SPECIAL ELEMENTS OF LATTICES OF SEMIGROUP VARIETIES

B.M.Vernikov

Ural Federal University

Algebraic theory of semigroups and applications

Porto, 2015

x is a *neutral* element

$\forall y, z$: the sublattice generated by x, y and z is distributive

or (equivalently)

$$\forall y, z: (x \vee y) \wedge (y \vee z) \wedge (z \vee x) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$$

Abstract lattices

x is a *neutral* element
$\forall y, z$: the sublattice generated by x, y and z is distributive
or (equivalently)
$\forall y, z\colon (x \vee y) \wedge (y \vee z) \wedge (z \vee x) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$

↓ ↓

x is a *standard* element
$\forall y, z\colon (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$

x is a *costandard* element
$\forall y, z\colon (x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$

x is a *neutral* element
$\forall y, z$: the sublattice generated by x, y and z is distributive
or (equivalently)
$\forall y, z\colon (x \vee y) \wedge (y \vee z) \wedge (z \vee x) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$

x is a *standard* element
$\forall y, z\colon (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$

x is a *costandard* element
$\forall y, z\colon (x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$

x is a *distributive* element
$\forall y, z\colon x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

x is a *codistributive* element
$\forall y, z\colon x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Abstract lattices

x is a *neutral* element
$\forall y, z$: the sublattice generated by x, y and z is distributive
or (equivalently)
$\forall y, z\colon (x \vee y) \wedge (y \vee z) \wedge (z \vee x) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$

x is a *standard* element
$\forall y, z\colon (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$

x is a *costandard* element
$\forall y, z\colon (x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$

x is a *distributive* element
$\forall y, z\colon x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

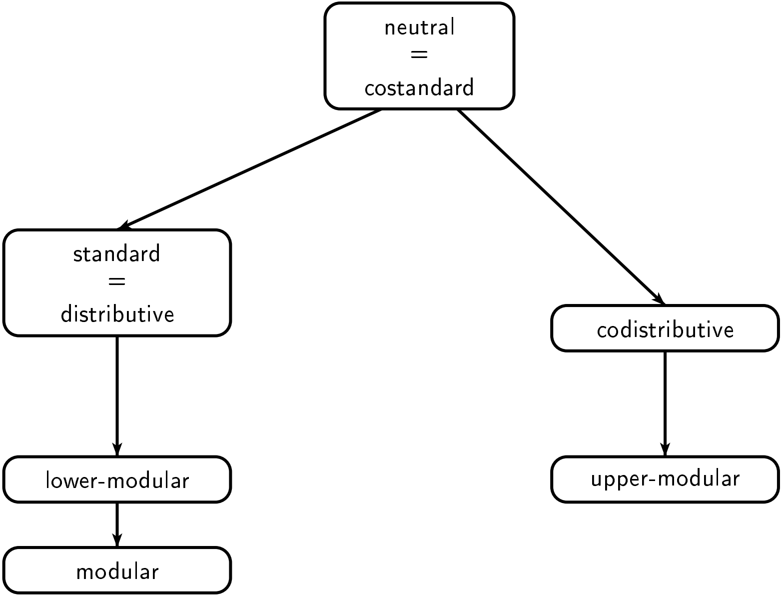
x is a *codistributive* element
$\forall y, z\colon x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

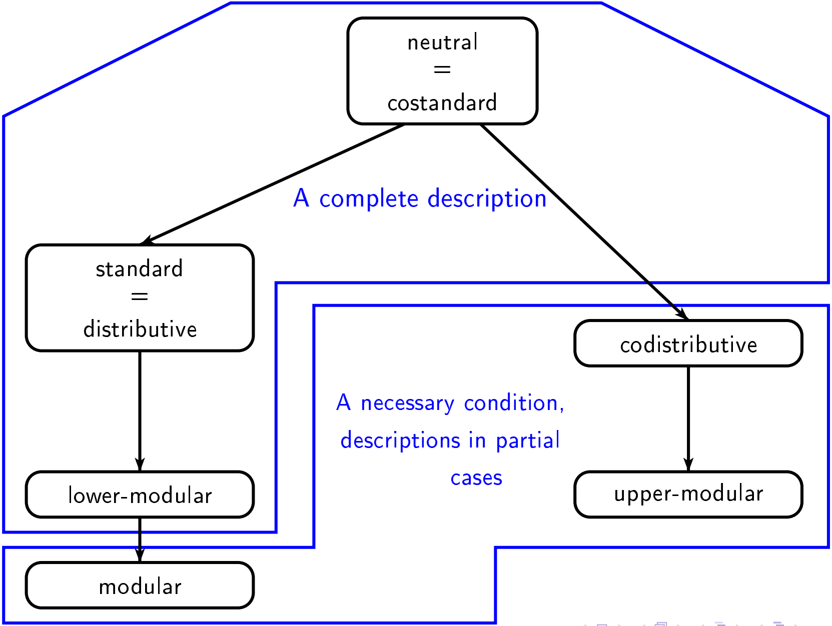
x is a *lower-modular* element
$\forall y, z\colon x \le y \to (z \vee x) \wedge y = (z \wedge y) \vee x$

