SPECIAL ELEMENTS OF THE LATTICE OF EPIGROUP VARIETIES

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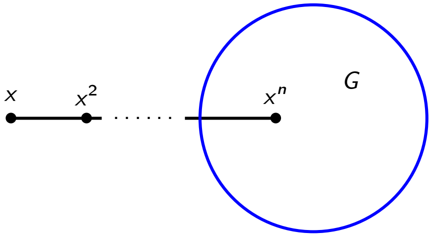
A semigroup S is called an *epigroup* if some power of every element x in S lies in a subgroup G of S.



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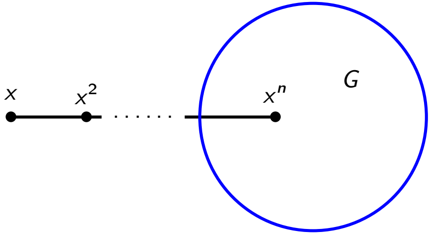
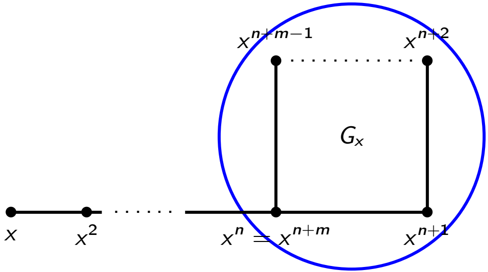


For any x there is the maximal subgroup containing x. It is denoted by G_x.

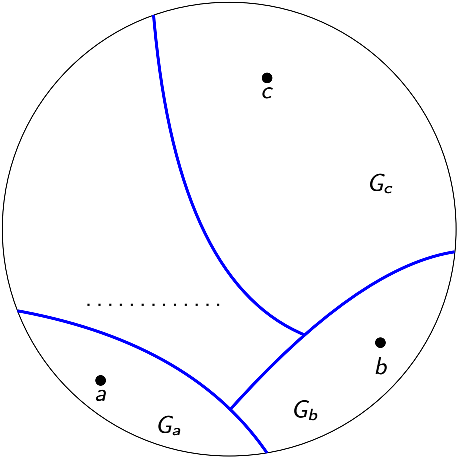
S is a periodic semigroup $\Longleftrightarrow$ S satisfies $x^n = x^{n+m}$ for some $n, m \geq 1$.



$$G_x = \{x^n, x^{n+1}, \dots, x^{n+m-1}\}$$

$$S = \bigcup_{x \in S} G_x$$

$$x \in G_x$$



Epigroups as unary semigroups

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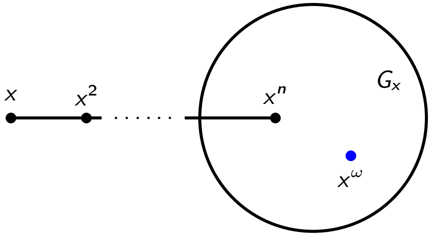

x^ω is a unit element of G_x.

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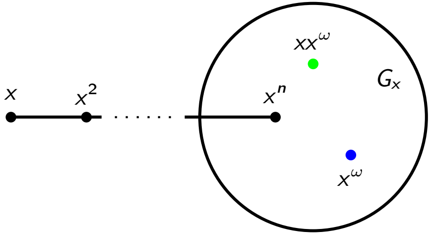

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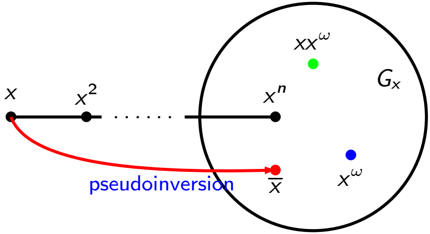

x^ω is a unit element of G_x; $xx^\omega = x^\omega x \in G_x$; $\overline{x} = (xx^\omega)^{-1}$ in G_x

$\overline{x}$ is *pseudoinverse* to x

If S is completely regular then $\overline{x} = x^{-1}$.

