COMMUTATIVE SEMIGROUP VARIETIES WITH MODULAR SUBVARIETY LATTICES

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ABSTRACT

All commutative semigroup varieties with modular subvariety lattices are described.

1. INTRODUCTION

Many papers have been devoted to study the lattice L_c of all commutative semigroup varieties in the late sixties and early seventies[3,4,7,11,12,14,15]. Two reasons have inspired investigators' enthusiasm: the significance of the class of commutative semigroups and Perkins' result[13] that L_c is countable which has generated hope that L_c may be described completely. However, the faith in a pre-ordered harmony mostly leads to disappointment. The lattice L_c turned out to be very complex: in particular it contains a dual of every finite partition lattice[3], and by well-known result by Pudlak and Tuma, this means that every finite lattice is embeddable in L_c. These circumstances led to sad conclusion (formulated explicitly e.g. in[9]) that the complete description of L_c is impossible. As a consequence, only few papers about L_c have been published during the last 15 years[1,2,9].

The authors of the present paper have developed some technique (published partially in[18]) which reanimates the hope that L_c is "treatable". Here we apply this technique to solve (for commutative case) an old problem mentioned already in Evans' survey[5]. Namely, we describe commutative semigroup varieties with modular subvariety lattices.

The main result of the paper is formulated in section 2 below. There we give also the part of its proof which does not depend on[18]. Results and constructions from[18] we need are collected in section 3. Then in section 4 we finish the proof of the main result.

Let us fix some definitions and notation. We will say that a semigroup variety $\mathcal{V}$ is *modular* if the lattice $L(\mathcal{V})$ of its subvarieties is modular. If $\mathcal{X}$ and $\mathcal{Y}$ are two semigroup varieties and $\mathcal{X} \subseteq \mathcal{Y}$, then $[\mathcal{X}, \mathcal{Y}] = \{\mathcal{Z} \mid \mathcal{X} \subseteq \mathcal{Z} \subseteq \mathcal{Y}\}$. Variety given by a system of identities Σ (generated by a semigroup S) will be denoted by varΣ (respectively, varS). If u is a word, then $l(u)$ is the length of u, $c(u)$ is the set of all letters occuring in u. For a letter $x \in c(u)$, the number of occurences of x in u is denoted by $l_x(u)$. We say that words u and v are *similar* and write $u \approx v$ if v may be obtained by renaming of some letters of u. As usually we use a short form $u = 0$ denoting the system of identities $ux = xu = u$ where $x \notin c(u)$. We write $u \blacktriangleleft v$ if $u \equiv av^{\alpha}b$ for some (maybe empty) words a, b and some substitution α. We also fix notation for some "standard" varieties. (Here and throughout sections 2 and 4 we will take for granted that all varieties discussed satisfy the commutative law.)

$\mathcal{A}_n = \mathrm{var}\{x^n y = y\}$;

$\mathcal{C} = \mathrm{var}\{x^2 = x^3\}$;

$\mathcal{S} = \mathrm{var}\{x^2 = x\}$;

$\mathcal{N}^* = \mathrm{var}\{x^2 y = 0\}$;

$\mathcal{T} = \mathrm{var}\{x = y\}$;

$\mathcal{M}_0 = \mathrm{var}\{x^3 y^2 = x^4 = x_1 \ldots x_6 = 0\}$;

$\mathcal{X}_0 = \mathrm{var}\ \{x^3 = xyzt = 0,\ xyx = yxy\}$.

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2. THE MAIN RESULT AND THE BEGINNING OF THE PROOF

THEOREM. Let V be a commutative semigroup variety. The following conditions are equivalent:

a) V is modular;

b) V does not contain the varieties M_0, $X_0 \vee A_p$ for all prime numbers p, and $X_0 \vee C$;

c) V satisfies one of the following systems of identities:

$$x^3yz = xy^3z, \quad x^6 = x^7; \tag{1}$$

$$x^3yz = xy^2z^2; \tag{2}$$

$$x^2y^2z = xy^2z^2; \tag{3}$$

$$x^{n+2}y = x^2y. \tag{4}$$

Proof. a) —> b). The non-modularity of the varieties $X_0 \vee A_p$ and $X_0 \vee C$ follows from Corollaries of Lemma 8 of[19]. It remains to prove that M_0 is not modular.

Let us consider varieties $M_1 - M_5$ which are given in M_0 by the following systems of identities:

M_1 : $x^3yz = xy^3z$;

M_2 : $x^2y^2z = xy^2z^2$;

M_3 : $x^3yz = xy^2z^2$;

M_4 : $x^3yz = xy^3z$, $x^2y^2z = xy^2z^2$;

M_5 : $x^3yz = x^2y^2z$.

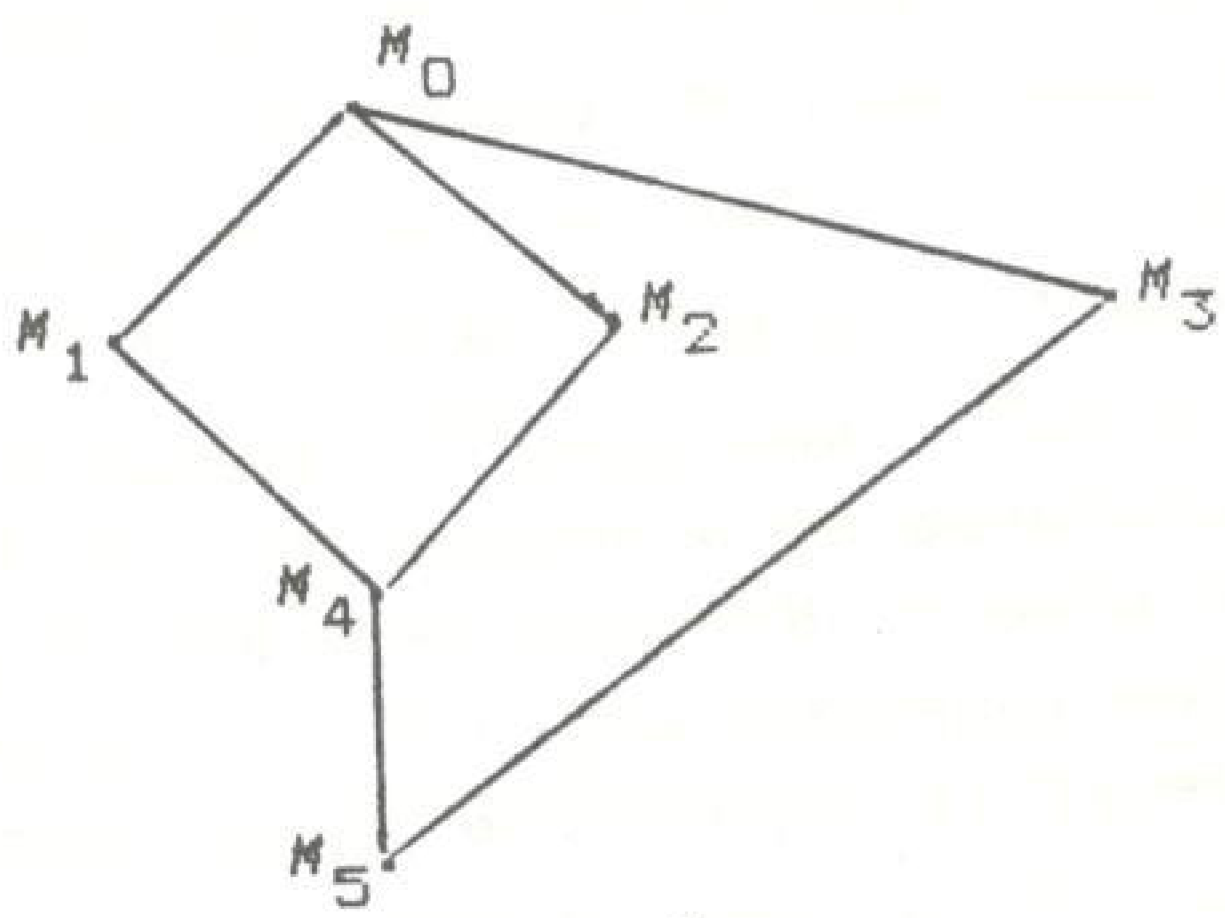


Figure 1

We are going to prove that the varieties $M_0 - M_5$ form the non-modular sublattice of $L(M_0)$ shown in Fig.1. Firstly, we verify that these varieties are different. Let F be the M_0-free semigroup with generators a,b,c. It is clear that F generates M_0. Further, let

$$A = \{a^3bc,\ ab^3c,\ abc^3,\ a^2b^2c,\ a^2bc^2,\ ab^2c^2\}.$$

Clearly, A is the set of all non-zero words of the length 5 from F. Consider partitions $\pi_1 - \pi_5$ of A with the following non-singleton classes:

π_1 : $\{a^3bc, ab^3c, abc^3\}$;

π_2 : $\{a^2b^2c, a^2bc^2, ab^2c^2\}$;

π_3 : $\{a^3bc, ab^2c^2\}$, $\{ab^3c, a^2bc^2\}$, $\{abc^3, a^2b^2c\}$;

π_4 : $\{a^3bc, ab^3c, abc^3\}$, $\{a^2b^2c, a^2bc^2, ab^2c^2\}$;

π_5 : A.