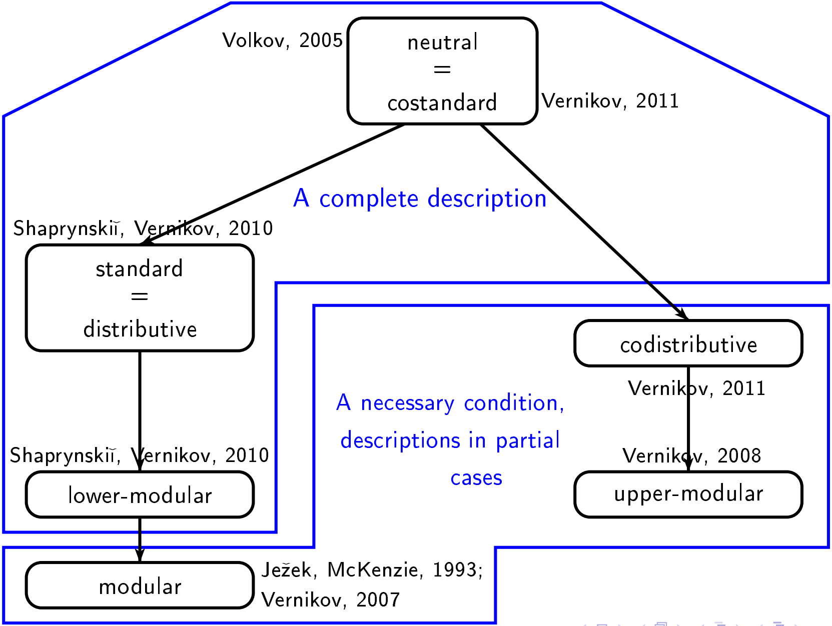
x is an *upper-modular* element
$\forall y, z\colon y \le x \to (z \vee y) \wedge x = (z \wedge y) \vee x$

x is a *modular* element
$\forall y, z\colon y \le z \to (x \vee y) \wedge z = (x \wedge z) \vee y$

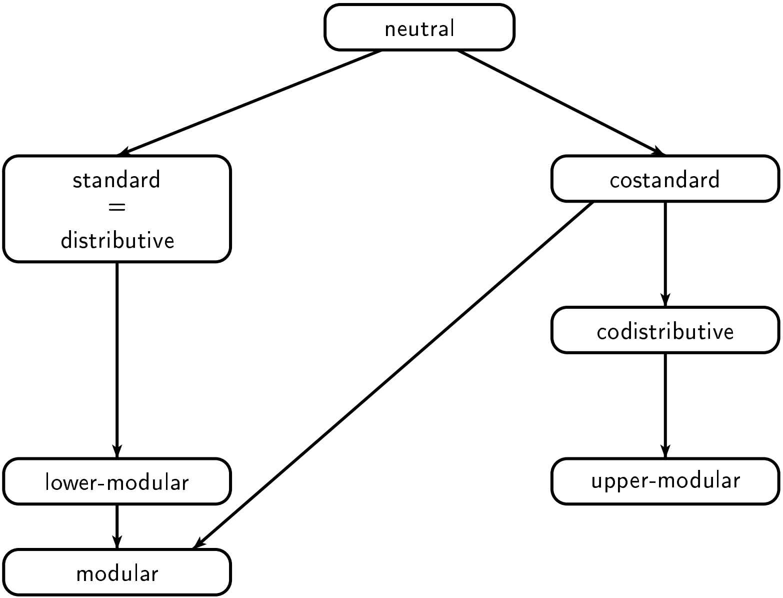
The lattice SEM of all semigroup varieties



