STRUCTURE OF REGULAR SEMIGROUPS, II
CROSS–CONNECTIONS

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PREFACE

This book is intended as a sequel to the author's memoir published by the American Mathematical Society in 1979. The author presented part of this material in a talk in the conference on 'Theory of Regular Semigroups and Applications' conducted in the Department as well as in the talk given at the semigroup theory session conducted along with the American Mathematical Society Conference in the fall of 1987 in Lincoln. Also a series of seminars were conducted in the Department of Mathematics, University of Kerala on this material. The author wishes to acknowledge the encouragement and help of several colleagues, especially Professor John Meakin of the University of Nebraska, Professor Francis Pastijn of the Marquette University, and colleagues and students of the Department of Mathematics, University of Kerala. Part of this work was done during the author's visit to the University of Nebraska, Lincoln, U. S. A, and the author wishes to acknowledge the encouragement and help received from the colleagues and other staff of the Department of Mathematics, University of Nebraska. The present form of this book is entirely due to the initiative and the help of Professor A. M. Mathai who edited the manuscript. I wish to express my sincere gratitude to him. I would like to thank Mr. C. S. Ling of McGill University, Canada, a student of Professor A. M. Mathai for the painful task of computer-setting the manuscript, Mr. K. Gopinatha Panicker and Miss K. T. Martiakutty of the Centre for Mathematical Sciences for looking after the printing and the Centre for Mathematical Sciences, Trivandrum-14, Kerala State, India and its Director for bringing out this monograph in their publication series.

K. S. S. Nambooripad

CONTENTS

STRUCTURE OF REGULAR SEMIGROUPS, II CROSS–CONNECTIONS

K. S. S. NAMBOORIPAD

Introduction

At present there are two established procedures for analysing the structure of an arbitrary regular semigroup—the theory of *biordered* sets (cf. [3] and [22]) and the theory of *cross–connections* (cf. [9] and [21]). The concept of a biordered set gives a concrete form for the structure of the set of idempotents of a semigroup and thus helps to visualize this important structural component as a mathematical object on its own right. The 'four spiral biordered set' [2] is an especially nice example of a biordered set which can be visualized in this manner. Other interesting examples of biordered sets can be found in [7]. We refer the reader to [22] for the definition of biordered sets where it is also proved that every regular biordered set comes from a regular semigroup. The result that every biordered set comes from a semigroup is due to Easdown (cf. [6]). On the other hand, a cross–connection characterizes the ideal structure of a regular semigroup. These theories are complementary in the sense that, while the theory of biordered sets is more elementary and explicit, the theory of cross–connections is essentially abstract and so it is more natural when one studies semigroups that arise in diverse contexts, such as semigroups of matrices, linear operators, etc. The scope of these theories is limited by the fact that they mainly apply to study fundamental regular semigroups. Recall that a semigroup S is *fundamental* if the only congruence on S contained in the Green's relation $\mathcal{H}$ is the identity congruence. However, the concept of *inductive groupoids*, discussed in [22], is useful in studying arbitrary regular semigroups. In this paper we shall consider a generalization of Grillet's theory of cross–connections that will be complementary to the theory of inductive groupoids.

Let S be a regular semigroup. Hall [11] showed that there is a representation ϕ of S by pairs of mappings on the partially ordered sets $I = S/\mathcal{R}$ and $\Lambda = S/\mathcal{L}$, where $\mathcal{R}$ and $\mathcal{L}$ are Green's relations, defined as follows: for each $x \in S$

$$\phi(x) = (r_x, l_x)$$

where $r_x: \Lambda \to \Lambda[l_x: I \to I]$ is the map defined by $L_u r_x = L_{ux}[l_x R_u = R_{xu}]$. He showed that ϕ is faithful if and only if S is fundamental. In [11], Hall also gave a construction of the maximum fundamental regular semigroup associated with S.

Grillet's theory of cross–connections may be viewed as a refinement of Hall's ideas. Grillet characterized the partially ordered sets I and Λ as *regular partially ordered sets* and the relations between them in terms of *cross–connections*. A cross–connection between I and Λ consists of two mappings $\Gamma: I \to \Lambda^\circ$ and $\Delta: \Lambda \to I^\circ$ of I and Λ into two regular partially ordered sets Λ° and I° of regular equivalence relations on Λ and I respectively satisfying Grillet's axioms (cf. [10]). He showed that the maximum fundamental regular semigroup determined by this data is the set of all pairs of normal mappings that are compatible with the given cross–connection (cf. [21], [26]). In order to distinguish Grillet's concept of cross–connections from the more general concept considered in this paper henceforth we shall refer to the former as *fundamental cross–connections*.

Now pairs of normal mappings compatible with a given cross–connection can be viewed as an abstract description of the pairs of mapping that appear in the image of Hall's representation ϕ of the regular semigroup S. On the other hand, since S is weakly reductive, the representation ψ defined by:

$$\psi(x) = (\rho_x, \lambda_x)$$

where $\rho_x[\lambda_x]$ is the right [left] translation by $x \in S$, is faithful and is equivalent to Hall's representation ϕ when S is fundamental. In order to characterize the pairs of mappings (ρ_x, λ_x) abstractly, we replace the partially ordered sets $I = S/\mathcal{R}$ and $\Lambda = S/\mathcal{L}$ in Grillet's theory by small categories $\mathbb{L}(S)$ and $\mathbb{R}(S)$, where $\mathbb{L}(S)$ [$\mathbb{R}(S)$] is the category whose objects are principal left [right] ideals of S and whose morphisms are right [left] translations. Thus $\rho \in \mathrm{Hom}_{\mathbb{L}(S)}(Sx, Sy)$ [$\lambda \in \mathrm{Hom}_{\mathbb{R}(S)}(xS, yS)$] if and only if $\rho: Sx \to Sy\,[\lambda: xS \to yS]$ is a mapping such that for all $u, v \in S$, $(uv)\rho = u(v)\rho\,[\lambda(uv) = \lambda(u)v]$. We show that when S is regular $\mathbb{L}(S)$ and $\mathbb{R}(S)$ are *normal reductive* categories (cf. Theorem 3.8). Conversely, if $\mathcal{C}$ is any normal, reductive category, then the semigroup $T\mathcal{C}$ of all normal cones in $\mathcal{C}$ is a regular semigroup such that $\mathbb{L}(T\mathcal{C})$ is isomorphism to $\mathcal{C}$ (cf. Theorems 3.3 and 3.11). Moreover in this case, there is an embedding of $\mathbb{R}(T\mathcal{C})$ into the functor category $\mathcal{C}^* = [\mathcal{C}, \mathbf{Set}]$ of all set valued functors on $\mathcal{C}$ (cf. Theorem 3.14). The image $N^*\mathcal{C}$ of this embedding is a normal reductive subcategory of $\mathcal{C}^*$ and is called the *normal dual* of $\mathcal{C}$.

Let $\mathcal{C}$ and $\mathcal{D}$ be two normal reductive categories. A cross-connection between $\mathcal{C}$ and $\mathcal{D}$ is a local isomorphism $\Gamma: \mathcal{D} \to N^*\mathcal{C}$ such that the image of Γ is *total* in $N^*\mathcal{C}$ (cf. Definition 4.2). Notice that a cross-connection is a category equivalence (cf. [28], p. 55). Furthermore, there is a local isomorphism $\Gamma^*: \mathcal{C} \to N^*\mathcal{D}$ and a natural isomorphism $\chi_\Gamma: \Gamma(-,-) \to \Gamma^*(-,-)$, where $\Gamma(-,-)$ denotes the bifunctor associated with Γ (cf. Theorem 4.5). If $c \in v\mathcal{C}$ and $d \in v\mathcal{D}$, then $\Gamma(c,d)$ is a set of normal cones in $\mathcal{C}$ with vertex c and $\Gamma^*(c,d)$ is a set of cones in $\mathcal{D}$ with vertex d. We say that the pair (ρ, λ), $\rho \in \Gamma(c,d)$, $\lambda \in \Gamma^*(c,d)$, is *linked* if $\chi_\Gamma(c,d)(\rho) = \lambda$. The set of all linked pairs of normal cones, under a suitable multiplication is a regular semigroup $\hat{S}\Gamma$ such that $\mathbb{L}(\hat{S}\Gamma)$ is isomorphic to $\mathcal{C}$ and $\mathbb{R}(\hat{S}\Gamma)$ is isomorphic to $\mathcal{D}$. Every regular semigroup is up to isomorphism of this form (cf. Theorems 4.13 and 4.15).

Let $\Gamma: \mathcal{D} \to N^*\mathcal{C}$ and $\Gamma': \mathcal{D}' \to N^*\mathcal{C}'$ be two cross-connections. A morphism $m: \Gamma \to \Gamma'$ is a pair $m = (F, G)$ where $F: \mathcal{C} \to \mathcal{C}'$ and $G: \mathcal{D} \to \mathcal{D}'$ are (inclusion-preserving) functors satisfying some compatibility conditions (cf. Definition 5.1). We prove that a morphism $m: \Gamma \to \Gamma'$ induces a homomorphism $\hat{S}m: \hat{S}\Gamma \to \hat{S}\Gamma'$ and that the assignment

$$\hat{S}: \Gamma \mapsto \hat{S}\Gamma \qquad m \mapsto \hat{S}m$$

is an equivalence of the category of regular semigroups with the category of cross-connections. In §6 we investigate the relation between the concept of cross-connections introduced in this paper and that of fundamental cross-connections (of Grillet).

We remark that the theory of cross-connections presented here provides a unified framework for studying various classes of regular semigroups. The use of abstract categories to formulate the theory is particularly useful in this context. Thus, for example, to study semigroups of matrices, it is natural to interpret the normal categories involved, as categories of vector spaces and it can be seen that such interpretation would lead to a natural interpretation of the dual category and the cross-connection in terms of the vector space dual and the bilinear form associated with the dual. An examination of the axioms for normal categories would show that natural examples of such categories come from small subcategories of Abelian categories such as categories of vector spaces, modules etc. and the concept of a normal, reductive category can be regarded as a generalization of the concept of *semisimplicity* or *reducibility* of classical algebra. In fact, every semisimple object gives rise to a normal and reductive category of its subobjects; however there are normal and reductive categories that do not come up in this way. In §7, we discuss some examples to illustrate these points. Another consequence of this theory is that it enables us to understand the nature of Grillet's cross-connections better. In fact, we can see that the two mappings $\Gamma: I \to \Lambda^\circ$ and $\Delta: \Lambda \to I^\circ$ are local isomorphisms whose images are total (cf. §6).

The concept of normal categories and cross-connections represent a combination as well as development of ideas of Grillet and those of [20] used to describe arbitrary regular semigroups (see also [17]). In [20] and [17] structure mappings were used to describe global multiplications in an arbitrary regular semigroup. It is easy to see that each $\mathcal{L}$-structure mapping [$\mathcal{R}$-structure mapping] extends uniquely to a morphism of the corresponding ideals in $\mathbb{L}(S)$ [$\mathbb{R}(S)$]. These mappings (together with inclusions) generate the normal category $\mathbb{L}(< E(S) >)$ [$\mathbb{R}(< E(S) >)$], where $< E(S) >$ denotes the idempotent generated subsemigroup of S. Moreover, the cross-connection between $\mathbb{L}(< E(S) >)$ and $\mathbb{R}(< E(S) >)$ specify the relations between $\mathcal{L}$- and $\mathcal{R}$-structure mappings which are given as axioms in [17] and [20].

In view of our definition of cross-connections, this paper may be regarded as a study of small categories of certain type; viz., small categories that are normal and reductive. We have tried, as far as possible to use standard results and constructions of the subject (cf. [15], [28] and [32]). However, it should be emphasized that our treatment of these objects are as algebraic objects rather than as categories in the usual sense. Thus while naturally equivalent categories and functors are indistinguishable, they are not so when treated as algebraic objects. Note that a connected groupoid is naturally equivalent to a group. Also we have already observed that a cross-connection is always a category equivalence. Similar use of categories have also been made by Rhodes and Tilson (cf. [34]), though their point of view is different from the one adopted here.

We also remark that it is possible to give an alternate formulation of the concept of cross-connections, especially an alternative representation of the cross-connection semigroup using Lallement's representation [15] of regular semigroups (see also [27]). Basic infrastructure required to develop this version is the concept of a complex in a category (see §2) which is also of independent interest. However, since Lallement's representation uses partial transformations, it appears that proofs would be more involved. For a more comprehensive discussion of questions related to structure theory of regular semigroups, we refer the reader to [18], [23] and [31] where one can also find a more extensive bibliography of papers on related questions.

Finally we make few remarks regarding notations used in the paper. Notations and terminology related to semigroups used in the paper are as in [4], while those related to biordered sets, fundamental cross–connections, etc. are those of [9], [21] and [22]. As in [12], small categories will be regarded as partial algebras and the symbol $\mathcal{C}$ for a small category will also denote its morphism set; its vertex set (set of objects) will be denoted by $v\mathcal{C}$. As in [22], compositions in the category will be written in the order in which it appears in commutative diagrams. Other notations related to categories used in the paper are those of [16].

1. Normal Categories

In this paper, for any notation or terminology not explicity defined, the reader should refer to [4], [12], [16] and / or [22].

Here as in [22], we shall regard small categories as partial algebras, with the underlying set as the morphism set of the category and operation as the composition. If $\mathcal{C}$ is a small category, $\mathcal{C}$ will also denote this partial algebra and $v\mathcal{C}$ will denote the set of vertices (set of objects) of $\mathcal{C}$. Often we shall identify vertices with the corresponding identities so that $v\mathcal{C} \subseteq \mathcal{C}$. Then a functor $F: \mathcal{C} \to \mathcal{D}$ will be a partial algebra homomorphism such that $vF = F|v\mathcal{C}$ maps $v\mathcal{C}$ to $v\mathcal{D}$.

Suppose that a canonical choice of subobjects has been provided in the category $\mathcal{C}$. This gives a relation $\subseteq$ called inclusions relations among objects of $\mathcal{C}$. If $\subseteq$ is a partial order (as in categories of **Sets**, **Groups** etc.), then we shall say that $\mathcal{C}$ *has subobjects* or that $\mathcal{C}$ is a *category with subobjects*. If j_A^B denotes the inclusion monomorphism of $A \to B$, (where $A \subseteq B$), then this implies that $j_A^B \circ j_B^C = j_A^C$ if $A \subseteq B \subseteq C$. Hence, $j_A^A \circ j_A^A = j_A^A = j_A^A \circ 1_A$ and since j_A^A is a monomorphism, we have $j_A^A = 1_A$. Antisymmetry of $\subseteq$ implies that when $A \subseteq B \subseteq A$, $j_A^B = j_B^A = 1_A = 1_B$. When $\mathcal{C}$ has subobjects, $v\mathcal{C}$ is a partially ordered set, and hence a category, and the assignments

$$j: A \mapsto A, \qquad A \subseteq B \mapsto j_A^B$$

give an embedding of $v\mathcal{C}$ into $\mathcal{C}$ as a subcategory. Further, if $\mathcal{C}$ and $\mathcal{D}$ are categories with subobjects, a functor $F: \mathcal{C} \to \mathcal{D}$ is said to be *inclusion preserving* if $F|v\mathcal{C}$ is a functor to $v\mathcal{D}$; i.e. $vF: v\mathcal{C} \to v\mathcal{D}$ is a functor.

A morphism $e \in \mathcal{C}(A, B)$ in a category $\mathcal{C}$ with subobjects, is called a *retraction* if it is a right inverse of an inclusion; that is, $B \subseteq A$ and $j_B^A \circ e = 1_B$, where j_B^A is the inclusions of B in A. Here, as in [22], we have written the composition in the diagram order; we will use this convention throughout this paper. If $f \in \mathcal{C}(A, B)$ is any morphism, a *normal factorization* of f is a factorization of the form $f = euj$ where e is a retraction, u is an isomorphism and j is an inclusion. A morphism may not have a normal factorization; even when it has, its factorization need not be unique. However, we have the following:

LEMMA 1.1. *Let $f = euj = e'u'j'$ be two normal factorizations of a morphism f. Then $j = j'$ and $eu = e'u'$.*

PROOF: Let j_1 and j_1' be inclusions that are left-inverses of e and e' respectively. Then

$$j = (u^{-1} j_1 e' u') j', \qquad j' = (u'^{-1} j_1' u) j$$

Hence j and j' are equivalent as monomorphisms and so $j = j'$. If $\bar{e}$ is the right inverse of j, we have

$$eu = euj\bar{e} = f\bar{e} = e'u'j\bar{e} = e'u'.$$

When the morphism f has a normal factorization $f = euj$, we write

$$\operatorname{Im} f = \operatorname{dom} j, \quad f^\circ = eu \quad \text{and} \quad \operatorname{coim} f = \operatorname{codom} e.$$

By Lemma 1.1, $\operatorname{Im} f$ and f° are independent of the normal factorization chosen to define these. It is clear that f° has a left-inverse and so it is an epimorphism. f° will be called, the *epimorphism associated with* f. Note that a normal factorization of f° is given by $f^\circ = eu1_A$ where $A = \operatorname{Im} f$. Hence $\operatorname{Im} f^\circ = \operatorname{Im} f$. In particular, $\operatorname{coim} f = \operatorname{Im} e$. Note that $\operatorname{coim} f$ is not uniquely determined, but two coimages are always isomorphic. Hence $\operatorname{coim} f$ denotes one object arising as images of e in a given normal factorization.

DEFINITION 1.1. *Let $\mathcal{C}$ be a category with subobjects. We say that $\mathcal{C}$ is normal if it satisfies the following:*

(1) *Every inclusion in $\mathcal{C}$ is a section; that is, if $j: A \subseteq B$, there is $e: B \to A$ such that $je = 1_A$.*

(2) *Every morphism in* $\mathcal{C}$ *has a normal factorization.*

There are several natural examples for normal categories. Some of them are:

Ex. 1: **Set**, the category of sets and mappings.

Ex. 2: $\mathbf{Vct}_k$, the category of vector spaces over a field k.

Ex. 3: **Bool**, the category of Boolean algebras and complete lattice homomorphisms that preserve zero.

In categories of Examples 1 and 2, a morphism, in general has more than one normal factorization, while in the category **Bool** (Example 3), every morphism has a unique normal factorization.

Let $\mathcal{C}$ be a category with subobjects and $A \in v\mathcal{C}$. We denote by (A) (or $(A)_{\mathcal{C}}$ if necessary), the full subcategory of $\mathcal{C}$ whose objects are subobjects of A. A subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is called an *ideal* if for every $A \in v\mathcal{C}'$, $(A)_{\mathcal{C}} \subseteq \mathcal{C}'$ and $\mathcal{C}'$ is full. Evidently, for each $A \in v\mathcal{C}$, (A) is an ideal; it is called the *principal ideal* of $\mathcal{C}$ generated by A.

An arbitrary subcategory of normal category need not be normal. However, we have the following:

LEMMA 2. *Let* $\mathcal{C}$ *be a normal category. Then:*

(i) *every ideal of* $\mathcal{C}$ *is normal.*

(ii) *If* $\mathcal{C}'$ *is a subcategory such that* $v\mathcal{C}' = v\mathcal{C}$ *and* $\mathcal{C}'$ *contains all inclusions and retractions of* $\mathcal{C}$, *then* $\mathcal{C}'$ *is normal.*

PROOF: Statement (i) is clear. To prove (ii), we observe that the given conditions immediately show that the subcategory $\mathcal{C}'$ satisfies axiom (1) of Definition 1.1. To prove axiom (2), it is sufficient to show that given a normal factorization of $f \in \mathcal{C}'$, say $f = euj$, $u \in \mathcal{C}'$. Now if e' is the right inverse of j and j' is the left inverse of e then, by the given conditions, j', $e' \in \mathcal{C}'$. Hence $j'fe' = u \in \mathcal{C}'$.

The smallest subcategory of $\mathcal{C}$ satisfying conditions of (ii) will be denoted by *cor* $\mathcal{C}$. Moreover, a functor $F: \mathcal{C} \to \mathcal{D}$ of normal categories is inclusion-preserving if and only if

$$cor\ F = F \mid cor\ \mathcal{C}$$

is a functor of *cor* $\mathcal{C}$ to *cor* $\mathcal{D}$. If **Ncat** denotes the category of all small normal categories with morphisms as inclusion preserving functors, then it is clear that

$$cor\colon \mathbf{Ncat} \to \mathbf{Ncat}; \quad \mathcal{C} \to cor\ \mathcal{C}; \quad F \to cor\ F$$

is a functor.

The rest of this section is devoted to proving some properties of normal categories that will be useful later on.

LEMMA 3. *In a normal category* $\mathcal{C}$, *an inclusion* $j_A^B: A \subseteq B$ *is an isomorphism if and only if* $A = B$. *Dually a retraction* $e: B \to A$ *is an isomorphism if and only if* $A = B$. *In this case,* $e = 1_B$.

PROOF: Let j_A^B be an isomorphism. Then there is $\alpha: B \to A$ such that $\alpha j_A^B = 1_B$, $j_A^B \alpha = 1_A$. From $j_A^B 1_B = j_A^B$, $\alpha j_A^B = 1_B$, we conclude that j_A^B and 1_B are equivalent as monomorphisms and since $1_B = j_B^B$, we have, $j_A^B = j_B^B = 1_B$ and so j_A^B is an isomorphism. If $e: B \to A$ is an isomorphism and $\alpha: A \to B$ is its inverse, then $\alpha = 1_A \alpha = (j_A^B e)\alpha = j_A^B 1_B$. Hence $A = B$ by the first part. So $\alpha = j_B^B = 1_B$ and so $e = e1_B = e\alpha = 1_B$. Conversely if $A = B$, e is a right-inverse of 1_B and so $e = 1_B$ so that e is an isomorphism.

LEMMA 4. *In a normal category, a morphism* $f: A \to B$ *is a monomorphism if and only if* f *is a section and* f *is an epimorphism if and only if it has a left-inverse (that is,* f *is a retraction in the usual sense; see* [16], [28]). *Consequently a normal category is balanced.*

PROOF: The 'if' part of both statements hold in an arbitrary category. Hence it is sufficient to prove the only if part.

Assume that f is a monomorphism and $f = euj$, where $e: A \to A'$ is retraction and $j = j_{B'}^B$ with $B' = \operatorname{Im} f$, is a normal factorization of f. If $h, k: c \to A$ are morphisms such that $he = ke$, then $hf = heuj = keuj = kf$. Since f is a monomorphism, we have $h = k$. Hence e is an monomorphism. But $e = e1_{A'} = (ej_{A'}^A)e = 1_A e$ and so $ej_{A'}^A = 1_A$. This proves that e is an isomorphism and hence $A' = A$ by Lemma 3. So, $f = uj$ and if $g = e'u^{-1}$ where $e': B \to B'$ is a retraction such that $je' = 1_{B'}$, then $fg = 1_A$. Thus f is a section.