Let μ_i be the fully invariant congruence on F generated by π_i, i = 1,...,5. It is clear that $A \subseteq \mathrm{Ann}F$ and partitions $\pi_1 - \pi_5$ are AutF-invariant. Hence for each i, non-singleton μ_i-classes coincide with non-singleton π_i-classes. It means that $F/\mu_i \notin M_j$ if $j \neq i$. Since $F/\mu_i \in M_i$, all M_i are different.

Further, it is evident that $M_1 \cap M_2 = M_4$ and $M_1 \cap M_3 = M_2 \cap M_3 = M_4 \cap M_3 = M_5$. It remains to check that $M_1 \vee M_2 = M_0 = M_3 \vee M_4$. Since $\pi_1 \cap \pi_2$ and $\pi_3 \cap \pi_4$ are trivial, F is a subsemigroup of $F/\mu_1 \times F/\mu_2$ as well as of $F/\mu_3 \times F/\mu_4$, but as we have mentioned F generates M_0 and $F/\mu_i \in M_i$. ■

b) —> c). Let N be the largest nilsubvariety of V.

CASE 1. $X_0 \not\subseteq N$. Then $N \subseteq N^*$ by Lemma 7 of [19]. As it follows from[8] (see also[19], Proposition 1), $V = N \vee M$ where M is generated by a monoid. In its turn, it follows from[7] that for some n $M = A_n \vee K$ where K is generated by a nilsemigroup with identity adjoined. It is shown in[19] that either $X_0 \subseteq K$ (that is impossible in our case) or $K \subseteq C$. Thus, $V \subseteq N^* \vee A_n \vee C$. Since each of the varieties N^*, A_n, C satisfies the identity (4), V satisfies (4), too. ■

CASE 2. $X_0 \subseteq N$. By condition $C \not\subseteq V$ and $A_p \not\subseteq V$ for all prime numbers p. It implies that $V \subseteq S \vee N$ (see[19], proof of Theorem 1). Since S satisfies (1),(2),(3), it remains to check that N satisfies one of these systems.

Now we need two remarks concerning nilvarieties. The first of them is obvious.

LEMMA 1. If a variety N of nilsemigroups satisfies an identity $u = v$ such that $c(u) \neq c(v)$, then N satisfies also $u = 0$ and $v = 0$. ■

LEMMA 2. If a variety N of commutative nilsemigroups satisfies an identity $u = v$ such that $l(u) < l(v)$ and $u \triangleleft v$, then N satisfies also $u = 0$ and $v = 0$.

Proof. By the previous Lemma we may (and will) assume that $c(u) = c(v)$. Since $u \triangleleft v$, there exist (possibly, empty) words a,b and a substitution α such that $v \equiv au^\alpha b$. Let F be the N-free semigroup with $|c(u)|$ generators. Since $u = v$ holds in N, we have the following equalities in F:

$$u = v \equiv au^\alpha b = av^\alpha b \equiv a(au^\alpha b)^\alpha b = aa^\alpha u^{\alpha^2} b^\alpha b = \ldots,$$

and it is easy to see that repeating this process, we can get $u = w$ with $l(w)$ so large as we want. Being finitely generated commutative nilsemigroup, F is nilpotent. Therefore $w = 0$ in F for any word w with sufficiently large $l(w)$. Thus, $u = 0$ in F (and in N). ■

Let us continue the proof of the implication b)——>c). By condition $M_0 \not\subseteq N$. The variety M_0 satisfies all identities of the kind $u = 0$ where either $l(u) \geq 6$ or $l_x(u) \geq 4$ for some x or $l_x(u) \geq 3$, $l_y(u) \geq 3$ for some different x and y. Therefore N should satisfy a non-trivial identity $u = v$ such that $u \in W$ where

$$W = \{x,\ xy,\ x^2,\ xyz,\ x^2y,\ x^3,\ xyzt,\ x^2yz,\ x^2y^2,\ x^3y,\ xyztu,\ x^2yzt,\ x^2y^2z,\ x^3yz\}.$$

It is clear that if $u \in W$, then $u = 0$ implies one of the systems (1)-(3). By Lemmas 1 and 2 we may assume that $c(u) = c(v)$ and if $l(u) \neq l(v)$, then u and v are incomparable under $\triangleleft$. Therefore the only possibilities for $u = v$ are

$$x^2y = xy^2; \tag{5}$$

$$x^2yz = xy^2z; \tag{6}$$

$$x^2y^2 = x^3y; \tag{7}$$

$$x^3y = xy^3; \tag{8}$$

$$x^2yzt = xy^2zt; \tag{9}$$

$$x^2y^2z = xy^2z^2; \tag{10}$$

$$x^2y^2z = x^3yz; \tag{11}$$

$$xy^2z^2 = x^3yz; \tag{12}$$

$$x^3yz = xy^3z; \tag{13}$$

$$x^2y^2z^2 = x^3yz. \tag{14}$$

Evidently, each of the identities (5),(6),(9),(10) implies (3), (7) or (12) imply (2), and (8) or (13) imply (1). It remains to consider (11) and (14). Using the substitution which permutes x and y, we see that each of these identities implies (1). ■

3. LATTICES OF NILPOTENT VARIETIES

Here we recall some results and constructions from[18].

1. #-product. Let L and L' be lattices and $\delta: L \longrightarrow L'$ be a meet-homomorphism. The set

$$L \underset{\delta}{\#} L' = \{(x,y) \mid x \in L,\ y \in L',\ \delta(x) \geq y\}$$

with the natural ordering is called *#-product* of L and L' over δ. Clearly, $L \underset{\delta}{\#} L'$ is a sublattice of $L \times L'$.

2. Insertion. Let $(M, \leq)$, (P, σ) be p.o. sets, and $M \cap P$ is empty. Let us fix two isotone maps $\alpha, \beta : P \longrightarrow M$ such that $\beta(x) < \alpha(x)$ for each $x \in P$. Put

$$\sigma' = \{(x,y) \mid x,y \in P,\ \alpha(x) \leq \beta(y)\}.$$

Now we extend the relation $\leq$ on $P \cup M$ by the way:

a) $\leq|_P = \sigma \cup \sigma'$;

b) if $x \in M$, $y \in P$ then $x \leq y$ iff $x \leq \beta(y)$ and $y \leq x$ iff $\alpha(y) \leq x$.

It is easy to check that $(P \cup M, \leq)$ is a p.o. set. It is denoted by $\mathrm{In}\,(\alpha,\beta,P,M)$ and is called *insertion* of P in M by α, β.

Let F be the absolutely free semigroup of countable rank. A variety V of nilsemigroups will be called *homogeneous* if all elements of an arbitrary class A of the fully invariant congruence α corresponding to $\mathcal{V}$ have equal length excepting the case when A is the zero element of F/α.

Let us fix a homogeneous variety $\mathcal{V}$ of nilsemigroups. The variety given in $\mathcal{V}$ by the identity $x_1 \dots x_n = 0$ will be denoted by $\mathcal{V}(n)$. A variety $\mathcal{M} \in [\mathcal{V}(k), \mathcal{V}]$ is called *k-split* if $\mathcal{M}$ satisfies the identities $u = 0$ and $v = 0$ whenever it satisfies an identity $u = v$ with $l(u) \leq k$ and $l(v) > k$. A variety which is not k-split is called *k-non-split*. The set of all k-split (k-non-split) varieties of $[\mathcal{V}(k), \mathcal{V}]$ is denoted by L_k (respectively, P_k). It is easy to see that L_k is a sublattice of $[\mathcal{V}(k), \mathcal{V}]$. As it was proved in[18], for any $\mathcal{N} \in P_k$ varieties $\alpha_k(\mathcal{N}) = \bigcap\{\mathcal{A} \in L_k \mid \mathcal{N} \subseteq \mathcal{A}\}$ and $\beta_k(\mathcal{N}) = \bigcup\{\mathcal{A} \in L_k \mid \mathcal{A} \subseteq \mathcal{N}\}$ belong to L_k. Moreover, it is possible to describe identity bases of these varieties.