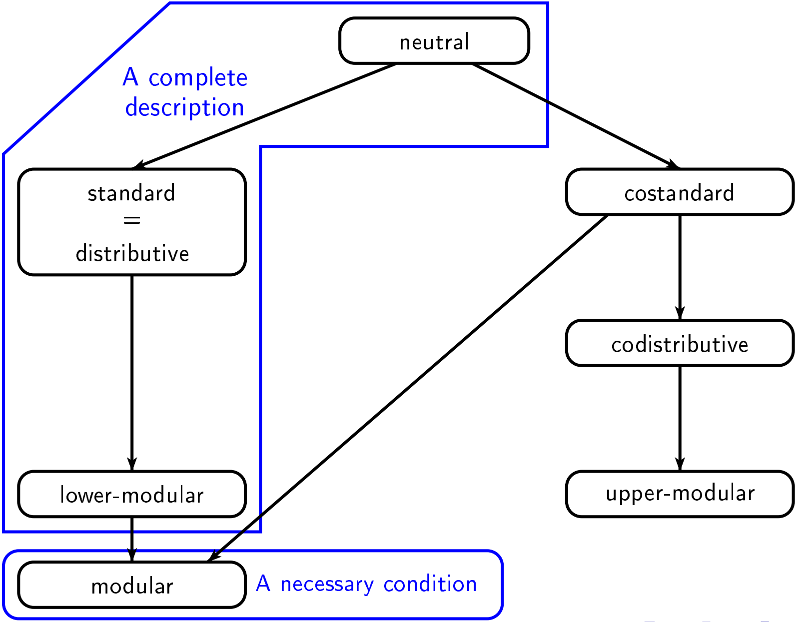

neutral
=
costandard
A complete description
standard
=
distributive
codistributive
A necessary condition, descriptions in partial cases
lower-modular
upper-modular
modular

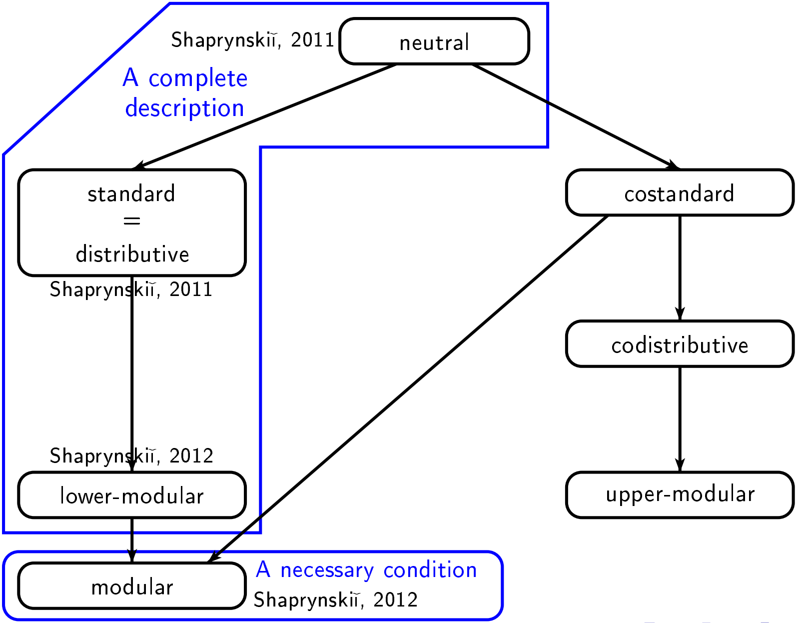

Volkov, 2005
neutral
=
costandard
Vernikov, 2011
A complete description
Shaprynskiĭ, Vernikov, 2010
standard
=
distributive
codistributive
Vernikov, 2011
A necessary condition,
descriptions in partial
cases
Shaprynskiĭ, Vernikov, 2010
Vernikov, 2008
lower-modular
upper-modular
modular
Ježek, McKenzie, 1993;
Vernikov, 2007

The lattice Com of commutative semigroup varieties (6 months ago)

