Spectrum of Joint Action of Factors in the Binary Theory of Sufficient Causes

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This work presents basic constructions of the Boolean formalization of the binary theory of sufficient causes, which is a two-level version of a commonly accepted causality model in epidemiology, toxicology and evidence-based medicine.

The theory of sufficient causes is appeared almost simultaneously in philosophy [J. L. Mackie (1965, 1980), D. Lewis (1973, 1987)] and in epidemiology [D. MacMahon and T. F. Pugh (1967), K. Rothman (1976)] in the 1960s of the 20th century. Further development of these ideas continued mainly in epidemiology [O. Miettinen (1982), S. Greenland and C. Poole (1988) T. J. VanderWeele and M. Robins (2006, 2007), T. J. VanderWeele and T. S. Richardson (2012)], where the name of the theory was given.

One can note that the semantic structure of the binary theory of sufficient causes fully corresponds to the axioms of Boolean algebra. Boolean formalization allows using of such concepts as graph, Hamming distance, Boolean cube, group, Boolean function support and others to study joint action of binary factors in the epidemiological theory of sufficient causes [authors (2015, 2018, 2019)].

Often responses used to describe epidemiological data have no natural ordering (for example, gender, race, religion, nature of work, area of residence). Therefore, their numerical encoding can be arbitrary. However, the type of joint action of variables should not depend on an encoding. In other words, type of joint action has to be invariant with respect to an encoding of the acting variables' levels.

Since an encoding of acting factors' levels is insignificant for the study of their joint action, a group of experiment symmetries [authors (2015, 2018, 2019)] emerges, which is the group G_n of automorphisms of the Boolean cube $\mathbb{B}^n$, $\mathbb{B} = \{0, 1\}$, considered as a graph whose two vertices are connected by an edge if and only if the Hamming distance between them is 1.

These automorphisms can be naturally extended to the Boolean algebra $\mathbb{B}(x_1, x_2, ..., x_n)$ of all Boolean functions of variables $x_1, x_2, ..., x_n$.

The action of the group G_n on the algebra $\mathbb{B}(x_1, x_2, ..., x_n)$ generates a partition of this algebra into orbits $\langle f \rangle$, $f \in \mathbb{B}(x_1, x_2, ..., x_n)$, which are the types of joint action of variables $x_1, x_2, ..., x_n$ [authors (2015, 2018, 2019)].

The action of the group G_n on each Boolean function $f \in \mathbb{B}(x_1, x_2, ..., x_n)$ consists in permutation of these variables and replacing variables with their negation.

Unless otherwise stated, we consider only nonzero Boolean functions (responses) presented in the form of DNF (disjunctive normal form), and, in addition, we assume that $2 \leq k \leq n$.

Definition 1 [authors (2019)]

We say that in the response $f \in \mathbb{B}(x_1, x_2, ..., x_n)$ *there is joint action of the factors* $x = (x_1, x_2, ..., x_n)$, if for some vector $\alpha \in \mathbb{B}^n$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, the conjunction $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}$ is a prime implicant of the Boolean function f. In this case, we say that the joint action of factors in the response f is attained at $x = \alpha$.

Example 1. Let $n = 3$. There is joint action of factors x_1, x_2, x_3 in the response $f = \bar{x}_1 \bar{x}_2 \bar{x}_3$, which is attained at levels of $x_1 = x_2 = x_3 = 0$. In the response $f = x_1 x_2 \bar{x}_3 \vee \bar{x}_1 \bar{x}_2$ joint action of three factors is attained at levels $x_1 = x_2 = 1$, $x_3 = 0$.

The presence of joint action is invariant with respect to the action of the group G_n [authors (2019)] that can be regarded as a property of the class $\langle f \rangle$.

