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Permutability of fully invariant congruences on relatively free semigroups

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Abstract. We describe semigroup varieties, on whose relatively free members either all fully invariant congruences or all fully invariant congruences contained in the least semilattice congruence permute.

0. Introduction and summary

Let α and β be congruences of an algebra A. By $\alpha\beta$ we denote their relational product, that is, the relation

$$\{(a, b) \in A \times A \mid a\ \alpha\ c \text{ and } c\ \beta\ b \text{ for some } c \in A\}.$$

The congruences α and β are said to *permute* if $\alpha\beta = \beta\alpha$. It is well-known that, exactly in this case, $\alpha\beta$ is again a congruence on A that coincides with the *lattice join* $\alpha \vee \beta$ of α and β, that is, the least congruence containing both α and β.

The family of *congruence permutable varieties* (that is, varieties in whose algebras all congruences permute) is very rich and important. Unfortunately, proper semigroup varieties fail to belong to this noble family. Saying so we refer to the following well-known result that first appeared in 1960 in E. J. Tully's dissertation [14], see also his report [15]: every congruence permutable semigroup variety must consist entirely of groups. This result was then rediscovered and strengthened several times (see, for instance, [2, p. 35], [4, Theorem 1.2(iii)], [7, Corollary 0]), and

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it was shown that imposing even a weaker form of the congruence permutability on a semigroup variety leads to the same discouraging conclusion. However, if we restrict to *fully invariant* congruences, the situation considerably improves since there already exist interesting varieties in whose semigroups all fully invariant congruences are permutable. For example, F. Pastijn [9, Theorem 4] has observed that fully invariant congruences on completely simple semigroups permute.

Considering fully invariant congruences is most natural for *relatively free* semigroups. Indeed, if $\mathcal{V}$ is a semigroup variety and $F_{\mathcal{V}}(X)$ is the $\mathcal{V}$-free semigroup over an infinite set X, then the lattice of fully invariant congruences on $F_{\mathcal{V}}(X)$ is known to be dually isomorphic to the lattice $L(\mathcal{V})$ of all subvarieties of the variety $\mathcal{V}$ (see [2, p. 33]). Thus any "positive" information about the fully invariant congruences on $F_{\mathcal{V}}(X)$ contributes to clarifying the structure of the lattice $L(\mathcal{V})$. In particular, the permutability of fully invariant congruences on relatively free semigroups reflects in the very important arguesian property of the corresponding subvariety lattice. (Recall that any lattice of permuting equivalences is arguesian [5] and that the class of arguesian lattices is self-dual [6].) There have been several papers (see [9], [10], [20]), where the approach via the permutability of (not necessarily all) fully invariant congruences on relatively free semigroups has been successfully employed to prove that certain lattices of semigroup varieties enjoy the arguesian property. The objective of the present paper is to determine all semigroup varieties that, roughly speaking, admit the approach. First, we describe varieties, on whose relatively free members all fully invariant congruences permute. In the following formulation of this description we adopt the usual agreement of writing $w = 0$ as a short form of the identity system $wu = uw = w$ where u runs over the set of all words.

Theorem 1. *Let $\mathcal{V}$ be a semigroup variety. The fully invariant congruences on all relatively free members of $\mathcal{V}$ permute if and only if $\mathcal{V}$ either consists of completely simple semigroups, or coincides with the variety $\mathcal{SL}$ of all semilattices, or satisfies one of the following identity systems:*

(0.1) $$xyz = 0;$$

(0.2) $$x^2 = 0,\ xyz = yxz;$$

(0.3) $$x^2 = 0,\ xyz = xzy;$$

(0.4) $$x^2 = 0,\ xyz = yzx;$$

(0.5) $$x^2 = 0,\ xyz = zyx,\ xyx = 0;$$

(0.6) $$x^2 = 0,\ xyz = zyx,\ xyzt = 0.$$

We also analyze a larger class of varieties arising from the following considerations. On every semigroup S, the least congruence σ such that S/σ is a semilattice is fully invariant. It is pretty easy to observe (see Lemma 1.5 below) that, whenever S is relatively free, σ cannot commute with every fully invariant congruence on S except the extreme cases when σ is either the universal relation δ or the equality relation ε on S. The good news is that this bad behavior of the least semilattice congruence does not play any negative role in applications of the permutability of fully invariant congruences to the lattice identities in subvariety lattices. Indeed, for any semigroup variety $\mathcal{V}$ containing the variety $\mathcal{SL}$, the lattice $L(\mathcal{V})$ is known to be a subdirect product of the two-element chain and the interval $[\mathcal{SL}, \mathcal{V}]$ of all varieties between $\mathcal{SL}$ and $\mathcal{V}$ (this follows from a result in [8], for instance). This interval is dually isomorphic to the interval $[\varepsilon, \sigma]$ of the lattice of fully invariant congruences on the $\mathcal{V}$-free semigroup over an infinite set. Hence, even the permutability of fully invariant congruences under σ already guarantees the arguesian property in the whole lattice $L(\mathcal{V})$. Taking this motivation into account, we look for semigroup varieties, on whose relatively free members the fully invariant congruences under σ permute. To formulate a description of these varieties, we denote by $\operatorname{var}\Theta$ the variety of all semigroups satisfying the identity system Θ and by $\mathcal{X} \vee \mathcal{Y}$ the *lattice join* of the varieties $\mathcal{X}$ and $\mathcal{Y}$, that is, the least variety containing both $\mathcal{X}$ and $\mathcal{Y}$.

Theorem 2. *Let $\mathcal{V}$ be a semigroup variety. The fully invariant congruences under the least semilattice congruence on the relatively free members of $\mathcal{V}$ permute if and only if $\mathcal{V}$ satisfies one of the following conditions:*

1) *$\mathcal{V}$ consists of completely regular semigroups;*
2) *$\mathcal{V} = \mathcal{C} \vee \mathcal{N}$ where $\mathcal{C} = \operatorname{var}\{x^2 = x^3,\ xy = yx\}$ and $\mathcal{N}$ satisfies one of the identity systems* (0.2)–(0.5)*;*
3) *$\mathcal{V} \subseteq \mathcal{SL} \vee \mathcal{M}$ where $\mathcal{M}$ satisfies one of the identity systems* (0.1)–(0.6)*;*
4) *$\mathcal{V}$ coincides with one of the varieties $\mathcal{P} = \operatorname{var}\{xy = x^2y,\ x^2y^2 = y^2x^2\}$ or $\mathcal{P}^* = \operatorname{var}\{xy = xy^2,\ x^2y^2 = y^2x^2\}$.*

Theorems 1 and 2 are proved in Sections 1 and 2, respectively.

To conclude with the Introduction, we should clarify the relationship between the present paper and the earlier paper [17] by the second author. This earlier paper was basically aimed at the problem now solved by our Theorem 1, that is, at a description of varieties with permuting fully invariant congruences on relatively free semigroups. It contained a reduction of the problem to the nilsemigroup case, a proof of a (rather unexpected) result saying that the subvariety lattice of any nilsemigroup variety with the required property is distributive, and a complete classification of all varieties of nilsemigroups with distributive subvariety lattice. Then the claim was made that the fully invariant congruences on relatively free semigroups of every nilsemigroup variety with distributive subvariety lattice permute. As a consequence of this claim and the results mentioned above, a description of varieties with permuting fully invariant congruences on relatively free semigroups was formulated. Unfortunately, the claim is wrong, and hence the description in [17] is wrong too. It would be quite easy to fix the error thus deducing Theorem 1 of the present paper from the (correct part of the) results in [17]. However, for reader's convenience, we have preferred to make the present paper be self-contained and so to avoid using [17] at all.

1. Proof of Theorem 1

We start with a general remark which can be straightforwardly checked.

Lemma 1.1. *Let α, β, and ν be equivalences on a set S such that $\alpha, \beta \supseteq \nu$. Then α and β permute if and only if the equivalences α/ν and β/ν on the quotient set S/ν do so.*

Lemma 1.1 shows that, when studying permuting fully invariant congruences, we may consider congruences on the absolutely free semigroup X^+ that contain the fully invariant congruence ν on X^+ corresponding to a variety $\mathcal{V}$ instead of congruences on the relatively free semigroup $F_{\mathcal{V}}(X) \cong X^+/\nu$. This is convenient for it is sometimes easier to deal with elements of X^+ (that is, words) than with elements of an arbitrary relatively free semigroup. Lemma 1.1 also implies that, for a fixed semigroup variety $\mathcal{V}$, the fully invariant congruences on all relatively free members of $\mathcal{V}$ permute if and only if the fully invariant congruences on the $\mathcal{V}$-free semigroups do so, a formally weaker requirement.