Further let $\mathcal{A} \in [\mathcal{V}(k), \mathcal{V}(k+1)]$. Then $\mathcal{A}$ may be given in $\mathcal{V}$ by a system of identities of the following kind:

$$x_1 \dots x_{k+1} = 0, \quad u_i = 0, \quad v_j = w_j \quad (i \in I,\ j \in J)$$

where $l(u_i) = l(v_j) = l(w_j) = k$ for all i, j. We denote by $\mu_k(\mathcal{A})$ the variety given in $\mathcal{V}$ by the identities

$$u_i = 0, \quad v_j = w_j \quad (i \in I,\ j \in J).$$

(It was verified in[18] that $\mu_k(\mathcal{A})$ does not depend on the choice of a basis of $\mathcal{A}$). Put $\delta_k(\mathcal{A}) = \mu_k(\mathcal{A}) \vee \mathcal{V}_{k+1}$. The mapping $\delta_k : [\mathcal{V}(k), \mathcal{V}(k+1)] \longrightarrow [\mathcal{V}(k+1), \mathcal{V}]$ was proved to be a meet-homomorphism in[18].

The results of[18] can be summarized in the following

PROPOSITION. a) $[\mathcal{V}(k), \mathcal{V}] \cong \mathrm{In}\,(\alpha_k, \beta_k, P_k, L_k)$.

b) $L_k \cong [\mathcal{V}(k), \mathcal{V}(k+1)] \underset{\delta_k}{\#} [\mathcal{V}(k+1), \mathcal{V}]$. ∎

Let now $\mathcal{V}$ be a homogeneous and nilpotent of index s variety. Then $\mathcal{V}(s) = \mathcal{V}$, and our Proposition reduces the description of the lattice $L(\mathcal{V})$ to the description of the intervals $[\mathcal{V}(1), \mathcal{V}(2)], \dots, [\mathcal{V}(s-2), \mathcal{V}(s-1)]$, $[\mathcal{V}(s-1), \mathcal{V}(s)]$ and of the p.o.sets $P_1, \dots, P_{s-2}$. In particular if $\mathcal{V}$ does not contain k-non-split varieties for each $k \leq s$ then $L(\mathcal{V})$ is embeddable into the direct product of the intervals $[\mathcal{V}(1), \mathcal{V}(2)], [\mathcal{V}(2), \mathcal{V}(3)], \dots, [\mathcal{V}(s-1), \mathcal{V}(s)]$.

4. THE MAIN RESULT: THE END OF THE PROOF

c) ——> a). Let us consider two cases.

CASE 1. V satisfies (4). We are going to prove that in this case $L(V)$ is even distributive. Let N be the largest nilsubvariety of V. By Lemma 2 N satisfies $x^2y = 0$. Repeating the proof of the case 1 of the implication b)——>c), we obtain that $V \subseteq N^* \vee A_n \vee C$ for some n. Thus, it sufficies to verify that the lattice $L(N^* \vee A_n \vee C)$ is distributive. By Proposition 2 of[17], it is isomorphic to the direct product of $L(N^* \vee C)$ and of $L(A_n)$. Since the subvariety lattice of any abelian group variety is known to be distributive, it remains to verify that the same is true for $N^* \vee C$.

For a variety X let mon$X$ denote the *monoid part* of X, i.e. the subvariety of V generated by all its monoids. Using the classifisation of commutative monoid varieties given by Head[7], we can see that there are only three possibilities for the monoid part of a variety $X \subseteq N^* \vee C$:

$$\text{mon}X = T;$$
$$\text{mon}X = S;$$
$$\text{mon}X = C.$$

Let Q denote the three-element chain $\{T, S, C\}$ and L denote the subset of the direct product $L(N^*) \times Q$ consisting of all pairs (A, B) such that $B \cap N^* \subseteq A$. It is easy to see that L is, in fact, a sublattice of $L(N^*) \times Q$.

The distributivity of the lattice $L(N^* \vee C)$ follows now from[16] (where distributivity of $L(N^*)$ was proved) and from the following

LEMMA 3. <u>The mapping</u>

$$f: V \longrightarrow (V \cap N^*, \text{mon}V)$$

<u>is an isomorphism of the lattice</u> $L(N^* \vee C)$ <u>on the lattice</u> L.

<u>Proof</u>. It is well known that every isotone bijection of lattices having isotone inverse is a lattice isomorphism. Since f is obviously isotone, it remains to check that f is injective and surjective, and that f^{-1} is isotone.

1. f <u>is injective</u>. Since C satisfies $x^2 = x^3$, $N^* \vee C$ satisfies $x^2y = x^3y$. By Lemma 2 every commutative nilsemigroup with the last identity satisfies $x^2y = 0$. Therefore every nilsemigroup of $N^* \vee C$ lies in N^*.

Now let X be a subvariety of $N^{*}\vee C$. By[8] X should equal the join of some variety of nilsemigroups A and of a variety B generated by a monoid. Clearly, $B \subseteq \mathrm{mon}X$ and by above-proved remark $A \subseteq X \cap N^{*}$. Thus,

$$X = A \vee B \subseteq (X \cap N^{*}) \vee \mathrm{mon}X \subseteq X,$$

and $X = (X \cap N^{*}) \vee \mathrm{mon}X$ is uniquely determined by $f(X)$. ■

2. f *is surjective*. Let $(A,B) \in L$. Proving injectivity, we showed that

$$A \vee B = ((A \vee B) \cap N^{*}) \vee \mathrm{mon}(A \vee B).$$

However, in the proof of Proposition 2 of the paper[17] it was verified that if $A \subseteq N^{*}, B \in Q$ and $B \cap N^{*} \subseteq A$, then A and B are determined uniquely by their join $A \vee B$. This implies that $A = (A \vee B) \cap N^{*}$ and $B = \mathrm{mon}(A \vee B)$, i.e $(A,B) = f(A \vee B)$. ■

3. f^{-1} *is isotone*. Proving surjectivity, we showed that $f^{-1}((A,B)) = A \vee B$; this means, in particular, that f^{-1} is isotone. ■

CASE 2. V *satisfies either* (1) *or* (2) *or* (3). It is clear that $\mathrm{mon}V \subseteq S$. Using[8], we see that $V \subseteq S \vee N$ where N is the largest nilsubvariety of V. By well-known result of[10], $L(S \vee N)$ is isomorphic to the direct product of $L(N)$ and of two-element chain. Therefore it sufficies to prove that N is modular. By the following Lemma, we may assume additionally that N is nilpotent.

LEMMA 4. *Let N be a non-modular variety of commutative nilsemigroups. Then N contains a non-modular nilpotent subvariety.*

Proof. Let the modular law fail for $X, Y, Z \in L(N)$, i.e. $Y \subseteq X$ and $X \cap (Y \vee Z) \neq Y \vee (X \cap Z)$. Since N is locally nilpotent, every subvariety of N is generated by its finite members. Hence there exists a finite semigroup S such that $S \in X \cap (Y \vee Z)$, but $S \notin Y \vee (X \cap Z)$. Since $S \in Y \vee Z$, S should be a homomorphic image of a $(Y \vee Z)$-free semigroup of a finite rank. This semigroup is nilpotent (because N is locally nilpotent), and it is a subdirect product of semigroups $A \in Y$ and $B \in Z$. Put $X' = \mathrm{var}(S \times A)$, $Y' = \mathrm{var}A$, $Z' = \mathrm{var}B$. Then $Y' \subseteq X'$, and $S \in X' \cap (Y' \vee Z')$. On the other hand, $X' \subseteq X$, $Y' \subseteq Y$, $Z' \subseteq Z$, and hence $S \notin Y' \vee (X' \cap Z')$. We see that $X' \vee Z'$ is a non-modular nilpotent subvariety of N. ■

Thus, let N be nilpotent of index s.

SUBCASE 2.1. N <u>satisfies</u> (3). Clearly, we may assume that $N = \mathrm{var}\{x_1 \cdots x_s = 0,\ x^2y^2z = xy^2z^2\}$. It sufficies to prove the following two statements (see remark at the end of section 3):

(a) there are no k-non-split subvarieties in N for any $k < s$;

(b) all intervals of the kind $[N(k), N(k+1)]$ $(k < s)$ are modular.

Let us prove (a) at first. Note that $x^2y^2zt = 0$ in N. Indeed, substituting zt for z in (3), we obtain

$$x^2y^2zt = xy^2z^2t^2$$

and $x^2y^2zt = 0$ in N by Lemma 2. Hence if $u = 0$ does not hold in N, then u is similar to a word from W where

$$W = \{x_1 \cdots x_n,\ x_1^2x_2 \cdots x_n,\ x_1^3x_2 \cdots x_n,\ x^2y^2,\ x^2y^2z,\ x^3y,\ x^3y^2,\ x^4,\ x^4y,\ x^5\}.$$

It is easy to verify that if $u \approx u' \in W$, $v \approx v' \in W$, $c(u) = c(v)$ and $l(u) < l(v)$, then $u \blacktriangleleft v$. By Lemma 2, $u = v$ implies $u = 0$.■

Let us prove (b). If $k \leq 4$, then (b) follows from [11]. The interval $[N(5), N(6)]$ is shown in Fig.2. The verification here is routine, and we allow ourselves to omit it. We see that $[N(5), N(6)]$ is modular.