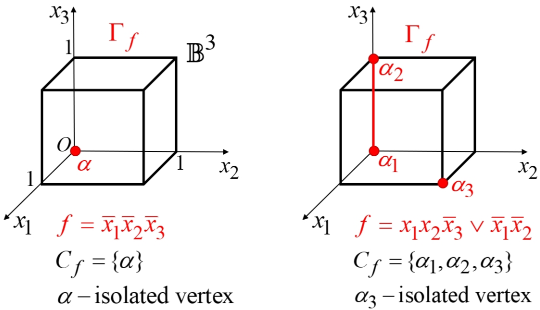
Example 2. There is joint action of three factors for the following classes (assuming that $n = 3$) $\langle \bar{x}_1 \bar{x}_2 \bar{x}_3 \rangle = \langle x_1 x_2 x_3 \rangle$ and $\langle x_1 x_2 \bar{x}_3 \vee \bar{x}_1 \bar{x}_2 \rangle = \langle x_1 x_2 x_3 \vee \bar{x}_1 \bar{x}_2 \rangle$.

Let us give a geometric interpretation of the Definition 1 in terms of graph theory.

Let C_f is a support of a Boolean function f, i.e. $C_f = \{\alpha \in \mathbb{B}^n \mid f(\alpha) = 1\}$ and Γ_f is a section of the graph $\mathbb{B}^n$ whose vertices are points from C_f and the edges are corresponding edges of $\mathbb{B}^n$.

Theorem 1 [authors (2019)]

The joint action of factors x attained at $x = \alpha$ is present in the response f if and only if the point α is an isolated vertex of the graph Γ_f.



Definition 2 [authors (2019)]

We call *degree of joint action of factors* $x = (x_1, ..., x_n)$ in a response $f \in \mathbb{B}(x_1, x_2, ..., x_n)$ at values of factors $x = \alpha$, where $\alpha \in C_f$, a number $\mu_f(\alpha)$ defined by equality $\mu_f(\alpha) = d(\alpha, C_f \backslash \{\alpha\}) - 1$, if $|C_f| > 1$, and $\mu_f(\alpha) = n$ if $|C_f| = 1$.

Here $|C_f|$ is the cardinality of the set C_f, $d(\alpha, \beta)$ is the Hamming distance between points α and β and $d(\alpha, C_f \backslash \{\alpha\})$ — distance from the point α to the set of all other points from C_f if $|C_f| > 1$.

To determine the degree of joint action of n factors in a given response, we are interested in the strongest joint action of n factors across all values of factors' levels for which the response is 1. Therefore, we have the following definition

Definition 3 [authors (2019)]

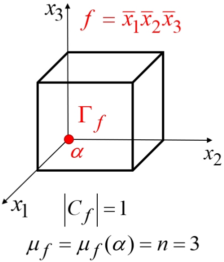
Let's call *degree of joint action of the factors* $x_1, x_2, \ldots, x_n$ in a response $f \in \mathbb{B}(x_1, x_2, ..., x_n)$ a number $\mu_f = \max\{\mu_f(\alpha) \mid \alpha \in C_f\}$. For a zero response, $f = 0$, it is convenient to assume that $\mu_f = 0$.

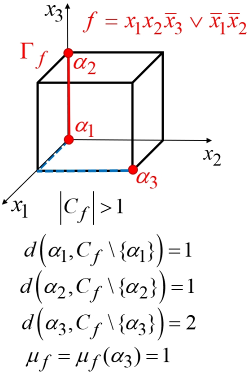
Let $\mu_f = \mu_f(\alpha)$ for some $\alpha \in C_f$. Then $\mu_f = n$ if $|C_f| = 1$, and the vertex α is the outermost point from other ones in C_f if $|C_f| > 1$.

Theorem 2 [authors (2019)]

There is joint action of all the factors $x_1, x_2, \ldots, x_n$ in a response f if and only if the inequality $\mu_f \geq 1$ holds.

Example 3. Let $n = 3$ and $f = \bar{x}_1\bar{x}_2\bar{x}_3$. Then the degree $\mu_f = \mu_f(\alpha) = 3$ for $\alpha = (0, 0, 0)$. For the response $f = x_1x_2\bar{x}_3 \vee \bar{x}_1\bar{x}_2$ degree $\mu_f = \mu_f(\alpha) = 1$ for $\alpha = (1, 1, 0)$. We see that the joined action of three factors is stronger in the first response than in the second one.





Note that for many classes there will be no joint action of all the factors on which a response depends, although there may be joint action of a smaller number of factors. This work is devoted to the study of the case.