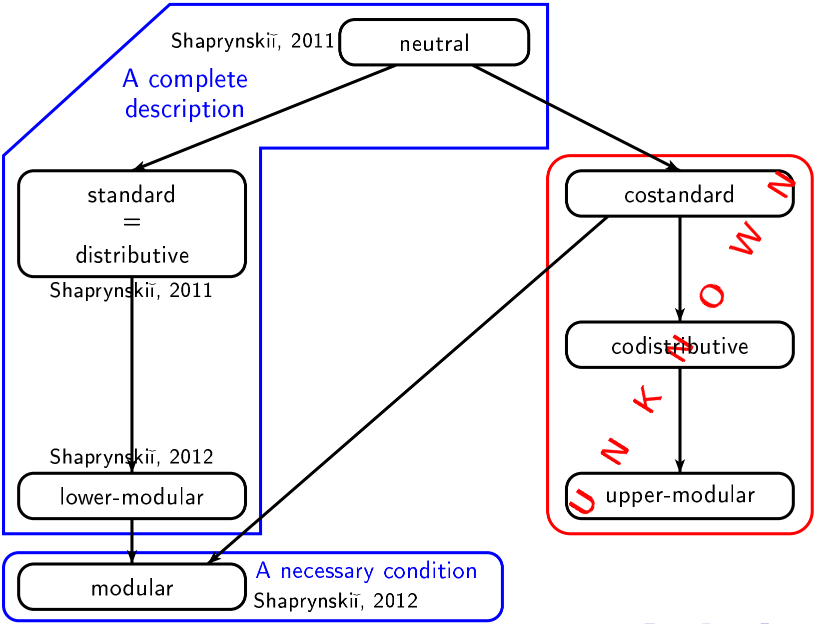


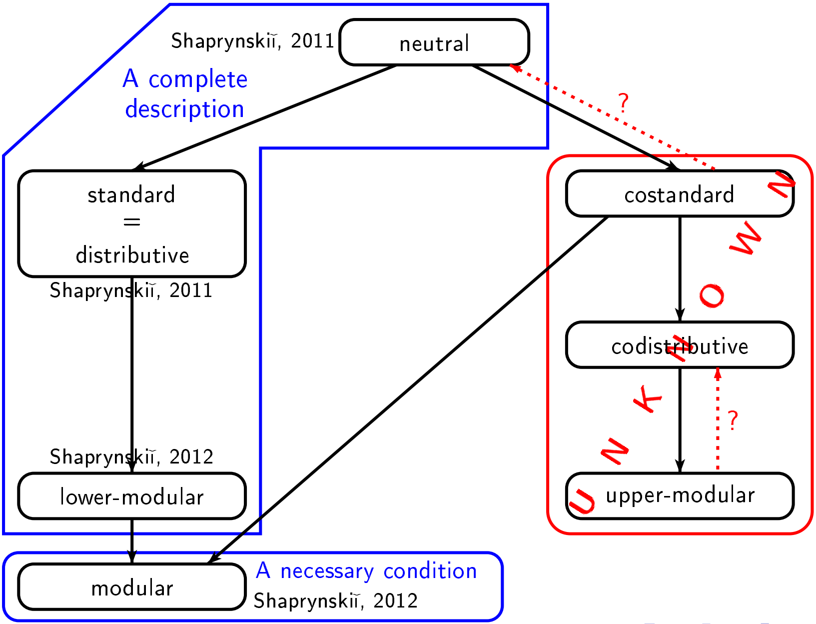
The lattice Com of commutative semigroup varieties (6 months ago)

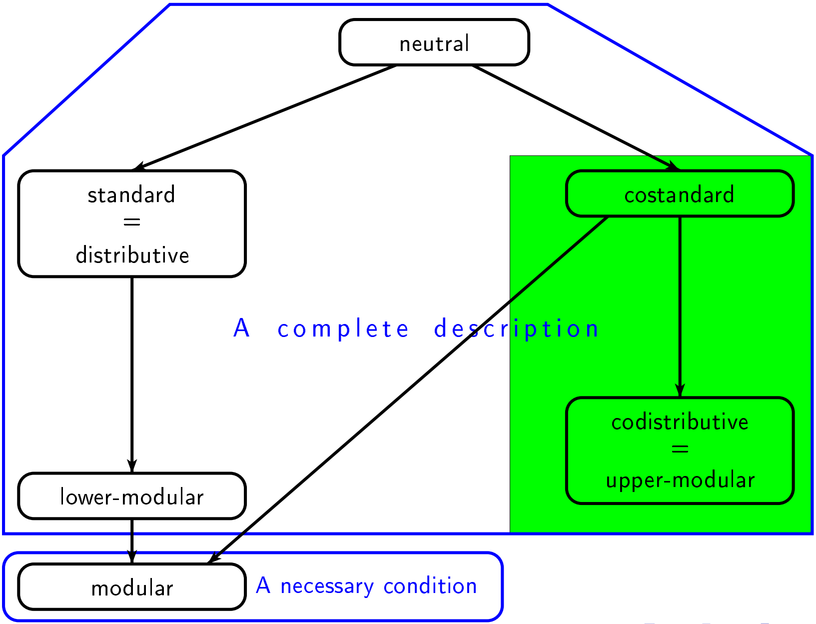



Shaprynskiĭ, 2011
neutral
A complete description
standard = distributive
costandard
Shaprynskiĭ, 2011
codistributive
Shaprynskiĭ, 2012
lower-modular
upper-modular
modular
A necessary condition
Shaprynskiĭ, 2012