Let $f = euj$ when $e: A \to A'$ is retraction and $j = j_{B'}^B$, be an epimorphism. If $h, k: B \to D$ are morphisms such that $j_{B'}^B h = j_{B'}^B k$, then $fh = fk$. Since f is *epi*, $h = k$. Hence $j_{B'}^B$ is an epimorphism. Let $e': B \to B'$ be a retraction with $j_{B'}^B e' = 1_{B'}$. Then $j_{B'}^B \cdot 1_B = j_{B'}^B(e' j_{B'}^B)$ and so $1_B = e' j_{B'}^B$. Hence $j_{B'}^B$ is an isomorphism and so, by Lemma 3, $B = B'$. and so $j_{B'}^B = 1_B$. So $f = eu$. If $j' = j_{A'}^A$ and $g = u^{-1} j'$, $gf = 1_B$. Hence g is a left-inverse of f.

We can restate the lemma above as follows:

Corollary 5. *A morphism f in a normal category is a monomorphism if and only if f° is an isomorphism. f is an epimorphism if and only if $f^\circ = f$. Moreover, every monomorphism has a unique normal factorization.*

Proof: If f is a monomorphism, the proof of Lemma 4 shows that the retraction factor in any normal factorization of f is the identity. Hence any normal factorization of f is of the form $f = uj$. If $uj = u'j'$, by Lemma 1.1, $j = j'$ and so $u = u'$. Hence the normal factorization of f is unique. Moreover, it follows that in this case f° is an isomorphism, $f = f^\circ j$ is obviously a monomorphism.

If f is an epimorphism, the proof of Lemma 4 shows that the inclusion factor in any normal factorization of f is identity. Hence $f = f^\circ 1_B = f^\circ$. Since f° is always an epimorphism, the converse is clear.

2. Projective Complexes

In the following, $\mathcal{C}$, $\mathcal{D}$ etc. denote normal categories.

Definition 2.1. *Let $\mathcal{C}_i \subseteq \mathcal{C}$, $i = 1,2$ be subcategories. A transformation $T : \mathcal{C}_1 \to \mathcal{C}_2$ consists of three maps*

$$vT : v\mathcal{C}_1 \to v\mathcal{C}_2, \quad MT : \mathcal{C}_1 \to \mathcal{C}_2, \quad \eta_T : v\mathcal{C}_1 \to \underline{\mathcal{C}}$$?

such that:

(a) $F_T = (vT, MT) : \mathcal{C}_1 \to \mathcal{C}_2$ *is a functor; and*

(b) $\eta_T : i_{\mathcal{C}_1} \to F_T \circ i_{\mathcal{C}_2}$, *where $i_{\mathcal{C}_i} : \mathcal{C}_i \subseteq \mathcal{C}$, $i = 1,2$ are the inclusion functors, is a natural transformation.*

In the following, if no confusions is likely, all the three maps determined by T will be denoted by T itself.

If $\mathcal{C}_1 \subseteq \mathcal{C}_2$, T defined by $F_T = i_{\mathcal{C}_1}^{\mathcal{C}_2}$ and $(\eta_T)_A = 1_A$ for each $A \in v\mathcal{C}_1$ is a transformation. This will be called the inclusion and will be denoted by the $i_{\mathcal{C}_1}^{\mathcal{C}_2}$ (i.e. we shall use the same symbol as the associated functor). Clearly $i_{\mathcal{C}_1}^{\mathcal{C}_1} = 1_{\mathcal{C}_1}$ is a transformation from $\mathcal{C}_1$ to $\mathcal{C}_1$.

If $T : \mathcal{C}_1 \to \mathcal{C}_2$, $S : \mathcal{C}_2 \to \mathcal{C}_3$ on the transformations of subcategories of $\mathcal{C}$, then we define $T \circ S$ by:

$$(2.1) \qquad F_{T\circ S} = F_T F_S, \quad (\eta_{T\circ S})_A = (\eta_T)_A (\eta_S)_{F_T(A)} \quad \text{for every} \quad A \in v\mathcal{C}_1 .$$

Clearly $T \circ S: \mathcal{C}_2 \to \mathcal{C}_3$ is a transformation. This composition is associative. $T: \mathcal{C}_1 \to C_2$ is an isomorphism if there is an $S: \mathcal{C}_2 \to C_1$ such that $T \circ S = 1_{\mathcal{C}_1}$ and $S \circ T = 1_{\mathcal{C}_i}$. This is true if and only if $F_T: \mathcal{C}_1 \to \mathcal{C}_2$ is a functional isomorphism and η_T is a natural isomorphism. If $\mathcal{C}'_1 \subseteq \mathcal{C}_1$ and $T: \mathcal{C}_1 \to \mathcal{C}_2$, we write

$$T|\mathcal{C}'_1 = i^{\mathcal{C}_1}_{\mathcal{C}'_1} \circ T, \quad \text{Im } T = \text{Im } F_T; \quad \text{and} \quad T(\mathcal{C}'_1) = \text{Im } (T|\mathcal{C}'_1).$$

Recall that, if $f: A \to B$ and $A' \subseteq A$, then $f|_{A'} = j^A_{A'} f$ and $f(A')$ in the uniqre object such that

$$j^A_{A'} f = (j^A_{A'} f)^\circ j^B_{f(A')}$$

(cf. Corollary 1.5). Since subobject relation in $\mathcal{C}$ is transitive, we have

$$f(A'') \subseteq f(A') \quad \text{if} \quad A'' \subseteq A'.$$

If f is an isomorphism, for each $A' \subseteq A$, $(j^A_{A'} f)^\circ$ is an isomorphism and the map $A' \to f(A')$ is an order isomorphism of the partially ordered set $< A >$ of subobjects of A to $< B >$. We have:

LEMMA 2.1. *Let $\sigma: A \to B$ be an isomorphism in $\mathcal{C}$. For $A' \subseteq A$ and $f: A_1 \to A_2 \in (A)$, define*

$$T^\sigma(A) = \sigma(A'), \quad T^\sigma_{A'} = (\sigma|A')^\circ = (j^A_{A'}\sigma)^\circ, \quad \text{and} \quad T^\sigma(f) = (T^\sigma_{A'})^{-1} f T^\sigma_{A_2}.$$

Then $T^\sigma: (A) \to (B)$ is a transformation which is an isomorphism.

In the following we shall consider only those transformations for which η_T is a natural isomorphism. If $T: \mathcal{C}_1 \to \mathcal{C}_2$ has this property, the naturality of η_T implies that for all $f: A \to B \in b_1$,

$$T(f) = T_A^{-1} f T_B \tag{2.2}$$

This shows that T is completely determined by vT and η_T. We have the following lemma whose routine verification is omitted.

LEMMA 2.2. *Let $\mathcal{C}_i \subseteq \mathcal{C}$, $i = 1,2$ be subcategories of $\mathcal{C}$, $\theta: v\mathcal{C}_1 \to v\mathcal{C}_2$, $\phi: v\mathcal{C}_1 \to \mathcal{C}$ maps such that*

(1) *θ is inclusion preserving;*

(2) *for each* $A \in v\mathcal{C}_1$, $\phi(A): A \to \theta(A)$ *is an isomorphism; and*
(3) *for* A', $A \in v\mathcal{C}_1, A' \subseteq A$ *implies* $\phi(A') = (\phi(A)|A')^\circ$.
Then there is a unique transformation $T = T(\theta,\phi): \mathcal{C}_1 \to \mathcal{C}_2$ *such that* $vT = \theta$, $\eta_T = \phi$ *and for each* $f: A \to B \in b_1$, $T(f) = \phi(A)^{-1} f \phi(B)$.

Conversely if $T: \mathcal{C}_1 \to \mathcal{C}_2$ *is a transformation such that* η_T *is an isomorphism, then the maps* vT *and* η_T *satisfies conditions* (1), (2) *and* (3). *Further, we have* $T = T(vT, \eta_T)$.

PROPOSITION 2.3. *Let* $T: \mathcal{C}' \to \mathcal{C}''$ *be a transformation of subcategories of* $\mathcal{C}$. *If* C' *is an ideal in* $\mathcal{C}$, Im T *is an ideal in* $\mathcal{C}$ *and for every* $A \in v\mathcal{C}'$,

$$T|(A) = T^{T_A}$$

where T^{T_A} *is the isomorphism of* (A) *to* $(T(A))$ *defined in Lemma 2.1.*

PROOF: Since $\mathcal{C}'$ is an ideal, it is full in $\mathcal{C}$ and so for $A, B \in v\mathcal{C}'$, $g \in \mathcal{C}(T(A), T(B))$ implies $T_A g T_B^{-1} \in \mathcal{C}'(A,B)$ and $T(T_A g T_B^{-1}) = g$. Hence Im T is full. If $A, B \in v\mathcal{C}'$, $g \in \mathcal{C}(T(A),T(B))$, $h \in \mathcal{C}(T(B),T(C))$, then $T(T_A g h T_C^{-1}) = gh$ and so Im T is a subcategory of $\mathcal{C}''$. Suppose $A \in v\mathcal{C}'$, $B = T(A)$ and $B' \subseteq B$. If $j_{B'}^{B} T_A^{-1} = u j_{A'}^{A}$ is a normal factorization of the monomorphism $j_{B'}^{B} T_{A'}^{-1}$ then by Corollary 1.5, $u^{-1} = T_{A'}$ and $B' = T(A')$. Thus Im T is an ideal in $\mathcal{C}$. For $A' \subseteq B$, we have $j_{A'}^{A} T_A = T_{A'} j_{T(A')}^{T(A)}$ and so $T_{A'} = T_A|A' = T_{A'}^{T_A}$ and $\eta_{T|(A)} = \eta_{T^{T_A}}$. Hence by Lemma 2.2, $T|(A) = T^{T_A}$.

discrete? category

A *preorder* $\mathcal{A}$ in a category $\mathcal{C}$ is subcategory such that for any two objects A, B of $\mathcal{A}$, $\mathcal{A}(A,B)$ contains at most one morphism. If $\mathcal{A}(A,B) \neq \phi$, the unique morphism in $\mathcal{A}(A,B)$ will be denoted by K_A^B (or $K_A^B(\mathcal{A})$). We shall write: $K_A^B: A < B(\mathcal{A})$ (read A is related to B modulo $\mathcal{A}$). $v\mathcal{C}$ and for each $\mathcal{A} \in v\mathcal{C}$, $< A >$ are examples of preorders in $\mathcal{C}$. As is usual for partially ordered sets, we shall use the symbol $\mathcal{A}$ also to denote the set of objects of the preorder $\mathcal{A}$. Then the relation $<$ on $\mathcal{A}$ is reflexive and transitivities and so is a quasiorder on $\mathcal{A}$. The equivalence relation $< \circ <^{-1}$ will be denoted by $\equiv$. Thus $A \equiv B$ in $\mathcal{A}$ if and only if $A < B$ and $B < A$. Transitivity of $<$ implies that K_A^B is an isomorphism if $A \equiv B(\mathcal{A})$. ?

By a complex in a category $\mathcal{C}$ we mean an order ideal $\mathcal{A}$ of $v\mathcal{C}$ which is small as a category. Thus a complex $\mathcal{A}$ is a preorder in which morphisms are inclusions and such that for each $A \in \mathcal{A}$, $< A > \subseteq \mathcal{A}$. The requirement that $\mathcal{A}$ is small implies that $\mathcal{C}$ is 'locally small'. In this case, for each $A \in v\mathcal{C}$, $< A >$ is in particular, a complex. If $\mathcal{C}$ is small, then $v\mathcal{C}$ is also a complex. We observe that the usual complexes (see for example [36]) are complexes in **Set**. A *projection* of a complex $\mathcal{A}$ is an idempotent transformation $P: \mathcal{A} \to \mathcal{A}$ such that $\text{Im } P =< A >$ for some $A \in \mathcal{A}$. The unique object A generating $\text{Im } P$ will be denoted by A_P. A *chamber* in a complex $\mathcal{A}$ is a maximal element in $\mathcal{A}$.

LEMMA 2.4. *For a complex $\mathcal{A}$ in the normal category $\mathcal{C}$ we have the following:*
(1) *$\mathcal{A}$ has a projection if and only if there is a surjective transformation $T: \mathcal{A} \to$ $< B >$ for some $B \in v\mathcal{C}$.*
(2) *If P is projection of $\mathcal{A}$, A_P is a chamber in $\mathcal{A}$.*

PROOF: If P is a projection then $P: \mathcal{A} \to < A_P >$ is a surjective transformation. Conversely if $T: \mathcal{A} \to < B >$ is a surjective transformation, then $T(A) = B$ for some $A \in \mathcal{A}$. By Lemma 2.2, T induces a transformation $T: (A) \to (B)$ when (A) is the ideal generated by A and by Proposition 2.3, $T|(A) = T^{T_A}$. Since T^{T_A} is an isomorphism with inverse $T^{T_A^{-1}}$, $T \circ T^{T_A^{-1}}$ is an idempotent, surjective transformation of (A) to (A). Since $T \circ T^{T_A^{-1}}$ is inclusion preserving this induces a transformation $P: \mathcal{A} \to \mathcal{A}$ with $P^2 = P$ and $\text{Im } P =< A >$; i.e. P is a projection.

(2) If $A_P \leq A$, $A \in \mathcal{A}$, $P_A: A \to P(A) \subseteq A_P$ is an isomorphism. Also $P_{A_P} = 1_{A_P}$ and so $1_{A_P} = P_{A_P} = P_A|A_P = j^A_{A_P} P_A$. Hence P_A is a retraction and so, by Lemma 1.3, $P_A = 1_A$, i.e. $A \subseteq A_P$.

Note that there are complexes that does not have projections. For a polygon with n sides, considered as a complex in the usual way has projection if and only if n is even.

DEFINITION 2.2. *A projective complex in a normal category $\mathcal{C}$ is a (small) preorder $\mathcal{A}$ satisfying the following:*
(1) *If $\mathcal{A} < B(\mathcal{A})$, $(K_A^B)^\circ: A \equiv \text{Im } K_A^B(A)$.*
(2) *There is $C \in \mathcal{A}$ such that $A < C(A)$ for all $A \in \mathcal{A}$.*
(3) *If $A \in \mathcal{A}$, then $< A > \subseteq \mathcal{A}$ and the inclusion is an embedding.*

It follows from (1) that each morphism K_A^B of $\mathcal{A}$ is a monomorphism and (3) shows that $\mathcal{A}$ with respect to inclusions is a complex. The relation $<$ on $\mathcal{A}$ together with associated morphisms $\{K_A^B: A < B,\ A, B \in \mathcal{A}\}$ will be called the *projective fructure* of $\mathcal{A}$. In view of (2), a complex $\mathcal{A}$ will be a projective complex (with respect to inclusion) if and only if $\mathcal{A} = < A >$ for some $A \in \mathcal{A}$.

Next result shows that the projection structure of a projective complex $\mathcal{A}$ is completely determined by a projection of the underlying complex $\mathcal{A}$.

PROPOSITION 2.5. *Let $\mathcal{A}$ be a projective complex and $c \in \mathcal{A}$ be such that $A < c(\mathcal{A})$ for all $A \in \mathcal{A}$. Thus P^c defined by*

$$P^c_A = (K^c_A)^\circ, \quad P^c(A) = \operatorname{Im} K^c_A \tag{2.3}$$

is a projection of the complex A such that:

$$A < B(\mathcal{A}) \Leftrightarrow P^c(A) \subseteq P^c(B); \quad \text{and} \tag{2.4}$$

$$K^B_A = P^c_A j^{P(B)}_{P(A)} P^{-1}_B, \quad \text{for} \quad A < B(\mathcal{A}). \tag{2.5}$$

Conversely if $\mathcal{A}$ is a complex with a projection P, then Equations (2.4) and (2.5) defines a projective structure on $\mathcal{A}$ such that $A < A_P(\mathcal{A})$ for all $A \in \mathcal{A}$ and $P = P^{A_P}$.

PROOF: Let $\mathcal{A}$ be a projective complex. We first prove (2.4) and (2.5). Let $A < B(\mathcal{A})$. Since $A < B \equiv P^c(B)$, $A < P^c(B)$. Let $K_A^{P^c(B)} = u j_{B_1}^{P^c(B)}$ be a normal factorization of $K_A^{P^c(B)}$. Then $K^c_A = u j^c_{B_1} = P^c_A j^c_{P^c(A)}$. By Corollary 1.5, $B_1 = P^c(A)$ and $u = P^c_A$. Hence $P^c(A) \subseteq P^c(B)$ and $K^B_A P^c_B j^{P^c(B)}_{P^c(A)}$. Thus $K^B_A = P^c_A j^{P^c(B)}_{P^c(A)} P^{c-1}_B$. Hence (2.5) holds. If $P^c(A) \subseteq P^c(B)$, then $A \equiv P^c(A) < P^c(B) \equiv B$; i.e. $A < B$. This proves (2.4). It follows immediately that the two maps $A \to P^c(A)$ and $A \to P^c_A$ satisfies conditions (1), (2) and (3) of Lemma 2.2 and hence defines a transformation $P^c : \mathcal{A} \to < C >$. If $A \subseteq c$, $K^c_A = j^c_A$ and so $P^c(A) = A$, $P^c_A = 1_A$. Hence P^c is a projection.

Conversely let P be a projection on the complex $\mathcal{A}$. Clearly $< \alpha$ defined by (2.4) is a quasiorder and if $A < B < c$, then $K^B_A K^c_B = K^c_A$. Thus $\mathcal{A}$ with morphisms defined by (2.5) is a preorder in $\mathcal{C}$. Let $A_P = c$. Then $P(c) = c$, $P_c = 1_c$. Since $P(A) \subseteq c$ for all $A \in \mathcal{A}$, $A < c$ for all $A \in \mathcal{A}$. Further by (2.5), $K^c_A = P_A j^c_{P(A)}$. Let $A < B$. By the definition K^B_A is a monomorphism and so $u = (K^B_A)^\circ$ is an isomorphism of A into $B_1 = \operatorname{Im} K^B_A$. Hence $K^c_A =$

$K_A^B K_B^c = u j_{B_1}^B P_B j_{P(B)}^c = u P_{B_1} j_{P(B_1)}^c$. Hence by Corollary 1.5, $P(B_1) = P(A)$ and $uP_{B_1} = P_A$. So by (2.5), $u = P_A P_{B_1}^{-1} = K_A^{B_1}$ and $u^{-1} = K_{B_1}^A$. Hence $(K_A^B)^\circ = K_A^{B_1}: A \equiv B_1$. This proves axioms (1) and (2) of Definition 2.2. To prove (3), consider $A \in \mathcal{A}$. If $A' \subseteq A$, $P(A') \subseteq P(A)$ and $P_{A'} = P_A | A'$. Hence $K_{A'}^A = P_{A'} j_{P(A')}^{P(A)} P_A^{-1} = j_{A'}^A$. Hence $< A >$ is a subcategory of $\mathcal{A}$. Let $A_1, A_2 \in < A >$ and $A_1 < A_2$. Then $P(A_1) \subseteq P(A_L)$. By Proposition 2.3, $P| < A >$ is an isomorphism onto $< P(A) >$ and so $A_1 \subseteq A_2$. Thus $< A > \subseteq \mathcal{A}$ is an embedding. It follows from the equality $K_A^c = P_A j_{P(A)}^c$, that $P_A^c = P_A$ and $P(A) = P^c(A)$ by (2.3). This completes the proof.

The proposition above shows that every projective structure on the complex A determines a projection on $\mathcal{A}$ and conversely. However, two different projections may determine the same projective structure. We have the following:

LEMMA 2.6. *Two projections P and P' of a complex $\mathcal{A}$ determines the same projective structure if and only if $PP' = P'$, $P'P = P$.*

PROOF: Suppose that P and P' determine the same projective structure on $\mathcal{A}$. Then for $A \in \mathcal{A}$, $P_A = K_A^{A_P}$, $P'_A = K_A^{P'_A}$. Hence $P'(P(A)) = P'(A)$ and $P'_A = P_A P'_{P(A)}$. (Since $P'(P(A))$, $P'(A) \subseteq A_{P'}$, we have $P'(P(A)) = P'(A)$ and $P'_A = P_A P'_{P(A)}$.) Hence by (2.1), we have $PP' = P'$. Similarly. $P'P = P$.

Conversely assume that $PP' = P'$, $P'P = P$. Let $<$ and $\{K_A^B\}$ be quasiorder and morphisms defined using P by Equations (2.4) and (2.5); and $<'$ and $\{K_A'^B\}$ be those defined using P'. Now

$$A < B \Leftrightarrow P(A) \subseteq P(B) \Leftrightarrow P'(P(A)) \subseteq P'(P(B))$$

since P' preserves inclusions. Since $P'(P(A)) = P'(A)$, $P'(P(B)) = P'(B)$ we get $A < B \Leftrightarrow P'(A) \subseteq P'(B) \Leftrightarrow A <' B$. Hence $< = <'$. Also $K_A'^B = P'_A j_{P'(A)}^{P'(B)} P_B'^{-1} = P_A P'_{P(A)} j_{P'(A)}^{P'(B)} P_B'^{-1} P_B^{-1} = P_A j_{P(A)}^{P(B)} P_B^{-1}$ since $P'_{P(A)} j_{P'(A)}^{P'(B)} P_{P(B)}'^{-1} = K_{P(A)}^{P(B)} = j_{P(A)}^{P(B)}$. Hence $K_A^B = K_A'^B$ for all $A, B \in \mathcal{A}$ with $A < B$. This proves the lemma.

The conditions $PP' = P'$, $P'P = P$ in the lemma above defines an equivalence relation on the set of all projections on the complex $\mathcal{A}$ and each equivalence class determines a unique projective structure on $\mathcal{A}$. Thus given the complex $\mathcal{A}$, a projective complex on $\mathcal{A}$ may be represented as $(\mathcal{A}, [P])$ where $[P]$ denote the equivalence class of projections in $\mathcal{A}$ containing P.