Varieties of completely regular semigroups are varieties of epigroups.

Neutral elements of lattices

a is called *neutral* in L if

$$\forall x, y \in L: \quad a, x, y \text{ generate a distributive sublattice in } L$$

or, equivalently

$$\forall x, y \in L: \quad (a \vee x) \wedge (x \vee y) \wedge (y \vee a) = (a \wedge x) \vee (x \wedge y) \vee (y \wedge a).$$

If a is neutral in L then L is a subdirect product of $(a]$ and $[a)$ where $(a] = \{x \in L \mid x \leq a\}$ and $[a) = \{x \in L \mid a \leq x\}$.

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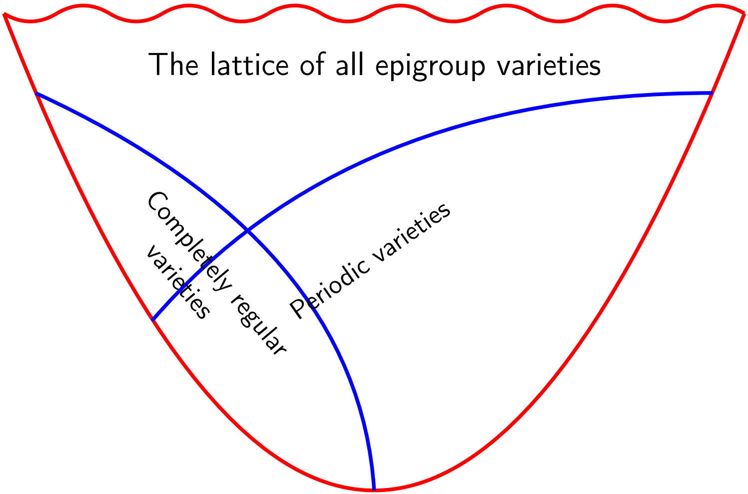
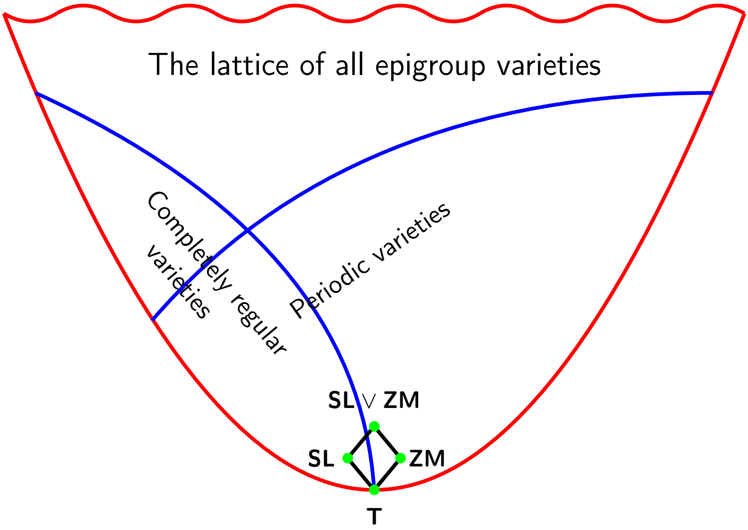
Theorem

Neutral elements of the lattice of epigroup varieties are

1. *the trivial variety* $\mathbf{T}$,
2. *the variety of all semilattices* $\mathbf{SL}$,
3. *the variety of all semigroups with zero multiplication* $\mathbf{ZM}$,
4. *the variety* $\mathbf{SL} \vee \mathbf{ZM}$

and only they.