Recall that a semigroup variety $\mathcal{V}$ is said to be an *atom* if it is minimal non-trivial. A classification of the atoms is well-known (see, for example, [2, Section IV]). Here we shall deal with the two following atoms: the variety $\mathcal{SL}$ of all semilattices and the variety $\mathcal{ZM}$ of all zero multiplication semigroups. We recall the well-known solution to the word problem in these varieties. For a word w, let $c(w)$ denote the set of all letters occurring in w and $\ell(w)$ the length of w.

Lemma 1.2.
(i) *The identity $u = v$ holds in the variety $\mathcal{SL}$ if and only if $c(u) = c(v)$.*
(ii) *The identity $u = v$ holds in the variety $\mathcal{ZM}$ if and only if $\ell(u), \ell(v) \geq 2$ or u and v coincide with the same letter.*

The following result is also a part of the semigroup variety folklore. In fact, it easily follows from Lemma 1.2(ii):

Lemma 1.3. *A variety consists of completely regular semigroups if and only if it does not contain the variety $\mathcal{ZM}$.*

Every non-trivial semigroup variety contains an atom. A non-trivial variety is called *precomplete* if it contains a unique atom. If $\mathcal{V}$ is a precomplete variety and $\mathcal{A}$ is the atom contained in $\mathcal{V}$, then $\mathcal{V}$ is said to be $\mathcal{A}$*-precomplete.*

Lemma 1.4. ([1], [13]).
(i) *A variety $\mathcal{V}$ is $\mathcal{SL}$-precomplete if and only if $\mathcal{V} = \mathcal{SL}$.*
(ii) *A variety $\mathcal{V}$ is $\mathcal{ZM}$-precomplete if and only if $\mathcal{V}$ consists of nilsemigroups.*

Now we are ready to start proving Theorem 1.

Lemma 1.5. *If the fully invariant congruences on relatively free semigroups in a variety $\mathcal{V}$ containing the atom $\mathcal{SL}$ permute, then $\mathcal{V} = \mathcal{SL}$.*

Proof. In view of Lemma 1.4(i), it suffices to show that $\mathcal{V}$ is $\mathcal{SL}$-precomplete. Arguing by contradiction, suppose that $\mathcal{V}$ contains, besides $\mathcal{SL}$, another atom $\mathcal{A}$. Let X be an infinite set and let α and σ denote the fully invariant congruence on the free semigroup X^+ corresponding to the varieties $\mathcal{A}$ and $\mathcal{SL}$, respectively. Since the intersection $\mathcal{SL} \cap \mathcal{A}$ is trivial, the fully invariant congruence $\sigma \vee \alpha$ corresponding to it must be equal to the universal relation δ on X^+.

By Lemma 1.1, σ and α permute whence already the product $\sigma\alpha$ is equal to δ. In particular, taking any two different elements $x, y \in X$, we should have $x \ \sigma\alpha \ y$, that is, $x \ \sigma \ w \ \alpha \ y$ for some word w. Since $x \ \sigma \ w$, Lemma 1.2(i) applies showing that $c(w) = \{x\}$, and therefore, w coincides with x^n for some positive integer n. Then $x^n \ \alpha \ y$. Substituting x for y, we deduce $x^n \ \alpha \ x$ and so $x \ \alpha \ y$, contradicting the fact that $\mathcal{A}$ is non-trivial.

■

Similarly, we obtain

Lemma 1.6. *If the fully invariant congruences on relatively free semigroups in a variety $\mathcal{V}$ containing the atom $\mathcal{ZM}$ permute, then $\mathcal{V}$ consists of nilsemigroups.*

Proof. By Lemma 1.4(ii), we have to prove that $\mathcal{V}$ is $\mathcal{ZM}$-precomplete. Suppose that $\mathcal{V}$ contains an atom $\mathcal{A}$ different from $\mathcal{ZM}$. Consider an infinite set X and let α and ζ denote the fully invariant congruences on X^+ corresponding to $\mathcal{A}$ and $\mathcal{ZM}$, respectively. As the intersection $\mathcal{ZM} \cap \mathcal{A}$ is trivial, the fully invariant congruence $\zeta \vee \alpha$ equals the universal relation δ. Since, by Lemma 1.1, ζ and α permute, we conclude that $\zeta\alpha = \delta$ whence, in particular, $x \;\zeta\alpha\; y$ for some different elements $x, y \in X$. This means that $x \;\zeta\; w \;\alpha\; y$ for some word w. By Lemma 1.2(ii), $x \;\zeta\; w$ implies that w must coincide with x. Now using $w \;\alpha\; y$, we conclude that $x \;\alpha\; y$, a contradiction.

■

Summarizing, we arrive to the following

Proposition 1.7. *If the fully invariant congruences on relatively free semigroups in a variety $\mathcal{V}$ permute, then $\mathcal{V}$ either consists of completely simple semigroups, or coincides with the variety $\mathcal{SL}$ of all semilattices, or consists of nilsemigroups.*

Proof. In view of Lemmas 1.5 and 1.6 we may assume that $\mathcal{V}$ contains neither $\mathcal{SL}$ nor $\mathcal{ZM}$. By Lemma 1.3, $\mathcal{V}$ consists of completely regular semigroups, that is, of semilattices of completely simple semigroups. As $\mathcal{V}$ contains only trivial semilattices, all its semigroups must be completely simple.

■

It is obvious that the fully invariant congruences on every relatively free semilattice permute for there are no more than two such fully invariant congruences: the equality and the universal relations. The fact from [9] that the fully invariant congruences on every (not necessarily relatively free) completely simple semigroup permute was already mentioned in the Introduction. Thus, in order to deduce Theorem 1 from Proposition 1.7, it remains to classify nilsemigroup varieties with permutable fully invariant congruences on relatively free members. We start moving towards the classification with an easy observation being a "fully invariant" version of an argument mentioned already in [15] (exactly this observation has been so unfortunately overseen in [17]).

Lemma 1.8. *Let the fully invariant congruences on a semigroup S permute. Then the fully invariant ideals in S form a chain under inclusion.*

Proof. For an ideal I in S, let ρ_I denote the corresponding Rees congruence. Obviously, ρ_I is a fully invariant congruence whenever the ideal I is fully invariant. Now suppose that I and J are non-comparable fully invariant ideals in S. We then show that the corresponding Rees congruences ρ_I and ρ_J do not permute. Indeed, take an element $s \in I \setminus J$ and an element $t \in J \setminus I$; clearly, $s \neq t$. The product st belongs to $I \cap J$ whence $s\ \rho_I\ st\ \rho_J\ t$, that is, $s\ \rho_I \rho_J\ t$. If ρ_I and ρ_J permute, we must have $s\ \rho_J\ u\ \rho_I\ t$ for some $u \in S$. Since $s \notin J$, the elements s and u can be ρ_J-related only if $s = u$. Analogously, since $t \notin I$, the elements t and u can be ρ_I-related only if $t = u$. Thus, $s = t$, a contradiction.

■

To successfully apply this lemma to nilsemigroup varieties, we need two technical remarks. The first of them is fairly obvious, and the second follows from [12, Lemma 1].

Lemma 1.9.

(i) *If a variety of nilsemigroups satisfies an identity $u = v$ with $c(u) \neq c(v)$, then it satisfies also the identity $u = 0$.*

(ii) *If a variety of nilsemigroups satisfies an identity $x_1 \cdots x_n = w$ with $\ell(w) > n$, then it satisfies also the identity $x_1 \cdots x_n = 0$.*

Now we are ready to describe restrictions holding in nilsemigroup varieties with permuting fully invariant congruences on relatively free semigroups.