Now let $k \geq 6$. Then the interval $[N(k), N(k+1)]$ is the chain shown in the Fig.3. Indeed, let $N(k) \subseteq X \subset N(k+1)$ and $u = v$ be an identity of X such that $l(u) = k$. We may (and will) suppose that $u \in W$ and either $v \equiv 0$ or $l(v) = k$ and $v \approx v' \in W$. It is clear that under this conditions $u = v$ is equivalent to one of the following identities:

$$x_1 \cdots x_k = 0,\quad x_1^2x_2 \cdots x_{k-1} = 0,\quad x_1^2x_2 \cdots x_{k-1} = x_1x_2^2 \cdots x_{k-1},$$

$$x_1^3x_2 \cdots x_{k-2} = 0,\ x_1^3x_2 \cdots x_{k-2} = x_1x_2^3 \cdots x_{k-2}.$$

In this sequence of identities every identity implies the next one (it is clear for all the cases excepting maybe $x_1^2x_2 \cdots x_{k-1} = x_1x_2^2 \cdots x_{k-1}$; but here it is sufficient to substitute x_1x_2 for x_2 and use $x^2y^2zt = 0$). It is obvious that these identities determine different varieties. Hence $[N(k), N(k+1)]$ is really a 6-element chain and therefore is modular. ■