The lattice of all epigroup varieties
Completely regular varieties
Periodic varieties


The lattice of all epigroup varieties
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Periodic varieties
SL ∨ ZM
SL
ZM
T

Completely reegular case

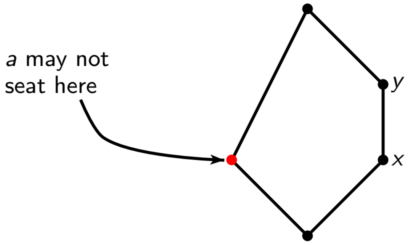
The lattice of completely regular varieties has infinitely many neutral elements including

- all varieties of bands;
- the variety of all groups;
- the variety of all completely simple semigroups;
- the variety of all orthodox semigroups

and some others (Trotter, 1989).

a is called *modular* in L if

$$\forall x, y \in L: \quad x \leq y \longrightarrow (a \vee x) \wedge y = (a \wedge y) \vee x$$



0-reduced identity: $w = 0$, that is $wx = xw = w$ where x is a letter that does not occur in the word w.

Substitutive identity: $v = w$ where w is obtained from v by renaiming of tletters

Examples: $xy = yx$, $xyz = yxz$, $x^2y = y^2x$, $xyx = yxy$ etc.

Theorem

If an epigroup variety $\mathbf{V}$ *is a modular element of the lattice of epigroup varieties then* $\mathbf{V} = \mathbf{X} \vee \mathbf{N}$ *where* $\mathbf{X}$ *is one of the varieties* $\mathbf{T}$ *or* $\mathbf{SL}$*, and* $\mathbf{N}$ *is a nil-variety given by* 0*-reduced and substitutive identities only.*

$\mathbf{X}$ is modular $\Longleftrightarrow$ $\mathbf{SL} \vee \mathbf{X}$ is modular.

The Theorem gives a complete reduction to nilvarieties.

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A commutative epigroup variety $\mathbf{V}$ *is a modular element of the lattice of epigroup varieties if and only if* $\mathbf{V} = \mathbf{X} \vee \mathbf{N}$ *where* $\mathbf{X}$ *is one of the varieties* $\mathbf{T}$ *or* $\mathbf{SL}$ *while* $\mathbf{N}$ *satisfies the identity* $x^2y = 0$.

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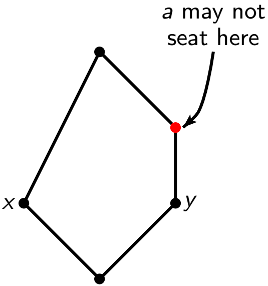

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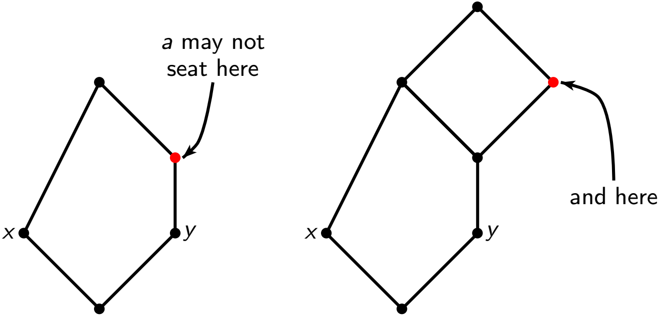

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A commutative epigroup variety $\mathbf{V}$ *is an upper-modular element of the lattice of epigroup varieties if and only if either* $\mathbf{V} \subseteq \mathbf{G} \vee \mathbf{C} \vee \mathbf{D}$ *or* $\mathbf{V} \subseteq \mathbf{SL} \vee \mathbf{E}$ *where* $\mathbf{G}$ *is an abelian group variety,* $\mathbf{C} = var\{x^2 = x^3, xy = yx\}$, $\mathbf{D} = var\{x^2y = 0, xy = yx\}$ *and* $\mathbf{E} = var\{x^2y = xy^2, x^2yz = 0, xy = yx\}$.

Corollary

If a commutative epigroup variety $\mathbf{V}$ *is a modular element of the lattice of epigroup varieties then it is an upper-modular element of this lattice.*

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