It follows from Lemma 2.4 and Proposition 2.5 that the object c of axiom (2) of Definition 2 is a chamber of the complex $\mathcal{A}$. We set:

$$\mathcal{M}\mathcal{A} = \{c \in \mathcal{A} : A < c \quad \text{for all} \quad A \in \mathcal{A}\}. \tag{2.6i}$$

Thus $\mathcal{M}\mathcal{A}$ is a set of chambers of $\mathcal{A}$. But not all chambers of $\mathcal{A}$ need to be in $\mathcal{M}\mathcal{A}$. Those chambers c that are in $\mathcal{M}\mathcal{A}$ are characterized by the fact that there is a projection P with $A_P = c$. Also given $c \in \mathcal{M}\mathcal{A}$, we have

$$[P^c] = \{P^{c'} : c' \in \mathcal{M}\mathcal{A}\} \quad \text{and} \tag{2.6ii}$$

$$\mathcal{M}\mathcal{A} = \{c' \in \mathcal{A} : K^c_{c'} \quad \text{is an isomorphism}\}. \tag{2.6iii}$$

PROPOSITION 2.7. *Let $\mathcal{A}$ be a projective complex in the normal category $\mathcal{C}$, $c \in \mathcal{M}\mathcal{A}$ and $h: c \to c'$ be a retraction. Then there is a unique projective complex $\mathcal{A}_h$ whose object set is given by:*

$$\mathcal{A}_h = \{A \in \mathcal{A} : K^c_A h \quad \text{is a monomorphism}\} \tag{2.7i}$$

and whose projective structure is determined by the projection $P^{c'}$ of $\mathcal{A}_h$ defined as follows: for each $A \in \mathcal{A}_h$,

$$P^{c'}_A = (K^c_A h)^\circ, \quad P^{c'}(A) = \operatorname{Im}(K^c_A h). \tag{2.7ii}$$

Moreover, $\mathcal{A}_h$ satisfy the following:
(a) *$A < B(\mathcal{A})$, $A, B \in \mathcal{A}_h \Rightarrow A < B(\mathcal{A}_h)$ and $K^B_A(\mathcal{A}) = K^B_A(\mathcal{A}_h)$.*
(b) *If $c_1 \in \mathcal{M}\mathcal{A}$ then there is $c'_1 \in \mathcal{M}\mathcal{A}_h$ with $c'_1 \subseteq c_1$ and for every such pair (c_1, c'_1), there is a retraction $h_1: c_1 \to c'_1$ such that $\mathcal{A}_h = \mathcal{A}_{h_1}$.*

PROOF: If $A' \subseteq A$, $A \in \mathcal{A}_h$, then $K^c_{A'} h = K^A_{A'} K^c_A h = j^A_{A'} K^c_A h$. Since $K^c_A h$ is monomorphism, so is $K^c_{A'} h$. Hence $A' \in \mathcal{A}_h$ and so $\mathcal{A}_h$ is a complex. Also, if $A' \subseteq A$, $A', A \in \mathcal{A}_h$, then

$$P^{c'}_{A'} = (K^c_{A'} h)^\circ = (j^A_{A'} K^c_A h)^\circ = (K^c_A h | A')^\circ = P^{c'}_A | A'$$

and so $P^{c'}(A') \subseteq P^{c'}(A)$. Thus the maps $A \to P^{c'}(A)$, $A \to P^{c'}_A$ defined by (2.7ii) satisfies conditions (1) and (3) of Lemma 2.2. Since $K^c_A h$ is a monomorphism, by Corollary 1.5, $P^{c'}_A$ is an isomorphism for each $A \in \mathcal{A}_h$. Thus $P^{c'}$

is a transformation. Clearly Im $P^{c'} = < c' >$. If $A \subseteq c'$ then, $A \subseteq c$ so, $K_A^{c'}(\mathcal{A}_h) = K_A^c h = j_A^{c_h} = j_A^{c'} j_{c'}^{c_h} = j_A^{c'}$. Hence $P^{c'}(A) = A$ and $P_A^{c'} = 1_A$. Therefore $P^{c'}$ is a projection on $\mathcal{A}_h$ and so $(\mathcal{A}_h, [P^{c'}]) = \mathcal{A}_h$ is a projective complex.

To prove statements (a) and (b), first consider $A \in \mathcal{A}_h$. Then $P_A^{c'} j_{P^{c'}(A)}^{c'} = K_A^{c'}(\mathcal{A}_h) = K_A^c h = P_A^c j_{P^c(A)}^c h = P_A^c P_{P^c(A)}^{c'} j_{P^{c'}(P^c(A))}^{c'}$. Hence by Corollary 1.5 we have $P_A^{c'} = P_A^c P_{P^c(A)}^{c'}$ and $P^{c'}(A) = P^{c'}(P^c(A))$. Therefore if $A < B(\mathcal{A})$, $A, B \in \mathcal{A}_h$, then $P^c(A) \subseteq P^c(B)$ and this implies $P^{c'}(A) = P^{c'}(P^c(A)) \subseteq P^{c'}(B)$ and so $A < B(\mathcal{A}_h)$. Further,

$$\begin{aligned} K_A^B(\mathcal{A}_h) &= P_A^{c'} j_{P^{c'}(A)}^{P^{c'}(B)} P_B^{c'-1} = P_A^c P_{P^c(A)}^{c'} j_{P^{c'}(P^c(A))}^{P^{c'}(P^c(B))} P_{P^c(B)}^{P^{c'}-1} P_B^{c-1} \\ &= P_A^c (K_{P^c(A)}^{P^c(B)}(\mathcal{A}_h)) P_B^{c-1} = P_A^c j_{P^c(A)}^{P^c(B)} P_B^{c-1} = K_A^B(\mathcal{A}). \end{aligned}$$

This proves (a). To prove (b) consider $c_1 \in \mathcal{MA}$. Then there is $c_1' \subseteq c_1$ such that $P^c(c_1') = c'$, since $P^c | < c_1 >$ is an isomorphism of $< c_1 >$ onto $< c >$ by (2.6iii). Thus $c_1' \equiv c'(\mathcal{A})$ and so $c_1' \equiv c'(\mathcal{A}_h)$. Hence $c_1' \in \mathcal{MA}_h$. Let $h_1 = P_{c_1}^c h P_{c_1'}^{c-1}$ Then $j_{c_1'}^{c_1} h_1 = j_{c_1'}^{c_1} P_{c_1}^c h P_{c_1'}^{c-1} = K_{c_1'}^c h P_{c_1'}^{c-1} = P_{c_1'}^{c'} P_{c_1'}^{c-1}$. Since $c_1', c' \in \mathcal{A}_h$ we have $P_{c_1'}^c = P_{c_1'}^{c'}$ and so $j_{c_1'}^{c_1'} h_1 = 1_{c_1'}$. Hence h_1 is a retraction. Also $K_A^{c_1} h_1$ is a monomorphism if and only if $K_A^c h$ is a monomorphism. Hence $\mathcal{A}_h = \mathcal{A}_{h_1}$ as complexes. Further, since $P_{c_1'}^c = P_{c_1'}^{c'}$, we have $P_{c_1'}^{c_1} = K_{c'}^{c_1'}(\mathcal{A}_{h_1}) = K_{c'}^{c'}(\mathcal{A}_h) P_{c_1'}^{c'} = P_{c_1'}^{c'-1}$. Hence we obtain from the equation above:

$$P_A^{c_1'} j_{P^{c_1'}(A)}^{c_1'} = P_A^{c'} j_{P^{c'}(A)}^{c'} P_{c'}^{c_1'} = P_A^{c'} P_{P^{c'}(A)}^{c_1'} j_{P^{c_1'}(P^{c'}(A))}^{c_1'}$$

for any $A \in \mathcal{A}_h$ and so by Corollary 1.5,

$$P^{c_1'}(A) = P^{c_1'}(P^{c'}(A)), \quad P_A^{c_1'} = P_A^{c'} P_{P^{c'}(A)}^{c_1'}$$

It follows that $P^{c'} P^{c_1'} = P^{c_1'}$. Similarly $P^{c_1'} P^{c'} = P^{c'}$ and so $P^{c'}$ and $P^{c_1'}$ determine the same projective structure by Lemma 2.6. Hence $\mathcal{A}_h = \mathcal{A}_{h_1}$ as projective complexes. This also proves that for each $c_1 \in \mathcal{MA}$, there is $c_1' \subseteq c_1$ and retraction $h_1 : c_1 \to c_1'$ such that $\mathcal{A}_h = \mathcal{A}_{h_1}$. Now to complete the proof of (b) it is sufficient to show that if $c'' \in \mathcal{MA}_h$, $c'' \subseteq c'$, there is $h' : c \to c''$ such that $\mathcal{A}_h = \mathcal{A}_{h'}$. Define $h' = h K_{c'}^{c''}(\mathcal{A}_h)$. Then $j_{c''}^c h' = (j_{c''}^c h) K_{c'}^{c''}(\mathcal{A}_h) =$

$K^{c'}_{c''}(\mathcal{A}_h)K^{c''}_{c'}(\mathcal{A}_h) = 1_{c''}$. Hence h' is a retraction. Since $K^{c''}_{c'}(\mathcal{A}_h) = P^{c'-1}_{c''}$ we have $K^{c}_{A}h' = (K^{c}_{A}h)P^{c'-1}_{c''} = K^{c'}_{A}(\mathcal{A}_h)P^{c'-1}_{C''}$. As before, it follows that the projective complexes $\mathcal{A}_h$ and $\mathcal{A}_{h'}$ are the same. This completes the proof.

We define a relation $\leq$ among projective complexes of a normal category $\mathcal{C}$ as follows:

$$\mathcal{A}' \leq \mathcal{A} \Leftrightarrow \mathcal{A}' = \mathcal{A}_h \quad \text{for some retraction} \quad h: c \to c', \quad c \in \mathcal{MA}. \tag{2.8}$$

If $c \in v\mathcal{C}$ and $\mathcal{A}$ is a preorder in $\mathcal{C}$, we denote by $(c) \cap \mathcal{A}$ the preorder whose objects are those common to $(\mathcal{C})$ and $\mathcal{A}$ and morphisms are those of $\mathcal{A}$. Note that $< c >$ is a projective complex in $\mathcal{C}$.

Definition 2.3. *Let $\mathcal{C}$ be a small normal category. A projective complex $\mathcal{A}$ is residual in $\mathcal{C}$ if for all $c \in v\mathcal{C}$, $(c) \cap \mathcal{A}$ is a projective complex in $\mathcal{C}$ such that $(c) \cap \mathcal{A} \leq < c >$.*

When $\mathcal{C}$ is small we denote by $\underline{Proj}\ \mathcal{C}$ the set of all projective complexes in $\mathcal{C}$ and $\mathcal{PC}$ those complexes of $\underline{Proj}\ \mathcal{C}$ that are residual in $\mathcal{C}$.

Proposition 2.8. *If $\mathcal{C}$ is a small normal category, $\underline{Proj}\ \mathcal{C}$ is a partially ordered set with respect to $\leq$ and $\mathcal{PC}$ is an order ideal of $\underline{Proj}\ \mathcal{C}$.*

Proof: For any $\mathcal{A} \in \underline{Proj}\ \mathcal{C}$ and $c \in \mathcal{MA}$, $\mathcal{A} = \mathcal{A}_{r_c}$ and so the relation $\leq$ defined by (2.8) is reflexive. Let $\mathcal{A}'' \leq \mathcal{A}'$, $\mathcal{A}' \leq \mathcal{A}$ where $\mathcal{A}, \mathcal{A}', \mathcal{A}'' \in \underline{Proj}\ \mathcal{C}$. Then by Proposition 2.7, given $c \in \mathcal{MA}$, we can find $c' \in \mathcal{MA}'$ and $c'' \in \mathcal{MA}''$ such that there exist retractions $h': c \to c'$, $h'': c' \to c''$ with $\mathcal{A}' = \mathcal{A}'_h$, $\mathcal{A}'' = \mathcal{A}'_{h''}$. Now $h = h'h'': c \to c''$ is a retraction and $(\mathcal{A}_{h'})_{h''} = \mathcal{A}_{h'h''} = \mathcal{A}_h = \mathcal{A}''$. Hence $\mathcal{A}'' \leq \mathcal{A}$ and so $\leq$ is transitive. If $\mathcal{A}' \leq \mathcal{A}$, $\mathcal{A} \leq \mathcal{A}'$ then for $c \in \mathcal{MA}$, we can find $c' \in \mathcal{MA}'$, $c'' \in \mathcal{MA}$ and retractions $h: c \to c'$, such that $\mathcal{A}' = \mathcal{A}_h$, $\mathcal{A} = \mathcal{A}_{h'}$. Thus $c'' \subseteq c' \subseteq c$ and so $K^{c}_{c''} = j^{c}_{c''}$. Since c'', $c \in \mathcal{MA}$, $K^{c}_{c''} = j^{c}_{c''}$ is an isomorphism and so by Lemma 1.3, $c'' = c$. Thus $c'' = c' = c$ and so $h = h' = 1_c$. Hence $\mathcal{A} = \mathcal{A}'$.

To prove that $\mathcal{PC}$ is an order ideal of $\underline{Proj}\ \mathcal{C}$, consider $\mathcal{A} \in \mathcal{PC}$ and $\mathcal{A}' \leq \mathcal{A}$. If $c \in v\mathcal{C}$, by Definition 2.3, there is a retraction $k: c \to c'$ with $(c) \cap \mathcal{A} = < c >_k$. Let $\bar{c} \in \mathcal{MA}$ and $h: \bar{c} \to \bar{c}'$ be a retraction with $\mathcal{A}' = \mathcal{A}_h$. Since $c' \in (c) \cap \mathcal{A}$, $c' < \bar{c}(\mathcal{A})$. Let $K^{\bar{c}}_{c'}h = k'uj$ be a normal factorization of $k^{\bar{c}}_{c'}h$. If $k': c' \to c''$ then

$k'' = kk' : c \to c''$ is a retraction and $k^{\bar{c}}_{c''} h = j^{c'}_{c''} k^{\bar{c}}_{c'} = j^{c'}_{c''} k' u j = uj$. Hence $k^{\bar{c}}_{c''} h$ is a monomorphism and so $c'' \in (c) \cap \mathcal{A}$. Also for any $c_1 \in (c) \cap \mathcal{A}$, $c_1 < c'$ and so

$$k^{\bar{c}}_{c_1} h = k^{c'}_{c_1} k^{\bar{c}}_{c'} h = j^{c}_{c_1} k k' u j = j^{c}_{c_1} k'' h$$

which is a monomorphism if and only if $j^{c}_{c_1} k''$ is a monomorphism. Hence $c_1 \in (c) \cap \mathcal{A}_h$ if and only if $c_1 \in < c >_{k''}$. Further, we have

$$P^{\bar{c}'}_{c_1} j^{\bar{c}'}_{P^{\bar{c}'}(c_1)} = k^{\bar{c}}_{c_1} h = P^{c''}_{c_1} j^{c''}_{P^{c''}(c_1)} P^{\bar{c}'}_{c''} j^{\bar{c}'}_{P^{\bar{c}'}(c'')} = P^{c''}_{c_1} P^{\bar{c}'}_{P^{c''}(c_1)} j^{\bar{c}'}_{P^{\bar{c}'}(P^{c''}(c_1))}$$

and this implies by Corollary 1.5, $P^{\bar{c}'}(c_1) = P^{\bar{c}'}(P^{c''}(c_1))$, $P^{\bar{c}'}_{c_1} = P^{c''}_{c_1} P^{\bar{c}'}_{P^{c''}(c_1)}$. As in the proof of Proposition 2.7, it now follows that the preorder on $(c) \cap \mathcal{A}_h$ induced by $P^{\bar{c}'}$ coincides with the projective structure on $< c >_{k''}$ induced by the projection $P^{c''}$ of $< c >_{k''}$. Hence $(c) \cap \mathcal{A}_h =< c >_{k''}$. This proves that $\mathcal{A}' = \mathcal{A}_h$ is residual in b.

The following observation is useful.

PROPOSITION 2.9. *Let $\mathcal{A} \in \mathcal{PC}$ and $A \in \mathcal{A}$. Then there is $\mathcal{A}' \in \mathcal{PC}$ such that $\mathcal{A}' \leq \mathcal{A}$ and $A \in \mathcal{M}\mathcal{A}'$.*

PROOF: Let $c \in \mathcal{M}\mathcal{A}$ and $c' = P^c(A)$. Since $c' \subseteq c$, there is a retraction $h : c \to c'$. Let $\mathcal{A}' = \mathcal{A}_h$. Thus

$$K^{c'}_A(A_h) = K^c_A h = P^c_A j^c_{P^c(A)} h = P^c_A$$

and so $K^{c'}_A(A_h)$ is an isomorphism. Hence $A \in \mathcal{M}\mathcal{A}_h = \mathcal{M}\mathcal{A}'$ by (2.6iii). By (2.8), $\mathcal{A}' \leq \mathcal{A}$ and by Proposition 2.8, $\mathcal{A}' \in \mathcal{PC}$.

3. Normal Duals

In the following $\mathcal{C}$, $\mathcal{D}$ etc. will stand for normal categories.

DEFINITION 3.1. *By a normal cone γ in a normal category $\mathcal{C}$, we shall mean a mapping $\gamma : v\mathcal{C} \to Mor\ \mathcal{C}$ such that*

(i) *there is $c_\gamma \in v\mathcal{C}$ such that for each $A \in v\mathcal{C}$, $\gamma_A : A \to c_\gamma$;*

(ii) *there exists at least one $c \in v\mathcal{C}$ such that $\gamma_c : c \to c_\gamma$ is an isomorphism.*

Recall that in a normal category $\mathcal{C}$, $v\mathcal{C}$ is a partially ordered set and is a subcategory of $\mathcal{C}$. The definition above implies that a normal cone γ is a cone from the base $v\mathcal{C}$ to the vertex c_γ such that at least one component of γ is an isomorphism. In the following we shall denote by $M\gamma$ the set of all $c \in v\mathcal{C}$ such that γ_c is an isomorphism. Also $T\mathcal{C}$ will denote the set of all normal cones in $\mathcal{C}$.

Recall from Lemma 1.1 that each morphism f in a normal category $\mathcal{C}$ has a unique factorization $f = f^\circ j$ where f° is an epimorphism and j is an inclusion.

LEMMA 3.1. *Let $\gamma \in T\mathcal{C}$ and*

$$(3.1) \qquad \mathcal{A}_\gamma = (\{A \in v\mathcal{C}: \gamma_A \ \text{ is a monomorphism}\}, <)$$

where $A < B \Leftrightarrow \operatorname{Im}\gamma_A \subseteq \operatorname{Im}\gamma_B$. Then $\mathcal{A}_\gamma \in \mathcal{PC}$ and $M\mathcal{A}_\gamma = M\gamma$. For each $c \in M\mathcal{A}_\gamma$, the projection P^c of $\mathcal{A}_\gamma$ is defined by:

$$(3.2) \quad P^c(A) = \operatorname{Im}(\gamma_A\gamma_c^{-1}) = \gamma_c^{-1}(\operatorname{Im}\gamma_A), \quad P^c_A = (\gamma_A\gamma_c^{-1})^\circ = \gamma_A^\circ \circ (\gamma_A^{-1}|\operatorname{Im}\gamma_A)$$

for each $A \in \mathcal{A}_\gamma$.

PROOF: If $A \in \mathcal{A}_\gamma$ and $A' \subseteq A$, $\gamma_{A'} = j^A_{A'}\gamma_A$ is a monomorphism since γ_A is a monomorphism. Hence $\mathcal{A}_\gamma$ is a complex. Let $c \in M\gamma$. By Proposition 2.3, the map $A \to P^c(A)$ defined by (3.2) is inclusion preserving, since γ_c^{-1} is an isomorphism and $\operatorname{Im}\gamma_{A'} \subseteq \operatorname{Im}\gamma_A$ if $A' \subseteq A$. Also $P^c_A: A \to P^c(A)$ is clearly an isomorphism for each $A \in \mathcal{A}_\gamma$. Moreover if $A' \subseteq A$, then $P^c_{A'} = (j^A_{A'}P^c_A)^\circ = (P^c_A|A')^\circ$. Hence by Lemma 2.2, $P^c: \mathcal{A}_\gamma \to < c >$ is a transformation. From (3.2), it is clear that for $c_1 \subseteq c'$, $P^c(c_1) = c_1$ and $P^c_{c_1} = (j^c_{c_1})^\circ = 1_{c_1}$. Hence P^c is a projection of $\mathcal{A}_\gamma$. Hence $M\gamma \subset M\mathcal{A}_\gamma$. If $D \in M\mathcal{A}_\gamma$, $P^c_D: D \to c$ is an isomorphism. Hence $P^c_D = \gamma_D\gamma_c^{-1}$ is an isomorphism and so $\gamma_D = P^c_D\gamma_c$ is an isomorphism. Thus $M\mathcal{A}_\gamma \subseteq M\gamma$.

To prove that $\mathcal{A}_\gamma$ is residual in $\mathcal{C}$, consider $B \in v\mathcal{C}$. Let $h: B \to B'$ be a retraction such that $\gamma_B = huj$ is a normal factorization of γ_B. Then $\gamma_{B'} = j^B_{B'}\gamma_B = uj$ and so $\gamma_{B'}$ is a monomorphism; i.e. $B' \in \mathcal{A}_\gamma$. (If $B'' \in (B) \cap \mathcal{A}_\gamma$, then $\gamma_{B''} = j^B_{B''}\gamma_B$ and so $\operatorname{Im}\gamma_{B''} \subseteq \operatorname{Im}\gamma_B = \operatorname{Im}\gamma_{B'}$.) Thus $B'' < B'$ in $\mathcal{A}_\gamma$.

Also, by Equation 2.5,

$$
\begin{aligned}
K_{B''}^{B'} &= P_{B''}^{c}\, j_{P^c(B'')}^{P^c(B')}\, P_{B'}^{c-1} = (\gamma_{B''}\gamma_c^{-1})^{\circ}\, j_{P^c(B'')}^{P^c(B')}((\gamma_{B'}\gamma_c^{-1})^{\circ})^{-1} \\
&= \gamma_{B''}^{\circ}\gamma_c^{-1}|\mathrm{Im}\,\gamma_{B''}\, j_{P^c(B'')}^{P^c(B')}(\gamma_c|P^c(B'))\gamma_{B'}^{\circ-1} \\
&= \gamma_{B''}^{\circ}\gamma_c^{-1}|\mathrm{Im}\,\gamma_{B''}\gamma_c|P^c(B'')\, j_{\mathrm{Im}\,\gamma_{B''}}^{\mathrm{Im}\,\gamma_{B'}}\gamma_{B'}^{\circ-1} \\
&= \gamma_{B''}^{\circ}\, j_{\mathrm{Im}\,\gamma_{B''}}^{\mathrm{Im}\,\gamma_{B'}}\gamma_{B'}^{\circ-1}
\end{aligned}
$$

Now $j_{B''}^{B}\gamma_B^{\circ} j_{\mathrm{Im}\,\gamma_B}^{c_\gamma} = \gamma_{B''} = \gamma_{B''}^{\circ} j_{\mathrm{Im}\,\gamma_{B''}}^{c_\gamma} = \gamma_{B''}^{\circ} j_{\mathrm{Im}\,\gamma_{B''}}^{\mathrm{Im}\,\gamma_B} j_{\mathrm{Im}\,\gamma_B}^{c_\gamma}$ and so $j_{B''}^{B}\gamma_B^{\circ} = \gamma_{B''}^{\circ} j_{\mathrm{Im}\,\gamma_{B''}}^{\mathrm{Im}\,\gamma_B}$. Hence $\gamma_{B''}^{\circ} j_{\mathrm{Im}\,\gamma_{B''}}^{\mathrm{Im}\,\gamma_B} = j_{B''}^{B} h\gamma_{B'}^{\circ}$ and so, $K_{B''}^{\circ} = \gamma_{B''}^{\circ} j_{\mathrm{Im}\,\gamma_{B''}}^{\mathrm{Im}\,\gamma_B}(\gamma_{B'}^{\circ})^{-1} = j_{B''}^{B} h\gamma_{B'}^{\circ}\gamma_{B'}^{\circ-1} = j_{B''}^{B} h$. It follows that $(B) \cap \mathcal{A}_\gamma = < B >_h$ and this proves that $\mathcal{A}_\gamma$ is residual. This completes the proof.

REMARK 3.1. Observe that for $\gamma \in T\mathcal{C}$, we have a transformation $T^\gamma : \mathcal{A}_\gamma \to < c >$ defined by:

$$T^\gamma : A \to \mathrm{Im}\,\gamma_A, \qquad A \to \gamma_A^{\circ}.$$

LEMMA 3.2. *Let $\gamma \in T\mathcal{C}$ and $f : c_\gamma \to D$ be an epimorphism. Then the map*

$$\gamma^* f : A \to \gamma_A f \tag{3.3}$$

is an element of $T\mathcal{C}$ such that $C_{\gamma^* f} = D$ and $\mathcal{A}_{\gamma^* f} \le \mathcal{A}_\gamma$.

