a \in A(L)$. Then $a \wedge x = 0$, $a \wedge y = 0$ and $a \wedge (x \vee y) = a \wedge 1 = a$ which contradicts $\mathcal{O}$-distributivity of L. Putting $y^* = y$, we see that (a) holds.

Theorem is proved.

One can note that (b) generalizes simultaneously several natural lattice conditions such as: L satisfies ascending chain condition (this implies (d) too); $A(L)$ is finite; L has a finite width or a finite breadth. Thus an upper semicomplemented lattice of $\mathbb{Z}$ satisfying one of these conditions is complemented.

Observing that a lattice L is upper semicomplemented if $\bigwedge C(L) = 0$ and using Theorem 1(c), we have

COROLLARY 1. *If* $L \in \mathbb{Z}$ *and* $\bigwedge C(L) = 0$ *then* L *is complemented.*

COROLLARY 2. *Let* $L \in \mathbb{Z}$ *is* $\mathcal{O}$*-distributive and* $\mathcal{O}$*-modular. The following are equivalent*:

(1) L *is upper semicomplemented*;
(2) L *is complemented*;
(3) L *is a finite Boolean algebra*;
(4) $\bigvee A(L) = 1$ *and* $A(L)$ *is finite.*

Proof. (1) → (2) by Theorem 1(e) or (f).
(2) → (1) is evident.

(2) → (3). It is easy to see that a $\mathcal{O}$-distributive and $\mathcal{O}$-modular complemented lattice is uniquely complemented. Further $\mathcal{O}$-modular uniquely complemented lattice is Boolean [1]. Finally a Boolean lattice $L \in \mathbb{Z}$ must be finite because 0 is cocompact in L.

(3) → (4) is evident.

(4) → (2). Let $x \in L \backslash \{0, 1\}$. Using the $\mathcal{O}$-distributivity of L and the fact that $A(L)$ is finite it is easy to see that $x' = \bigvee \{a \in A(L) \mid a \nleq x\}$ is a complement to x. Corollary is proved.

It is evident that the analogue of Corollary 2 is not valid for an arbitrary lattice L from $\mathbb{Z}$. Furthermore it is not valid even for an arbitrary lattice L of $\mathbb{V}$. This follows, e.g., from the fact that $\mathbb{V}$ contains the 5-element non-modular lattice (see [4], e.g.).

Let us turn from the lattice language to the varietal one.

THEOREM 2. *If $L = L(\mathfrak{X}) \in \mathbb{V}$ is upper semicomplemented then each of the following conditions implies that L is complemented:*

(a) *$\mathfrak{X}$ is a locally finite variety of a finite type;*

(b) *$\mathfrak{X}$ is congruence-modular;*

(c) *$\mathfrak{X}$ is a semigroup variety;*

(d) *for each $A \in \mathfrak{X}$ and each non-trivial $\alpha \in \operatorname{Con}(A)$ there exists a non-singleton α-class being a subalgebra of A.*

Proof. It is well known that $A(L)$ is finite in case (a), L is modular in case (b) and L is $\mathcal{O}$-distributive in case (c). In case (d) L is $\mathcal{O}$-distributive too by Lemma 1 of [6]. It remains to take into account Theorem 1(b), (e), (f).

One can note case (d) of Theorem 2 embraces some interesting classes of varieties which are not covered by (a)–(c), e.g. varieties of completely regular semigroups, inverse semigroups, idempotent groupoids.

COROLLARY 3. *Let $L = L(\mathfrak{X}) \in \mathbb{V}$ and either $\mathfrak{X}$ is congruence-permutable or $\mathfrak{X}$ is a semigroup variety or $\mathfrak{X}$ satisfies condition* (d) *of Theorem* 2. *Then conditions* (1)–(4) *of Corollary* 2 *are equivalent.*

Proof. If $\mathfrak{X}$ is congruence-permutable (satisfies condition (d) of Theorem 2) then L satisfies implication $x \wedge z = 0 \rightarrow (x \vee y) \wedge z = x \wedge y$ by [5], Lemma 3 (resp., [6], Lemma 1). In both cases it remains to take into account Corollary 2. For semigroup varieties the equivalence of (1) and (2) is guaranteed by Theorem 2(c) and equivalence of (2)–(4) was proved in [6].

At the conclusion let us formulate two open questions.

QUESTION 1. Are conditions (1)–(4) of Corollary 2 equivalent for an arbitrary congruence-modular variety $\mathfrak{X}$?

QUESTION 2. Has an arbitrary congruence-modular variety a $\mathcal{O}$-distributive subvariety lattice?

Corollary 2 shows that if the answer to Question 2 is affirmative then the answer to Question 1 is affirmative too. Note that not all varieties have $\mathcal{O}$-distributive subvariety lattices. Corresponding example (of a variety of 3-unary algebras) was constructed by P. P. Palfy (unpublished).

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