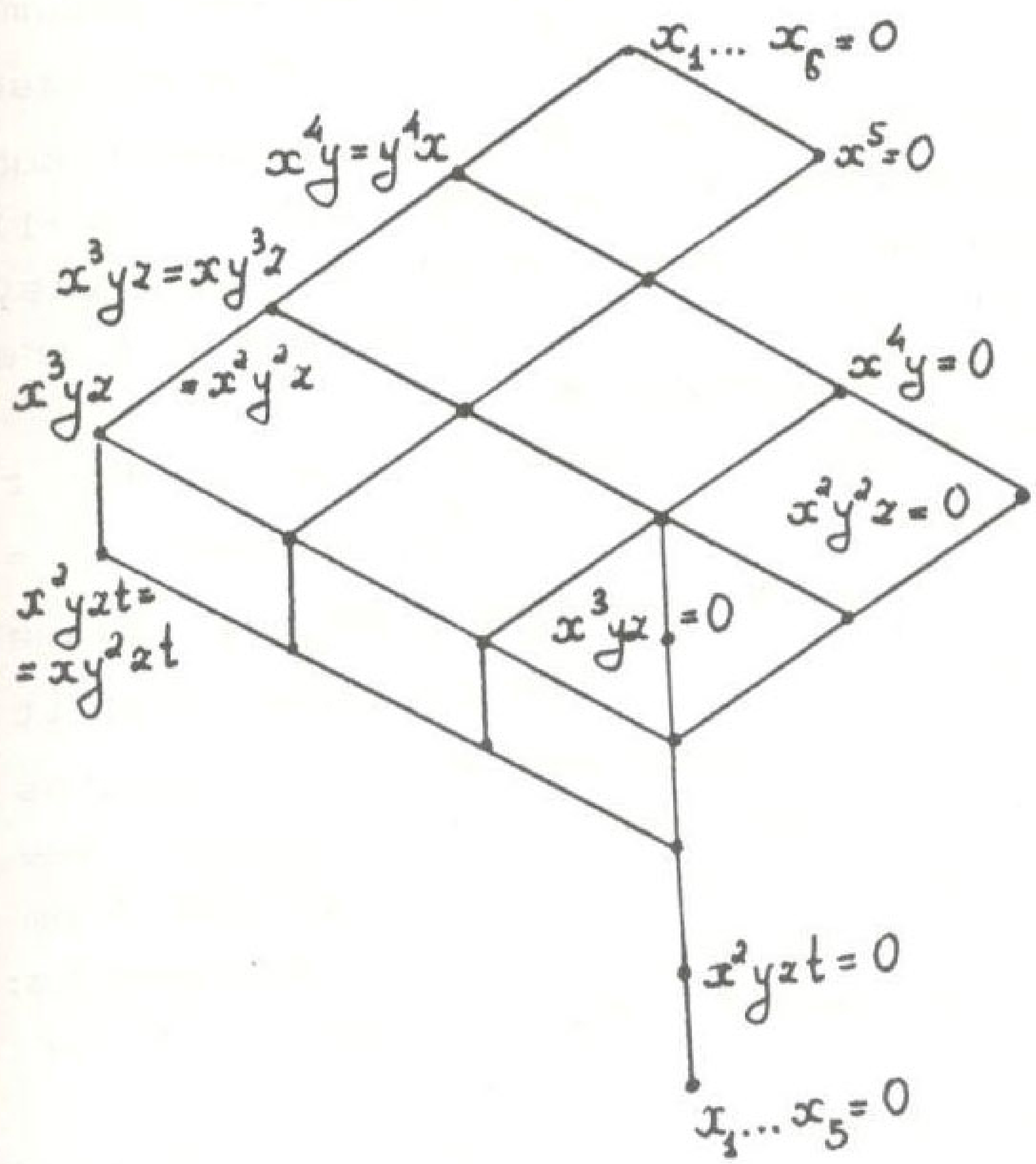


Figure 2

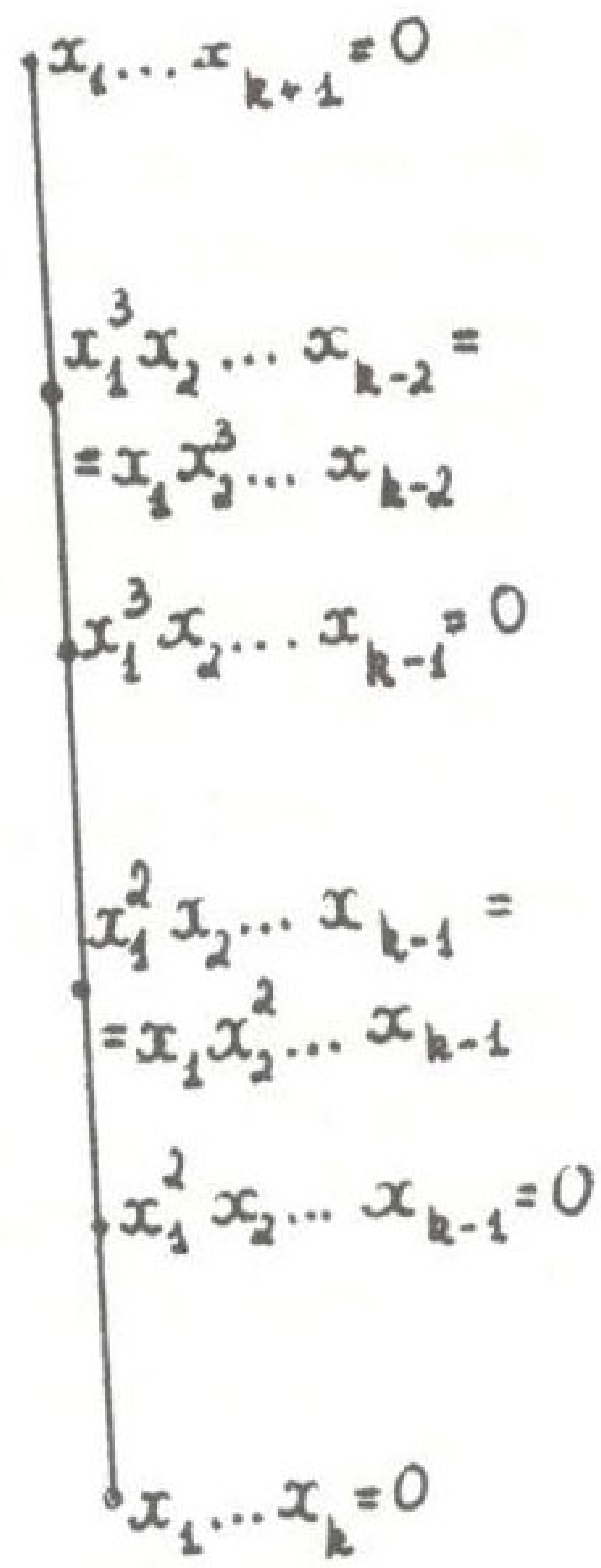


Figure 3

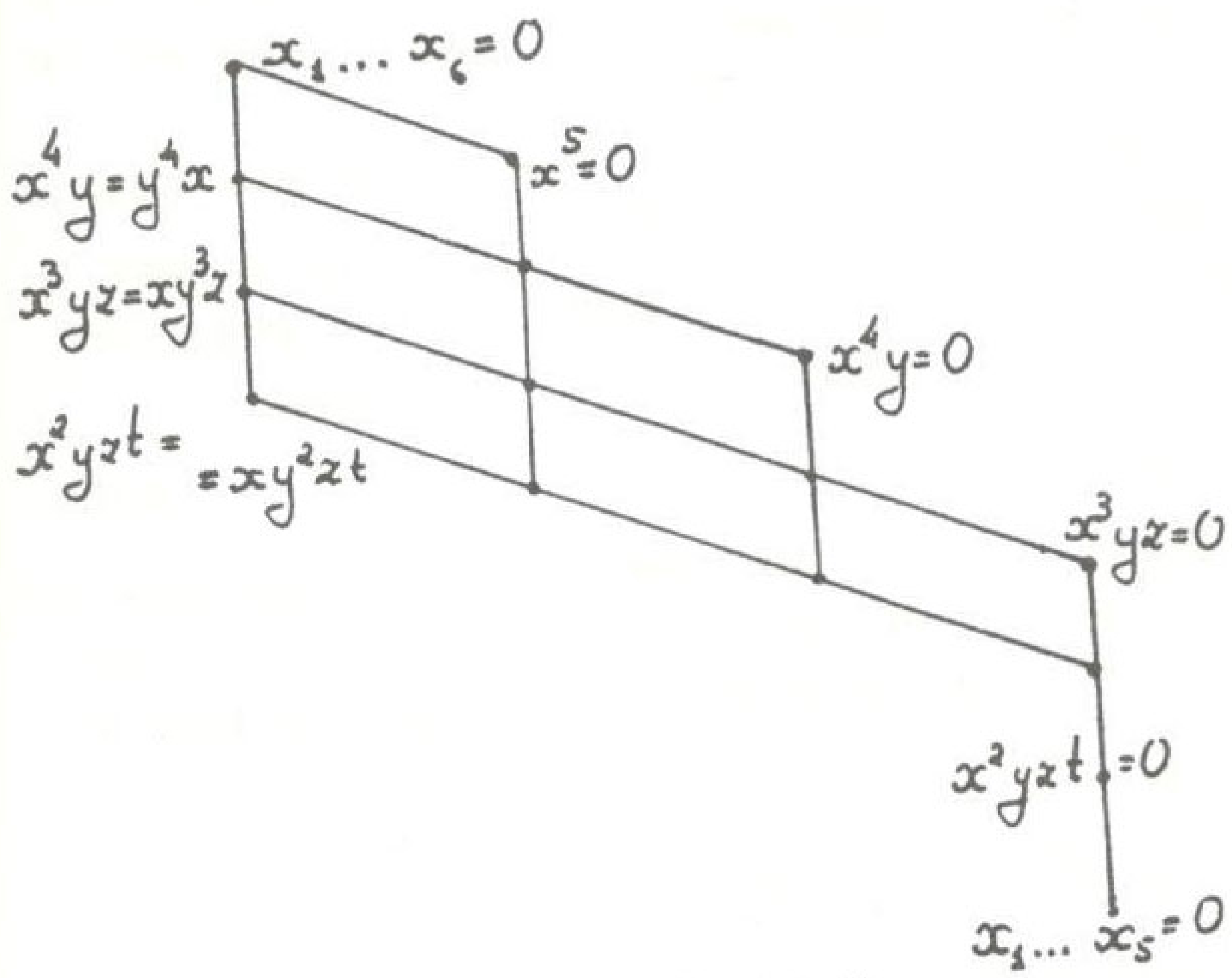


Figure 4

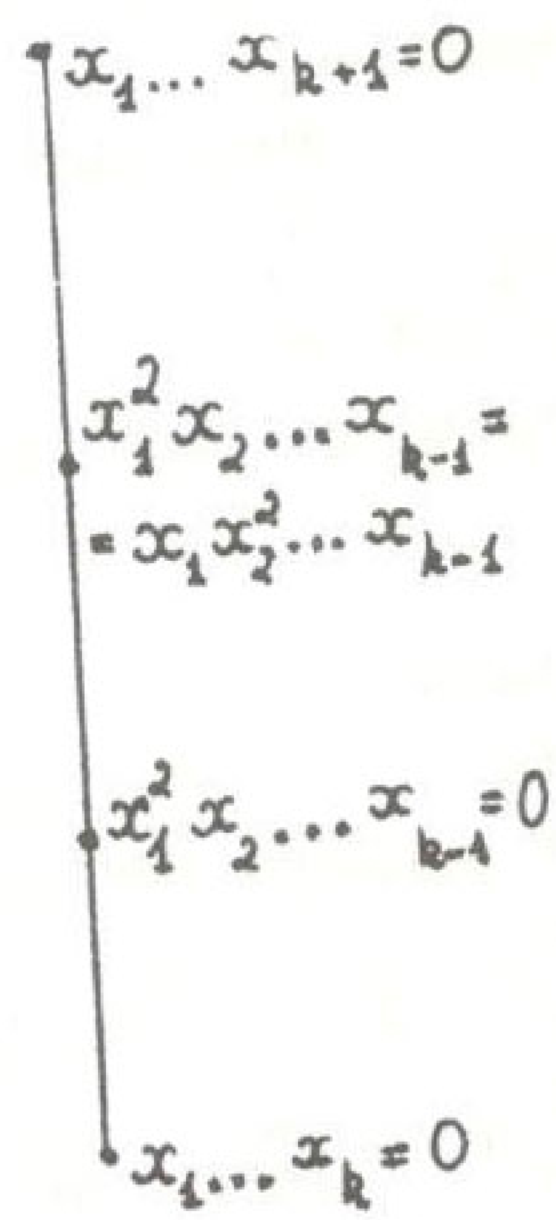


Figure 5

SUBCASE 2.2. N <u>satisfies</u> (2). Clearly, we may assume that $N = \mathrm{var}\ \{x_1 \ldots x_s = 0,\ x^3yz = xy^2z^2\}$. In this subcase as well as in the previous one there are no k-non-split subvarieties of N and all intervals of the kind $[N(k), N(k+1)]$ are modular. We omit the verification which is very easy. The intervals $[N(5), N(6)]$ and $[N(k), N(k+1)]$, $k \geq 6$ are shown in the Fig.4 and 5 respectively. ■

SUBCASE 2.3. N <u>satisfies</u> (1). Substituting x for z and x^2 for y in the identity $x^3yz = xy^3z$, we get $x^6 = x^8$, and by Lemma 2 $x^6 = 0$ in N. Therefore we may assume that $N = \mathrm{var}\ \{x_1 \ldots x_s = 0,\ x^3yz = xy^3z\}$. It is clear that it is homogeneous. Unfortunately, in this subcase N contains non-split subvarieties, namely 5-non-split subvarieties (see Lemma 5 below). Nevertheless, N is modular. It follows from the results of section 3 and from the next three statements:

(a') there are no k-non-split subvarieties in N for $k < 5$;

(b) all intervals of the kind $[N(k), N(k+1)]$ $(k < s)$ are modular;

(c) the interval $[N(5), N]$ is modular.

The statement (a') can be checked easily. Let us prove (b). For $k \leq 4$ the modularity of $[N(k), N(k+1)]$ follows from[11)]. The interval $[N(5), N(6)]$ is shown in Fig.6. The verification here is routine. We see that this interval is modular.

Now let $k \geq 6$. Substituting yt for y in the identity $x^3yz = xy^3z$, we obtain $x^3yzt = xy^3zt^3$, and by Lemma 2 $x^3yzt = 0$ in N. Hence if $l(u) = k$ and $u = 0$ does not hold in N, then u is similar to a word of the type