PROOF: If $A' \subseteq A$, then $j_{A'}^{A}(\gamma^* f)_A = j_{A'}^{A}\gamma_A f = \gamma_{A'} f = (\gamma^* f)_{A'}$ and so $\gamma^* f$ is a cone from $v\mathcal{C}$ to the vertex D. Let $c \in M\gamma$. Since γ_c is an isomorphism, $(\gamma^* f)_c = \gamma_c f$ is an epimorphism. Let $h : c \to c'$ be a retraction such that $\gamma_c f = hu$ is a normal factorization. Then $\gamma_{c'} f = (\gamma^* f)_{c'} = u$ is an isomorphism and so $\gamma^* f \in T\mathcal{C}$. Now let $A \in \mathcal{A}_{\gamma^* f}$. Then $(\gamma^* f)_A = \gamma_A f$ is a monomorphism. Hence γ_A is also a monomorphism; i.e. $A \in \mathcal{A}_\gamma$. Further, from (3.2), we have $\gamma_A = K_A^c(\mathcal{A}_\gamma)\gamma_c$ and so $(\gamma^* f)_A = \gamma_A f = K_A^c(\mathcal{A}_\gamma)\gamma_c f = K_A^c(\mathcal{A}_\gamma)h(\gamma^* f)_{c'}$. But $(\gamma^* f)_A = K_A^{c'}(\mathcal{A}_{\gamma^* f})(\gamma^* f)_{c'}$ and so $K_A^c(\mathcal{A}_\gamma)h = K_A^{c'}(\mathcal{A}_{\gamma^* f})$. On the otherhand, if $K_A^c h$ is a monomorphism, then $K_A^c h\gamma_{c'} f = K_A^c\gamma_c f = \gamma_A f = (\gamma^* f)_A$ is a monomorphism and hence $A \in \mathcal{A}_{\gamma^* f}$. Thus $\mathcal{A}_{\gamma^* f} = (\mathcal{A}_\gamma)_h \le \mathcal{A}_\gamma$. This completes the proof.

THEOREM 3.3. *Let $\mathcal{C}$ be a normal category. For $\gamma_1, \gamma_2 \in T\mathcal{C}$ define:*

$$\gamma_1 \cdot \gamma_2 = \gamma_1^*(\gamma_2)^{\circ}_{c_{\gamma_1}} \tag{3.4}$$

Then $T\mathcal{C}$ is a semigroup with binary operation defined by (3.4) such that:
(a) $\gamma \in T\mathcal{C}$ is an idempotent if and only if $\gamma_{c_\gamma} = 1_{c_\gamma}$;
(b) $\gamma \in T\mathcal{C}$ is a regular element if and only if $c_\gamma \in \mathcal{A}_{\gamma'}$ for some idempotent $\gamma' \in T\mathcal{C}$.
In particular, the set of regular elements of $T\mathcal{C}$ is a subsemigroup of $T\mathcal{C}$.

PROOF: Let $\gamma^i \in T\mathcal{C}$, $i = 1,2,3$. By (3.4), we have , $c_{\gamma^1\cdot\gamma^2} = \gamma^2(c_{\gamma^1})$ and for any $A \in v\mathcal{C}$, $(\gamma^1 \cdot \gamma^2)_A = \gamma^1_A(\gamma^2_{c_{\gamma^1}})^{\circ}$ by (3.3) and (3.4). Hence

$$((\gamma^1\cdot\gamma^2)\cdot\gamma^3)_A = (\gamma^1\cdot\gamma^2)_A(\gamma^3)^{\circ}_{c_{\gamma^1\cdot\gamma^2}} = \gamma^1_A(\gamma^2)^{\circ}_{c_{\gamma^1}}(\gamma^3)^{\circ}_{c_{\gamma^1\cdot\gamma^2}} = \gamma^1_A(\gamma^2)^{\circ}_{c_{\gamma^1}}(\gamma^3)^{\circ}\gamma^2(c_{\gamma^1})$$

and $(\gamma^1 \cdot (\gamma^2 \cdot \gamma^3))_A = \gamma^1_A(\gamma^2 \cdot \gamma^3)^{\circ}_{c_{\gamma^1}}$.
Now

$$(\gamma^2 \cdot \gamma^3)_{c_{\gamma^1}} = \gamma^2_{c_{\gamma^1}}(\gamma^3)^{\circ}_{c^2_\gamma} = (\gamma^2)^{\circ}_{c_{\gamma^1}} j^{c_{\gamma^2}}_{\gamma^2(c_{\gamma^1})}(\gamma^3)^{\circ}_{c_{\gamma^2}} = (\gamma^2)^{\circ}_{c_{\gamma^1}}(\gamma^3)^{\circ}_{c_{\gamma^2(c_{\gamma^1})}} j^{\gamma^3(c_{\gamma^2})}_{\gamma^3(c_{\gamma^1\gamma^2})}.$$

Since every morphism in $\mathcal{C}$ has a unique factorization $f = f^{\circ}j$, we have $(\gamma^2 \cdot \gamma^3)^{\circ}_{c_{\gamma^1}}$ $= (\gamma^2)^{\circ}_{c_{\gamma^1}}(\gamma^3)^{\circ}_{c_{\gamma^1\cdot\gamma^2}}$ (since $\gamma^2(c_{\gamma^1}) = C_{\gamma^1\cdot\gamma^2}$). Hence for all $A \in v\mathcal{C}$, we have $((\gamma^1 \cdot \gamma^2) \cdot \gamma^3)_A = (\gamma^1 \cdot (\gamma^2 \cdot \gamma^3))_A$. Thus (3.4) defines an associative binary operation in $T\mathcal{C}$ (which is single–valued by Lemma 3.2). To prove (a), let γ be an idempotent in $T\mathcal{C}$. Then $\gamma \cdot \gamma = \gamma$ and so, $\gamma_{c_\gamma} = \gamma_{c_\gamma} \cdot \gamma^{\circ}_{c_\gamma}$. Hence $\gamma_{c_\gamma} = 1_{c_\gamma}$. Conversely if $\gamma_{c_\gamma} = 1_{c_\gamma}$, then by (3.3) and (3.4) $\gamma \cdot \gamma = \gamma$ and so γ is an idempotent. Suppose that $c_\gamma \in \mathcal{A}_{\gamma'}$ for some idempotent γ'. By (a) $1_{c_{\gamma'}} = \gamma'_{c_{\gamma'}}$ and so $c_{\gamma'} \in M\gamma' = M\mathcal{A}_{\gamma'}$. Since $C_\gamma \in \mathcal{A}_{\gamma'}$, $\gamma'_{c_\gamma} = K^{c_{\gamma'}}_{c_\gamma}\gamma'_{c_{\gamma'}} = K^{c_{\gamma'}}_{c_\gamma}$. Let g be a right inverse of the monomorphism $K^{c_{\gamma'}}_{c_\gamma}$. Then if $\gamma'' = \gamma'^*g$, $\gamma''_{c_\gamma} = \gamma'_{c_\gamma}g = K^{c_{\gamma'}}_{c_\gamma}g = 1_{c_\gamma}$. Hence $c_{\gamma''} = c_\gamma$ and γ'' is an idempotent. Let $c \in M\gamma$ and let $\tilde{\gamma} = \gamma''^*\gamma_c^{-1}$. Then by Lemma 3.2, $\tilde{\gamma} \in T\mathcal{C}$ for any $A \in v\mathcal{C}$,

$$(\gamma \cdot \tilde{\gamma} \cdot \gamma)_A = \gamma_A\tilde{\gamma}_{c_\gamma}\gamma_c = \gamma_A \quad \text{since} \quad \tilde{\gamma}_{c_\gamma} = \gamma_c^{-1} \ ; \text{ and}$$

$$(\tilde{\gamma} \cdot \gamma \cdot \tilde{\gamma})_A = \tilde{\gamma}_A \cdot \gamma_c\gamma_{c_\gamma} = \tilde{\gamma}_A$$

Thus $\gamma\tilde{\gamma}\gamma = \gamma$ and $\tilde{\gamma}\gamma\tilde{\gamma} = \tilde{\gamma}$. Hence γ in regular. Conversely if γ is regular, there is an idempotent $\gamma' \in T\mathcal{C}$ with $\gamma \cdot \gamma' = \gamma$. Then for all $A \in v\mathcal{C}$, $\gamma_A(\gamma')^\circ_{c_\gamma} = (\gamma \cdot \gamma')_A = \gamma_A$ and hence $(\gamma')^\circ_{c_\gamma} = 1_{c_\gamma}$. Hence $c_\gamma \subseteq c_{\gamma'}$ and so $c_\gamma \in \mathcal{A}_{\gamma'}$. This proves (b).

Finally if γ^1 and γ^2 are regular elements of $T\mathcal{C}$, then for some idempotent γ', $c_{\gamma^2} \in \mathcal{A}_{\gamma'}$. Now $c_{\gamma^1 \cdot \gamma^2} = \gamma^2(c_{\gamma^1}) \subseteq c_{\gamma^2}$ and so $c_{\gamma^1 \cdot \gamma^2} \in \mathcal{A}_{\gamma'}$. By (b), we conclude that $\gamma^1 \cdot \gamma^2$ is regular.

DEFINITION 3.2. *A normal category $\mathcal{C}$ is said to be reductive (reducible) if for each $A \in v\mathcal{C}$, there is some $\gamma \in T\mathcal{C}$ such that $A \in \mathcal{A}_\gamma$.*

REMARK 3.2. It is immediate from Theorem 3.3 that $\mathcal{C}$ is reductive if and only if $T\mathcal{C}$ is a regular semigroup. In general the full subcategory *red* $\mathcal{C}$ generated by all those objects belonging to some complex $\mathcal{A}_\gamma$ for $\gamma \in T\mathcal{C}$ is clearly reductive. Also each $\gamma \in T\mathcal{C}$ induces, by restriction a unique *red* $\gamma \in T$ *red* $\mathcal{C}$ if $c_\gamma \in V$ *γed* $\mathcal{C}$ and it is easy to see that

$$\gamma \to red\ \gamma$$

is a homomorphism of the subsemigroup of regular elements of $T\mathcal{C}$ into the regular semigroup $T\,red\ \mathcal{C}$.

REMARK 3.3. If $\mathcal{C}$ has a largest object c (is if the *poset* $v\mathcal{C}$ has a largest element), then $\mathcal{C} \in \mathcal{PC}$ with c as a chamber (unique). It is clear that each $\gamma \in T\mathcal{C}$ induces a unique endomorphism $\gamma_c j^c_{c_\gamma} : c \to c$. Also it can be seen that the map $\gamma \to \gamma_c j^c_{c_\gamma}$ is an isomorphism of $T\mathcal{C}$ and $End\ c = \mathcal{C}(c,c)$. Thus, in this case $T\mathcal{C}$ may be identified with $\mathcal{C}(c,c)$.

LEMMA 3.4. *Let $\gamma \in T\mathcal{C}$. Then there is a bijection between $M\gamma$ and the factorization of $\gamma = \epsilon^* f$ with f, an isomorphism and ϵ, an idempotent in $T\mathcal{C}$.*

PROOF: Suppose $\gamma = \epsilon^* f$ where ϵ is an idempotent, and f is an isomorphism. If $c = c_\epsilon$, then $\gamma_c = \epsilon_c f = 1_c f = f$. It follows that $c \in M\gamma$. Conversely if $c \in M\gamma$, and if $\epsilon = \gamma^* \gamma_c^{-1}$, then $\epsilon_c = \gamma_c \gamma_c^{-1} = 1_c$. Hence ϵ is an idempotent in $T\mathcal{C}$ by Theorem 3.3(a). Also for any $A \in v\mathcal{C}$, $(\epsilon^* \gamma_c)_A = (\gamma^* \gamma_c^{-1} \gamma_c)_A = \gamma_A$, i.e. $\epsilon^* \gamma_c = \gamma$. Further, for $c \in M\gamma$, if $\gamma = \epsilon^* \gamma_c = \epsilon^{1*} \gamma_c$, then it is clear that $\gamma^* \gamma_c^{-1} = \epsilon = \epsilon'$. The factorization corresponding to each $c \in M\gamma$ is unique.

LEMMA 3.5. *Let $\gamma \in T\mathcal{C}$ and ϵ be an idempotent in $T\mathcal{C}$. Then we have:*
(1) *$\epsilon \cdot \gamma = \gamma$ if and only if $\gamma = \epsilon^* f$, for some epimorphism f.*

In particular, γ is a regular element of $T\mathcal{C}$ if and only if

$$E(R_\gamma) = \{\epsilon^c : c \in M\gamma\}$$

where ϵ^c is the unique idempotent such that $\gamma = \epsilon^{c*}\gamma_c$.
(2) *$\gamma \cdot \epsilon = \gamma$ if and only if $c_\gamma \subseteq c_\epsilon$ and $\gamma \mathcal{L} \epsilon$ if and only if $c_\gamma = c_\epsilon$.*

PROOF: Let $\epsilon \cdot \gamma = \gamma$. Then if $c = c_\epsilon$, $\gamma_A = \epsilon_A \gamma_c$ for all $A \in v\mathcal{C}$. If $c' \in M\gamma$, $\gamma_{c'}$ is an isomorphism. Therefore, since $\gamma_{c'} = \epsilon_{c'}\gamma_c$, γ_c must be an epimorphism and we have $\gamma = \epsilon^*\gamma_c$. Conversely if $\gamma = \epsilon^* f$ for some epimorphism f, then we must have $\gamma_c = \epsilon_c f = 1_c f = f$. Further for any $A \in v\mathcal{C}$, $(\epsilon \cdot \gamma)_A = \epsilon_A \gamma_c = \epsilon_A f = (\epsilon^* f)_A = \gamma_A$. Thus $\epsilon \cdot \gamma = \gamma$.

If $E(R_\gamma) = \{\epsilon^c : c \in M\gamma\}$, then clearly γ is regular. Conversely assume that γ is regular. If $\mathcal{C} \in M\gamma$ by Lemma 3.4, $\gamma = \epsilon^{c*}\gamma_c$. Since γ is regular, as in the proof of Theorem 3.3, there is $\epsilon' \in E(T\mathcal{C})$ such that $c_{\epsilon'} = c_\gamma$. Let $\bar{\gamma} = \epsilon'^* \gamma_c^{-1}$. Then $\gamma \cdot \bar{\gamma} = \epsilon^c$ and so $\epsilon^c \mathcal{R} \gamma$. Hence $E(R_y) \subseteq \{\epsilon^c : c \in M\gamma\}$. If $\epsilon \in E(R_\gamma)$, then $\epsilon\gamma = \gamma$ and so $\gamma = \epsilon^*\gamma_c$ where $c = c_\epsilon$ and γ_c is an epimorphism. Now there is $\gamma' \in T\mathcal{C}$ with $\gamma \cdot \gamma' = \epsilon$. Then $1_c = \epsilon_c = \gamma_c(\gamma'_{c_\gamma})^\circ$. Since γ_c is an epimorphism, this implies that γ_c is an isomorphism, so that $\epsilon = \epsilon^c$, $c \in M\gamma$.

If $\gamma \cdot \epsilon = \gamma$, then for any $A \in v\mathcal{C}$, $\gamma_A \epsilon^\circ_{c_\gamma} = \gamma_A$. Hence $\epsilon_{c_\gamma} = j^{c_\epsilon}_{c_\gamma}$ and so $c_\gamma \subseteq c_\epsilon$. Conversely if this holds, $\gamma_A \epsilon^\circ_{c_\gamma} = \gamma_A 1_{c_\gamma} = \gamma_A$ and so $\gamma \cdot \epsilon = \gamma$. The last statement is obvious.

As an immediate consequence of the result above, we have:

COROLLARY 3.6. *Let $\mathcal{C}$ be reductive. Then*
(1) *$L_\gamma \to C_\gamma$ is an order isomorphism of $T\mathcal{C}/\mathcal{L}$ onto $v\mathcal{C}$.*
(2) *$R_\gamma \to \mathcal{A}_\gamma$ is an order-preserving map of $T\mathcal{C}/\mathcal{R}$ into $\mathcal{P}\mathcal{C}$.*

PROOF: Statement (1) is immediate from Lemma 3.5. If $\gamma \mathcal{R} \gamma'$, then by Lemma 3.5, $\gamma' = \gamma^* f$ for some isomorphism f. It follows from the definition of $\mathcal{A}_\gamma$ that $\mathcal{A}_{\gamma'}$. If $R_\gamma \leq R_{\gamma'}$, then by Lemma 3.6, $\gamma = \gamma'^* f$ for some epimorphism f and by Lemma 3.2. $\mathcal{A}_\gamma \leq \mathcal{A}_{\gamma'}$.

Let S be a regular semigroup. Define the category $\mathbb{L} = \mathbb{L}(S)$ as follows:

$$v\mathbb{L}(S) = (\{sx : x \in S\}, \subseteq),$$
$$\mathrm{Hom}_{\mathbb{L}(S)}(Sx, Sy) = \{\rho : Sx \to Sy : (st)\rho = s(t\rho),\ s, t \in Sx\}. \tag{3.5}$$

It is easily verified that $\mathbb{L}(S)$ is a category with subobjects in which, the inclusion is the usual 'set-inclusion'. Dually, we have:

$$(3.5^*)\qquad \begin{gathered} v\mathbb{R}(S) = (\{xS: x \in S\}, \subseteq), \\ \mathrm{Hom}_{\mathbb{R}(S)}(xS, yS) = \{\lambda\lambda: xS \to yS: \lambda(st) = (\lambda s)f, s, f \in xS\} \end{gathered}$$

is a category $\mathbb{R}(S)$ with subobjects.

Let $e, f \in E(S)$. For $u \in eSf$, define

$$(3.6)\qquad \rho(e,u,f): x \to xu$$

It is easily seen that $\rho(e,u,f): Se \to sf$ is a morphism of $L(S)$. If $\rho: Se \to Sf$ is any other morphism and if $u = e\rho$, then $\rho = \rho(e,u,f)$. In fact it is easy to see that the map $u \to \rho(e,u,f)$ is a bijection of eSf onto $\mathrm{Hom}_{\mathbb{L}(S)}(Se, Sf)$. Observe that the representation of $\rho: Se \to Sf$ as $\rho = \rho(e,u,f)$ is not unique; one can easily verify that

$$(3.7)\qquad \rho(e,u,f) = \rho(e', e'u, f') \quad \text{if} \quad u \in eSf,\ e'\mathcal{L}e \quad \text{and} \quad f'\mathcal{L}f.$$

For later reference, we collect the foregoing statements as:

LEMMA 3.7. *Let S be a regular semigroup. Then:*
(1) *The morphism $\rho(e,u,f): Se \to Sf$ is injective if and only if $e\mathcal{R}u$. In this case $x\mathcal{R}x\rho$ for each $x \in Se$.*
(2) *$\rho(e,u,f): Se \to Sf$ is surjective if and only if $u\mathcal{L}f$.*
(3) *Se and Sf are isomorphic in $\mathbb{L}(S)$ if and only if $e\mathcal{D}f$. In this case the set of isomorphisms of Se into Sf is in one-to-one correspondence with the $\mathcal{H}$-class $R_e \cap L_f$.*
(4) *If $Se \subseteq Sf$, then $j_{Se}^{Sf} = \rho(e,e,f)$ and $\rho: Sf \to Se$ is a retraction if and only if $\rho = \rho(f,g,e)$ for some $g \in E(L_3) \cap \omega(f)$. In particular $\rho(f, fe, e): Sf \to Se$ is a retraction.*

If $\rho(e,u,f): Se \to Sf$, $\rho(g,v,h): Sg \to Sh$ are composable morphisms of $L(S)$, we must have $Sf = Sg$, i.e. $f\mathcal{L}g$. Then $\rho(g,v,h) = \rho(f, fv, h)$ by (3.7) and $\rho(e,u,f)\rho(f,fu,h) = \rho(e,ufv,h) = \rho(e,uv,h)$. Hence when $\rho(e,u,f)$ and $\rho(g,v,h)$ are composable, the composition is given by the rule:

$$(3.8)\qquad \rho(e,u,f)\rho(g,v,h) = \rho(e,uv,h) \quad \text{if} \quad f\mathcal{L}g.$$

It follows that for any morphism $\rho(e,u,f)$, we have

(3.9)
$$\rho(e,u,f)=\rho(e,g,g)\rho(g,u,h)\rho(h,h,f) \quad \text{for} \quad g\in E(R_u)\cap\omega(e), \quad h\in E(L_u).$$

Since $u\in eSf$, for any $g'\in E(R_u)$, $g'we$ and so $g'e\in E(R_u)\cap w(e)$. Hence we can always find g and h as required for (3.9). The Lemma 3.7 above shows that $\rho(e,g,g)$ is a retraction, $\rho(g,u,h)$ is an isomorphism and $\rho(h,h,f)$ is the inclusion $Sh\subseteq Sf$. Hence the right hand side of (3.9) gives a normal factorization of $\rho(e,u,f)$. Thus $\mathbb{L}(S)$ is a normal category.

Now consider $a\in S$. For each $Se\in v\mathbb{L}(S)$, define

(3.10)
$$\rho^a_{Se}=\rho(e,ea,f) \quad \text{where} \quad f\in E(L_a).$$

If $Se'\subseteq Se$, then by (3.8) $\rho(e',e',e)\rho(e,ea,f)=\rho(e',e'ea,f)=\rho(e',e'a,f)$. Since by Lemma 3.7, $\rho(e',e',e)=j^{Se}_{Se'}$, we have $j^{Se}_{Se'}\rho^a_{Se}=\rho^a_{Se'}$. Hence

$$\rho^a: Se\to\rho^a_{Se}$$

is a cone from $v\mathbb{L}(S)$ to the vertex $Sf=Sa$. Also if $e\in E(R_a)$, then ρ^a_{Se} is an isomorphism by Lemma 2.7. Hence $\rho^a\in TL(S)$. In fact by Lemma 2.7, we have

(3.11)
$$M\rho^a=\{Se: e\in E(R_a)\}.$$

In particular for each $Se\in v\mathbb{L}(S)$ (with $e\in E(S)$), $Se\in M\rho^e\subseteq\mathcal{A}_{\rho^e}$ and so $\mathbb{L}(S)$ is reductive.

As usual, let $\rho: S\to T_s,\ a\to\rho_a$ be the right–regular representations of S. Then $\rho: S\to S_\rho=\{\rho_a: a\in S\}$ is a surjective homomorphism. Define

(3.12)
$$\psi: S_\rho\to T\mathbb{L}(S);\ \rho_a\to\rho^a.$$

Clearly ψ is a well–defined injective map. We show that ψ is a homomorphism. Indeed, if $a,g\in S$, thus by (3.4), for any $Se\in v\mathbb{L}(S)$,

$$(\rho^a\rho^b)_{Se}=\rho^a_{Se}(\rho^b_{Sa})^\circ=\rho(e,ea,f)\rho(f,fg,g)^\circ$$

if $f\in E(L_a)$, $g\in E(L_g)$. Now if $h\in E(L_{fg})$, by (3.9), $\rho(f,fg,g)^\circ=\rho(f,fg,h)$. Since $fg\,\mathcal{L}fg$, we have

$$(\rho^a\rho^b)=\rho(e,ea,f)\rho(f,fb,h)=\rho(e,eafb,h)=\rho(e,eab,h)=\rho^{ab}_{Se}$$

Hence

(3.13)
$$\rho^a\rho^b=\rho^{ab}$$

and so $\psi(\rho_a\rho_b)=\psi(\rho_{ab})=\rho^{ab}=\rho^a\rho^b=\psi(\rho_a)\psi(\rho_b)$.