Lemma 1.10. *Let $\mathcal{N}$ be a variety of nilsemigroups such that, on relatively free members of $\mathcal{N}$, the fully invariant congruences permute. Then $\mathcal{N}$ satisfies one of the identities*

(0.1) $$xyz = 0,$$

(1.1) $$x^2 = 0;$$

and one of the identities

(1.2) $$xyx = 0,$$

(1.3) $$xyzt = 0.$$

Proof. Let $X = \{x, y, z\}$ and consider in the free semigroup X^+ the fully invariant ideals I and J generated (as fully invariant ideals) by the words x^2 and xyz, respectively. The universal property of free semigroups guarantees that the images of I and J under the natural homomorphism from X^+ onto the $\mathcal{N}$-free semigroup $F_{\mathcal{N}}(X)$ over X are fully invariant ideals in $F_{\mathcal{N}}(X)$. By Lemma 1.8, these images must be comparable. If the image of I is contained in the image of J, then the variety $\mathcal{N}$ satisfies an identity of the kind $x^2 = u$ where $u \in J$. It should be clear that such an identity implies the identity (1.1). Indeed, if $c(u) \neq \{x\}$, we may refer to Lemma 1.9(i). If $c(u) = \{x\}$, that is, $u = x^n$ for some positive integer n, we make use of the fact that the length of every word in J is at least 3 (in fact, J is exactly the set of all words of length ≥ 3). In particular, $n \geq 3$ and, applying the identity $x^2 = x^n$ to itself, we obtain

$$x^2 = x^2 \cdot x^{n-2} = x^n \cdot x^{n-2} = x^2 \cdot x^{2(n-2)} = x^n \cdot x^{2(n-2)} = \dots$$

Since $\mathcal{N}$ is a nilsemigroup variety, $x^m = 0$ in $\mathcal{N}$ for all sufficiently large m, and therefore, $x^2 = 0$ in $\mathcal{N}$.

Now suppose that the image of J is contained in the image of I. In this case, the variety $\mathcal{N}$ satisfies an identity of the kind $xyz = v$ where $v \in I$. We aim to show that such an identity implies the identity (0.1). Taking into account Lemma 1.9(i), we may assume that $c(v) = \{x, y, z\}$. Since the word v belongs to the fully invariant ideal generated by x^2, it contains a square of a word as a subword. This implies that one of the letters appears in v at least twice, and therefore, $\ell(v) \geq 4$. Now Lemma 1.9(ii) applies.

The second claim of our lemma is obvious if $\mathcal{N}$ satisfies the identity (0.1). We therefore assume that $\mathcal{N}$ satisfies the identity (1.1). Let $X = \{x, y, z, t\}$ and denote by I and J the fully invariant ideals of X^+ generated by the words xyx and $xyzt$, respectively. Again Lemma 1.8 applies, showing that the images of I and J under the natural homomorphism from X^+ onto $F_{\mathcal{N}}(X)$ are comparable. If the image of I is contained in the image of J, then $\mathcal{N}$ satisfies an identity of the kind $xyx = u$ where $u \in J$. Such an identity implies (modulo the identity (1.1) holding in $\mathcal{N}$) the identity (1.2). Indeed, in view of Lemma 1.9(i), it suffices to consider the case $c(u) = \{x, y\}$. The length of every word in J is at least 4, and it is easy to check that any word of length 4 over two letters has a square of a word as a subword. Therefore, $u = 0$ in $\mathcal{N}$, and hence $xyx = 0$ in $\mathcal{N}$.

If the image of J is contained in the image of I, then $\mathcal{N}$ satisfies an identity of the kind $xyzt = v$ where $v \in I$. Such an identity implies the identity (1.3). First, by Lemma 1.9(i), we may assume that $c(v) = \{x, y, z, t\}$. As the word v belongs to the fully invariant ideal generated by xyx, it contains a subword of the form

$w_1w_2w_1$ whence one of the letters appears in v at least twice. Therefore, $\ell(v) \geq 5$ and Lemma 1.9(ii) applies.

■

Lemma 1.11. *Let $\mathcal{N}$ be a variety of nilsemigroups such that, on relatively free members of $\mathcal{N}$, the fully invariant congruences permute. Then $\mathcal{N}$ satisfies an identity of the kind*

$$x_1x_2x_3 = x_{1\pi}x_{2\pi}x_{3\pi}, \tag{1.4}$$

where π is a non-trivial permutation.

Proof. First, we note that it suffices to show that $\mathcal{N}$ satisfies a non-trivial identity of the form $xyz = w$. Indeed, if such an identity implies the identity (0.1), then it obviously implies (1.4) for every π. By Lemma 1.9, $xyz = w$ does not imply (0.1) only provided that $c(w) = \{x, y, z\}$ and $\ell(w) = 3$ but, in this case, the identity $xyz = w$ has exactly the form (1.4) for some non-trivial permutation π.

Looking for a contradiction, we now suppose that every identity of the form $xyz = w$ holding in $\mathcal{N}$ is trivial. Let $X = \{x_1, x_2, x_3\}$ and consider the $\mathcal{N}$-free semigroup $F_{\mathcal{N}}(X)$ over X. Denote by $W(3,3)$ the set $\{x_{1\pi}x_{2\pi}x_{3\pi}\}$ where π runs over the symmetric group $\mathbb{S}_3$. Our assumption implies that each element of $W(3,3)$ is equal in $F_{\mathcal{N}}(X)$ to no other word over X.

Now consider the fully invariant congruences α and β on $F_{\mathcal{N}}(X)$ generated by the pairs $(x_1x_2x_3, x_2x_1x_3)$ and $(x_1x_2x_3, x_1x_3x_2)$, respectively. Then the restrictions of α and β to the set $W(3,3)$ induce the following "benzol structure" on $W(3,3)$ (the doubled lines connect α-related elements while the single lines represent the relation β):

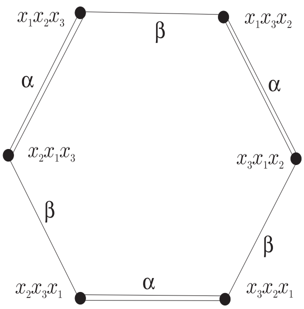

Figure 1. The restrictions of α and β to $W(3,3)$

Using this, we immediately observe that, say, the pair $(x_1x_2x_3, x_2x_3x_1)$ belongs to the product $\alpha\beta$ but fails to belong to the reverse product $\beta\alpha$. This yields the desired contradiction.

■

We are now well prepared to prove the following proposition which, as discussed above, completes the proof of the "only if" part of Theorem 1.

Proposition 1.12. *Let $\mathcal{N}$ be a variety of nilsemigroups such that, on relatively free members of $\mathcal{N}$, the fully invariant congruences permute. Then $\mathcal{N}$ satisfies one of the identity systems* (0.1)–(0.6).

Proof. By Lemma 1.10 $\mathcal{N}$ satisfies one of the identities (0.1) or (1.1). In the former case, we are done. In the latter case, we appeal to Lemma 1.11 which shows that $\mathcal{N}$ satisfies an identity of the kind (1.4) for some non-trivial permutation π of the set $\{1, 2, 3\}$. If π is one of the transpositions (12) or (23), then we conclude that $\mathcal{N}$ satisfies one of the identity systems (0.2) or (0.3), respectively. If π is one of the cycles (123) or (132), then $\mathcal{N}$ satisfies the identity system (0.4). The only subcase that remains is that when π is the transposition (13). By Lemma 1.10 $\mathcal{N}$ also satisfies one of the identities (1.2) or (1.3); this yields one of the identity systems (0.5) or (0.6), respectively.

■

In order to prove the "if" part of Theorem 1, we need a conversion of Proposition 1.12. Again we start with a pretty obvious technical observation. Namely, below we list the words w such that the identity $w = 0$ can fail in a variety $\mathcal{N}$ satisfying one of the identity systems (0.1)–(0.6), in which case, for brevity, we write $w \neq 0$ in $\mathcal{N}$. Clearly, this property does not depend on the names of the letters in the word w. We therefore call two words u and v *similar* (and write $u \approx v$) if u differs from v by the names of the letters only, and the list mentioned is given up to similarity. It is contained in the following table:

If $\mathcal{N}$ satisfies one of the identity systems:	then every word w such that $w \neq 0$ in $\mathcal{N}$ is similar to one of the words:
(0.1)	x, xy, x^2
(0.2), (0.3), (0.4), (0.5)	x, xy, $x_1x_2x_3$, ..., $x_1x_2\cdots x_n$, ...
(0.6)	x, xy, xyz, xyx

Table 1

Combining the information in Table 1 with Lemma 1.9, we immediately obtain the first statement of our next lemma.