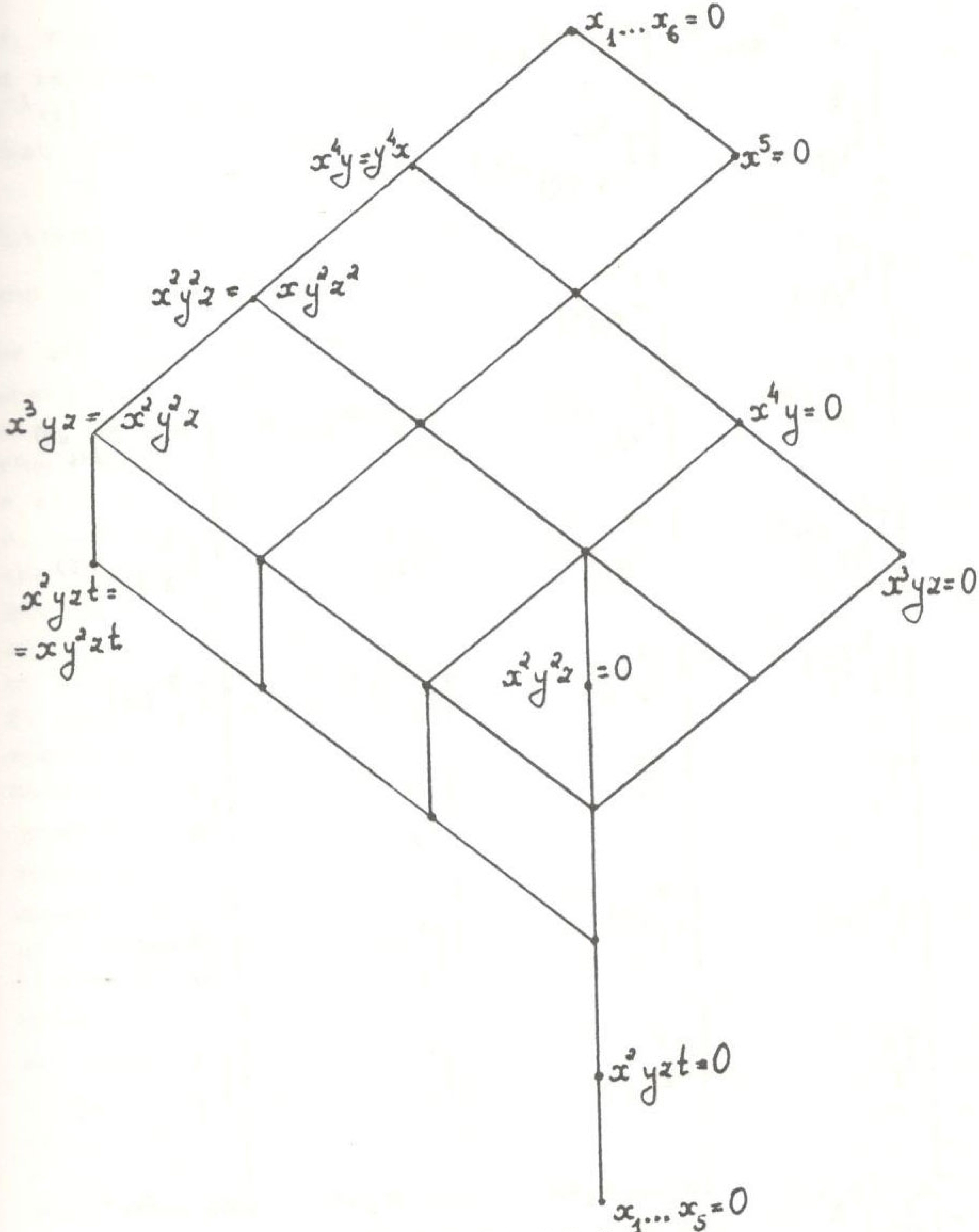
$$x_1^2 \ldots x_h^2 x_{h+1} \ldots x_d \tag{15}$$

where $0 \leq h \leq k/2$ and $d = k-h$. We see that if an identity $u = v$ holds in $N(k)$, but does not hold in $N(k+1)$, then it should be equivalent to one of the following identities:

I_h^k: $x_1^2 \ldots x_h^2 x_{h+1} \ldots x_d = 0$,

$I_h^k(\sigma)$: $x_1^2 \ldots x_h^2 x_{h+1} \ldots x_d = x_{\sigma(1)}^2 \ldots x_{\sigma(h)}^2 x_{\sigma(h+1)} \ldots x_{\sigma(d)}$,

where $\sigma \in \Sigma(1,\ldots,d)$, but $\sigma \notin \Sigma[h,d] = \Sigma(1,\ldots,h) \times \Sigma(h+1,\ldots,d)$.



$x_1 \ldots x_6 = 0$
$x^4y = y^4x$
$x^5 = 0$
$x^2y^2z = xy^2z^2$
$x^3yz = x^2y^2z$
$x^4y = 0$
$x^2yzt = = xy^2zt$
$x^3yz = 0$
$x^2y^2z = 0$
$x^2yzt = 0$
$x_1 \ldots x_5 = 0$