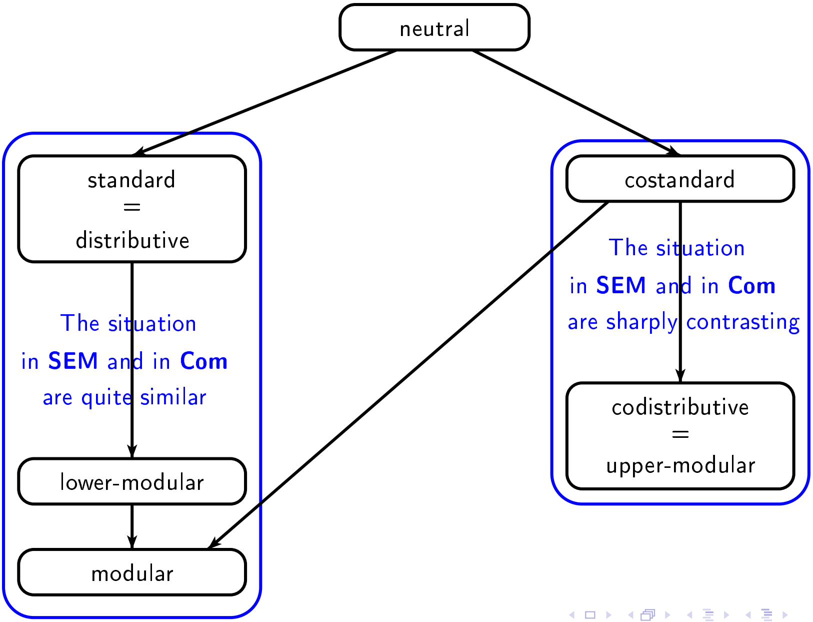
The lattice Com of commutative semigroup varieties (6 months ago)




Shaprynskiĭ, 2011
neutral
A complete description
standard = distributive
Shaprynskiĭ, 2011
Shaprynskiĭ, 2012
lower-modular
modular
A necessary condition
Shaprynskiĭ, 2012
costandard
codistributive
upper-modular
?
?
UNKNOWN


neutral
standard
=
distributive
costandard
A complete description
codistributive
=
upper-modular
lower-modular
modular
A necessary condition

The lattice Com (now)



In both the lattices **SEM** and **Com**:

- an element is distributive if and only if it is standard;
- an element is modular whenever it is lower-modular.

Results concerning modular or lower-modular or distributive (= standard) elements in these two lattices have quite similar formulations. For instance:

Lower-modular elements in SEM (Shaprynskiĭ and Vernikov, 2010)

A semigroup variety $\mathcal{V}$ is a lower-modular element in **SEM** $\Longleftrightarrow$ *either $\mathcal{V}$ is the variety $\mathcal{SEM}$ of all semigroups or $\mathcal{V} = \mathcal{N}$ or $\mathcal{V} = \mathcal{SL} \vee \mathcal{N}$ where $\mathcal{N}$ is given by identities of the form $w = 0$ only, and $\mathcal{SL}$ is the variety of semilattices.*

Lower-modular elements in in Com (Shaprynskiĭ, 2012)

A commutative semigroup variety $\mathcal{V}$ is a lower-modular element in **Com** $\Longleftrightarrow$ *either $\mathcal{V}$ is the variety $\mathcal{COM}$ of all commutative semigroups or $\mathcal{V} = \mathcal{N}$ or $\mathcal{V} = \mathcal{SL} \vee \mathcal{N}$ where $\mathcal{N}$ is given within $\mathcal{COM}$ by identities of the form $w = 0$ only, and $\mathcal{SL}$ is the variety of semilattices.*

There are other pairs of analog statements of such a kind (for instance, necessary conditions for modular elements in **SEM** and **Com**).

Special elements in SEM and Com: the similarities

In both the lattices **SEM** and **Com**:

- an element is distributive if and only if it is standard;
- an element is modular whenever it is lower-modular.

Results concerning modular or lower-modular or distributive (= standard) elements in these two lattices have quite similar formulations. For instance:

Lower-modular elements in SEM (Shaprynskiĭ and Vernikov, 2010)

A semigroup variety $\mathcal{V}$ is a lower-modular element in **SEM** $\Longleftrightarrow$ *either $\mathcal{V}$ is the variety $\mathcal{SEM}$ of all semigroups or $\mathcal{V} = \mathcal{N}$ or $\mathcal{V} = \mathcal{SL} \vee \mathcal{N}$ where $\mathcal{N}$ is given by identities of the form $w = 0$ only, and $\mathcal{SL}$ is the variety of semilattices.*

Lower-modular elements in in Com (Shaprynskiĭ, 2012)

A commutative semigroup variety $\mathcal{V}$ is a lower-modular element in **Com** $\Longleftrightarrow$ *either $\mathcal{V}$ is the variety $\mathcal{COM}$ of all commutative semigroups or $\mathcal{V} = \mathcal{N}$ or $\mathcal{V} = \mathcal{SL} \vee \mathcal{N}$ where $\mathcal{N}$ is given within $\mathcal{COM}$ by identities of the form $w = 0$ only, and $\mathcal{SL}$ is the variety of semilattices.*

There are other pairs of analog statements of such a kind (for instance, necessary conditions for modular elements in **SEM** and **Com**).

In both the lattices **SEM** and **Com**:

- an element is distributive if and only if it is standard;
- an element is modular whenever it is lower-modular.