Lemma 1.13. *Let Θ be one of the identity systems* (0.1)–(0.6)*, and let a variety $\mathcal{N}$ satisfy Θ. If $u = v$ is an identity of $\mathcal{N}$ such that $u \neq 0$ in $\mathcal{N}$, then $u \approx v$ and $c(u) = c(v)$. If, besides that, the identity $u = v$ does not follow from Θ, then the variety $\mathcal{N}$ satisfies every identity $u = w$ such that $u \approx w$ and $c(u) = c(w)$.*

Proof. It remains to verify the second claim of the lemma. If u is similar to one of the words x, xy, x^2, xyx, then there exists no more than one word w different from u and such that $u \approx w$ and $c(u) = c(w)$. This makes our claim trivial in all these cases. We may therefore assume that $u \approx x_1x_2\cdots x_n$ with $n \geq 3$. The identity to be analyzed is then equivalent to the identity

$$x_1x_2\cdots x_n = x_{1\tau}x_{2\tau}\cdots x_{n\tau}, \tag{1.5}$$

where τ is a permutation of the set $\{1, 2, \ldots, n\}$, and we have to show that the variety $\mathcal{N}$ satisfies every identity of this form provided that the particular identity (1.5) does not follow from the identity system Θ. It is well-known (see [11]) and easy to verify that, for every positive integer n and for every semigroup variety $\mathcal{V}$, the collection of all permutations ξ such that $\mathcal{V}$ satisfies the identity $x_1x_2\cdots x_n = x_{1\xi}x_{2\xi}\cdots x_{n\xi}$ forms a subgroup $\mathrm{Perm}_n(\mathcal{V})$ in the symmetric group $\mathbb{S}_n$. With this notation, our aim is to prove that $\mathrm{Perm}_n(\mathcal{N}) = \mathbb{S}_n$.

Recall that Θ contains an identity of the kind (1.4) for some non-trivial permutation π of the set $\{1, 2, 3\}$. If $n = 3$, then we can suppose that τ does not belong to the subgroup of $\mathbb{S}_3$ generated by π (otherwise the identity (1.5) would be a consequence of (1.4)). Now the claim follows from the obvious fact that $\mathbb{S}_3$ is generated by every pair of its non-trivial subgroups.

Let now $n \geq 4$. If the permutation π in the identity (1.4) belonging to Θ is either the transposition (13) or one of the cycles (123) or (132), then by [11, Theorem 1], (1.4) implies every identity of the kind (1.5) so our claim becomes trivial. Thus we assume that π is one of the transpositions (12) or (23). By symmetry, it suffices to consider either of the two possibilities, say the first one. Clearly, the identity $x_1x_2x_3 = x_2x_1x_3$ implies every identity of the form $x_1x_2\cdots x_n = x_{1\xi}x_{2\xi}\cdots x_{n\xi}$ such that $n\xi = n$. Since the identity (1.5) does not follow from Θ, the subgroup $\mathrm{Perm}_n(\mathcal{N})$ should strictly contain the stabilizer of n, that is the subgroup $\mathrm{Stab}(n) = \{\xi \in \mathbb{S}_n | \; n\xi = n\}$. However $\mathrm{Stab}(n)$ is obviously a maximal proper subgroup in $\mathbb{S}_n$ whence $\mathrm{Perm}_n(\mathcal{N}) = \mathbb{S}_n$ as desired.

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For any set X, we introduce a quasi-order $\triangleleft$ on the free semigroup X^+ by putting $u \triangleleft v$ if v contains a factor being a value of u under a substitution $X \to X^+$ (in other words, if v belongs to the fully invariant ideal of X^+ generated by u). We notice that the similarity relation as defined above is nothing but the equivalence associated with this quasi-order. Looking at Table 1, one can immediately observe that the words in each row of that table form, as they are listed, an increasing $\triangleleft$-chain. The last remark we need is a fairly easy consequence of this observation.

Lemma 1.14. *Let Θ be one of the identity systems* (0.1)–(0.6)*, and let $\mathcal{N}$ be a variety satisfying Θ. Then the fully invariant ideals of $\mathcal{N}$-free semigroups form a chain under inclusion.*

Proof. Every fully invariant ideal of the $\mathcal{N}$-free semigroup $F_{\mathcal{N}}(X)$ over a set X is generated (as an ideal) by all values of some collection of words from X^+. Of course, only words admitting a non-zero value in $F_{\mathcal{N}}(X)$ really come into the play, and moreover, any set $W \subset X^+$ produces the same fully invariant ideal as any cross-section of the set of all minimal elements of the ordered set $\langle W/\approx, \triangleleft\rangle$. Combining these arguments with the observation made before the formulation of the lemma, we conclude that every non-zero fully invariant ideal of $F_{\mathcal{N}}(X)$ is generated by all values of a word w listed in Table 1. Since $u \triangleleft v$ obviously implies that the fully invariant ideal produced by u contains that produced by v, the claim of our lemma follows.

■

Now we are ready to complete the proof of Theorem 1. In fact, we do this by proving a result which is much stronger than the promised conversion of Proposition 1.12.

Proposition 1.15. *Let Θ be one of the identity systems* (0.1)–(0.6)*, and let $\mathcal{N}$ be a variety satisfying Θ. Then the fully invariant congruences on $\mathcal{N}$-free semigroups permute, moreover, the product of any two fully invariant congruences coincides with their set-theoretical union.*

Proof. Let α and β be two arbitrary fully invariant congruences on the $\mathcal{N}$-free semigroup $F_{\mathcal{N}}(X)$ over a set X, and $a\ \alpha\beta\ b$. We have to show that the pair (a,b) belongs to either α or β.

Denote by I_α (respectively, I_β) the α-class (respectively, β-class) of 0. Clearly, I_α and I_β are fully invariant ideals in $F_{\mathcal{N}}(X)$. Now we consider 4 cases:

Case 1. $a \in I_\alpha,\ b \in I_\beta$. By Lemma 1.14, the ideals I_α and I_β are comparable whence both the elements a and b belong to the bigger of these ideals. Therefore $(a,b) \in \alpha$ or $(a,b) \in \beta$.

Case 2. $a \notin I_\alpha,\ b \in I_\beta$. There exists an element c such that $a\ \alpha\ c\ \beta\ b$. Let the words u and v be some preimages of a and c, respectively, by the canonical projection of X^+ onto $F_{\mathcal{N}}(X)$, and let $\mathcal{A}$ be the subvariety of $\mathcal{N}$ generated by the semigroup $F_{\mathcal{N}}(X)/\alpha$. Then the identity $u = v$ holds in $\mathcal{A}$ while $u \neq 0$ in $\mathcal{A}$. Now Lemma 1.13 applies, yielding, in particular, $u \approx v$. The corresponding renaming of the letters of v induces an automorphism of $F_{\mathcal{N}}(X)$ that sends c to a. Since $c\ \beta\ b$ and $b \in I_\beta$, we have $c \in I_\beta$ whence $a \in I_\beta$ for I_β is a fully invariant ideal. Therefore $(a,b) \in \beta$.

Case 3. $a \in I_\alpha,\ b \notin I_\beta$. This case is dual to the previous one.

Case 4. $a \notin I_\alpha,\ b \notin I_\beta$. There exists an element c such that $a\ \alpha\ c\ \beta\ b$. Similarly to Case 2, choose some words u, v, and w to be preimages of the elements a, c, and b, respectively, by the canonical projection of X^+ onto $F_{\mathcal{N}}(X)$ and denote by $\mathcal{A}$ and $\mathcal{B}$ the subvarieties of $\mathcal{N}$ generated by the semigroups $F_{\mathcal{N}}(X)/\alpha$ and $F_{\mathcal{N}}(X)/\beta$, respectively. Then the identity $u = v$ holds in $\mathcal{A}$ while $u \neq 0$ in $\mathcal{A}$, and the identity $w = v$ holds in $\mathcal{B}$ while $w \neq 0$ in $\mathcal{B}$. By Lemma 1.13, we may conclude that $u \approx v \approx w$ and $c(u) = c(v) = c(w)$. If the identity $u = v$ follows from the identity system Θ, then $a = c$, and therefore, $(a,b) \in \beta$. If, however, $u = v$ does not follow from Θ, the second claim of Lemma 1.13 applies showing that the variety $\mathcal{A}$ satisfies the identity $u = w$. This means that $(a,b) \in \alpha$. ∎

Thus, we have proved Theorem 1.

2. Proof of Theorem 2

As was explained in the Introduction, every semigroup variety $\mathcal{V}$ such that, on $\mathcal{V}$-free semigroups, fully invariant congruences under the least semilattice congruence permute has arguesian, and therefore, modular subvariety lattice. This suggests to deduce Theorem 2 from a complete description of semigroup varieties with modular subvariety lattice found by the second author. The description, however, is quite involved, and even its precise formulation (see [18, Theorems 1–3]) would require several pages. Fortunately, we do not need it in its full strength, and the following necessary condition of modularity will quite suffice for our purposes. (This necessary condition was basically found in [16] but here we formulate it in the form first published in [19].)