Figure 6

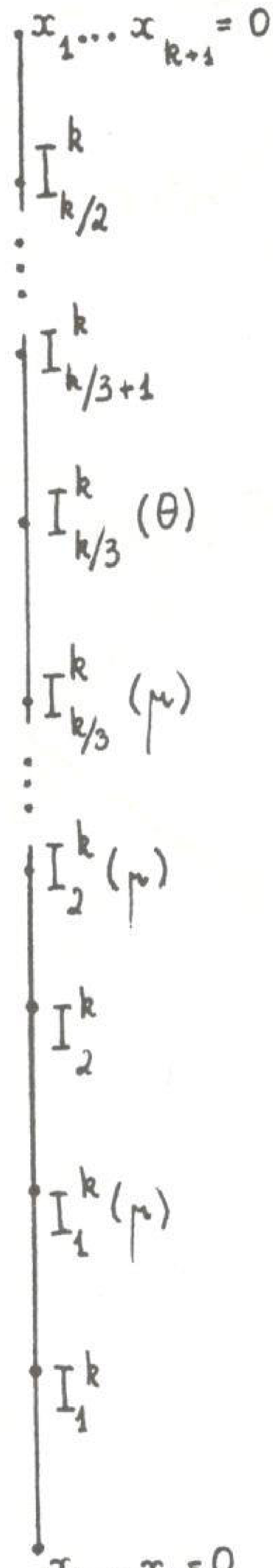


Figure 7

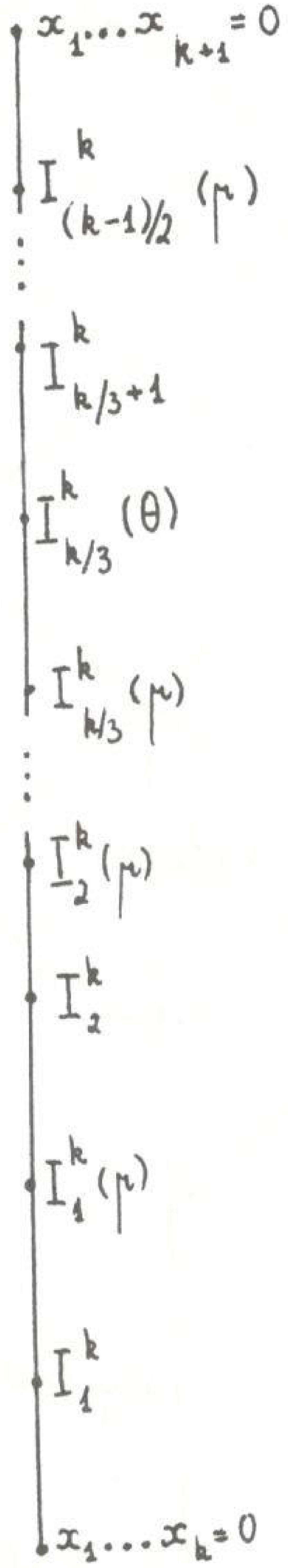


Figure 8

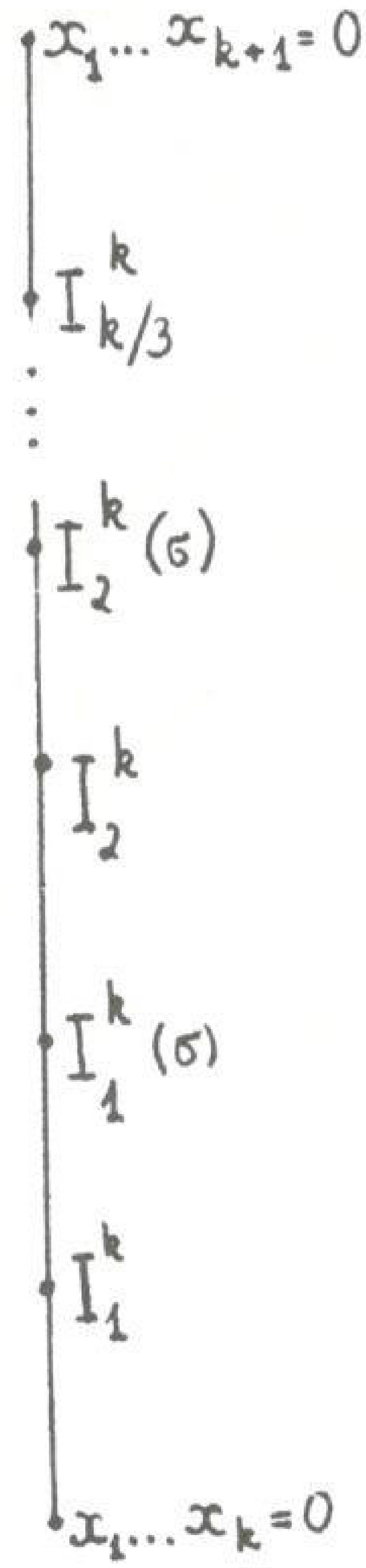


Figure 9

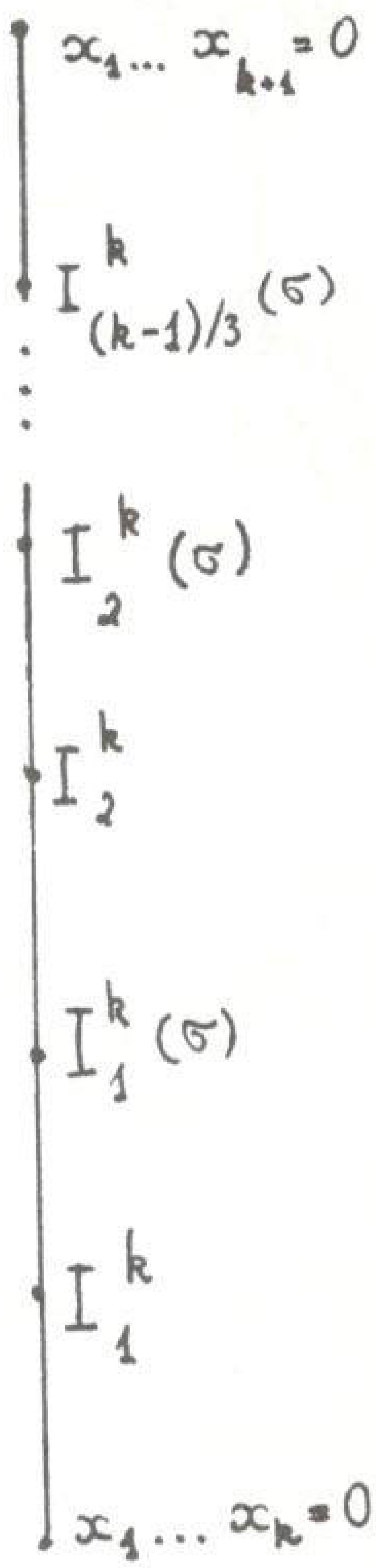


Figure 10

(Here $\Sigma(p,\ldots,q)$ denotes the permutation group on the set $\{p,p+1,\ldots,q\}$.) It is clear that I_r^k implies I_h^k and $I_h^k(\sigma)$ if $r \leq h$. Let us prove that $I_r^k(\sigma)$ implies I_h^k if $r < h$. It is clear that there is an index i such that $i \leq r$ and $\sigma^{-1}(i) > r$. We may assume also that there exists $j \neq i$ such that $\sigma^{-1}(j) > r$ because $r < h$ implies

$$(k - r) - r > k - 2h + 1 \geq k - 2(k/2) + 1 = 1.$$

Putting $x_j = x_i$ in I_r^k, we obtain x_i^2 in its right part and x_i^3 (or x_i^4) in its left part. Since $x^3yzt = 0$ in N, we get I_{r+1}^k. Now we prove that all the identities $I_h^k(\sigma)$ where h is fixed and σ varies, are equivalent excepting the case when $h = k/3$ and $\sigma(i) > h$ for each $i \leq h$. Indeed, there exists an index j such that $h < j \leq d$ and $1 \leq i = \sigma(j) \leq h$. Let us suppose that there exists also an index n such that $1 \leq n \leq h$ and $1 \leq m = \sigma(n) \leq h$. It is clear that $\sigma(im)\sigma^{-1} = (jn)$ and $(qr) = (nr)(jn)(qj)(jn)(nr)$ for any $q > h$ and $r \leq h$. It means that all transpositions of $\Sigma(1,\ldots,d)$ belong to the subgroup generated by σ and by $\Sigma[h,d]$, and hence this subgroup coincides with $\Sigma(1,\ldots,d)$. By duality, the same is true also when there is an index n' such that $h < n' \leq d$ and $h < m' = \sigma(n') \leq d$. It is clear, however, that if a permutation μ belongs to the subgroup generated by σ and by $\Sigma[h,d]$, then $I_h^k(\sigma)$ implies $I_h^k(\mu)$. Therefore, if both σ and μ satisfy one of the assumptions above, the corresponding identities are equivalent. If none of these assumptions holds, then we obtain the exception mentioned above. All exceptional permutations form (together with $\Sigma[k/3,2k/3]$) a subgroup Σ^*. It is easy to see that all identities $I_{k/3}^k(\theta)$ where $\theta\in\Sigma^*\backslash\Sigma[k/3,2k/3]$ are mutually equivalent and do not imply $I_{k/3}^k(\mu)$ for any $\mu\notin\Sigma^*$. Now it is clear that the interval $[N(k), N(k+1)]$ is the chain shown in Fig.7 (if 6 divides k) or in Fig.8 (if 3 divides k and 2 does not) or in Fig.9 (if 2 divides k and 3 does not) or in Fig.10 (if 2 and 3 do not divide k). There σ,μ and θ have the same sense as above, i.e. $\sigma\in\Sigma(1,\ldots,d)\backslash\Sigma[h,d]$, $\mu\in\Sigma(1,\ldots,d)\backslash\Sigma^*$ and $\theta\in\Sigma^*\backslash\Sigma[k/3,2k/3]$. ■

To prove (c) we verify firstly the interval $[N(6), N]$ is modular. Thanks to (b) and to the results of section 3, it is sufficient to prove that for each $k \geq 6$, any variety in $[N(k), N]$ is k-split. Suppose that X is a k-non-split subvariety of $[N(k), N]$ where $k \geq 6$. Then X satisfies an identity $u = v$ such that $l(u) = k$, $l(v) = r > k$ and $u = 0$, $v = 0$ do not hold in X (and hence in N). We noted above that in such a case $u \approx u'$ for some u' of the type (15) and $v \approx x_1^2 \ldots x_q^2 x_{q+1} \ldots x_p$ where $0 \leq q \leq r/2$ and $p = r - q$. If $d \neq p$, then $c(u) \neq c(v)$, and $u = v = 0$ in X by Lemma 1. If $d = p$, then $q > h$ and therefore $u \blacktriangleleft v$. This implies $u = v = 0$ in X by Lemma 2. ■

Now we are going to describe 5-non-split subvarieties of $[N(5), N]$. Such a variety should satisfy an identity $u = v$ such that $l(u) = 5$, $l(v) = k > 5$, $u = 0$, $v = 0$ does not hold in N. Then we may assume that

$$u \in \{x_1 \ldots x_5, x^2yzt, x^2y^2z, x^3yz, x^3y^2, x^4y, x^5\}$$

and v is similar to a word of the type (15). Using Lemmas 1 and 2, it is easy to see that the only possibility for u is $u \equiv x^3yz$, and hence $v \equiv x^2y^2z^2$. Now we can prove

LEMMA 5. *If M is a 5-non-split variety from $[N(5), N]$, then M may be given in N by the identities (14), I_0^n, I_1^m or $I_1^m(\sigma)$, I_2^k or $I_2^k(\sigma)$, where $7 \leq k \leq m \leq n$.*

Proof. The fact that M satisfies (14) is proved above. Substituting zt for z in (14), we get $x^3yzt = x^2y^2z^2t^2$. On the other hand, multyplying (14) by t, we have $x^3yzt = x^2y^2z^2t$. Hence $x^2y^2z^2t^2 = x^2y^2z^2t$, and I_3^7 holds in M by Lemma 2. We mentioned above that every identity $u = v$ where $l(u) \geq 7$ and $l(v) \geq 7$ is equivalent in N to I_h^k or to $I_h^k(\sigma)$ for some k, h and σ. Since I_3^7 holds in M, only identities with $h = 0$, 1 or 2 should be included in a basis of this variety. It remains to consider identities $u = v$ where $l(u) \leq 6$. It is easy to see, however, that such an identity either follows from (14) or implies $x^2y^2z^2 = 0$. The latter case is impossible because M is 5-non-split. ■

Now we can prove modularity of $[N(5), N]$ by direct verifying of the modular law. Suppose that

$$X \subseteq Y \longrightarrow (X \vee Z) \cap Y = X \vee (Y \cap Z)$$

fails for some $X, Y, Z \in [N(5), N]$. By section 3, the lattice of all 5-split subvarieties of N is embeddable in the direct product $[N(5), N(6)] \times [N(6), N]$, and hence is modular. Therefore at least one of the varieties X, Y and Z is 5-non-split. Since N is nilpotent, the interval $[N(5), N]$ is finite. By Exersize 3 of section 4.1 of[6] we may assume that Y covers X. It is clear that under this assumption,

$$(X \vee Z) \cap Y = Y \quad \text{and} \quad X \vee (Y \cap Z) = X. \tag{16}$$

There are four possibilities to consider:

(i) X is 5-non-split and Y is 5-split;

(ii) X is 5-split and Y is 5-non-split;

(iii) X and Y are 5-non-split;

(iv) X and Y are 5-split and Z is 5-non-split.

(i) X _is 5-non-split and_ Y _is 5-split_. Suppose that (14) holds in Z. Then it holds in $X \vee Z$ by Lemma 5. By (16), $Y \subseteq X \vee Z$, and hence Y satisfies (14). But it is impossible because Y is split and satisfies $x^3yz = 0$ whenever it satisfies (14). Thus it remains to prove that (14) holds in Z.

