Special elements in lattices of semigroup varieties

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Neutral elements

Definition

An element x of a lattice L is called *neutral* if x, y and z generate a distributive sublattice of L for all $y, z \in L$.

Theorem

(Volkov, 2005) The following varieties and only they are neutral elements of the lattice **SEM**: $\mathcal{T}$, $\mathcal{SEM}$, $\mathcal{SL}$, $\mathcal{ZM}$, $\mathcal{SL} \vee \mathcal{ZM}$.

Theorem

(Shaprynskii, 2011) A variety of commutative semigroups $\mathcal{V}$ is a neutral element of the lattice **Com** if and only if either $\mathcal{V} = \mathcal{COM}$ or $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$, while $\mathcal{N}$ satisfies the identities $x^2y = 0$ and $xy = yx$.

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Distributive elements in **SEM**

Definition

An element x of a lattice L is called *distributive* if

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

for all $y, z \in L$.

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(Shaprynskii, Vernikov, 2010) A variety of semigroups $\mathcal{V}$ is a distributive element of the lattice **SEM** if and only if either $\mathcal{V} = \mathcal{SEM}$ or $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$, while $\mathcal{N}$ is a 0-reduced variety and $\mathcal{N}$ satisfies the identities $x^2y = xyx = yx^2 = 0$.

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Modular and lower-modular elements (definitions)

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An element x of a lattice L is called *modular* if

$$(x \vee y) \wedge z = (x \wedge z) \vee y$$

for all $y, z \in L$ with $y \leq z$.

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An element x of a lattice L is called *lower-modular* if

$$x \vee (y \wedge z) = y \wedge (x \vee z)$$

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Observation

Every distributive element is lower-modular.

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Modular elements in **SEM**

Definition

An identity $u = v$ is called *substitutive* if u and v depend on the same letters and v may be obtained from u by renaming of letters

Theorem

a) (Jezek, McKenzie, 1993; reproved in a shorter and simpler way by Shaprynskii) If a variety of semigroups $\mathcal{V}$ is a modular element of the lattice **SEM** and $\mathcal{V} \neq \mathcal{SEM}$ then $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$, while $\mathcal{N}$ is a nil-variety.

b) (Vernikov, 2007) If a nil-variety $\mathcal{N}$ is a modular element of the lattice **SEM** then $\mathcal{N}$ may be given by 0-reduced and substitutive identities only.

c) (Vernikov, Volkov, 1988; independently Jezek, McKenzie, 1993) If $\mathcal{N}$ is a 0-reduced variety then $\mathcal{N}$ is a modular element of the lattice **SEM**.

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Modular elements in **SEM**, a continuation

Corollary

A lower-modular element of the lattice **SEM** is a modular element of this lattice.

Theorem

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