Results concerning modular or lower-modular or distributive (= standard) elements in these two lattices have quite similar formulations. For instance:

Lower-modular elements in SEM (Shaprynskiĭ and Vernikov, 2010)

A semigroup variety $\mathcal{V}$ is a lower-modular element in **SEM** $\Longleftrightarrow$ *either $\mathcal{V}$ is the variety $\mathcal{SEM}$ of all semigroups or $\mathcal{V} = \mathcal{N}$ or $\mathcal{V} = \mathcal{SL} \vee \mathcal{N}$ where $\mathcal{N}$ is given by identities of the form $w = 0$ only, and $\mathcal{SL}$ is the variety of semilattices.*

Lower-modular elements in in Com (Shaprynskiĭ, 2012)

A commutative semigroup variety $\mathcal{V}$ is a lower-modular element in **Com** $\Longleftrightarrow$ *either $\mathcal{V}$ is the variety $\mathcal{COM}$ of all commutative semigroups or $\mathcal{V} = \mathcal{N}$ or $\mathcal{V} = \mathcal{SL} \vee \mathcal{N}$ where $\mathcal{N}$ is given within $\mathcal{COM}$ by identities of the form $w = 0$ only, and $\mathcal{SL}$ is the variety of semilattices.*

There are other pairs of analog statements of such a kind (for instance, necessary conditions for modular elements in **SEM** and **Com**).

Special elements in SEM and Com: the similarities

In both the lattices **SEM** and **Com**:

- an element is distributive if and only if it is standard;
- an element is modular whenever it is lower-modular.

Results concerning modular or lower-modular or distributive (= standard) elements in these two lattices have quite similar formulations. For instance:

Lower-modular elements in SEM (Shaprynskiĭ and Vernikov, 2010)

A semigroup variety $\mathcal{V}$ is a lower-modular element in **SEM** $\Longleftrightarrow$ *either $\mathcal{V}$ is the variety $\mathcal{SEM}$ of all semigroups or $\mathcal{V} = \mathcal{N}$ or $\mathcal{V} = \mathcal{SL} \vee \mathcal{N}$ where $\mathcal{N}$ is given by identities of the form $w = 0$ only, and $\mathcal{SL}$ is the variety of semilattices.*

Lower-modular elements in in Com (Shaprynskiĭ, 2012)

A commutative semigroup variety $\mathcal{V}$ is a lower-modular element in **Com** $\Longleftrightarrow$ *either $\mathcal{V}$ is the variety $\mathcal{COM}$ of all commutative semigroups or $\mathcal{V} = \mathcal{N}$ or $\mathcal{V} = \mathcal{SL} \vee \mathcal{N}$ where $\mathcal{N}$ is given within $\mathcal{COM}$ by identities of the form $w = 0$ only, and $\mathcal{SL}$ is the variety of semilattices.*

There are other pairs of analog statements of such a kind (for instance, necessary conditions for modular elements in **SEM** and **Com**).

In the lattice **SEM**:

- an element is neutral if and only if it is costandard;
- the properties of being upper-modular and codistributive elements are not equivalent.

In the lattice **Com**:

- the properties of being neutral and costandard elements are not equivalent;
- an element is upper-modular if and only if it is codistributive.

In the lattice **SEM**:

- an element is neutral if and only if it is costandard;
- the properties of being upper-modular and codistributive elements are not equivalent.

In the lattice **Com**:

- the properties of being neutral and costandard elements are not equivalent;
- an element is upper-modular if and only if it is codistributive.

$$\mathcal{I} = \mathrm{var}\{x^2yz = 0,\ x^2y = xy^2,\ xy = yx\},\quad \mathcal{J} = \mathrm{var}\{x^2y = 0,\ xy = yx\}$$

If $\mathcal{X} \subseteq \mathcal{I}$ then $\mathcal{X}$ is given within $\mathcal{I}$

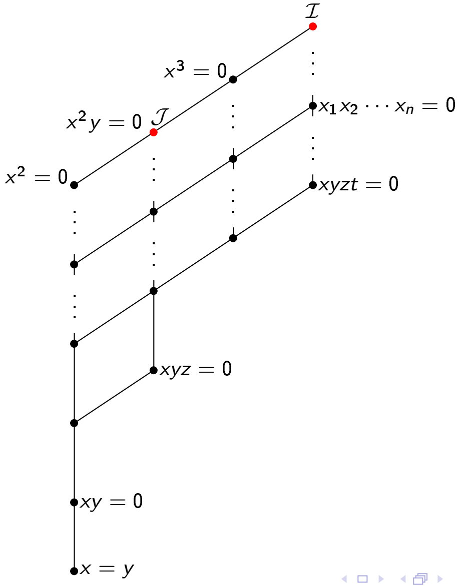
- either by the identity $w = 0$ where $w \in \{x^2, x^3, x^2y\}$,
- or by the identity $x_1x_2 \cdots x_n = 0$ for some n,
- or by these two identities together.