We thus have the following theorem:

THEOREM 3.8. *Let S be a regular semigroup. Then $\mathbb{L}(S)$ is a normal reductive category and there is a homomorphism $\tilde{\rho}: S \to T\mathbb{L}(S)$ such that the congruence ker ρ, when $\rho: S \to S_\rho$ is the right regular representation.*

We observe that the dual of all the results above regarding the category $\mathbb{L}(S)$, holds for the category $\mathbb{R}(S)$. Thus, for $e, f \in E(S)$, $u \in fSe$, (3.6*) $\lambda(e,u,f): x \to ux \in \mathbb{R}(S)(eS, fS) = \mathrm{Hom}_{\mathbb{R}(S)}(eS, fS)$ and the mapping $u \to \lambda(e,u,f)$ is a bijection of fSe onto $\mathbb{R}(S)(eS, fS)$. Also Equation (3.7*) holds with $\lambda(e,u,f) = \lambda(e', ue', f')$ if $e\mathcal{R}e'$ and $\mathcal{R}f'$ and Equation (3.8*) becomes $\lambda(e,u,f)\lambda(f,v,g) = \lambda(e, vu, g)$. The morphism $\lambda(e,u,f)$ is injective if (monomorphism) $e\mathcal{L}f$ surjective (epimorphism) if $u\mathcal{R}f$ and (bijective) is an isomorphism if $u \in L_e \cap R_f$. If $gS \subseteq eS$, then $\lambda(g, ge) = j_{gS}^{eS}$ and λ is a retraction if and only if $\lambda = \lambda(e,g,h)$ where $g \in E(R_h) \cap w(e)$. A normal factorization of $\lambda = \lambda(e,u,f)$ is given by:

$$(3.9^*) \qquad \lambda = \lambda(e,g,g)\lambda(g,u,h)\lambda(h,h,j), \quad g \in E(L_u) \cap w(e), \quad h \in E(R_u)$$

Analogously, for $a \in S$ it can be shown that

$$(3.10^*) \qquad \lambda^a: eS \mapsto \lambda^a_{eS} = \lambda(e, ae, f), \quad f \in E(R_a).$$

is a cone from $v\mathbb{R}(S)$ to the vertex as and that $M\lambda^a = \{eS : e \in E(L_a)\}$. If $a \to \lambda_a : S \to S_\lambda$ is the antirepresentation of S by left translations, then the map $\psi^*: S_\lambda \to T\mathbb{R}(S)$, $\lambda_a \to \lambda^a$ is an injective homomorphism. Finally dual statements (3.7*) and (3.8*) of Lemma 3.7 and Theorem 3.8 holds for $\mathbb{R}(S)$. Observe that corresponding to the isomorphism $\tilde{\rho}: S \to T\mathbb{L}(S)$, we have an anti-homomorphism $\tilde{\lambda}: S \to T\mathbb{R}(S)$ such that $ker\ \tilde{\lambda} = ker\ \lambda$.

It follows immediately from Theorem 3.8 [(3.8*)] that S is [anti-] isomorphic to a subsemigroup of $T\mathbb{L}(S)[T\mathbb{R}(S)]$ if S is right-[left-] reductive. In particular, we have:

COROLLARY 3.9. *If S is a regular monoid, then S is [anti-] isomorphic to $T\mathbb{L}(S)$ $[T\mathbb{R}(S)]$.*

PROOF: By Theorem 3.8 and the remark above $\tilde{\rho}: S \to T\mathbb{L}(S)$ is an injective homomorphism. If $\gamma \in T\mathbb{L}(S)$, and if $a = 1_{\gamma_{1S}}$, the image of 1 under the component of γ corresponding to $1S = S$, then $\gamma_{eS} = P_{a/eS} = P^a_{eS}$. Hence $\tilde{\rho}(a) = p^a = \gamma$ and so $\tilde{\rho}$ is surjective. Dually $\tilde{\lambda}$ is an antimorphism.

If S does not have an identity, it is well-known that the right regular representation of S^1 induces a faithful representation of S into S^1_ρ and $S^1_\rho \approx T\mathbb{L}(S')$ by the corollary above. Hence

COROLLARY 3.10. *Every regular semigroup is [anti-] isomorphic to a subsemigroup of $T\mathcal{C}$ for some suitable reduction category $\mathcal{C}$.*

If S is a regular semigroup, we have seen that $\mathbb{L}(S)$ is a normal reductive category. The following theorem shows that every such category arises this way:

THEOREM 3.11. *If $\mathcal{C}$ is a normal, reductive category, then $\mathcal{C}$ is isomorphic to $\mathbb{L}(T\mathcal{C})$.*

PROOF: For each $C \in v\mathcal{C}$ and $f \in \mathcal{C}(C,D)$ define

(3.14i) $vF(C) = T\mathcal{C}\epsilon, \quad \epsilon \in E(T\mathcal{C}), \quad {}'C_\epsilon = C;$

(3.14ii) $F(f) = \rho(\epsilon, \epsilon^* f^\circ, \epsilon'), \quad \epsilon, \epsilon' \in (T\mathcal{C})$ with $C_\epsilon = C, \quad C_{\epsilon'} = D.$

By Corollary 3.6, vF is an order-isomorphism of $v\mathcal{C}$ onto $v\mathbb{L}(T\mathcal{C})$. By Lemma 3.5, $\epsilon^* f^\circ \in \epsilon T\mathcal{C}\epsilon'$. Since by Lemma 1.1, $f \mapsto f^\circ$ is a bijection of $\mathcal{C}(C,D)$ with $\epsilon T\mathcal{C}\epsilon'$ and so $f \to \rho(\epsilon, \epsilon^* f^\circ, \epsilon')$ is a bijection of $\mathcal{C}(C,D)$ with $\mathbb{L}(T\mathcal{C})(S\epsilon, S\epsilon')$. Also if $F(f) = p(\epsilon, \epsilon^* f^\circ, \epsilon')$, $F(g) = p(\epsilon'_1, \epsilon'^*_1 g^\circ, \epsilon'')$, and if fg exists in $\mathcal{C}$, we have $\epsilon' \mathcal{L} \epsilon'_1$ and so we may replace $\rho(\epsilon'_1, \epsilon'_1 g^\circ, \epsilon'')$, with $\rho(\epsilon', \epsilon'^* g^\circ, \epsilon'')$ by (3.7) and by (3.8),

$$\rho(\epsilon, \epsilon^* f^\circ, \epsilon')\rho(\epsilon', \epsilon'^* g^\circ, \epsilon'') = \rho(\epsilon, (\epsilon^* f^\circ)(\epsilon'^* g^\circ), \epsilon'').$$

Now $(\epsilon^* f^\circ)(\epsilon'^* g^\circ) = \epsilon^* f^\circ (j^{c_{\epsilon'}}_{c_1} g^\circ)^\circ$, $c_1 = \operatorname{Im} f^\circ$.
Also $(fg)^\circ = f^\circ (j^{c_{\epsilon'}}_{c_1} g)^\circ = f^\circ (j^{c_{\epsilon'}}_{c_1} g^\circ)^\circ$.
Hence $\rho(\epsilon, \epsilon^* f^\circ, \epsilon')\rho(\epsilon', \epsilon'^* g^\circ, \epsilon'') = \rho(\epsilon, \epsilon^*(fg)^\circ, \epsilon'')$.
Thus $F(f)F(g) = F(fg)$.
Clearly F preserves identities and so $F: \mathcal{C} \to \mathbb{L}(T\mathcal{C})$ is an isomorphism.

Theorems 3.8 and 3.11 together give:

COROLLARY 3.12. *A normal category $\mathcal{C}$ is reductive if and only if $\mathcal{C}$ is isomorphic to the category $\mathbb{L}(S)$ for some regular semigroup S.*

We now proceed to characterize the category $\mathbb{R}(T\mathcal{C})$. Let $\epsilon \in E(T\mathcal{C})$. For $c \in V\mathcal{C}$ and $g \cdot \mathcal{C}(c, c')$, define

(3.15i) $\qquad H(\epsilon, c) = \{\epsilon^* f^\circ : f \in \mathcal{C}(C_\epsilon, C)\};$ and

(3.15ii) $\qquad H(\epsilon; g) = \epsilon^* f^\circ \to \epsilon^*(fg)^\circ$

Since $fg \in \mathcal{C}(c_\epsilon, c')$, $H(\epsilon; g)$ is a map from $H(\epsilon; c)$ to $H(\epsilon; c')$. Clearly $H(\epsilon; 1_c) = 1_{H(\epsilon;c)}$. If $g \in \mathcal{C}(c, c')$, $h \in \mathcal{C}(c', c'')$, then

$$H(\epsilon; g)H(\epsilon; h)(\epsilon^* f^\circ) = H(\epsilon; h)(H(\epsilon; g)(\epsilon^* f^\circ)) = H(\epsilon, h)(\epsilon^*(fg)^\circ) = \epsilon^*(fgh)^\circ,$$
$$H(\epsilon, gh)(\epsilon^* f^\circ) = \epsilon^*(fgh)^\circ$$

Hence $H(\epsilon; -): \mathcal{C} \to \mathbf{Set}$ is a functor. In the following $\mathcal{C}^*$ denotes the functor category $[\mathcal{C}, \mathbf{Set}]$.

LEMMA 3.13. *For each $\epsilon \in E(T\mathcal{C})$, $H(\epsilon; -) \in \mathcal{C}^*$ and $H(\epsilon; -)$ is naturally equivalent to the home-functor $\mathcal{C}(c; -)$ for each $c \in M\epsilon$. Also $H(\epsilon; -) = H(\epsilon'; -)$ if and only if $\epsilon \mathcal{R} \epsilon'$.*

PROOF: By (3.15i), $\epsilon \in H(\epsilon; c_\epsilon)$ and if $\epsilon^* f^\circ \in H(\epsilon; c)$, $f \in \mathcal{C}(c_\epsilon, c)$, then $H(\epsilon; f)(\epsilon) = \epsilon^* f^\circ$. Hence ϵ is a universal element for the functor $H(\epsilon; -)$ and so by Yoneda Lemma, there is a unique natural isomorphism $\eta_\epsilon: H(\epsilon; -) \to \mathcal{C}(c_\epsilon; -)$ such that $(\eta_\epsilon)_{c_\epsilon}(\epsilon) = 1_{c_\epsilon}$. In view of Lemma 3.5(i), the proof of the Lemma will be complete if we show that $H(\epsilon; -) = H(\epsilon'; -)$ if and only if $\epsilon \mathcal{R} \epsilon'$.

First suppose that $H(\epsilon; -) = H(\epsilon'; -)$. Since $\epsilon \in H(\epsilon; c_\epsilon)$, $\epsilon \in H(\epsilon'; c_\epsilon)$ and so there is an epimorphism $g: c_{\epsilon'} \to c_\epsilon$ such that $\epsilon \in \epsilon'^* g$. Then by Lemma 3.5 $\epsilon \cdot \epsilon' = \epsilon'$. Similarly $\epsilon' \cdot \epsilon = \epsilon$ and so $\epsilon \mathcal{R} \epsilon'$. Conversely, assume that $\epsilon \mathcal{R} \epsilon'$. Then by Lemma 3.5, $\mathcal{M}\epsilon = \mathcal{M}\epsilon'$ and $\epsilon_{c_{\epsilon'}}: c_{\epsilon'} \to c_\epsilon$ is an isomorphism with $\epsilon_{c_{\epsilon'}}^{-1} = \epsilon'_{c_\epsilon}$. It follows that $\epsilon^* f^\circ = \epsilon'^* \epsilon_{c_{\epsilon'}} f^\circ = \epsilon'^* (\epsilon_{c_\epsilon} f)^\circ$. Since $f \to \epsilon_{c_{\epsilon'}} f$ is a bijection of $\mathcal{C}(c_\epsilon, c)$ onto $\mathcal{C}(c_{\epsilon'}, c)$ it follows that $H(\epsilon; c) = H(\epsilon', c)$ for all $c \in V\mathcal{C}$. If $g: c \to c'$ is any morphism, we have for $f \in \mathcal{C}(c_{\epsilon'}, c)$,

$$\begin{aligned} H(\epsilon; g)(\epsilon^* f^\circ) &= \epsilon^*(fg)^\circ = \epsilon'^* (\epsilon_{c_{\epsilon'}} fg)^\circ = H(\epsilon'; g)(\epsilon'^* \epsilon_{C_{\epsilon'}} f^\circ) = H\epsilon' \\ &= H(\epsilon'; g)(\epsilon^* f^\circ) \end{aligned}$$

Hence $H(\epsilon; g) = H(\epsilon'; g)$ for all morphisms. Thus $H(\epsilon; -) = H(\epsilon'; -)$.

Suppose that $\epsilon \omega^r \epsilon'$ in $E(T\mathcal{C})$. If $C \in V\mathcal{C}$ and $\epsilon^* f^\circ \in H(\epsilon; C)$, where $f \in \mathcal{C}(C_\epsilon, C)$, then by Lemma 3.5, $\epsilon^* f^\circ = \epsilon'^* g^\circ$ for some $g \in \mathcal{C}(C_{\epsilon'}, C)$ and so $\epsilon^* f^\circ \in H(\epsilon'; C)$. Thus $H(\epsilon; C) \subseteq H(\epsilon'; C)$ for all $C \in V\mathcal{C}$. If $g: C \to C'$, then for $\epsilon^* f^\circ \in H(\epsilon; C)$,

$$H(\epsilon; g)(\epsilon^* f^\circ) = \epsilon^*(fg)^\circ = \epsilon'^* (\epsilon_{C_\epsilon} fg)^\circ = H(\epsilon'; g)(\epsilon'^* (\epsilon_{C_{\epsilon'}} f)^\circ).$$

Since $\epsilon^* f^\circ = \epsilon'^*(\epsilon_{C_{\epsilon'}} f)^\circ$, it follows that the inclusion $H(\epsilon; C) \subseteq H(\epsilon'; C)$ is a natural transformation. Conversely if $H(\epsilon; -) \subseteq H(\epsilon'; -)$, then $\epsilon \in H(\epsilon; C_\epsilon) \subseteq H(\epsilon'; C_\epsilon)$ and so $\epsilon = \epsilon'^* f^\circ$ for some $f \in \mathcal{C}(C_{\epsilon'}, C_\epsilon)$ and so by Lemma 3.5 $\epsilon' \cdot \epsilon = \epsilon$; i.e. $\epsilon \omega^r \epsilon'$. It follows the map $v\mathcal{C}$ defined by the following equation is an order–isomorphism of $v\mathbb{R}(T\mathcal{C})$ onto the set $\{H(\epsilon; -) : \epsilon \in E(T\mathcal{C})\} \subseteq V\mathcal{C}^*$:

(3.16i) $$vG(\epsilon S) = H(\epsilon, -), \qquad S = T\mathcal{C} \quad \text{and} \quad \epsilon \in E(S).$$

Let $N^*\mathcal{C}$ denote the full subcategory of $\mathcal{C}^*$ such that $vN^*\mathcal{C} = \operatorname{Im} vG$. We proceed to show that the map vG is the vertex map of an isomorphism $a: \mathbb{R}(T\mathcal{C}) \to N^*\mathcal{C}$. To this end, consider $\lambda = \lambda(\epsilon, \gamma, \epsilon') : \epsilon S \to \epsilon' S$. By equation (3.6*), $\gamma \in \epsilon' S \epsilon$ and so $C_\gamma \subseteq C_\epsilon$ and $\gamma = \epsilon'^* \gamma_{C_{\epsilon'}}$ by Lemma 3.5. Let $\tilde{\gamma} = \gamma_{C_{\epsilon'}} j_{C_\gamma}^{C_\epsilon}$. Then $\tilde{\gamma} \in \mathcal{C}(C_{\epsilon'}, C_\epsilon)$. Define

(3.16ii) $$G(\lambda) = \eta_\epsilon \cdot \mathcal{C}(\tilde{\nu}, -) \cdot \eta_{\epsilon'}^{-1}$$

when $\eta_\epsilon : H(\epsilon, -) \to G(C_\epsilon, -)$ is the natural isomorphism sending ϵ to 1_{C_ϵ} and $G(\tilde{\gamma}, -)$ is the natural transformation induced by $\tilde{\gamma} : C_{\epsilon'} \to C_\epsilon$. Then $G(\lambda) : H(\epsilon; -)$ is natural transformation. We now show that $G(\lambda)$ does not depend on the choice of ϵ and ϵ' in the representation of γ. If $\epsilon' \mathcal{R} \epsilon''$, then $\epsilon' S = \epsilon'' S$ and so clearly $G(\lambda)$ does not change. If $\epsilon_1 \mathcal{R} \epsilon$, by (3.7*) $\lambda = \lambda(\epsilon_1, \gamma \cdot \epsilon_1, \epsilon')$. Then $(\gamma_1)_{C_{\epsilon'}} = (\gamma \cdot \epsilon_1)_{\epsilon_{\epsilon'}} = \gamma_{C_{\epsilon'}} (\epsilon_1)^\circ_{C_{\gamma'}}$. Since $\epsilon_{1C_\gamma} = j_{C C_\gamma}^{C_\epsilon} (\epsilon_1)_{C_\epsilon} (\gamma_1)_{C_\epsilon} = (\gamma_{C_{\epsilon'}} j_{C_\gamma}^{C_\epsilon} (\epsilon_1)_{C_\epsilon})^\circ = (\tilde{\gamma}(\epsilon_1)_{C_\epsilon})^\circ$. So $\gamma_1 = (\gamma_1)_{C_{\epsilon'}} j_{C_{\gamma_1}}^{C_{\epsilon_1}} = \tilde{\gamma}(\epsilon_1)_{C_\epsilon}$. Also, $(\eta_{\epsilon_1})_{C_\epsilon}(\epsilon) = (\epsilon)_{C_{\epsilon_1}}$ and $(\eta_\epsilon \cdot \mathcal{C}(\epsilon_{C_{\epsilon_1}}, -))_{C_\epsilon}(\epsilon) = ((\eta_\epsilon)_{C_\epsilon} \mathcal{C}(\epsilon_{C_{\epsilon_1}}, C_\epsilon))(\epsilon) = \epsilon_{C_{\epsilon_1}}$. It follows that

(3.17) $$\eta_{\epsilon_1} = \eta_\epsilon \cdot \mathcal{C}(\epsilon_{C_{\epsilon_1}}, -)$$

Hence $\eta_{\epsilon_1} \cdot \mathcal{C}(\tilde{\gamma}_1, -) \cdot \eta_{\epsilon'}^{-1} = \eta_\epsilon \cdot \mathcal{C}(\epsilon_{C_{\epsilon_1}}, -) \mathcal{C}(\tilde{\gamma}_1, -) \cdot \eta_{\epsilon'}^{-1} = \eta_\epsilon \cdot \mathcal{C}(\tilde{\gamma}, \epsilon_{C_{\epsilon_1}}, -) \cdot \eta_{\epsilon'}^{-1} = \eta_\epsilon \cdot \mathcal{C}(\tilde{\gamma}, -) \cdot \eta_{\epsilon'}^{-1}$. Since $\epsilon_{C_{\epsilon_1}} = (\epsilon_1)_{C_\epsilon}^{-1}$. This proves that $G(\lambda)$ is well–defined. Since $f \to \lambda(\epsilon, \epsilon'^* f^\circ, \epsilon')$ is a bijection of $\mathcal{C}(C_{\epsilon'}, C_\epsilon)$ with $\mathbb{R}(S)(\epsilon S, \epsilon' S)$ and since contravariant Yoneda embedding is fully faithful, $\lambda \mapsto G(\lambda)$ is a bijection of $\mathbb{R}(\epsilon S, \epsilon' S)$ onto $N^*\mathcal{C}(H(\epsilon, -), H(\epsilon', -))$. To show that G is a functor, let $\lambda = \lambda(\epsilon, \gamma, \epsilon')$, $\lambda' = \lambda(\epsilon', \gamma', \epsilon'')$. Now $(\gamma' \gamma)^\circ = (\gamma' \cdot \gamma)_{C_{\epsilon''}} = \gamma'_{C_{\epsilon''}} \gamma^\circ_{C_{\gamma'}}$ since $\gamma'_{C_{\epsilon''}}$ is surjective. Hence

$$\begin{aligned}\gamma' \gamma &= \gamma'_{C_{\epsilon''}} \cdot \gamma^\circ_{C_\gamma} j_{C_{\gamma' \cdot \gamma}}^{C_\epsilon} = \gamma'_{C_{\epsilon''}} \cdot \gamma^\circ_{C_\gamma} j_{C_{\gamma' \cdot \gamma}}^{C'_\gamma} \cdot j_{C_{\gamma'}}^{C_\epsilon} = \gamma'_{C_{\epsilon''}} \gamma_{C_{\gamma'}} j_{C_{\gamma'}}^{C_\epsilon} \\ &= \gamma'_{C_{\epsilon''}} j_{C_{\gamma'}}^{C_{\epsilon'}} \gamma_{C_{\epsilon'}} j_{C_{\gamma'}}^{C_\epsilon} = \tilde{\gamma}' \tilde{\gamma}.\end{aligned}$$

Therefore

$$\begin{aligned} G(\lambda\lambda') &= \eta_\epsilon \cdot \mathcal{C}(\gamma'\gamma, -) \cdot \eta_{\epsilon''}^{-1} \\ &= \eta_\epsilon \cdot \mathcal{C}(\tilde{\gamma}'\tilde{\gamma}, -)\eta_{\epsilon''}^{-1} \\ &= \eta_\epsilon \cdot \mathcal{C}(\tilde{\gamma}', -) \cdot \eta_{\epsilon'}^{-1} \cdot \eta_{\epsilon'} \cdot \mathcal{C}(\tilde{\gamma}, -) \cdot \eta_{\epsilon''}^{-1} \\ &= G(\lambda)G(\lambda') \end{aligned}$$

Hence:

THEOREM 3.14. *If $\mathcal{C}$ is a normal reductive category, then there is an isomorphism $G: \mathbb{R}(T\mathcal{C}) \to N^*\mathcal{C}$.*

DEFINITION 3.3. *When $\mathcal{C}$ is a normal reductive category, the category $N^*\mathcal{C}$ will be called the normal dual of $\mathcal{C}$.*

4. Cross-connections

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. If $\mathcal{C}^*$ denotes the functor category $[\mathcal{C}, \mathbf{Set}]$, it is well-known that there are isomorphisms of functor categories:

$$[\mathcal{C}, \mathcal{D}^*] \approx (\mathcal{C} \times \mathcal{D})^* \approx [\mathcal{D}, \mathcal{C}^*].$$

In fact if $F: \mathcal{C} \to \mathcal{D}^*$ the corresponding element $F(-,-) \in (\mathcal{C} \times \mathcal{D})^*$ is defined by:

$$F(C, D) = F(C)(D), \quad F(f, g) = F(C)(g)F(f)(D') = F(f)(D)F(C')(g)$$

where $f: C \to C' \in \mathcal{C}$ and $g: D \to D' \in \mathcal{D}$. Here $F(f)(D)$ is the D-component of the natural transformation $F(f): F(C) \to F(C')$. Conversely if $F(-,-): \mathcal{C} \times \mathcal{D} \to \mathbf{Set}$,

$$F(C) = F(C, -), \qquad F(f) = F(f, -)$$

for $f: C \to C' \in \mathcal{C}$, defines the functor $F: \mathcal{C} \to \mathcal{D}^*$ corresponding to $F(-,-)$.