For an ordered k-element subset I of the set $\mathbb{N}_n = \{1, 2, \dots, n\}$ and a vector $\beta \in \mathbb{B}^{n-k}$ we denote by $f_{I,\beta}$ a restriction of the mapping f to the k-dimensional face (subcube or k-face) $\mathbb{B}_J^\beta = \{\xi \in \mathbb{B}^n \mid (\xi_j)_{j \in J} = \beta\}$ of the cube $\mathbb{B}^n$, where $J = \bar{I}$ is the ordered complement of the subset I in the set $\mathbb{N}_n$.

Definition 4

We say that *joint action of k factors is present in a response f, depending on n factors*, if there exist such an ordered k-element subset I of the set $\mathbb{N}_n$ and the set $\beta \in \mathbb{B}^{n-k}$ that joint action of factors $x_I = (x_i)_{i \in I}$ is present in the response $f_{I,\beta}$.

Definition 4 means that joint action of some k factors in a given response is present, when the other factors take some fixed values. This definition generalizes Definition 1 for n factors [authors, (2019)] and is a rigorous form of a similar construction from [T.J. VanderWeele and M. Robins (2006, 2007)].

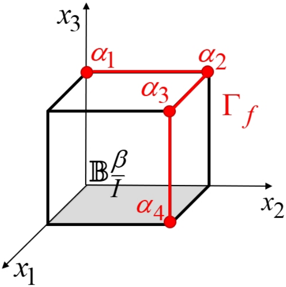
Example 4. There are no isolated vertices in the graph Γ_f for the response $f = x_1x_2 \vee \bar{x}_1x_3, n = 3$. According to Theorem 1, there is no joint action of three factors x_1, x_2, x_3 in f. However, a joint action of two factors x_1, x_2 is present in this response. Indeed, for $I = \{1, 2\}$ and $\beta = 0$, the restriction of the mapping f to the 2-dimensional face by the condition $x_3 = 0$ is equal to $f_{I,\beta} = x_1x_2$, in which, the joint action of two factors takes place according to the Theorem 1.

The following statement is a geometric interpretation of Definition 4 in terms of graph theory and is an analogue of the Theorem 1.

Theorem 3

In a response f which depends on n factors, there is joint action of $k \leq n$ factors if and only if the inequality $k \leq n - \delta_f$ holds for the minimal degree δ_f of vertices of the graph Γ_f.

It follows from Theorem 3 that if a given response has joint action of $k > 2$ factors, then it has joint action of fewer factors (not less than two).



$$f = x_1 x_2 \vee \bar{x}_1 x_3$$

$$I = \{1,2\},\ \beta = 0$$

$$\mathbb{B}_I^\beta = \left\{ x \in \mathbb{B}^n \middle| x_3 = 0 \right\}$$

$$f_{I,\beta} = x_1 x_2$$

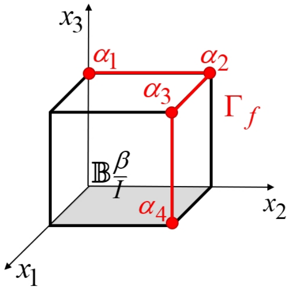
$$n = 3,\ \delta_f = 1 \Rightarrow k = 2$$

We now introduce a concept of degree of joint action for k factors in a response depending on $n \geq k$ factors so that it generalizes the concept of the degree of joint action of n factors.

Definition 5

We call *the degree of joint action of k factors* in a response f that depends on $n \geq k$ factors a number $\mu_{f,k} = \max\{\mu_{f_{I,\beta}} \mid I \subseteq \mathbb{N}_n, |I| = k, \beta \in \mathbb{B}^{n-k}\}$. We set $\mu_{f,0} = 0$ for every response f, and $\mu_{f,1} = 1$ for $f \neq 1$ and $\mu_{f,1} = 0$ for $f = 1$.