$$\mathcal{I} = \text{var}\{x^2yz = 0,\ x^2y = xy^2,\ xy = yx\}, \quad \mathcal{J} = \text{var}\{x^2y = 0,\ xy = yx\}$$

If $\mathcal{X} \subseteq \mathcal{I}$ then $\mathcal{X}$ is given within $\mathcal{I}$

- either by the identity $w = 0$ where $w \in \{x^2, x^3, x^2y\}$,
- or by the identity $x_1x_2 \cdots x_n = 0$ for some n,
- or by these two identities together.

$$\mathcal{I} = \mathrm{var}\{x^2yz = 0,\ x^2y = xy^2,\ xy = yx\}, \quad \mathcal{J} = \mathrm{var}\{x^2y = 0,\ xy = yx\}$$



Main results

Upper-modular and codistributive elements in Com

For a commutative semigroup variety $\mathcal{V}$, the following are equivalent:

a) *$\mathcal{V}$ is an upper-modular element of* **Com***;*

b) *$\mathcal{V}$ is a codistributive element of* **Com***;*

c) *one of the following holds:*

   (i) $\mathcal{V} = \mathcal{COM}$;
   (ii) $\mathcal{V} \subseteq \mathcal{A} \vee \mathcal{SL} \vee \mathcal{I}$ *where $\mathcal{A}$ is an Abelian periodic group variety;*
   (iii) $\mathcal{V} \subseteq \mathcal{C} \vee \mathcal{J}$ *where* $\mathcal{C} = \mathrm{var}\{x^2 = x^3, xy = yx\}$.

Costandard elements in Com

A commutative semigroup variety $\mathcal{V}$ is a costandard element of **Com** *if and only if either $\mathcal{V} = \mathcal{COM}$ or $\mathcal{V} \subseteq \mathcal{SL} \vee \mathcal{I}$.*

For comparison: $\mathcal{V}$ is a neutral element of **Com** if and only if either $\mathcal{V} = \mathcal{COM}$ or $\mathcal{V} \subseteq \mathcal{SL} \vee \mathcal{J}$ (Shaprynskiĭ, 2011).

Main results

Upper-modular and codistributive elements in Com

For a commutative semigroup variety $\mathcal{V}$, the following are equivalent:

a) *$\mathcal{V}$ is an upper-modular element of* **Com**;

b) *$\mathcal{V}$ is a codistributive element of* **Com**;

c) *one of the following holds:*

  (i) $\mathcal{V} = \mathcal{COM}$;

  (ii) $\mathcal{V} \subseteq \mathcal{A} \vee \mathcal{SL} \vee \mathcal{I}$ *where $\mathcal{A}$ is an Abelian periodic group variety;*

  (iii) $\mathcal{V} \subseteq \mathcal{C} \vee \mathcal{J}$ *where* $\mathcal{C} = \mathrm{var}\{x^2 = x^3, xy = yx\}$.

Costandard elements in Com

A commutative semigroup variety $\mathcal{V}$ is a costandard element of **Com** *if and only if either $\mathcal{V} = \mathcal{COM}$ or $\mathcal{V} \subseteq \mathcal{SL} \vee \mathcal{I}$.*

For comparison: $\mathcal{V}$ is a neutral element of **Com** if and only if either $\mathcal{V} = \mathcal{COM}$ or $\mathcal{V} \subseteq \mathcal{SL} \vee \mathcal{J}$ (Shaprynskiĭ, 2011).

Main results

Upper-modular and codistributive elements in Com

For a commutative semigroup variety $\mathcal{V}$, the following are equivalent:

a) *$\mathcal{V}$ is an upper-modular element of* **Com***;*

b) *$\mathcal{V}$ is a codistributive element of* **Com***;*

c) *one of the following holds:*

  (i) $\mathcal{V} = \mathcal{COM}$;

  (ii) $\mathcal{V} \subseteq \mathcal{A} \vee \mathcal{SL} \vee \mathcal{I}$ *where $\mathcal{A}$ is an Abelian periodic group variety;*

  (iii) $\mathcal{V} \subseteq \mathcal{C} \vee \mathcal{J}$ *where* $\mathcal{C} = \mathrm{var}\{x^2 = x^3, xy = yx\}$.

Costandard elements in Com

A commutative semigroup variety $\mathcal{V}$ is a costandard element of **Com** *if and only if either $\mathcal{V} = \mathcal{COM}$ or $\mathcal{V} \subseteq \mathcal{SL} \vee \mathcal{I}$.*

For comparison: $\mathcal{V}$ is a neutral element of **Com** if and only if either $\mathcal{V} = \mathcal{COM}$ or $\mathcal{V} \subseteq \mathcal{SL} \vee \mathcal{J}$ (Shaprynskiĭ, 2011).

