Identities and semimodularity in lattices of epigroup varieties

Boris M. Vernikov, Mikhail V. Volkov,
Vyacheslav Yu. Shaprynskii, Dmitry V. Skokov

Ural Federal University

Workshop on General Algebra AAA-82
Potsdam, 25 June, 2011

Definition

A semigroup S is called an *epigroup* if for any element x of S some power of x lies in some subgroup of S.

A *unary semigroup* is a semigroup with an additional unary operation.

Let S be an epigroup, e_a the identity element of the maximal subgroup G_a that contains some power of a.
It is known that $ae_a = e_a a \in G_a$.
The element $\overline{a}$ inverse to ae_a in G_a is said to be *pseudo-inverse* to a.

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Completely regular epigroup varieties are varieties of completely regular semigroups in the usual sense.

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Describe varieties of epigroups with modular subvariety lattice.

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Periodic case:

Semigroup varieties with the following restrictions to subvariety lattices were completely determined:

1. modularity (Volkov, 1989-92);
2. arguesity, upper or lower semimodularity (Vernikov, Volkov, 2002-2004) (arguesity = upper semimodularity = modularity $\neq$ lower semimodularity);
3. distributivity (modulo groups):
   - out of the case of the varieties of semigroups with completely regular square (Volkov, 1989-92);
   - for varieties of semigroups with orthodox square (Volkov, unpublished).

(A semigroup S is called *orthodox* if it is completely regular and the product of any two idempotents in S is an idempotent.)

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For a non-periodic epigroup variety $\mathcal{V}$ the following are equivalent:

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(M1) $\mathcal{V} = \mathcal{A} \vee \mathcal{K} \vee \mathcal{N}$ *where* $\mathcal{A}$ *is the variety of all Abelian groups,* $\mathcal{K}$ *is either the trivial variety or the variety of semilattices or the variety* $\mathrm{var}\{x^2 = x^3,\ xy = yx\}$*, while* $\mathcal{N}$ *satisfies the identities* $x^2y = xyx = yx^2 = 0$ *and* $x_1x_2x_3x_4 = x_{1\pi}x_{2\pi}x_{3\pi}x_{4\pi}$ *where* π *is an even permutation on the set* $\{1, 2, 3, 4\}$*;*

(M2) $\mathcal{V} = \mathcal{U} \vee \mathcal{W}$ *where* $\mathcal{U}$ *is one of the varieties* $\mathrm{var}\{xy = x^2y,\ x^2y^2 = y^2x^2\}$ *or* $\mathrm{var}\{xy = x^2y,\ xyz^2 = yxz^2,\ xyx = yx^2\}$*, while* $\mathcal{W}$ *is an orthodox variety and every band in* $\mathcal{W}$ *satisfies the identity* $xyz = xyxz$*;*

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The greatest group subvariety of a variety $\mathcal{V}$ is denoted by $\mathrm{Gr}(\mathcal{V})$.

Theorem 2

Let $\mathcal{V}$ be a non-periodic epigroup variety and $\mathcal{V}$ is not a variety of epigroups with completely regular square. The lattice $L(\mathcal{V})$ is distributive if and only if one of the following holds:

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The main results (Theorem 3)

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Let $\mathcal{V}$ be a variety of epigroups with orthodox square. If the lattice $L(\mathrm{Gr}(\mathcal{V}))$ is distributive that the lattice $L(\mathcal{V})$ is distributive too.