DEFINITION 4.1. *Let $F: \mathcal{C} \to \mathcal{D}$ be a functor of normal categories. We shall say that F is a local isomorphism if it is fully-faithful, inclusion preserving and for each $C \in v\mathcal{C}$, $f|(C)$ is an isomorphism onto $(F(C))$.*

For $\epsilon \in E(7\mathcal{C})$, let $H(\epsilon, -)$ be the functor defined by (3.15), we write:

$$MH(\epsilon, -) = \{C_{\epsilon'} : \epsilon' \mathcal{R} \epsilon\} \tag{4.1}$$

Note that $MH(\epsilon, -) = M\epsilon$ and the subset $MH(\epsilon, -)$ of $v\mathcal{C}$ is completely determined by the functor $H(\epsilon, -)$.

THEOREM 4.1. *Let S be a regular semigroup. For $fS \in v\mathbb{R}(S)$ and $\lambda = \lambda(e,u,f) \in \mathbb{R}(S)$, let*

$$(4.2) \qquad vT(fS) = H(\rho^f,-), \quad \Gamma(\lambda) = \eta_{\rho^e} L(S)(\rho(f,u,e),-) \cdot \eta_{\rho^f}^{-1}$$

where $H(\rho^f,-)$ is the functor defined by (3.15), η_{ρ^e} is the natural isomorphism of $H(\rho^e,-)$ to $\mathbb{L}(S)(Se,-)$ sending ρ^3 to 1_{Se}. Then $T:\mathbb{R}(S) \to N^*\mathbb{L}(S)$ is a local isomorphism. Dually

$$(4.2^*) \qquad v\Delta(se) = H(\lambda^3,-), \quad \Delta(\rho(e,u,f)) = \eta_{\lambda^e} \cdot \mathbb{R}(S)(\lambda(f,u,e),-) \cdot \eta_{\lambda^f}^{-1}$$

for $Se \in v\mathbb{L}(S)$ and $\rho(e,u,f) \in \mathbb{L}(S)$ defines a local isomorphism $\Delta:\mathbb{L}(S) \to N^*\mathbb{R}(S)$. Moreover $(Se,fS) \in v\mathbb{L}(S) \times v\mathbb{R}(S)$, the map defined by

$$(4.3) \qquad \mathfrak{X}(Se,fS): \rho^{f} \cdot \rho(f,u,e)^{\circ} \to \lambda^{e} \cdot \lambda(e,u,f)^{\circ}$$

is a natural isomorphism $\mathfrak{X}:\Gamma(-,-) \to \Delta(-,-)$, where $\Gamma(-,-)$ and $\Delta(-,-)$ are bifunctors associated with Γ and Δ respectively.

PROOF: If $fS = f'S$, then $f\mathcal{R}f'$ and so by Theorem 3.8, $\tilde\rho(f) = \rho^f\mathcal{R}\rho^{f'} = \tilde\rho(f')$. Hence by Lemma 3.13, $H(\rho^f,-) = H(\rho^{f'},-)$ and so $v\Gamma$ is well-defined by (4.2). Dually, $v\Delta$ is also well-defined. Let $\lambda = \lambda(e,u,f): eS \to fS$ so that $u \in fSe$. Then by Theorem 3.8, $\rho^u = \tilde\rho(u) \in \rho^f T\mathbb{L}(S)\rho^e$ and so $\bar\lambda = \lambda(\rho^e,\rho^u,\rho^f): \rho^e\mathbb{L}(S) \to \rho^f T\mathbb{L}(S)$. By (3.16ii) $G(\bar\lambda) = \eta_{\rho^e} \cdot \mathbb{L}(S)(\tilde\rho^u,-) \cdot \eta_{\rho^f}^{-1}: G(\rho^3 T\mathbb{L}(S)) \to G(\rho^f T\mathbb{L}(S))$, where $\eta_{\rho^e}: H(\rho^e,-) \to \mathbb{L}(S)(Se,-)$ is the natural isomorphism sending ρ^e to 1_{Se} and $\tilde\rho^u = \rho^u_{Sf} j^{Se}_{Su} = \rho(f,u,e)$. By (3.16i), $G(\rho^e T\mathbb{L}(S)) = H(\rho^e,-) = \Gamma(eS)$. Hence by (4.2) $\Gamma(\lambda) = G(\bar\lambda)$. If $\lambda = \lambda(e,u,f)$, $\lambda' = \lambda(f,u,g)$, by Equation (3.8), $\lambda\lambda' = \lambda(\rho^e,\rho^v,\rho^u,\rho^g) = \lambda(\rho^e,\rho^{uv},\rho^g) = \overline{\lambda\lambda'}$. Thus Γ is a functor. Since G is inclusion-preserving, it is clear that Γ is also inclusion-preserving. Let eS, $e'S \in v(fS)$, $\Gamma(eS) = \Gamma(e'S)$. Then by Lemma 3.13, $\rho^e\mathcal{R}\rho^{e'}$. Since $u \to \rho^u$ is a homomorphism, by Proposition 2.14 of [11], there is $e_1\omega^r e'$, such that $\rho^e = \rho^{e_1}$. Then $e_1e'\omega e'$ and $\rho^{e_1}\rho^{e'} = \rho^e \rho^{e'} = \rho^{e'}$. Hence $Se_1e' = Se'$ or $e_1e'\mathcal{L}e'$. Therefore $e_1e' = e'$; that is $e_1\mathcal{R}e'$. This implies that $e, e_1 \in \omega^r(f)$ and $\rho^e = \rho^{e_1}$. So $e = fe = f\rho^e = f\rho^{e_1} = fe_1 = e_1$. Thus $e\mathcal{R}e'$ and so $eS = e'S$. Then $v\Gamma|\Gamma(fS)$ is one-to-one. To prove that it is onto, consider $H(\epsilon,-) \subseteq H(\rho^f,-)$. Thus $\epsilon = \rho^f \cdot h$ for some $h: Sf \to C_\epsilon = Sg$ (say). Since ϵ and ρ^f are idempotents in $TL(S)$ such that $\epsilon\omega^r\rho^f$, $\epsilon\rho^f\mathcal{R}\epsilon$ and $\omega\rho^f\omega\rho^f$. Since $H(\epsilon,-) = H(\epsilon\rho^f,-)$, we may assume that $\epsilon\omega\rho^f$. Thus $Sg \subseteq Sf$ and g may be assumed to be in $\omega(f)$. Then $\epsilon = \rho^{f}\cdot h$. So $\epsilon_{Sg} = 1_{Sg} = \rho^f_{Sg}h = j^{Sf}_{Sg}h$. Hence $h: Sf \to Sg$ is a retraction. Hence by Lemma 3.7, $h = \rho(f, g|, g)$ where $g^*\mathcal{L}g'\omega f$. Hence by (3.10) $\epsilon_{Se} = \rho^f_{Se}\rho(f, g', g) = \rho(e, ef, f)\rho(f, g', g) = \rho(e, eg', g)$. Thus $\epsilon_{Se} = \rho^{g'}_{Se}$ i.e. $\epsilon = \rho^{g'}$. Thus $\Gamma(g's) = H(\rho^{g'}, -) = H(\epsilon, -)$. This proves that $v\Gamma$ is an order-isomorphism of $v(fS)$ onto $v(H(\rho^f, -))$. If $\Gamma(\lambda(e, u, f)) = \Gamma(\lambda(e, u', f))$, then by (4.2), $\rho(f, u, e) = \rho(f, u', e)$ and so $u = u'$; i.e. $\lambda(e, u, f) = \lambda(e, u', f)$. Thus Γ is faithful. Now let $\varphi: H(\rho^e, -) \to H(\rho^f, -)$ be any morphism. Then $\eta^{-1}_{\rho^e} \cdot \varphi \cdot \eta_{\rho^f}$ is a natural transformation from $\mathbb{L}(S)(Se, -)$ to $\mathbb{L}(S)(Sf, -)$. Since the *contravariant* Yoneda embedding is fully faithful, then is $\rho(f, u, e): Sf \to Se$ such that $\eta^{-1}_{\rho^e} \cdot \varphi \cdot \eta_{\rho^f} = \mathbb{L}(S)(\rho(f, u, e), -)$. Hence $\varphi = \eta_{\rho^e} \cdot L(S)(\rho(f, u, e), -) \cdot \eta^{-1}_{\rho^f} = \Gamma(\lambda(e, u, f))$. Hence Γ is full. This completes the proof that Γ is a local isomorphism.

Now if $S^{\circ p}$ is the left-right dual of S, then $\mathbb{L}(S^{\circ p}) = \mathbb{R}(S)$, $\lambda(S^{\circ p}) = \mathbb{L}(S)$. Hence Δ defined by (4.2^*) is the same as the functor Γ for $S^{\circ p}$. Hence Δ is a local isomorphism.

Now if $S^{\circ p}$ is the left-right dual of S, then $\mathbb{L}(S^{\circ p}) = \mathbb{R}(S^{\circ p}) = \mathbb{L}(S)$. Hence Δ defined by (4.2^*) is the same as the functor Γ for $S^{\circ p}$. Hence Δ is a local isomorphism.

By (3.15), $\rho^{f*}\rho(f, u, e)^\circ \to \rho(f, u, e)$ is a bijection of $H(\rho^f; Se)$ to $\mathbb{L}(S)(Sf, Se)$, and similarly $\lambda^{e*}\lambda(e, u, f)^\circ \to \lambda(e, u, f)$ is a bijection of $H(\lambda^e; fS)$ to $\mathbb{R}(S)(eS, fS)$. Now $\rho(f, u, e) \to u \to \lambda(e, u, f)$ is a bijection of $\mathbb{L}(S)(Sf, Se)$ to $\mathbb{R}(S)(eS, fS)$. Hence (4.3) defines a bijection of $\Gamma(fS)(Se) = \Gamma(Se, fS)$ with $\Delta(Se, fS)$. Let $x \in \Gamma(Se, fS)$ and $\rho(e, v, e'): Se \to Se!$. By the definition of $\Gamma(Se, fS)$, $x = \rho^{f*}\rho(f, u, e)^\circ$ where $\rho(f, u, e): Sf \to Se$. Then

$$\begin{aligned}\mathfrak{X}(Se, fS)\Delta(\rho(e, v, e'), fS)(x) &= \Delta(\rho(e, v, e'), fS)(\mathfrak{X}(Se, fS)(x)) \\ &= \Delta(\rho(e, v, e'), fS)(\lambda^{e}\cdot\lambda(e, u, f)^\circ) \quad \text{by (4.3)}\end{aligned}$$

and $\Delta(\rho(e, v, e'), fS)(\lambda^{e}\cdot\lambda(e, u, f)^\circ) = \Delta(\rho(e, v, e'))(fS)(\lambda^{e}\cdot\lambda(e, u, f)^\circ)$

$$\begin{aligned}&= \mathbb{R}(S)(\lambda(e', v, e), fS) \cdot (\eta^{-1}_{\lambda^{e'}})_{fS}((\eta_{\lambda^e})_{fS}(\lambda^{e}\cdot\lambda(e, u, f)^\circ) \\ &= (\eta^{-1}_{\lambda^{e'}})_{fS}(\mathbb{R}(S))(\lambda(e', v, e), fS)(\lambda(e, u, f)) \\ &= (\eta^{-1}_{\lambda^{e'}})_{fS}(\lambda(e', v, e)\lambda(e, u, f)) = (\eta^{-1}_{\lambda^{e'}})_{fS}(\lambda(e', uv, f)) = \lambda^{e'}\cdot\lambda(e', uv, f)^\circ.\end{aligned}$$

Hence, $\mathcal{X}(Se, fS)\Delta(\rho(e,v,e'), fS)(x) = \lambda^{e'}\cdot\lambda(e', uv, f)^{\circ}$.

$$\begin{aligned}\Gamma(\rho(e,v,e'), fS)\mathcal{X}(Se', fS)(x) &= \mathcal{X}(Se', fS)(\Gamma(\rho(e,v,e'), fS)(x)) \\ &= \mathcal{X}(Se', fS)(H(\rho^f, \rho(e,v,e'))(x)) \\ &= \mathcal{X}(Se', fS)(\rho^{f}\cdot\rho(f,u,e)\rho(e,v,e')^{\circ}) \\ &= \mathcal{X}(Se', fS)(\rho^{f}\cdot\rho(f,uv,e')^{\circ}) = \lambda^{e'}\cdot\lambda(e', uv, f)^{\circ}\end{aligned}$$

This proves that the first diagram below commutes. Similarly if $\lambda(f,w,f'): fS \to f'S$ is any morphism of $\mathbb{R}(S)$, we can show that the second diagram also commutes:

$$\begin{array}{ccc} \Gamma(Se,fS) & \xrightarrow{\mathcal{X}(Se,fS)} & \Delta(Se,fS) \\ \big\downarrow{\scriptstyle \Gamma(\rho(e,v,e'),fS)} & & \big\downarrow{\scriptstyle \Delta(\rho(e,v,e'),fS)} \\ \Gamma(Se',eS) & \xrightarrow{\mathcal{X}(Se',fS)} & \Delta(Se',fS) \end{array} \qquad \begin{array}{ccc} \Gamma(Sc,fS) & \xrightarrow{\mathcal{X}(Se,fS)} & \Delta(Se,fS) \\ \big\downarrow{\scriptstyle \Gamma(Se,\lambda(f,w,f'))} & & \big\downarrow{\scriptstyle \Delta(Se,\lambda(f,w,f'))} \\ \Gamma(Se,f'S) & \xrightarrow{\mathcal{X}(Se,f'S)} & \Delta(Se,f'S) \end{array}$$

It follows by bifunctor criterion that $\mathcal{X}: \Gamma(-,-) \to \Delta(-,-)$ is a natural isomorphism. This completes the proof.

Suppose $f: \mathcal{C} \to \mathcal{D}$ is an isomorphism of reductive categories. If $\epsilon \in E(T\mathcal{C})$, it is clear that the map $F(C) \to F(C_\epsilon)$ is a cone from $v\mathcal{D}$ to $F(C_\epsilon)$ satisfying the conditions of Definition 3.1; we denote this by $F(\epsilon)$. We define

$$\text{(4.4i)} \qquad N^*F(H(\epsilon,-)) = H(F(\epsilon),-) \quad \text{for all} \quad \epsilon \in E(T\mathcal{C}).$$

If $\eta: H(\epsilon,-) \to H(\epsilon',-)$ is any morphism of $N^*\mathcal{C}$, since C_ϵ and $C_{\epsilon'}$ are representing objects for $H(\epsilon,-)$ and $H(\epsilon',-)$ respectively, there is a unique morphism $f: C_{\epsilon'} \to C_\epsilon$ such that $\eta = \eta_\epsilon \cdot \mathcal{C}(f,-)\cdot \eta_{\epsilon'}^{-1}$. We set

$$\text{(4.4ii)} \qquad N^*F(\eta) = \eta_{F(\epsilon)} \cdot \mathcal{D}(F(f),-)\cdot \eta_{F(\epsilon')}^{-1}$$

It is easy to see that:

LEMMA 4.2. *If $F: \mathcal{C} \to \mathcal{D}$ is an isomorphism of reductive normal categories, $N^*F: N^*\mathcal{C} \to N^*\mathcal{D}$ defined by (4.4) is an isomorphism.*

If $\mathcal{C}$ is a reductive normal category, the restruction of the usual evaluation functor to $N^*\mathcal{C} \times \mathcal{C}$ induces a functor $E_{\mathcal{C}}: \mathcal{C} \to N^{**}\mathcal{C} = (N^*(N^*\mathcal{C}))$ as follows: for $f: c \to c'$ and $\eta: H(\epsilon,-) \to H(\epsilon', f) = H(\epsilon, f)\eta_{c'}$.

Proposition 4.3. *Suppose that $S = T\mathcal{C}$ where $\mathcal{C}$ is a normal reductive category. If F and G are isomorphisms defined by (3.14) and (3.16) respectively and if Γ and Δ are local isomorphisms defined by (4.2) and (4.2*) respectively, then we have $\Gamma = G \circ N^*F$. Further, $\theta = F \circ \Delta \circ N^*G : \mathcal{C} \to N^{**}\mathcal{C}$ is a local isomorphism which is naturally isomorphic to $E_{\mathcal{C}}$.*

Proof: By (3.16), (3.19) and (4.4), for any $\epsilon \in E(S)$,

$$N^*F(G(\epsilon S)) = N^*F(H(\epsilon, -)) = H(F(\epsilon), -).$$

Now $F(\epsilon)$ is the cone $F(c) \to F(C_\epsilon)$. By (3.14), $F(c) = \epsilon' S$, when $\epsilon' \in E(S)$ with $c_{\epsilon'} = c$, and $F(\epsilon_c) = \rho(\epsilon', \epsilon'^{*} \epsilon_c, \epsilon) = \rho(\epsilon', \epsilon' \epsilon, \epsilon) = \rho^{\epsilon}_{\epsilon' S}$. Hence $F(\epsilon) = \rho^\epsilon$ and so $N^*F(G(\epsilon S)) = H(\rho^\epsilon, -) = \Gamma(\epsilon S)$ by (4.2). Similarly, if $\lambda = \lambda(\epsilon, \gamma, \epsilon') : \epsilon S \to \epsilon' S$, by (3.16), (3.14) and (4.4),

$$\begin{aligned} N^*F(G(\lambda)) &= \eta_{F(\epsilon)} \cdot \mathbb{L}(S)(F(\tilde{\gamma}), -) \cdot \eta^{-1}_{F(\epsilon')}, \quad \tilde{\gamma} = \gamma_{c_{\epsilon'}} \cdot j^{c_\epsilon}_{c_\gamma} \\ &= \eta_{\rho^\epsilon} \cdot \mathbb{L}(S)(\rho(\epsilon', \epsilon'^{*} \gamma_{c_\gamma}, \epsilon), -) \cdot \eta^{-1}_{\rho^{\epsilon'}} \\ &= \eta_{\rho^\epsilon} \cdot \mathbb{L}(S)(\rho(\epsilon', \gamma, \epsilon), -) \cdot \eta_{\rho^{\epsilon'}} = \Gamma(\lambda) \quad \text{by (4.2)} \end{aligned}$$

This proves that $\Gamma = G \circ N^*F$.

It is clear that $\theta = F \circ \Delta \circ N^*G$ is a local isomorphism of $\mathcal{C}$ to $N^{**}\mathcal{C}$, since F and N^*G are isomorphisms (by Theorem 3.11 and Lemma 4.2) and Δ is a local isomorphism (by Theorem 4.1). From (3.14), (3.16), (4.2*) and (4.4), we have

$$\begin{aligned} &\theta(c) = H(G(\lambda^\epsilon), -), \quad \text{when} \quad c_\epsilon = c, \quad \text{and} \\ &\theta(g) = \eta_{G(\lambda^\epsilon)} \cdot \mathcal{C}(\eta_{\epsilon''} \cdot \mathcal{C}(g, -) \cdot \eta_\epsilon^{-1}, -) \cdot \eta^{-1}_{G(\lambda^{\epsilon''})}, \quad g : c'' = c_{\epsilon'}. \end{aligned}$$

For $c \in v\mathcal{C}$ and $H(\epsilon', -) \in vN^*\mathcal{C}$, define

(4.6) $\tilde{\omega}(c, H(\epsilon', -)) : \epsilon'^{*} f^0 \to G(\lambda^\epsilon)^*(\eta_\epsilon \cdot \mathcal{C}(f, -) \cdot \eta_{\epsilon'}^{-1})^0, \quad c_{\epsilon'} = c', \quad f \in \mathcal{C}(c'. c).$

This is independent of the choice of ϵ. For if $c_{\epsilon_1} = c_\epsilon$, $\epsilon_1 \mathcal{L} \epsilon$ and so $G(\lambda^\epsilon)\mathcal{R}G(\lambda^{\epsilon_1})$ and $G(\lambda^{\epsilon_1}) = G(\lambda^\epsilon)^*(\eta_\epsilon \cdot \eta_{\epsilon_1}^{-1})$. Hence

$$G(\lambda^{\epsilon_1})^*(\eta_{\epsilon_1} \cdot \mathcal{C}(f, -) \cdot \eta_{\epsilon'}^{-1})^0 = G(\lambda^\epsilon)^*(\eta_\epsilon \cdot \mathcal{C}(f, -) \cdot \eta_{\epsilon'}^{-1})^\circ.$$

Clearly $\tilde{\omega}(c, H(\epsilon', -))$ is a bijection of $E_{\mathcal{C}}(c, H(\epsilon', -)) = H(\epsilon', c)$ with $\theta(c, H(\epsilon', -)) = H(G(\lambda^{\epsilon}), H(\epsilon', -))$. To prove that it is natural, first consider $g: c \to c''$. Let $\epsilon'' \in E(T\mathcal{C})$ with $c_{\epsilon''} = c''$. Then

$$\theta(g, H(\epsilon', -))(\tilde{\omega}(c, H(\epsilon', -))(\epsilon'^{*} f^0) = \theta(g)(H(\epsilon', -))G(\lambda^{\epsilon})^{*}(\eta_{\epsilon} \cdot \mathcal{C}(f, -) \cdot \eta_{\epsilon'}^{-1})^0)$$
$$= G(\lambda^{\epsilon''})^{*}(\eta_{\epsilon''} \cdot \mathcal{C}(fg, -) \cdot \eta_{\epsilon'}^{-1})^0$$

and $\tilde{\omega}(c'', H(\epsilon', -))(E_{\mathcal{C}}(g, H(\epsilon', -))(\epsilon'^{*} f^0)) = \tilde{\omega}(c'', H(\epsilon', -))(\epsilon'^{*}(fg)^0)$

$$= G(\lambda^{\epsilon''})^{*}(\eta_{\epsilon''} \cdot \mathcal{C}(fg, -) \cdot \eta_{\epsilon'}^{-1})^0$$

Hence $\tilde{\omega}(c, H(\epsilon', -))\theta(g, H(\epsilon', -)) = E_{\mathcal{C}}(g, H(\epsilon', -))\tilde{\omega}(c'', H(\epsilon', -))$.