Proposition 2.1. [19, Theorem 1]. *Let $\mathcal{V}$ be a semigroup variety with modular subvariety lattice. Then $\mathcal{V}$ satisfies one of the following conditions:*

a) *for some $n > 1$, $\mathcal{V}$ satisfies one of the identities*

$$x^n y = xy, \tag{2.1}$$

$$xy^n = xy, \tag{2.2}$$

$$(xy)^n = xy; \tag{2.3}$$

b) $\mathcal{V} \subseteq \mathcal{N} \vee \mathcal{G} \vee \mathcal{C}$, *where $\mathcal{G}$ is a variety of periodic abelian groups, $\mathcal{C} = \mathrm{var}\{x^2 = x^3,\ xy = yx\}$, and $\mathcal{N}$ satisfies the identities*

$$x^2 y = xyx = yx^2 = 0 \tag{2.4}$$

and an identity of the kind

$$x_1 x_2 x_3 x_4 = x_{1\vartheta} x_{2\vartheta} x_{3\vartheta} x_{4\vartheta} \tag{2.5}$$

for a non-trivial even permutation ϑ;

c) $\mathcal{V} \subseteq \mathcal{M} \vee \mathcal{SL}$ *where $\mathcal{M}$ consists of nilsemigroups and satisfies an identity of the kind* (2.5).

■

Thus, looking for varieties $\mathcal{V}$ with the property that, on relatively free members of $\mathcal{V}$, the fully invariant congruences under the least semilattice congruence permute, we may select them among the varieties satisfying one of the conditions a)–c) of Proposition 2.1. Let us start this selection process with the varieties satisfying c):

Lemma 2.2. *If a semigroup variety $\mathcal{V}$ satisfies the condition* c) *of Proposition* 2.1 *and, on relatively free members of* $\mathcal{V}$*, the fully invariant congruences under the least semilattice congruence permute, then* $\mathcal{V}$ *satisfies the condition* 3) *of Theorem* 2*.*

Proof. We have $\mathcal{V} \subseteq \mathcal{M} \vee \mathcal{SL}$ for some variety $\mathcal{M}$ of nilsemigroups. It is known (see [8]) and easy to verify that the mapping $(\mathcal{X}, \mathcal{Y}) \mapsto \mathcal{X} \vee \mathcal{Y}$ is an isomorphism of the direct product $L(\mathcal{M}) \times L(\mathcal{SL})$ onto the lattice $L(\mathcal{M} \vee \mathcal{SL})$. In particular, $\mathcal{V} = \mathcal{X} \vee \mathcal{Y}$ for some $\mathcal{X} \subseteq \mathcal{M}$, $\mathcal{Y} \subseteq \mathcal{SL}$ whence $\mathcal{V} \subseteq \mathcal{X} \vee \mathcal{SL}$. In other words, we may assume that, in the inclusion $\mathcal{V} \subseteq \mathcal{M} \vee \mathcal{SL}$ we started with, $\mathcal{M}$ was a subvariety of $\mathcal{V}$. Then, on relatively free semigroups contained in $\mathcal{M}$, the fully invariant congruences under the least semilattice congruence permute. However $\mathcal{M}$ consists of nilsemigroups, and the least semilattice congruence on a nilsemigroup coincides with the universal relation. This means that $\mathcal{M}$ is a nilsemigroup variety with permuting fully invariant congruences on relatively free semigroups so Theorem 1 applies showing that $\mathcal{M}$ satisfies one of the identity systems (0.1)–(0.6). We have thus arrived at the condition 3) of Theorem 2.

■

Similarly, we can treat the varieties satisfying the condition b) of Proposition 2.1. However, let us first state an easy auxiliary result which will be useful also in studying the varieties satisfying a).

Lemma 2.3. *Let* $\mathcal{V}$ *be a semigroup variety such that, on relatively free members of* $\mathcal{V}$*, the fully invariant congruences contained in the least semilattice congruence permute. Then either* $\mathcal{V}$ *consists of completely regular semigroups or all completely simple semigroups in* $\mathcal{V}$ *are trivial.*

Proof. By Lemma 1.3, $\mathcal{V}$ either consists of completely regular semigroups (and we are done) or contains the variety $\mathcal{ZM}$ of all zero multiplication semigroups. In the second case, arguing by contradiction, suppose that $\mathcal{V}$ contains a non-trivial completely simple semigroup. Then $\mathcal{V}$ also contains a non-trivial variety $\mathcal{S}$ consisting entirely of completely simple semigroups. Then the variety join $\mathcal{ZM} \vee \mathcal{S} \subseteq \mathcal{V}$ contains only trivial semilattices, and the least semilattice congruence on a semigroup from $\mathcal{ZM} \vee \mathcal{S}$ coincides with the universal relation. Therefore, on relatively free members of $\mathcal{ZM} \vee \mathcal{S}$, all fully invariant congruences permute. By Lemma 1.6, a variety with the latter property consists of nilsemigroups whenever it contains $\mathcal{ZM}$. This forces $\mathcal{S}$ to be trivial, a contradiction.

■

Now we return to our selection procedure.

Lemma 2.4. *If a semigroup variety $\mathcal{V}$ satisfies the condition* b) *of Proposition* 2.1 *and, on relatively free members of $\mathcal{V}$, the fully invariant congruences under the least semilattice congruence permute, then $\mathcal{V}$ satisfies one of the conditions* 1)–3) *of Theorem* 2.

Proof. Since $\mathcal{C}$ contains the variety $\mathcal{C}_0 = \mathrm{var}\{xy = yx,\ x^2 = 0\}$ which obviously satisfies the identities (2.4) and every identity of the kind (2.5), we may without any loss assume that $\mathcal{N} \supseteq \mathcal{C}_0$. Let $\mathbf{C}_3$ denote the 3-element chain $\mathcal{T} \subset \mathcal{SL} \subset \mathcal{C}$, where $\mathcal{T}$ stands for the trivial variety. Consider in the direct product $L(\mathcal{N}) \times L(\mathcal{G}) \times \mathbf{C}_3$ the sublattice L consisting of all triples $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ such that $\mathcal{X}$ contains the variety $\mathcal{C}_0$ whenever $\mathcal{Z} = \mathcal{C}$. It was shown in [19, Section 2] that the mapping $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \mapsto \mathcal{X} \vee \mathcal{Y} \vee \mathcal{Z}$ is an isomorphism of the sublattice L onto the lattice $L(\mathcal{N} \vee \mathcal{G} \vee \mathcal{C})$. In particular, $\mathcal{V} = \mathcal{X} \vee \mathcal{Y} \vee \mathcal{Z}$ for some $\mathcal{X} \subseteq \mathcal{N}$, $Y \subseteq G$, $Z \in \{T, SL, C\}$.

If $\mathcal{V}$ contains a non-trivial group, then, by Lemma 2.3, $\mathcal{V}$ consists of completely regular semigroups and so satisfies the condition 1) of Theorem 2. We may therefore assume that the variety $\mathcal{Y}$ is trivial, and $\mathcal{V} = \mathcal{X} \vee \mathcal{Z}$. If $\mathcal{Z} \subseteq \mathcal{SL}$, we have $\mathcal{V} \subseteq \mathcal{X} \vee \mathcal{SL}$ where $\mathcal{X}$ is a variety of nilsemigroups satisfying an identity of the kind (2.5) for a non-trivial even permutation ϑ of the set $\{1, 2, 3, 4\}$. This means that $\mathcal{V}$ satisfies the condition c) of Proposition 2.1 and, by Lemma 2.2, $\mathcal{V}$ then satisfies the condition 3) of Theorem 2.

Finally, let $\mathcal{Z} = \mathcal{C}$. Then $\mathcal{V} = \mathcal{X} \vee \mathcal{C}$ where $\mathcal{X}$ is a variety of nilsemigroups containing the variety $\mathcal{C}_0$. Arguing as in the proof of Lemma 2.2, we conclude that, on the relatively free members of $\mathcal{X}$, all fully invariant congruences permute, and therefore, by Theorem 1, $\mathcal{X}$ satisfies one of the identity systems (0.1)–(0.6). The first and the last of these systems, however, fail in the variety $\mathcal{C}_0$ so $\mathcal{X}$ should satisfy one of the remaining identity systems (0.2)–(0.5). This means that the variety $\mathcal{V}$ satisfies the condition 2) of Theorem 2. ∎

To complete the proof of the "only if" part of Theorem 2, it remains to analyze the varieties satisfying the condition a) of Proposition 2.1. This will be the aim of our next lemma.