Example 5. As noted in Example 4, there is no joint action of three factors in the response $f = x_1x_2 \vee \bar{x}_1x_3$, and therefore, by Theorem 2, $\mu_f = 0$ which means that $\mu_{f,3} = 0$. For the set $I = \{1,2\}$ and $\beta = 0$ we have $f_{I,\beta} = x_1x_2$ and by Definition 3 $\mu_{f_{I,\beta}} = 2$. From inequalities $\mu_{f_{I,\beta}} \leq \mu_{f,2} \leq 2$ we obtain $\mu_{f,2} = 2$.



$$f = x_1 x_2 \vee \bar{x}_1 x_3$$

$$\mu_{f,3} = \mu_f = 0$$

$$\mu_{f,2} = \mu_{f_{I,\beta}} = \mu_{f_{I,\beta}}(\alpha_4) = 2$$

$$I = \{1,2\},\ \beta = 0\ (x_3 = 0)$$

$$f_{I,\beta} = x_1 x_2$$

Spectrum of joint action of the factors

Theorem 4

There is joint action of k factors in a response f depending on $n \geq k$ factors if and only if the inequality $\mu_{f,k} \geq 1$ holds.

In addition, just as for the degree of μ_f the following statement holds.

Theorem 5

The degree $\mu_{f,k}$ of join action of k factors is invariant with respect to the group G_n action.

To assess the strength of joint action of the factors in a given response as a whole, let us introduce a set $M_f = (\mu_{f,1}, \mu_{f,2}, \ldots, \mu_{f,n})$.

Definition 6

We call M_f the *spectrum* of joint action of the factors in a given response.

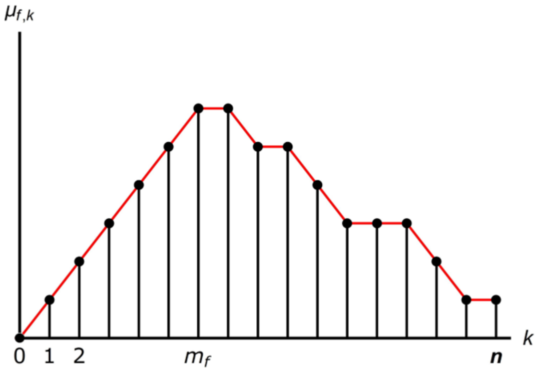
An exhaustive answer to the question about the relationship of degrees of joint action for different k, and thus about the structure of the spectrum M_f is the following

Theorem 6

For any (including zero) response f, there exists a single number $m_f \in \{1, 2, \dots, n\}$ such that inequalities (1) $\mu_{f,i} = i$ for any $i \in \{0, 1, \dots, m_f\}$; (2) if $m_f < n$, then $\mu_{f,j} \le \mu_{f,j-1}$ for any $j \in \{m_f + 1, \dots, n\}$ are satisfied.

Thus, the number m_f is in a sense "critical" for a given response f. Adding a factor to any $k \le m_f - 1$ factors increases the degree of joint action of these factors in a given response by one, and adding a factor to any number of $k \ge m_f$ factors does not change or decrease the degree of joint action.

Slide 19. Geometrical interpretation of the relationship of degrees of joint action for different k



It follows from the Theorem 5 that the spectrum M_f is invariant with respect to the group G_n action, and therefore is a characteristic of a type $\langle f \rangle$ of joint action.

This means that all types $\langle f \rangle$ can be ordered by the strength of their joint action using the reverse lexicographic ordering $\succeq$ for the tuples in M_f.

The use of the reverse lexicographic ordering is justified by the fact that joint action of a larger number of factors is more important than joint action of fewer ones.

Example 6. From the Example 3 and Theorem 6 we conclude that for the type $\langle x_1x_2x_3 \rangle$ its spectrum is $(1,2,3)$. Just as in Example 5, it can be shown that $\mu_{f,2} = 2$ for $f = x_1x_2\bar{x}_3 \vee \bar{x}_1\bar{x}_2$. Thus, for the type $\langle x_1x_2x_3 \vee \bar{x}_1\bar{x}_2 \rangle$ the spectrum is $(1,2,1)$. According to the Example 4 one can infer that for the type $\langle x_1x_2 \vee \bar{x}_1x_3 \rangle$ its spectrum is $(1,2,0)$. Since $(1,2,3) \succeq (1,2,1) \succeq (1,2,0)$, we conclude that these interaction types are ordered in descending order of their joint action.

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Thank you for your attention!