Now suppose that $\eta: H(\epsilon', -) \to H(\epsilon'', -)$. Then there is $g: c_{\epsilon''} \to c_{\epsilon}$, such that $\eta = \eta_{\epsilon'} \cdot \mathcal{C}(g, -) \cdot \eta_{\epsilon''}^{-1}$. Now, by (3.15)

$$\theta(c, \eta) = \theta(c)(\eta) = H(G(\lambda^{\epsilon}), \eta): G(\lambda^{\epsilon})^{*}\eta'^0 \to G(\lambda^{\epsilon})^{*}(\eta'\eta)^0 .$$

Hence for $\epsilon'^{*} f^{\circ} \in E_{\mathcal{C}}(c, H(\epsilon', -)) = H(\epsilon', c)$, where $f: c_{\epsilon'} \to c$, we have

$$\begin{aligned}\theta(c, \eta)(\tilde{\omega}(c, H(\epsilon', -))(\epsilon'^{*} f^0)) &= \theta(c, \eta)(G(\lambda^{\epsilon})^{*}(\eta_{\epsilon} \cdot \mathcal{C}(f, -), \eta_{\epsilon'}^{-1})^0 \\ &= G(\lambda^{\epsilon})^{*}((\eta_{\epsilon} \cdot \mathcal{C}(f, -) \cdot \eta_{\epsilon'}^{-1})(\eta_{\epsilon'} \cdot \mathcal{C}(g, -) \cdot \eta_{\epsilon''}^{-1}))^0 \\ &= G(\lambda^{\epsilon})^{*}(\eta_{\epsilon} \cdot \mathcal{C}(gf, -) \cdot \eta_{\epsilon''}^{-1})^0 ,\end{aligned}$$

$$\begin{aligned}\tilde{\omega}(c, H(\epsilon'', -))(E_{\mathcal{C}}(c, \eta)(\epsilon'^{*} f^0)) &= \tilde{\omega}(c, H(\epsilon'', -))(\eta(c)(\epsilon'^{*} f^0)) \quad \text{by (4.5)} \\ &= \tilde{\omega}(c, H(\epsilon'', -))(\epsilon''^{*}(gf)^0) \\ &= G(\lambda^{\epsilon})^{*}(\eta_{\epsilon} \cdot \mathcal{C}(gf, -) \cdot \eta_{\epsilon''}^{-1})^0 .\end{aligned}$$

Hence $E_{\mathcal{C}}(c, \eta)\tilde{\omega}(c, H(\epsilon'', -)) = \tilde{\omega}(c, H(\epsilon', -))\theta(c, \eta)$.
It now follows from bifunctor criterion that $\tilde{\omega}: (c, H(\epsilon', -)) \to \tilde{\omega}(c, H(\epsilon', -))$ is a natural isomorphism: $\tilde{\omega}: E_{\mathcal{C}}(-, -) \to \theta(-, -)$. It follows from the isomorphism $(\mathcal{C} \times N^{*}\mathcal{C})^{*}$ with $[\mathcal{C}, N^{**}\mathcal{C}]$ that there is a natural isomorphism $\tilde{\omega}: E_{\mathcal{C}} \to \theta$. This completes the p[illegible]of.

LEMMA 4.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be normal reductive category and $\Gamma: \mathcal{D} \to N^{*}\mathcal{C}$ to be a local isomorphism. Let $\bar{\Gamma}(\mathcal{D})$ be the full subcategory of $N^{**}\mathcal{C}$ whose objects* *are $H(\tau,-)$ where τ is a cone from $vN^*\mathcal{C}$ to $\Gamma(D)$ for some $D \in v\mathcal{D}$. Then there is an embedding $\tilde{\Gamma}: \tilde{\Gamma}(\mathcal{D}) \to N^*\mathcal{D}$.*

Proof: Let $\tau: vN^*\mathcal{C} \to \Gamma(D)$ be a normal cone (Definition 3.1) in $N^*\mathcal{D}$ for some $D \in v\mathcal{D}$. If $D' \in v\mathcal{D}$, there is a unique morphism $\tilde{\tau}_{D'}: D' \to D$ such that $\Gamma(\tilde{\tau}_{D'}) = \tau_{\Gamma(D')}$, since Γ is fully faithful. Since Γ is inclusion preserving and preserves and reflects isomorphisms, the map $\tilde{\tau}: D' \to \tilde{\tau}_{D'}$ is a normal cone in $\mathcal{D}$. Define

$$v\tilde{\Gamma}(H(\tau,-)) = H(\tilde{\tau},-). \tag{4.7i}$$

If $\tilde{\tau}: v\mathcal{D} \to D$ and $\tilde{\tilde{\tau}}: v\mathcal{D} \to D'$ on two cones in $\mathcal{D}$ such that $\Gamma(D) = \Gamma(D')$, and $\Gamma(\tilde{\tau}_{D''}) = \Gamma(\tilde{\tilde{\tau}}_{D''})$ for each $D'' \in v\mathcal{D}$, then $\tilde{\tilde{\tau}}_{D''} = \tilde{\tau}_{D''} f$ where $f: D \to D'$ is the unique isomorphism such that $\Gamma(f) = 1_{\Gamma(D)}$. Hence $\tilde{\tilde{\tau}} = \tilde{\tau} * f$ and so $\tilde{\tilde{\tau}} \mathcal{R} \tilde{\tau}$ in $T\mathcal{D}$. Hence $H(\tilde{\tau},-) = H(\tilde{\tilde{\tau}},-)$ by Lemma 3.13. Since Γ is fully faithful and inclusion preserving, $\Gamma(h)$ is an epimorphism in $N^*\mathcal{C}$ if and only if h is an epimorphism in $\mathcal{D}$. It follows that $\tau\omega^r\tau'$ in $E(TN^*\mathcal{C})$ if and only if $\tilde{\tau}\omega^r\tilde{\tau}'$, (where $\tilde{\tau}$ denoted some cone constructed as above for some $D \in v\mathcal{D}$ etc.). This proves, by (3.15) that the map $v\tilde{\Gamma}$ defined above is an inclusion–preserving bijection.

Let $\eta: H(\tau,-) \to H(\tau',-)$ be a morphism in $N^{**}\mathcal{C}$ when vertices of τ and τ' are $\Gamma(D)$ and $\Gamma(D')$ respectively. Then there is a unique morphism $f: D' \to D$ such that $\eta = \eta_\tau \cdot N^*\mathcal{C}(T(f),-) \cdot \eta_{\tau'}^{-1}$. Define

$$\tilde{\Gamma}(\eta) = \eta_{\tilde{\tau}} \cdot \mathcal{D}(f,-) \cdot \eta_{\tilde{\tau}'}^{-1} \tag{4.7ii}$$

It is obvious that this defines a bijection of $N^{**}\mathcal{C}(H(\tau,-),\ H(\tau',-))$ with $N^*\mathcal{D}$ $(H(\tilde{\tau},-),\ H(\tilde{\tau}',-)) = N^*\mathcal{D}(\tilde{\Gamma}(H(\tau,-)),\ \tilde{\Gamma}(H(\tau',-)))$. It is also to see that this defines a functor of $\tilde{\Gamma}(\mathcal{D})$ to $N^*\mathcal{D}$. Hence $\tilde{\Gamma}: \tilde{\Gamma}(\mathcal{D}) \to N^*\mathcal{D}$ is an embedding. This completes the proof.

Theorem 4.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be normal and reductive categories and $\Gamma: \mathcal{D} \to N^*\mathcal{C}$ be a local isomorphism. Let $\mathcal{C}'$ be the full subcategory of $\mathcal{C}$ such that $v\mathcal{C}' = \{c \in v\mathcal{C}$: For some $D \in v\mathcal{D}$, $c \in M\Gamma(D)\}$. Then we have:*

(1) *$\mathcal{C}'$ is an ideal in $\mathcal{C}$ and hence normal and reductive.*

(2) *There exists a local isomorphism $\Gamma^*: \mathcal{C}' \to N^*\mathcal{D}$ such that $c \in M\Gamma(D)$ for $c \in v\mathcal{C}'$, $D \in v\mathcal{D}$ if and only if $D \in M\Gamma^*(c)$.*

(3) *For $c \in v\mathcal{C}'$, $D \in v\mathcal{D}$ and $\epsilon'^* f^0 \in H(\epsilon',c) = \Gamma(c,D)$ define*

$$\mathfrak{X}_\Gamma(c, D)(\epsilon'^* f^0) = \tilde{G}(\lambda^\epsilon)^* g^0 \tag{4.8}$$

where $c_\epsilon = c$, $\tilde{G}(\lambda^\epsilon)$ *is the cone in* $\mathfrak{D}$ *to the vertex* $D' \in v\mathfrak{D}$ *with* $\Gamma(D') = H(\epsilon, -)$ *and* $\Gamma(\tilde{G}(\lambda^\epsilon, -))$ *and* $\Gamma(\tilde{G}(\lambda^\epsilon)) = \tilde{G}(\lambda^\epsilon)$, *and* $g: D' \to D$ *is the morphism such that* $\Gamma(g) = \eta_\epsilon \cdot \mathcal{C}(f, -) \cdot \eta_{f'}^{-1}$. *Then the map*

$$(c, D) \to \mathfrak{X}_\Gamma(c, D)$$

is a natural isomorphism $\mathfrak{X}_\Gamma: \Gamma(-,-) \to \Gamma^*(-,-)$.

Proof: (1) Let $c \in v\mathcal{C}'$ and $c' \subseteq c$. Choose $D \in v\mathfrak{D}$ such that $c \in M\Gamma(D)$. Then there is $\epsilon \in E(T\mathcal{C})$ with $c_\epsilon = c$ such that $\Gamma(D) = H(\epsilon, -)$. Let $h: c \to c'$ be a retraction. Then $\epsilon' = \epsilon^* h\omega\epsilon$ and so $H(\epsilon', -) \subseteq H(\epsilon, -) = \Gamma(D)$. Since Γ is a local isomorphism, there is $D' \subseteq D$ such that $\Gamma(D') = H(\epsilon', -)$. Thus $c' \in MH(\epsilon', -) = M\Gamma(D')$ and so $c' \in v\mathcal{C}'$. This proves that $\mathcal{C}'$ is an ideal of $\mathcal{C}$. (2) Let $c \in v\mathcal{C}'$ and choose $D \in v\mathfrak{D}$ with $c \in M\Gamma(D)$. Suppose that $\Gamma(D) = H(\epsilon, -)$ so that $c \in MH(\epsilon, -)$. By (4.1), we may assume that $c_\epsilon = c$. Then $G(\lambda^\epsilon)$ is a cone in $N^{**}\mathcal{C}$ to the vertex $H(\epsilon, -) = \Gamma(D)$ and so $\theta(c) = H(G(\lambda^\epsilon), -) \in v\tilde{\Gamma}(\mathfrak{D})$. Conversely if $H(\sigma, -) \in v\tilde{\Gamma}(\mathfrak{D})$, where the vertex of σ is $\Gamma(D)$, then for any $c \in M\Gamma(D)$, we have $\theta(c) = H(\sigma, -)$. Hence by Proposition 4.3, $\theta|\mathcal{C}'$ is a local isomorphism of $\mathcal{C}'$ onto $\tilde{\Gamma}(\mathfrak{D})$. Define

$$\Gamma^* = (\theta|\mathcal{C}') \cdot \tilde{\Gamma} \tag{4.9}$$

Then $\Gamma^*: \mathcal{C}' \to N^*\mathfrak{D}$ is a local isomorphism, since $\tilde{\Gamma}$ is an embedding by Lemma 4.4. Let $c \in M\Gamma(D)$. We can find $\epsilon \in E(T\mathcal{C})$ with $c_\epsilon = c$ and $H(\epsilon, -) = \Gamma(D)$. Then $\theta(c) = H(G(\lambda^\epsilon), -)$ where $G(\lambda^\epsilon)$ is the cone in $N^{**}\mathcal{C}$ to the vertex $H(\epsilon, -) = \Gamma(D)$. As in the proof of Lemma 4.4, there is a unique cone $\tilde{G}(\lambda^\epsilon)$ in $\mathfrak{D}$ to the vertex D so that $\tilde{\Gamma}(\theta(c)) = H(\tilde{G}(\lambda^\epsilon), -)$. Thus by (4.1), $D \in MH(\tilde{G}(\lambda^\epsilon), -) = M\Gamma^*(c)$. Conversely, let $D \in \Gamma^*(c)$. Clearly, $\Gamma(D) \in M\theta(c) = MH(G(\lambda^\epsilon), -)$, where $c_\epsilon = c$. Hence there is a cone $\sigma: vN^*\mathcal{C} \to \Gamma(D)$ such that $\sigma\mathcal{R}G(\lambda^\epsilon)$, and $H(\sigma, -) = H(G(\lambda^\epsilon), -)$ by (4.1) and Lemma 3.13. Since $G: \mathbb{R}(T\mathcal{C}) \to N^*\mathcal{C}$ is an isomorphism, there is a cone σ' in $\mathbb{R}(T\mathcal{C})$ such that $\sigma = G(\sigma')$. Since $\sigma = G(\lambda^\epsilon)^*\xi$ for some morphism $\xi: H(\epsilon, -) \to H(\epsilon', -) = \Gamma(D)$, it follows that $\sigma' = \lambda^{\epsilon} * g$ for a unique morphism g of $\mathbb{R}(T\mathcal{C})$ such that $G(g) = \xi$. Since σ is an idempotent, so is σ' and hence $\sigma' = \lambda^{\epsilon''}$ for some idempotent ϵ''

of $\mathcal{C}$. Now $\lambda: T\mathcal{C} \to (T\mathcal{C})$ is the antirepresentation and so there is an idempotent $\epsilon_1 \in E(T\mathcal{C})$ such that $\epsilon_1 \mathcal{L} \epsilon$ and $\lambda^{\epsilon''} = \lambda^{\epsilon_1}$. So $\sigma' = \lambda^{\epsilon_1}$ and so $\sigma = G(\lambda^{\epsilon_1})$. It follows that $\Gamma(D) = H(\epsilon_1, -)$. Since $\epsilon_1 \mathcal{L} \epsilon$, $c_{\epsilon_1} = c_\epsilon = c$. Therefore $c \in M\Gamma(D)$. This completes the proof of (2).

(3) Let $c \in v\mathcal{C}$ and $D \in v\mathcal{D}$. Define

$$\varphi(c, D)(G(\lambda^{\epsilon'})^*(\eta_{\epsilon'} \cdot \mathcal{C}(f, -) \cdot \eta_\epsilon^{-1})^0) = \tilde{G}(\lambda^{\epsilon'})^* g^0$$

where $c_{\epsilon'} = c$, $f \in \mathcal{C}(c, c_\epsilon)$ with $\epsilon, \epsilon' \in E(T\mathcal{C})$ and $g \in \mathcal{D}(D', D)$ such that $\Gamma(D) = H(\epsilon, -)$, $\Gamma(D') = H(\epsilon', -)$ and $\Gamma(g) = \eta_{\epsilon'} \mathcal{C}(f, -) \cdot \eta_\epsilon^{-1}$. φ is well-defined; that is, it does not depend on the choice of ϵ, ϵ' and D'. If ϵ' and ϵ'' are elements of $E(T\mathcal{C})$ such that $c_{\epsilon'} = c_{\epsilon''}$, then $\epsilon' \mathcal{L} \epsilon''$ and so $\lambda^{\epsilon'} \mathcal{R} \lambda^{\epsilon''}$. So $G(\lambda^{\epsilon'}) \mathcal{R} G(\lambda^{\epsilon''})$. So $G(\lambda^{\epsilon''})^*(\eta_{\epsilon''} \cdot \mathcal{C}(f, -) \colon \eta_\epsilon^{-1})^2 = G(\lambda^{\epsilon'})^*(\eta_{\epsilon'} \cdot \eta_{\epsilon''}^{-1} \eta_{\epsilon''} \cdot \mathcal{C}(f, -) \cdot \eta_\epsilon^{-1}) = G(\lambda^{\epsilon'})^*(\eta_{\epsilon'} \cdot \mathcal{C}(f, -) \cdot \eta_\epsilon^{-1})^0$. Similarly if $\epsilon, \epsilon_1 \in E(T\mathcal{C})$ with $H(\epsilon, -) = H(\epsilon_1, -)$, then there is a unique $f_1 \in \mathcal{C}(c_{\epsilon_1}, c)$ such that $\eta_{\epsilon'} \cdot \mathcal{C}(f, -) \cdot \eta_\epsilon^{-1} = \eta_{\epsilon'} \cdot \mathcal{C}(f_1, -) \cdot \eta_{\epsilon_1}^{-1}$. Finally if $D'', D' \in v\mathcal{D}$ with $\Gamma(D'') = \Gamma(D') = H(\epsilon', -)$, then there is a unique $g' \in \mathcal{D}(D'', D)$ with $\Gamma(g) = \Gamma(g')$. So, since $\tilde{G}(\lambda^\epsilon)$ is uniquely determined by $G(\lambda^\epsilon)$ and D', we have $\tilde{G}(\lambda^\epsilon)^* g^0 = \tilde{\tilde{G}}(\lambda^{\epsilon'})^* g_0$ where $\tilde{\tilde{G}}(\lambda^{\epsilon'})$ is the unique cone to the vertex D'' such that $\Gamma(\tilde{G}(\lambda^{\epsilon'})) = G(\lambda^{\epsilon'})$. Hence $\psi(c, D)$, is a well defined bijection of $\theta(c, \Gamma(D))$ to $\Gamma^*(c, D)$. Suppose that $f \in \mathcal{C}(c, c')$. Then for $\epsilon_1 \in E(T\mathcal{C})$ with $c_{\epsilon_1} = c'$ and $D_1' \in v\mathcal{D}$ with $\Gamma(D_1') = H(\epsilon_1', -)$, there is a unique $h: D_1' \to D'$ such that $\Gamma(h) = \eta_{\epsilon_1'} \cdot \mathcal{C}(f, -) \cdot \eta_{\epsilon'}^{-1}$. Then for $G(\lambda^{\epsilon'})^* \Gamma(g) \in \theta(c, \Gamma(D))$,

$$\theta(f, \Gamma(D))(G(\lambda^{\epsilon'})^* \Gamma(g)^0) = G(\lambda^{\epsilon_1'})^*(\Gamma(h)\Gamma(g))^0 = G(\lambda^{\epsilon_1'})^* \Gamma((hg)^0).$$

Hence $\varphi(c', D)(\theta(f, \Gamma(D)))(G(\lambda^{\epsilon'})^* \Gamma(g^0))) = \tilde{G}(\lambda^{\epsilon_1'})^*(hg)^0$. Similarly by the definitions of θ, Γ^* and (4.7), we have

$$\Gamma^*(f, D)(\tilde{G}(\lambda^{\epsilon'})^* g^0) = \tilde{G}(\lambda^{\epsilon_1'})^*(hg)^0.$$

This proves that the following diagram (the first) commute:

$$\begin{array}{ccc} \theta(c, \Gamma(D)) & \xrightarrow{\varphi(c,D)} & \Gamma^*(c, D) \\ \Big\downarrow{\scriptstyle \theta(f,\Gamma(D))} & & \Big\downarrow{\scriptstyle \Gamma^*(f,D)} \\ \theta(c', \Gamma(D)) & \xrightarrow{\varphi(c',D)} & \Gamma^*(c', D) \end{array} \qquad \begin{array}{ccc} \theta(c, \Gamma(D)) & \xrightarrow{\varphi(c,D)} & \Gamma^*(c, D) \\ \Big\downarrow{\scriptstyle \theta(c,\Gamma(h))} & & \Big\downarrow{\scriptstyle \Gamma^*(c,h)} \\ \theta(c, \Gamma(D')) & \xrightarrow{\varphi(c,D')} & \Gamma^*(c, D') \end{array}$$

Now if $h \in \mathcal{D}(D, D')$, then

$$\theta(c, \Gamma(h))(G(\lambda^{\epsilon'})^*\Gamma(g^0)) = G(\lambda^{\epsilon'})^*(\Gamma(g)\Gamma(h))^0 = G(\lambda^{\epsilon'})^*\Gamma(gh)^0 ;$$

and so $\varphi(c, D')(\theta(c, \Gamma(h))(G(\lambda^{\epsilon'})^*\Gamma(g)^0)) = \tilde{G}(\lambda^{\epsilon'})^*(gh)^0$. Also

$$\Gamma^*(c, h)(\tilde{G}(\lambda^{\epsilon'})^*g^0) = \tilde{G}(\lambda^{\epsilon'})^*(gh)^0$$

This proves that the second diagram also commutes. Hence $\varphi: (c, D) \to \varphi(c, D)$ is a natural isomorphism of $\theta(-, \Gamma(-))$ with $\Gamma^*(-,-)$. It is clear from (4.6) that the map $(c, D) \to \tilde{\omega}(c, \Gamma(D))$ is a natural isomorphism $\tilde{\omega}: \Gamma(-,-) \to \theta(-, \Gamma(-))$. From (4.6), (4.8) and the definition of φ we see that for each $c \in v\mathcal{C}$, $D \in v\mathcal{D}$,

$$\mathfrak{X}_\Gamma(c, D) = \tilde{\omega}(c, \Gamma(D))\varphi(c, D)$$

Hence $\mathfrak{X}_\Gamma: \Gamma^*(-,-) \to \Gamma^*(-,-)$ is a natural isomorphism.

Let $D \in v\mathcal{D}$. If $\Gamma(D) = H(\epsilon, -) \in vN^*\mathcal{C}$, where ϵ is a cone in $\mathcal{C}$ with vertex $c = c_\epsilon$, then the pair (c, D) determines the cone ϵ; we shall denote this unique cone by $\gamma(c, D)$. Thus given $D \in v\mathcal{D}$ and $c \in M\Gamma(D)$, $\gamma(c, D)$ is the unique cone such that

$$\Gamma(D) = H(\gamma(c, D), -), \quad c \in M\Gamma(D) . \tag{4.10}$$

Similarly, we write

$$\Gamma^*(c) = H(\gamma^*(c, D), -) \tag{4.10*}$$

Again the pair (c, D) with $c \in M\Gamma(D)$ (or $D \in M\Gamma^*(c)$) completely determines $\gamma^*(c, D)$. Now by the definition of Γ^*,

$$\Gamma^*(c) = H(\tilde{G}(\lambda^{\gamma(c,D)}), -)$$

where $\tilde{G}(\lambda^{\gamma(c,D)})$ is the unique cone such that $\Gamma(\tilde{G}(\lambda^{\gamma(c,D)}), -) = G(\lambda^{\gamma(c,D)})$. Hence, using the definition of G (cf. (3.16)), we get, for any $D' \in v\mathcal{D}$,

$$\Gamma(\gamma^*(c, D)_{D'}) \cdot \eta_{\gamma(c,D)} = \eta_{\gamma(c',D')} \cdot \mathcal{C}(\gamma(c', D')_c, -)c' \in M\Gamma(D') \tag{4.11}$$

This shows that, since Γ is faithful, the cone $\gamma^*(c, D)$ is completely determined by the functor Γ, and cones $\gamma(c, D)$ and $\gamma(c', D')$.

Proposition 4.6. *Let $\mathcal{C}$, $\mathcal{D}$ and Γ be as in Theorem 4.5. Let $c \in M\Gamma(D)$, $c' \in M\Gamma(D')$, $f \in \mathcal{C}(c,c')$ and $g \in \mathcal{D}(D',D)$. Then the first diagram below commutes if and only if the second also commutes.*

$$\begin{tikzcd}
\Gamma(D') \arrow[r, "\eta_{\gamma(c',D')}"] \arrow[d, "\Gamma(g))"] & \mathcal{C}(c',-) \arrow[d, "\mathcal{C}(f,-)"] \\
\Gamma(D) \arrow[r, "\eta_{\gamma(c,D)}"] & \mathcal{C}(c,-)
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma^*(c) \arrow[r, "\eta_{\gamma^*(c,D)}"] \arrow[d, "\Gamma^*(f))"] & \mathcal{D}(D,-) \arrow[d, "\mathcal{D}(g,-)"] \\
\Gamma^*(c') \arrow[r, "\eta_{\gamma^*(c',D')}"] & \mathcal{D}(D',-)
\end{tikzcd}$$

D.1 D.2

Further, the map which assigns to each $f \in \mathcal{C}(c,c')$ a morphism $g \in \mathcal{D}(D',D)$ making these diagrams commute, is a bijection.