Lemma 2.5. *If a semigroup variety $\mathcal{V}$ satisfies the condition* a) *of Proposition* 2.1 *and, on relatively free members of $\mathcal{V}$, the fully invariant congruences under the least semilattice congruence permute, then $\mathcal{V}$ satisfies one of the conditions* 1), 3) *or* 4) *of Theorem* 2.

Proof. We may suppose that $\mathcal{V}$ contains a semigroup which is not completely regular. Then, by Lemma 2.3, all completely simple semigroups in $\mathcal{V}$ are trivial. In particular, all groups in $\mathcal{V}$ are trivial whence, for every semigroup $S \in \mathcal{V}$, if $s \in S$ is a group element, then s should be an idempotent.

First consider the case of $\mathcal{V}$ satisfying the identity (2.3). Then, for every semigroup $S \in \mathcal{V}$ and for all elements $s, s' \in S$, the product ss' is a group element. Hence ss' is an idempotent, and the variety $\mathcal{V}$ satisfies the identity

$$(xy)^2 = xy. \tag{2.6}$$

Further, being a band, the ideal $S^2 = \{ss' \mid s, s' \in S\}$ is a semilattice of rectangular bands. However, all completely simple semigroups (in particular, all rectangular bands) in $\mathcal{V}$ are trivial. Therefore S^2 is a semilattice whence $\mathcal{V}$ also satisfies the identity

$$xy \cdot zt = zt \cdot xy. \tag{2.7}$$

The identities (2.6) and (2.7) easily imply that $\mathcal{V}$ is in fact commutative. Indeed,

$$xy = (xy)^2 = (xy)^3 = xyx \cdot yxy = yxy \cdot xyx = (yx)^3 = (yx)^2 = yx.$$

Since both s^2 and s^3 are idempotents, we have $s^2 = s^3$ for any $s \in S$. Therefore $\mathcal{V}$ satisfies the identity

$$x^2 = x^3. \tag{2.8}$$

Combined with (2.6) and the commutativity, (2.8) implies the identity

$$x^2y = xy. \tag{2.9}$$

Indeed,

$$xy = (xy)^2 = x^2y^2 = x^3y^2 = x \cdot x^2y^2 = x \cdot xy = x^2y.$$

Now let u be an arbitrary word and $c(u) = \{x_1, x_2, \ldots, x_n\}$. If $n \geq 2$, then using the commutativity and (2.9), we can reduce this word to the word $x_1x_2 \cdots x_n$; if $n = 1$, then either u coincides with the letter x_1 or we can apply (2.8) to reduce it to the word x_1^2. This analysis shows that two words u and v are equal in the variety $\mathcal{V}$ whenever $c(u) = c(v)$ and $\ell(u), \ell(v) \geq 2$ or u and v coincide with the same letter. Comparing the conclusion with Lemma 1.2, we can reformulate it saying that $u = v$ in $\mathcal{V}$ whenever $u = v$ in both $\mathcal{ZM}$ and $\mathcal{SL}$. The latter condition says exactly that $\mathcal{V} \subseteq \mathcal{ZM} \vee \mathcal{SL}$ so we get that $\mathcal{V}$ satisfies the condition 3) of Theorem 2.

The two remaining cases are symmetric so we may restrict ourselves to considering one of them. Suppose that $\mathcal{V}$ satisfies the identity (2.1). It implies the identity $x^2 = x^{n+1}$ which shows that, for every semigroup $S \in \mathcal{V}$ and for every element $s \in S$, the elements $s^2, s^3, \ldots, s^n$ form a subgroup in S. Therefore they all are equal to the same idempotent. In particular, we have $x^2 = x^n$ in $\mathcal{V}$; being combined with (2.1), this identity obviously implies (2.9). Now we aim to show that the set $E(S)$ of all idempotents of S is a subsemigroup in S. In fact, $E(S)$ is even a left ideal in S. Indeed, let $s \in S,\ e \in E(S)$. We then have

$$se = se \cdot e = (se)^2 e = (se)^2.$$

Being a band, $E(S)$ must be a semilattice. Since s^2 is an idempotent for every $s \in S$ and idempotents in S permute, the variety $\mathcal{V}$ satisfies the identity

$$x^2y^2 = y^2x^2. \tag{2.10}$$

However the identities (2.9) and (2.10) together define the variety $\mathcal{P}$. Thus we have proved that $\mathcal{V} \subseteq \mathcal{P}$. It is well-known and easy to verify that the subvariety lattice of the variety $\mathcal{P}$ has the following diagram:

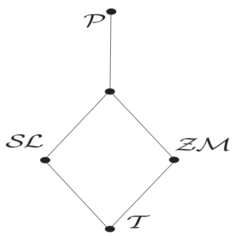

Figure 2. The lattice $L(\mathcal{P})$

We see that, being a subvariety of $\mathcal{P}$, the variety $\mathcal{V}$ either coincides with $\mathcal{P}$ or is contained in the join $\mathcal{ZM} \vee \mathcal{SL}$. Therefore $\mathcal{V}$ satisfies one of the conditions 3) or 4) of Theorem 2.

■

The "only if" part of Theorem 2 is thus proved. Let us check that, conversely, every variety satisfying one of the conditions 1)–4) has the property that, on its relatively free members, the fully invariant congruences under the least semilattice congruence permute. The following lemma and its dual cover the last of the 4 cases.

Lemma 2.6. *Let $S \in \mathcal{P}$ be a relatively free semigroup. Then the fully invariant congruences of S contained in the least semilattice congruence permute.*

Proof. Let $\mathcal{V}$ be any subvariety of $\mathcal{P}$ such that S is a $\mathcal{V}$-free semigroup. If $\mathcal{V} \subseteq \mathcal{ZM}$, then there are no more than two fully invariant congruences on S (the equality and the universal relations), and the claim is obvious. Otherwise we may deduce from the description of the lattice $L(\mathcal{P})$ presented on Fig. 2 that $\mathcal{V} \supseteq \mathcal{SL}$ and the interval $[\mathcal{SL}, \mathcal{V}]$ is a chain. Hence the fully invariant congruences of S contained in the least semilattice congruence also form a chain, and therefore, permute. ∎

The next case is easy to handle with the help of the additional information about nilsemigroup varieties with permuting fully invariant congruences on their relatively free members that we found in Proposition 1.15.

Lemma 2.7. *Let a variety $\mathcal{M}$ satisfy one of the identity systems* (0.1)–(0.6) *and let $S \in \mathcal{M} \vee \mathcal{SL}$ be a relatively free semigroup. Then the fully invariant congruences of S contained in the least semilattice congruence σ permute, moreover, the product of any two such congruences coincides with their set-theoretical union.*

Proof. Let $\mathcal{V}$ be any subvariety of $\mathcal{M} \vee \mathcal{SL}$ such that S is a $\mathcal{V}$-free semigroup. Recall that the lattice $L(\mathcal{M} \vee \mathcal{SL})$ is isomorphic to the direct product $L(\mathcal{M}) \times L(\mathcal{SL})$ (see the proof of Lemma 2.2), and, in particular, $\mathcal{V} = \mathcal{M}' \vee \mathcal{Y}$ for some $\mathcal{M}' \subseteq \mathcal{M}$, $\mathcal{Y} \in \{\mathcal{T}, \mathcal{SL}\}$. If the variety $\mathcal{Y}$ is trivial, then $\mathcal{V} = \mathcal{M}' \subseteq \mathcal{M}$, and, by Theorem 1, all fully invariant congruences of S permute. Thus we may assume that $\mathcal{V} = \mathcal{M}' \vee \mathcal{SL}$ for some variety $\mathcal{M}'$ satisfying one of the identity systems (0.1)–(0.6).