Corollaries

Upper modularity in Com and SEM

A commutative semigroup variety $\mathcal{V}$ with $\mathcal{V} \neq \mathcal{COM}$ is an upper-modular element of **Com** *if and only if it is an upper-modular element of* **SEM**.

A nil-case

For a commutative nil-variety of semigroups $\mathcal{N}$, the following are equivalent:

a) *$\mathcal{N}$ is an upper-modular element of* **Com**;
b) *$\mathcal{N}$ is a codistributive element of* **Com**;
c) *$\mathcal{N}$ is a costandard element of* **Com**;
d) *$\mathcal{V} \subseteq \mathcal{I}$.*

A hereditary property

If a commutative semigroup variety $\mathcal{V}$ is an upper-modular element of **Com** *and $\mathcal{V} \neq \mathcal{COM}$ then every subvariety of $\mathcal{V}$ is an upper-modular element of* **Com**.

Distributivity of subvariety lattice

If a commutative semigroup variety $\mathcal{V}$ is an upper-modular element of **Com** *and $\mathcal{V} \neq \mathcal{COM}$ then the subvariety lattice of $\mathcal{V}$ is distributive.*

Upper modularity in Com and SEM

A commutative semigroup variety $\mathcal{V}$ with $\mathcal{V} \neq \mathcal{COM}$ is an upper-modular element of **Com** *if and only if it is an upper-modular element of* **SEM**.

A nil-case

For a commutative nil-variety of semigroups $\mathcal{N}$, the following are equivalent:

a) *$\mathcal{N}$ is an upper-modular element of* **Com**;

b) *$\mathcal{N}$ is a codistributive element of* **Com**;

c) *$\mathcal{N}$ is a costandard element of* **Com**;

d) $\mathcal{V} \subseteq \mathcal{I}$.

A hereditary property

If a commutative semigroup variety $\mathcal{V}$ is an upper-modular element of **Com** *and $\mathcal{V} \neq \mathcal{COM}$ then every subvariety of $\mathcal{V}$ is an upper-modular element of* **Com**.

Distributivity of subvariety lattice

If a commutative semigroup variety $\mathcal{V}$ is an upper-modular element of **Com** *and $\mathcal{V} \neq \mathcal{COM}$ then the subvariety lattice of $\mathcal{V}$ is distributive.*

Upper modularity in Com and SEM

A commutative semigroup variety $\mathcal{V}$ with $\mathcal{V} \neq \mathcal{COM}$ is an upper-modular element of **Com** *if and only if it is an upper-modular element of* **SEM**.

A nil-case

For a commutative nil-variety of semigroups $\mathcal{N}$, the following are equivalent:

a) *$\mathcal{N}$ is an upper-modular element of* **Com**;

b) *$\mathcal{N}$ is a codistributive element of* **Com**;

c) *$\mathcal{N}$ is a costandard element of* **Com**;

d) $\mathcal{V} \subseteq \mathcal{I}$.

A hereditary property

If a commutative semigroup variety $\mathcal{V}$ is an upper-modular element of **Com** *and $\mathcal{V} \neq \mathcal{COM}$ then every subvariety of $\mathcal{V}$ is an upper-modular element of* **Com**.

Distributivity of subvariety lattice

If a commutative semigroup variety $\mathcal{V}$ is an upper-modular element of **Com** *and $\mathcal{V} \neq \mathcal{COM}$ then the subvariety lattice of $\mathcal{V}$ is distributive.*

Upper modularity in Com and SEM

A commutative semigroup variety $\mathcal{V}$ with $\mathcal{V} \neq \mathcal{COM}$ is an upper-modular element of **Com** *if and only if it is an upper-modular element of* **SEM**.

A nil-case

For a commutative nil-variety of semigroups $\mathcal{N}$, the following are equivalent:

a) *$\mathcal{N}$ is an upper-modular element of* **Com**;
b) *$\mathcal{N}$ is a codistributive element of* **Com**;
c) *$\mathcal{N}$ is a costandard element of* **Com**;
d) $\mathcal{V} \subseteq \mathcal{I}$.

A hereditary property

If a commutative semigroup variety $\mathcal{V}$ is an upper-modular element of **Com** *and $\mathcal{V} \neq \mathcal{COM}$ then every subvariety of $\mathcal{V}$ is an upper-modular element of* **Com**.

Distributivity of subvariety lattice

If a commutative semigroup variety $\mathcal{V}$ is an upper-modular element of **Com** *and $\mathcal{V} \neq \mathcal{COM}$ then the subvariety lattice of $\mathcal{V}$ is distributive.*