Let $W = \{x^3yz,\ x^2y^2z,\ x^2y^2z^2\}$. Recall that a word u is said to be *isoterm* in N for a variety A if A does not satisfy non-trivial (in N) identities of the kind $u = v$.

LEMMA 6. _All words of_ W _are isoterms in_ N _for_ Y.

Proof. Let $u \in W$ and Y satisfies $u = v$ for some v. If $l(u) \neq l(v)$, this identity implies $u = 0$ because Y is 5-split. However, Y does not satisfy identities of the kind $u = 0$ where $u \in W$. By the same reason, we may assume that $c(u) = c(v)$. Now let $u \equiv x^3yz$. Then our identity coincides with one of the identities (11)-(13). Since (13) holds in N, it may be excluded. Further, (11) implies (12). Suppose that Y satisfies (12). Substituting x^2 for x, we get $x^2y^2z^2 = 0$, but this identity does not hold in Y. If $u \equiv x^2y^2z$, our identity should coincide with (10). Substituting z^2 for z, we again get $x^2y^2z^2 = 0$. Finally, if $u \equiv x^2y^2z^2$, we get the same conclusion because all words v different from u and such that $l(v) = 6$ and $c(v) = \{x, y, z\}$, equal 0 in N. ■

Let τ and δ be fully invariant congruences on the N-free semigroup of countable rank corresponding to Y and to Z respestively. By (16) $Y \cap Z \subseteq X$, and by Lemma 5 $Y \cap Z$ satisfies (14). Therefore there exist $u_0, u_1, \ldots, u_{2n}$ such that $u_0 \equiv x^3yz$, $u_{2n} \equiv x^2y^2z^2$ and $u_i \tau u_{i+1} \delta u_{i+2}$ for $i = 0,2,\ldots,2n-2$. If all the words $u_0, u_1, \ldots, u_{2n}$ are similar to ones from W, then for all even i, we have $u_i = u_{i+1}$ in N by Lemma 6 and $u_0 \delta u_{2n}$, i.e. (14) holds in Z. As we mentioned above, it is enough to obtain a contradiction. Therefore we suppose that a word in our sequence is not similar to members of W. Let j be the least index such that u_j has this property. By Lemma 6 j should be even and $u_i = u_{i+1}$ in N for all even $i < j$. Hence $u_0 \delta u_j$, i.e. Z satisfies $x^3yz = u_j$. It is easy to see that either $u_j = 0$ in N or $c(u_j) \neq \{x,y,z\}$. In both the cases Z satisfies $x^3yz = 0$. Dually, considering the most index k such that u_k is not similar to words of W, we obtain that Z satisfies $x^2y^2z^2 = 0$. Thus (14) again holds in Z. ■

(ii) X <u>is 5-split and</u> Y <u>is 5-non-split</u>. Lemma 5 shows that X satisfies $x^3yz = 0$ and $x^2y^2z^2 = 0$. By (16) $Y \cap Z \subseteq X$, and hence these identities hold in $Y \cap Z$ too. Suppose that one of them holds in Z. Then it holds also in $Y = (X \vee Z) \cap Y$, but this impossible because Y is 5-non-split. Thus it remains to prove that Z satisfies $x^3yz = 0$ or $x^2y^2z^2 = 0$.

The next observation follows easily from Lemma 5:

LEMMA 7. <u>Let</u> M <u>be a 5-non-split variety of</u> $[N(5), N]$.

a) x^2y^2z <u>is isoterm in</u> N <u>for</u> M.

b) <u>If</u> $x^3yz = v$ <u>holds in</u> M, <u>then</u> $v = x^2y^2z^2$ <u>in</u> N.

c) <u>If</u> $x^2y^2z^2 = v$ <u>holds in</u> M, <u>then</u> $v = x^3yz$ <u>in</u> N. ■

Now let $u_0, \ldots, u_{2n}$ be a sequence of words such that $u_0 \equiv x^3yz$, $u_{2n} = 0$ in N and $u_i \tau u_{i+1} \delta u_{i+2}$ for all $i = 0,2,\ldots,2n-2$. Let j be the least index such that u_j is not similar to words of W. (Here W, δ and τ have the same sence as in (i) above.) By Lemma 7 j should be even, and $u_{j-1} \delta u_j$. Clearly, $u_j = 0$ in N or $c(u_j) \neq \{x,y,z\}$. In both the cases Z satisfies $u_{j-1} = 0$. However u_{j-1} is similar to a word from W, and hence either $x^3yz = 0$ or $x^2y^2z^2 = 0$ in Z. As we have seen, this leads to a contradiction. ■

(iii) X and Y are 5-non-split. There exists an identity J which holds in X, but fails in Y. By (16) $Y \cap Z \subseteq X$, and hence J holds in $Y \cap Z$. It is sufficient to prove that J holds in Z. Indeed, in this case it would hold in Y by (16), but this is impossible. By Lemma 5 $J \in \{I_0^n, I_1^m, I_1^m(\sigma), I_2^k, I_2^k(\sigma)\}$. Let us consider each of these possibilities.

1. $J = I_0^n$. Using Lemma 1, one can verify that $x_1 \cdots x_n$ is isoterm in N for any variety which does not satisfy I_0^n (in particular for Y). This implies easily that if I_0^n does not hold in Z, it also cannot hold in $Y \cap Z$. ■

2. $J = I_1^m$. Using Lemma 1 and the identity $x^3yzt = 0$ holding in N, it is easy to verify that if a subvariety of N does not satisfy I_1^m, the only non-trivial (in N) identity of the kind $x_1^2x_2 \cdots x_n = v$ which could hold in this subvariety is $I_1^m(\sigma)$. This remark is valid in particular for Y. This implies easily that if I_1^m does not hold in Z, it also cannot hold in $Y \cap Z$. ■

3. $J = I_2^k$. This case is completely analogous to the previous one. ■

4. $J = I_1^m(\sigma)$. As we mentioned, this identity does not really depend on σ and follows from every identity of the kind $x_1^2x_2 \cdots x_n = v$ which is non-trivial in N. Therefore the word $x_1^2x_2 \cdots x_n$ should be isoterm in N for Y. As above, this implies easily that if $I_1^m(\sigma)$ does not hold in Z, it cannot hold in $Y \cap Z$. ■

5. $J = I_2^k(\sigma)$. Since $k \geq 7$, this identity also does not depend on σ, and hence the case is completely analogous to the previous one. ■

(iv) X and Y are 5-split and Z is 5-non-split. We will use the notation introduced in section 3. Since the variety $P_5(Z)$ is 5-split and the lattice of 5-split varieties is proved to be modular, $(X\vee\mathcal{P}_5(Z))\cap Y = X\vee(\mathcal{P}_5(Z)\cap Y)$. Using (16), we get $X \subseteq (X\vee\mathcal{P}_5(Z))\cap Y \subseteq X\vee(Y\cap Z) = X$. Hence $(X\vee\mathcal{P}_5(Z))\cap Y = X$. Thus there exists an identity J which does not hold in Y, but holds in $X\vee\mathcal{P}_5(Z)$. If deducing J from identities of $\mathcal{P}_5(Z)$, we do not use the identities $x^3yz = 0$ and $x^2y^2z^2 = 0$, then J could be deduced already from identities of Z, and $(X\vee Z)\cap Y = X$ in contradiction with (16). It is easy to see that every non-trivial (in N) consequence of $x^3yz = 0$ (or of $x^2y^2z^2 = 0$) either is equivalent to it or follows from (14). Thus we may assume that $J\in\{x^3yz = 0, x^2y^2z^2 = 0\}$. Both the possibilities are completely analogous, therefore let us consider only first of them. By (16) $Y\cap Z$ satisfies $x^3yz = 0$. Let W, δ and τ have the same sence as in (i), and $u_0\tau u_1\delta u_2\tau\ldots\delta u_{2n}$ where $u_0 \equiv x^3yz$ and $u_{2n} = 0$ in N. If j is the least index such that u_j is not similar to words of W, then j should be odd by Lemma 7 (applied to Z), and $u_{j-1}\tau u_j$. Clearly, $u_j = 0$ in N or $c(u_j) \neq \{x,y,z\}$. In both the cases Y satisfies $u_{j-1} = 0$. However u_{j-1} is similar to a word from W, and hence either $x^3yz = 0$ or $x^2y^2z^2 = 0$ in Y. Since the former case is impossible in view of the choice of J, Y satisfies $x^2y^2z^2 = 0$. This implies that X satisfies both $x^3yz = 0$ and $x^2y^2z^2 = 0$. Then (14) holds in $X\vee Z$ and $x^3yz = x^2y^2z^2 = 0$ holds in $(X\vee Z)\cap Y$. We get a contradiction with (16). ■

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