We have to prove that $\alpha\beta = \alpha \cup \beta$ for any two fully invariant congruences α, β under the least semilattice congruence σ. Denote by μ' the fully invariant congruence on S corresponding to the variety $\mathcal{M}'$. Then α and β can be represented as $\alpha = \alpha' \cap \sigma$ and $\beta = \beta' \cap \sigma$, respectively, for some fully invariant congruences α' and β' containing the congruence μ'. Take a pair of elements $(a, c) \in \alpha\beta$, and let b be an element such that $a\ \alpha\ b\ \beta\ c$. Then we have $a\ \sigma\ b\ \sigma\ c$ whence $a\ \sigma\ c$ and $a\ \alpha'\ b\ \beta'\ c$ whence $a\ \alpha'\beta'\ c$. It follows from Proposition 1.15 that, on the relatively free semigroup S/μ' of the variety $\mathcal{M}'$, the product of any two fully invariant congruences is equal to their set-theoretical union. Hence, in S, the fully invariant congruences containing μ' enjoy the same property, and therefore, we have $a\ \alpha'\ c$ or $a\ \beta'\ c$. Since $a\ \sigma\ c$, this implies that $a\ \alpha\ c$ or $a\ \beta\ c$, respectively. ∎

Similarly, we obtain

Lemma 2.8. *Let $\mathcal{C} = \mathrm{var}\{x^2 = x^3,\ xy = yx\}$ and let $\mathcal{N}$ satisfy one of the identity systems* (0.2)–(0.5). *If $S \in \mathcal{N} \vee \mathcal{C}$ is a relatively free semigroup, then the fully invariant congruences of S contained in the least semilattice congruence σ permute, moreover, the product of any two such congruences coincides with their set-theoretical union.*

Proof. Since $\mathcal{C}$ contains the variety $\mathcal{C}_0 = \mathrm{var}\{xy = yx,\ x^2 = 0\}$ which satisfies each of the identity systems (0.2)–(0.5), we may (and shall) assume that $\mathcal{N}$ contains $\mathcal{C}_0$. Let $\mathbf{C}_3$ denote the 3-element chain $\mathcal{T} \subset \mathcal{SL} \subset \mathcal{C}$. Consider in the direct product $L(\mathcal{N}) \times \mathbf{C}_3$ the sublattice L consisting of all pairs $(\mathcal{M}, \mathcal{Y})$ such that $\mathcal{M}$ contains the variety $\mathcal{C}_0$ whenever $\mathcal{Y} = \mathcal{C}$. According to [19, Lemma 2], the mapping $(\mathcal{M}, \mathcal{Y}) \mapsto \mathcal{M} \vee \mathcal{Y}$ is an isomorphism of the sublattice L onto the lattice $L(\mathcal{N} \vee \mathcal{C})$. Fix a subvariety $\mathcal{V}$ in $\mathcal{N} \vee \mathcal{C}$ such that S is free in $\mathcal{V}$. We get, in particular, that $\mathcal{V} = \mathcal{M} \vee \mathcal{Y}$ for some $\mathcal{M} \subseteq \mathcal{N},\ \mathcal{Y} \in \{\mathcal{T}, \mathcal{SL}, \mathcal{C}\}$.

If the variety $\mathcal{Y}$ is trivial, then $\mathcal{V} = \mathcal{M} \subseteq \mathcal{N}$, and the claim of the lemma follows from Proposition 1.15. If $\mathcal{Y} = \mathcal{SL}$, then $\mathcal{V} = \mathcal{M} \vee \mathcal{SL}$ whence Lemma 2.7 applies. Thus we assume that $\mathcal{V} = \mathcal{M} \vee \mathcal{C}$ in which case $\mathcal{M}$ must contain the variety $\mathcal{C}_0$. Let μ, γ, γ_0 denote the fully invariant congruences on the semigroup S corresponding to the varieties $\mathcal{M}, \mathcal{C}, \mathcal{C}_0$, respectively. In order to clarify the relative location of these congruences and the least semilattice congruence σ, we show them on Figure 3 on the next page. Recall that δ and ε denote the universal relation and the equality relation on S, respectively.

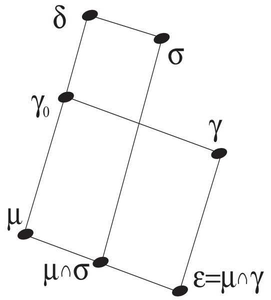

Figure 3. A sketch of the lattice of fully invariant congruences on S

The fully invariant congruences on S which are contained in σ correspond to the subvarieties of $\mathcal{V}$ containing $\mathcal{SL}$. These subvarieties are of the form $\mathcal{A} \vee \mathcal{SL}$ or $\mathcal{B} \vee \mathcal{C}$ where $\mathcal{A}, \mathcal{B} \subseteq \mathcal{M}$ and, besides that, $\mathcal{B} \supseteq \mathcal{C}_0$. Therefore the fully invariant congruences under σ can be represented as $\alpha = \alpha' \cap \sigma$ (*congruences of the first type*) or $\beta = \beta' \cap \gamma$ (*congruences of the second type*) where $\alpha', \beta' \supseteq \mu$ and $\beta' \subseteq \gamma_0$.

To prove that the product of any two fully invariant congruences under σ equals their set-theoretical union $\alpha_1 \cup \alpha_2$, we first analyze the case when they both are of the same type. So consider two fully invariant congruences $\alpha_1 = \alpha_1' \cap \rho, \alpha_2 = \alpha_2' \cap \rho$ where $\alpha_1', \alpha_2' \supseteq \mu$ and ρ is one of the congruences σ and γ. This subcase is quite analogous to the proof of Lemma 2.7. Indeed, take a pair of elements $(a, c) \in \alpha_1\alpha_2$, and let b be an element such that $a\ \alpha_1\ b\ \alpha_2\ c$. Then we have $a\ \rho\ b\ \rho\ c$ whence $a\ \rho\ c$ and $a\ \alpha_1'\ b\ \alpha_2'\ c$ whence $a\ \alpha_1'\alpha_2'\ c$. By Proposition 1.15, on the relatively free semigroup S/μ of the variety $\mathcal{M}$, the product of any two fully invariant congruences equals their set-theoretical union. Hence, in S, we have $a\ \alpha_1'\ c$ or $a\ \alpha_2'\ c$. As $a\ \rho\ c$, this implies that $a\ \alpha_1\ c$ or $a\ \alpha_2\ c$, respectively, so the product $\alpha_1\alpha_2$ coincides with $\alpha_1 \cup \alpha_2$.

Now consider the case of the product $\alpha\beta$ of two fully invariant congruences of different types. Let, for unambiguity, $\alpha = \alpha' \cap \sigma$ be a congruence of the first type and $\beta = \beta' \cap \gamma$ a congruence of the second type (it will be clear that the proof would work without any change for the opposite order of types as well). Again take a pair of elements $(a, c) \in \alpha\beta$, and let b be an element such that $a\ \alpha\ b\ \beta\ c$. Then $a\ \alpha'\ b\ \beta'\ c$, that is, $a\ \alpha'\beta'\ c$. As above, an application of Proposition 1.15 yields $a\ \alpha'\ c$ or $a\ \beta'\ c$. Besides that, it follows from $a\ \alpha\ b\ \beta\ c$ that $a\ \sigma\ b\ \gamma\ c$ whence $a\ \sigma\ c$ for $\sigma \supset \gamma$. Therefore, if $a\ \alpha'\ c$, then we have $a\ \alpha\ c$. Now assume that $a\ \beta'\ c$. Let $\mathcal{B}$ be the subvariety of $\mathcal{M}$ corresponding to the fully invariant congruence β', and let the words u and w be preimages of a and c, respectively, by the canonical projection from the appropriate absolutely free semigroup X^+ onto S. First suppose that $u \neq 0$ in $\mathcal{B}$. Using the classification of the words which may differ from 0 in a variety satisfying one of the identity systems (0.2)–(0.5) (see the second last row of Table 1), we observe that u must be similar to a word of the form $x_1x_2\cdots x_n$ for some n. By Lemma 1.13, we have $u \approx w$ and $c(u) = c(w)$ whence the identity $u = w$ is equivalent to the identity $x_1x_2\cdots x_n = x_{1\tau}x_{2\tau}\cdots x_{n\tau}$ where τ is a permutation of the set $\{1, 2, \ldots, n\}$. Clearly, for any τ, this identity follows from the commutativity, and therefore, holds in the variety $\mathcal{C}$. Thus, the identity $u = w$ holds in $\mathcal{C}$ whence $a\gamma c$ in S. As $a\ \beta'\ c$, we then have $(a, c) \in \beta = \beta' \cap \gamma$. It remains to consider the situation when $u = 0$ in $\mathcal{B}$. Since $\mathcal{B} \supseteq \mathcal{C}_0$, only words having a repeated appearance of a letter can be equal to 0 in $\mathcal{B}$. On the other hand, every word with a repeated appearance of a letter equals 0 in the variety $\mathcal{M}$. Thus, $u = 0$ (and then $v = 0$) in $\mathcal{M}$ whence $a\ \mu\ c$ in S. Above we noticed that $a\ \sigma\ c$ so $(a, c) \in \mu \cap \sigma \subseteq \alpha' \cap \sigma = \alpha$. We have proved that the product $\alpha\beta$ coincides with the set-theoretical join of α and β.

■

To complete the proof of the "if" part of Theorem 2, it remains to consider completely regular varieties. However the fact that, on each relatively free completely regular semigroup, the fully invariant congruences under the least semilattice congruence permute follows from results of the papers [9] or [10]. (In fact, both these papers dealt with completely regular semigroups in the unary semigroup setting but it is easy to see that the proofs work for plain semigroups as well.) More precisely, let S be a relatively free completely regular semigroup. If the least semilattice congruence on S equals the universal relation, then S is completely simple, and all fully invariant congruences on S permute by [9, Theorem 4] or [10, Theorem 3.1]. Otherwise we may apply [9, Theorem 14] or again [10, Theorem 3.1].

Thus, we have proved Theorem 2.

As discussed in the Introduction, the permutability of the fully invariant congruences on relatively free members of a variety $\mathcal{V}$ has been expected to be related to the arguesian identity in the subvariety lattice of $\mathcal{V}$. From the proof of Theorem 2 it follows that, beyond the completely regular case, our property turns out to imply a much stronger lattice identity, namely, distributivity — a fact that was hard to predict a priori.

Corollary 2.9. *Let $\mathcal{V}$ be a semigroup variety such that the fully invariant congruences under the least semilattice congruence on the relatively free members of $\mathcal{V}$ permute. Then either $\mathcal{V}$ consists of completely regular semigroups or the subvariety lattice of $\mathcal{V}$ is distributive.*

Proof. We may assume that $\mathcal{V}$ satisfies one of the conditions 2), 3) or 4) of Theorem 2. In the latter case, we can refer to Fig. 2. If $\mathcal{V}$ satisfies one of the conditions 2) or 3), then Lemmas 2.8 and 2.7, respectively, imply that, on the $\mathcal{V}$-free semigroup $F_{\mathcal{V}}(X)$ over an infinite set X, the lattice join $\alpha \vee \beta$ of any two fully invariant congruences α and β under the least semilattice congruence σ coincides with their set-theoretical union $\alpha \cup \beta$. Since the lattice meet of congruences always equals their set-theoretical intersection $\alpha \cap \beta$, we see that the interval $[\varepsilon, \sigma]$ of the lattice of all fully invariant congruences on $F_{\mathcal{V}}(X)$ is a sublattice in the subset lattice of $F_{\mathcal{V}}(X) \times F_{\mathcal{V}}(X)$, and therefore, is distributive. The interval $[\varepsilon, \sigma]$ is dually isomorphic to the interval $[\mathcal{SL}, \mathcal{V}]$ of the lattice $L(\mathcal{V})$ whence the latter interval is distributive as well. Therefore the whole lattice $L(\mathcal{V})$, being a subdirect product of this interval and the two-element chain, is distributive, too.

■

At the beginning of Section 1, we noticed that the property studied there was equivalent to a formally weaker restriction to a variety $\mathcal{V}$, namely, that the fully invariant congruences on $\mathcal{V}$-free semigroups (and not on all relatively free members of $\mathcal{V}$) permute. We conclude Section 2 with an example showing that the property considered here allows no similar reformulation. Namely, we exhibit a variety $\mathcal{V}$ such that:

(+) on $\mathcal{V}$-free semigroups, the fully invariant congruences under the least semilattice congruence permute;

(−) $\mathcal{V}$ contains a relatively free semigroup, on which the fully invariant congruences under the least semilattice congruence do not permute.

Example 2.10. *Let* $\mathcal{K} = \mathrm{var}\{xy = yx,\ xyx = 0,\ xyzt = 0\}$*. Then the variety* $\mathcal{V} = \mathcal{K} \vee \mathcal{SL}$ *satisfies both properties* (+) *and* (−)*.*

Proof. Clearly, $\mathcal{V}$ satisfies none of the conditions of Theorem 2, and therefore, contains a relatively free semigroup, on which the fully invariant congruences under the least semilattice congruence does not permute. It is easy to see that every semigroup of the form $F_{\mathcal{K}}(X)$ where X has ≥ 3 elements can play such a role: the least semilattice congruence on $F_{\mathcal{K}}(X)$ coincides with the universal relation, and the congruences α' and β' on $F_{\mathcal{K}}(X)$ corresponding to the varieties $\mathcal{A} = \mathrm{var}\{xy = yx,\ xyz = 0\}$ and $\mathcal{B} = \mathrm{var}\{xy = yx,\ x^2 = 0,\ xyzt = 0\}$, respectively, fail to permute.

Nevertheless, on $\mathcal{V}$-free semigroups, the fully invariant congruences under the least semilattice congruence σ permute. In order to show this, consider the subvariety lattice of $\mathcal{V}$; its diagram is shown on Fig. 4.

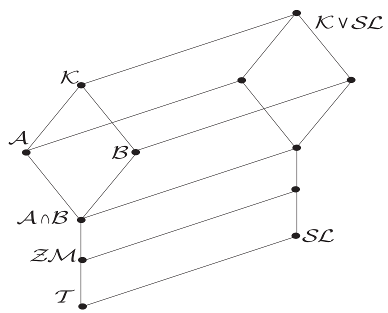

Figure 4. The lattice $L(\mathcal{K} \vee \mathcal{SL})$

The fully invariant congruences under σ correspond to the varieties from the interval $[\mathcal{SL}, \mathcal{K} \vee \mathcal{SL}]$. Since the interval contains only one pair of incomparable varieties and comparable congruences obviously permute, the only "suspicious" pair of the fully invariant congruences under σ consists of the fully invariant congruences $\alpha = \alpha' \cap \sigma$ and $\beta = \beta' \cap \sigma$ corresponding to the varieties $\mathcal{A} \vee \mathcal{SL}$ and $\mathcal{B} \vee \mathcal{SL}$, respectively. It is more convenient to lift the situation into the absolutely free semigroup X^+; to simplify notation, we use the same letters α and β to denote the corresponding fully invariant congruences on X^+.

It is easy to describe the congruences α and β on X^+:

$u \; \alpha \; v \iff u = v$ in X^+, or $c(u) = c(v)$ and $\ell(u), \ell(v) \geq 3$;

$u \; \beta \; v \iff u = v$ in X^+, or $c(u) = c(v)$, a letter appears in u at least twice and a letter appears in v at least twice.

Let now $u \; \alpha\beta \; w$ for some words u and w. We are going to prove that then $(u, w) \in \alpha \cup \beta$. Indeed, there exists a word v such that $u \; \alpha \; v \; \beta \; w$. If $u = v$ in X^+ or $v = w$ in X^+, then obviously $(u, w) \in \alpha \cup \beta$. Otherwise we have that $c(u) = c(v), \ell(u), \ell(v) \geq 3, c(v) = c(w)$, a letter appears in v at least twice and a letter appears in w at least twice. If $|c(u)| = |c(v)| \leq 2$, then the fact that $\ell(u), \ell(v) \geq 3$ implies that a letter appears in u at least twice and a letter appears in v at least twice. Thus, $u \; \beta \; v$, and therefore, $u \; \beta \; w$. If $|c(v)| = |c(w)| \geq 3$, then obviously $\ell(v), \ell(w) \geq 3$ whence $v \; \alpha \; w$, and therefore, $u \; \alpha \; w$.

We see that $\alpha\beta = \alpha \cup \beta$. Analogously, $\beta\alpha = \alpha \cup \beta$, so we have $\alpha\beta = \beta\alpha$, as desired. It remains to refer to Lemma 1.1.

■

We see that the condition (+) defines a larger class of varieties than that described in Theorem 2. This class, however, can be completely characterized in a similar manner. The only principal difference is that one should add to the varieties listed in Theorem 2 all varieties satisfying the identity $(xy)^2 = xy$ and containing the variety $\mathcal{SL}$; besides that, one only should slightly modify the conditions 2) and 3) of Theorem 2. It is interesting that the statement of Corollary 2.9 remains valid in this more general situation. We plan to devote a separate paper to the condition (+) and also to other natural multiplicative properties of the fully invariant congruences on relatively free semigroups.

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