Proof: Assume that $f \in \mathcal{C}(c,c')$, $g \in \mathcal{D}(D',D)$ with $c \in M\Gamma(D)$ and $c' \in M\Gamma(D')$. Suppose that the first diagram above commutes. Hence the following diagram commutes: $\eta_{\gamma(c',D')}(c')\mathcal{C}(f,-) = \Gamma(c',g)\eta_{\gamma(c,D)}(c')$. Now,

$$\mathcal{C}(f,-)(\eta_{\gamma(c',D')}(c')(\eta(c',D'))) = \mathcal{C}(f,-)(1_{c'}) = f.$$

Also $\Gamma(c',g)(\eta(c',D')) \in \Gamma(c',D)$ and so $\Gamma(c',g)(\eta(c',D')) = \eta(c,D)^* k^0$ where $k \in \mathcal{C}(c,c')$. Hence $\eta_{\gamma(c,D)}(\Gamma(c',g)(\gamma(c',D'))) = k$ and so $k = f$. Thus

$$\Gamma(c',g)(\gamma(c',D')) = \gamma(c,D)^* f^0 = \Gamma(f,D)(\gamma(c,D)). \tag{4.12}$$

In view of (4.10) and (4.10*), Equation (4.8) defining $\mathfrak{X}_\Gamma$ may be written as

$$\mathfrak{X}_\Gamma(c',D)(\gamma(c,D)^* f^0) = \gamma^*(c',D')^* g^0 \tag{4.8$'$}$$

where f and g satisfies the hypothesis above (i.e. makes the first diagram D.1 commutes). Now, let $\varphi(c,-) = \eta^{-1}_{\gamma(c,D)} \cdot \mathfrak{X}_\Gamma(-,D) \colon \mathcal{C}(c,-) \to \Gamma^*(-,D)$. Then for each c (and D with $c \in M\Gamma(D)$) $\phi(\psi,-)$ is a natural isomorphism. Also $\varphi(c,c)(1_c) = \mathfrak{X}_\Gamma(c,D)(\eta^{-1}_{\gamma(c,D)}(1_c)) = \mathfrak{X}_\Gamma(c,D)(\gamma(c,D)) = \gamma^*(c,D)$. Hence $\varphi(c,c')(f) = \varphi(c,c')(\mathcal{C}(c,f)(1_c)) = \Gamma^*(f,D)(\varphi(c,c)(1_c)) = \Gamma^*(f,D)(\gamma^*(c,D))$ and $\varphi(c,c')(f) = \mathfrak{X}_\Gamma(c',D)(\gamma(c,D)^* f^0) = \gamma^*(c',D')^* g^0 = \Gamma^*(c',g)(\gamma^*(c',D'))$. Thus:

$$\Gamma^*(f,D)(\gamma^*(c,D)) = \gamma^*(c',D')^* g^0 = \Gamma^*(c',g)(\gamma^*(c',D')). \tag{4 12*}$$

Now

$$(\eta)_{\gamma^*(c,D)}(D)\mathcal{D}(g,-))\gamma^*(c,D) = \mathcal{D}(g,-)(\eta_{\gamma^*(c,D)}(D)(\gamma^*(c,D)) = \mathcal{D}(g,-)(1_D) = g$$

and $(\Gamma^*(f,D)\eta_{\gamma^*(c',D')}(D))(\gamma^*(c,D)) = \eta_{\gamma^*(c',D')}(D)(\gamma^*(c',D')^*g^0 = g$. Hence by Yoneda Lemma, the two natural transformations $\eta_{\gamma^*(c,D)}\mathcal{D}(g,-)$, $\Gamma^*(f,D)$ $\eta_{\gamma^*(c',D')}\colon \Gamma^*(c) \to \mathcal{D}(D',-)$ coincide. Hence the diagram D2 commutes:

Conversely assume that f and g makes diagram D2 commute. By $(4.8')$, if $\mathfrak{X}_\Gamma(c',D)(\gamma(c,D)^*f^0) = \gamma^*(c',D')^*\bar{g}^0$, then f and $\bar{g}$ makes the diagram D1 commute and so by the above they also make D2 commute. Hence we get $\eta_{\gamma^*(c,D)}$ $\mathcal{D}(g,-) = \eta_{\gamma^*(c,D)}\mathcal{D}(\bar{g},-)$ from which we conclude that $g = \bar{g}$. Hence f and g make the first diagram commute. Since $\mathfrak{X}_\Gamma$ is a natural isomorphism, it follows from $(4.8')$ that $f \to g$ is a bijection of $\mathcal{C}(c,c')$ with $\mathcal{D}(D',D)$. This completes the proof.

When f and g satisfy the condition of the Proposition above, we shall denote g by f^* and f^* will be called the *transpose of* f from D' to D. Similarly given $g \in \mathcal{D}(D',D)$, the transpose $g^*\colon c \to c'$ of g will be the morphism f above. Note that $f^{**} = f$ and $g^{**} = g$. From the proposition above we have the following:

Corollary 4.7. *Let* $c_i \in M\Gamma(D_i)$, $i = 0,1,2$ *and* $f_i\colon c_{i-1} \to c_i$, $i = 1,2$. *Then* $(f_1f_2)^* = f_2^*f_1^*$.

Theorem 4.8. *Let* $\Gamma\colon \mathcal{D} \to N^*\mathcal{C}$ *be a local isomorphism of normal reductive categories. If for each* $c \in v\mathcal{C}$, *there is* $D \in v\mathcal{D}$ *such that* $c \in M\Gamma(D)$, *then* $\Delta^* = \Gamma$ *and* $\mathfrak{X}_\Delta = \mathfrak{X}_\Gamma^{-1}$, *where* $\Delta = \Gamma^*$.

Proof: By Theorem 4.5, $c \in M\Gamma(D)$ if and only if $D \in M\Gamma^*(c) = M\Delta(c)$; this is true if and only if $c \in M\Delta^*(D)$. Thus $M\Gamma(D) = M\Delta^*(D)$. Now if $c \in M\Gamma(D) = M\Delta^*(D)$, then there are cones ϵ and ϵ' to the vertex c such that $\Gamma(D) = H(\epsilon,-)$ and $\Delta^*(D) = H(\epsilon',-)$. Let $c' \in v\mathcal{C}$ and $D' \in M\Delta(c')$. Then $c' \in M\Gamma(D') = M\Delta^*(D')$ and it follows from (4.10) and Proposition 4.6 that $\epsilon^*_{c'}\colon D \to D' = \tau'_D$ where $\tau'\colon v\mathcal{D} \to D'$ is the cone such that $\Delta(c') = H(\tau',-)$. By the same reason $\tau'^*_D\colon c \to c' = \epsilon'_{c'}$. Hence $\epsilon'_{c'} = \epsilon^{**}_{c'} = \epsilon_{c'}$. Hence $\epsilon' = \epsilon$ and so $\Gamma(D) = \Delta^*(D)$ for every $D \in v\mathcal{D}$. Let $g\colon D \to D'$ be any morphism of $\mathcal{D}$. By Proposition 4.6,

$$\Gamma(g) = \eta_{\gamma(c,D)} \cdot \mathcal{C}(g^*,-) \cdot \eta^{-1}_{\gamma(c',D')} = \Delta^*(g^{**}) = \Delta(g),$$

where $c \in M\Gamma(D)$, $c' \in M\Gamma(D')$. Hence $\Gamma = \Delta^*$. Finally it follows immediately from $(4.8')$ that $\mathfrak{X}_\Gamma^{-1} = \mathfrak{X}_\Delta$. This completes the proof.

We observe that if $\Gamma: \mathcal{D} \to N^*\mathcal{C}$ is any local isomorphism, then the local isomorphism $\Gamma^*: \mathcal{C} \to N^*\mathcal{D}$ satisfies the hypothesis of Theorem 4.8. Hence we have:

COROLLARY 4.9. *If $\Gamma: \mathcal{D} \to N^*\mathcal{C}$ is any local isomorphism, then $\Gamma^{***} = \Gamma^*$.*

DEFINITION 4.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be normal reductive categories. A cross-connection from $\mathcal{D}$ to $\mathcal{C}$ is a local isomorphism $\Gamma: \mathcal{D} \to N^*\mathcal{C}$ such that for $c \in v\mathcal{C}$ there is $D \in v\mathcal{D}$ such that $c \in M\Gamma(D)$.*

In the following, by *cross-connection*, we shall mean a triplet of the form $(\mathcal{C}, \mathcal{D}; \Gamma)$ where $\Gamma: \mathcal{D} \to N^*\mathcal{C}$ is a cross-connection from $\mathcal{D}$ to $\mathcal{C}$. However, we shall often say that Γ '*is a cross-connection*' if the categories $\mathcal{C}$ and $\mathcal{D}$ are clear from the context.

When $\Gamma: \mathcal{D} \to N^*\mathcal{C}$ is a cross-connection, the category $\mathcal{C}'$ defined in Theorem 4.5 coincides with $\mathcal{C}$ and $\Gamma^*: \mathcal{C} \to N^*\mathcal{D}$ is a cross-connection from $\mathcal{C}$ to $\mathcal{D}$. We shall refer to $\Gamma^*(= (\mathcal{D}, \mathcal{C}; \Gamma^*))$ as the *dual of* $\Gamma(= (\mathcal{C}, \mathcal{D}; \Gamma))$ and the natural isomorphism $\mathfrak{X}_\Gamma: \Gamma(-,-) \to \Gamma^*(-,-)$ as the *duality* associated with the cross-connection Γ. Note that if $\Gamma: \mathcal{D} \to N^*\mathcal{C}$ is any local isomorphism, then $\Gamma^*: \mathcal{C}' \to N^*\mathcal{D}$ is a cross-connection.

PROPOSITION 4.10. *Let $\Gamma: \mathcal{D} \to N^*\mathcal{C}$ be a cross-connection and let*

$$U\Gamma = \cup\{\Gamma(c, D) : (c, D) \in v\mathcal{C} \times v\mathcal{D}\} \tag{4.13}$$

Then $\gamma \in T\mathcal{C}$ belongs to $U\Gamma$ if and only if $\gamma = \gamma(c', D)^* f$, where $f: c' \to c$ is an isomorphism in $\mathcal{C}$. Further, $U\Gamma$ is a regular subsemigroup of $T\mathcal{C}$ such that $\mathbb{L}(U\Gamma)$ is isomorphic to $\mathcal{C}$.

PROOF: If $\gamma = \gamma(c', D)^* f$ where $f: c' \to c$ is an isomorphism, then $\gamma \in H(\gamma(c', D); c) = \Gamma(c, D)$ and so $\gamma \in U\Gamma$. Conversely if $\gamma \in \Gamma(c, D) = H(\gamma(c', D); c)$, then $\gamma = \gamma(c', D)^* f^0$ with $f: c' \to c$. Then $\gamma \in \Gamma(c_\gamma, D)$ where $c_\gamma = \operatorname{Im} f$. Hence we may assume that $\gamma = \gamma(c', D)^* f$ where $f: c' \to c$ is an isomorphism. Let $f = hg$ be a normal factorization of f. If $\epsilon = \gamma(c', D)^* h$, then $\epsilon_{c_\epsilon} = \gamma(c', D)_{c_\epsilon} h = j_{c_\epsilon}^{c'} = 1_{c_\epsilon}$ since $h: c' \to c_\epsilon$ is a retraction. Thus ϵ is an idempotent such that $\epsilon \omega \gamma(c', D)$ (by Lemma 3.5). Hence $H(\epsilon; -) \subseteq H(\gamma(c', D); -) = \Gamma(D)$.

Since Γ is a local isomorphism, there is $D' \subseteq D$ such that $H(\epsilon; -) = \Gamma(D')$. Then $c_\epsilon \in M\Gamma(D')$ and so, $\epsilon = \gamma(c_{\epsilon'}, D')$ and $\lambda \in H(\gamma(c_\epsilon, D'), c)$. Now $\gamma = \gamma(c', D)^* hg = (\gamma(c', D)^* h)^* g = \gamma(c_\epsilon, D')^* g$ and $g: c_\epsilon \to c$ is an isomorphism.

By (3.4), a product $\gamma_1 \cdot \gamma_2 \in U\Gamma$ if $\gamma_1 \in U\Gamma$. Hence $U\Gamma$ is a right ideal of $T\mathcal{C}$ and hence, in particular, $U\Gamma$ is a semigroup. If $\gamma \in U\Gamma$, then $\gamma = \gamma(c', D)^* f$ where $f: c' \to c$ is an isomorphism. Since Γ is a cross-connection, there is $D' \in v\mathcal{D}$ such that $c \in M\Gamma(D')$. Let $\gamma' = \gamma(c, D')^* f^{-1}$. Then $\gamma' \in U\Gamma$ and it is easy to see (from (3.4)) that $\gamma \cdot \gamma' \cdot \gamma = \gamma$ and $\gamma' \cdot \gamma \cdot \gamma' = \gamma'$. Hence $U\Gamma$ is a regular subsemigroup of $T\mathcal{C}$.

Since each $\mathcal{L}$-class of $T\mathcal{C}$ contains an idempotent of the form $\gamma(c, D)$, it follows that $\gamma(c, D)T\mathcal{C} \to \gamma(c, D)U\Gamma$ is an order-isomorphism of $v\mathbb{L}(T\mathcal{C})$ onto $v\mathbb{L}(U\Gamma)$. Similarly the map $\rho(\gamma(c, D), \gamma(c, D)^* f^0, \gamma(c', D')) \to \rho(\gamma(c, D), \gamma(c, D)^* f^0, \gamma(c', D'))|U\Gamma$ is a bijection of $\mathbb{L}(T\mathcal{C})(T\mathcal{C}\gamma(c, D), T\mathcal{C}\gamma(c', D'))$ onto $\mathbb{L}(U\Gamma)(U\Gamma\gamma(c, D), U\Gamma\gamma(c', D'))$ and this bijective is functorial. Thus the restriction induces an isomorphism of $\mathbb{L}(T\mathcal{C})$ onto $\mathbb{L}(U\Gamma)$. Hence by Theorem 3.11, there is an isomorphism $F': \mathcal{C} \to \mathbb{L}(U\Gamma)$.

If $\Gamma: \mathcal{D} \to N^*\mathcal{C}$ is a cross-connection; we know that $\Gamma^*: \mathcal{C} \to N^*\mathcal{D}$ is also a cross-connection. Hence

$$U\Gamma^* = \cup\{\Gamma^*(c, D): c, D \in v\mathcal{C} \times v\mathcal{D}\} \tag{4.13*}$$

is a regular subsemigroup of $T\mathcal{D}$ such that $L(U\Gamma^*)$ is isomorphic to $\mathcal{D}$. We proceed to show that Γ determines a subdirect product $\hat{S}\Gamma$ of $U\Gamma$ and $U\Gamma^*$. We need the following lemma:

LEMMA 4.11. *Suppose that* $\gamma(c, D)^* f^0 = \gamma(c', D')^* g$ *where* $f: c' \to c_1'$ *is a morphism,* $g: c' \to c_1'$ *is an isomorphism and* $\gamma(c', D')\omega\gamma(c, D)$. *Then*

$$\mathfrak{X}_\Gamma(c_1, D)(\gamma(c, D)^* f^0) = \mathfrak{X}_\Gamma(c_1', D')(\gamma(c', D')^* g).$$

PROOF: The given equality implies that $\operatorname{Im} f = \operatorname{Im} f^0 = \operatorname{Im} g = c_1'$. Let $h = f^0 g^{-1}$. Then $\gamma(c, D)^* h = (\gamma(c, D)^* f^0)^* g^{-1} = (\gamma(c', D')^* g)^* g^{-1} = \gamma(c', D')$. Hence $1_{c'} = \gamma(c', D')_{c'} = \gamma(c, D)_{c'} h$. Now by the given condition we have $c' \subseteq c$ (and $D' \subseteq D$). Hence $\gamma(c, D)_{c'} = j_{c'}^{c}$ and so $h: c \to c'$ is a retraction. Hence $f = hgj_{c_1'}^{c_1}$ is a normal factorization of f. Since $\gamma(c, D)^* h = \gamma(c', D')$, $h = \gamma(c', D')_c$. Hence by (4.11) and Proposition 4.6, we have $h^* = \gamma^*(c, D)$

is the transpose of h from D' to D. Since $D' \subseteq D$, $\gamma^*(c, D)_{D'} = j^D_{D'}$. By similar arguments, we see that the transpose of $j^{c_1}_{c'_1}$ from D_1 to D'_1 (with $D_1 \in MD(c_1)$, $D'_1 \in MD(c'_1)$ and $D'_1 \subseteq D_1$) is a retraction $k: D_1 \to D'_1$. Hence $f^* = kg^* j^D_{D'}$ where $g^*: D'_1 \to D'$ is the transpose of $g: c' \to c'_1$ which is clearly an isomorphism. Therefore by (4.8),

$$\mathfrak{X}_\Gamma(c_1, D)(\gamma(c, D)^* f^0) = \gamma^*(c_1, D_1)^* kg^* = (\gamma^*(c_1, D_1)^* k)^* g^*$$

$$\begin{aligned}
\text{Now}\quad \gamma^*(c_1, D_1)^* k &= \Gamma^*(c_1, k)(\gamma^*(c_1, D_1)) = \Gamma^*(j^{c_1}_{c'_1}, D'_1)(\gamma^*(c'_1, D'_1)) \\
&= \Gamma^*(j^{c_1}_{c'_1}, D'_1)\gamma^*(c'_1, D'_1)) && \text{by } (4.12^*) \\
&= \Gamma^*(j^{c_1}_{c'_1}, D'_1)\mathfrak{X}_\Gamma(c'_1, D'_1)(\gamma(c'_1, D'_1)) && \text{by } (4.8) \\
&= \mathfrak{X}_\Gamma(c'_1, D'_1)(\Gamma(j^{c_1}_{c'_1}, D'_1)(\gamma(c'_1, D'_1))) \\
&= \mathfrak{X}_\Gamma(c'_1, D'_1)(\gamma(c'_1, D'_1)^*(j^{c_1}_{c'_1})^0) \\
&= \mathfrak{X}_\Gamma(c'_1, D'_1)(\gamma(c'_1, D'_1) = \gamma^*(c'_1, D'_1) && \text{(by (4.8))}.
\end{aligned}$$

Hence $\mathfrak{X}_\Gamma(c'_1, D'_1)(\gamma(c'_1, D'_1) = \gamma^*(c'_1, D'_1)^* g^* = \mathfrak{X}_\Gamma(c'_1, D')(\gamma(c', D')^* g)$ by (4.8). This completes the proof.

Given a cross-connection $\Gamma: \mathfrak{D} \to N^*\mathfrak{C}$, $\rho \in U\Gamma$ is said to be *linked* to $\lambda \in U\Gamma^*$ (or λ is *linked* to ρ) if there is $(c, D) \in v\mathfrak{C} \times v\mathfrak{D}$ such that

(4.14) $$\rho \in \Gamma(c, D) \quad \text{and} \quad \lambda = \mathfrak{X}_\Gamma(c, D)(\rho)$$

In this case the pair (ρ, λ) is referred to as a *linked pair*. We write

(4.15) $$S\hat{\Gamma} = \{(\rho, \lambda) \in U\Gamma \times U\Gamma^* : (\rho, \lambda) \quad \text{is a linked pair}\}.$$

Since every $\rho \in U\Gamma[\lambda \in U\Gamma^*]$ is linked to at least one $\lambda \in U\Gamma^*[\rho \in U\Gamma]$, and $U\Gamma^*$ respectively. Moreover, $S\hat{\Gamma}$ is clearly a relation from $U\Gamma$ to $U\Gamma^*$ and its inverse relation is $\hat{S}\Gamma^*$.

LEMMA 4.12. *If* $\rho_i \in \Gamma(c_i, D_i)$, *then* $\mathfrak{X}_\Gamma(c_1, D_1)(\rho_1)\mathfrak{X}_\Gamma(c_2, D_2) = \mathfrak{X}_\Gamma(c_1, D_2)(\rho_2 \cdot \rho_1)$.

PROOF: Suppose that $c'_i \in M\Gamma(D_i)$ and $D'_i \in M\Gamma^*(c_i)$, $i = 1, 2$. Then there exists $f_i: c'_i \to c_i$ such that $\rho_i = \gamma(c'_i, D_i)^* f_i^0$ and $\mathfrak{X}_\Gamma(c_i, D_i)(\rho_i) = \gamma^*(c_i, D'_i)^*(f^*)^0$,

$i = 1,2$ (by (4.8)). Now by (3.4), $\rho_1\rho_2 = \rho_1^*(\rho_2)^0_{c_{\rho_1}} = \gamma(c_1', D_1)^* f_1^0(\gamma(c_2', D_2)_{c_{\rho_1}} f_2)^0$ $= \gamma(c_1', D_1)^*(f_1\gamma(c_2', D_2)_{c_1} f_2)^0$. Hence by (3.8)

$$\mathfrak{X}_\Gamma(c_1, D_2)(\rho_1\rho_2) = \gamma^*(c_2, D_2')^*((f_1\gamma(c_2', D_2)_{c_1} f_2)^*)^0 .$$

$$\begin{aligned} \text{Now,}\quad (f_1\gamma(c_2', D_2)_c f_2)^* &= f_2^*(\gamma(c_2', D_2)_{c_1})^* f_1^* &&\text{by Corollary 4.7}\\ &= f_2^*\gamma(c_1, D_1')_{D_2}) f_1^* &&\text{by (4.11) and Proposition 4.6}\end{aligned}$$

Hence $\mathfrak{X}_\Gamma(c_1, D_2)(\rho_1\rho_2) = \gamma^*(c_2, D_2')^*(f_2^*\gamma^*(c_1, D_1')_{D_2} f_1^*)^0$
Now, as before, it is easy to see that

$$\mathfrak{X}_\Gamma(c_2, D_2)(\rho_2)\mathfrak{X}_\Gamma(c_1, D_1)(\rho_1) = \gamma^*(c_2, D_2')^*(f_2^*\gamma^*(c_1, D_1')_{D_2} f_1^*)^0$$

and so the lemma is proved.

THEOREM 4.13. *Let $\Gamma\colon \mathcal{D} \to N^*\mathcal{C}$ be a cross-connection. For $(\rho, \lambda), (\rho', \lambda') \in \hat{S}\Gamma$, define*

$$(\rho, \lambda)(\rho', \lambda') = (\rho\rho', \lambda'\lambda) . \tag{4.16}$$

This defines a binary operation in $\hat{S}\Gamma$ and $\hat{S}\Gamma$ is a regular semigroup which is a subdirect product of $U\Gamma$ and $U\Gamma^*$. Moreover there exist isomorphisms $F_\Gamma\colon \mathcal{C} \to \mathbb{L}(\hat{S}\Gamma)$ and $G_\Gamma\colon \mathcal{D} \to \mathbb{R}(\hat{S}\Gamma)$.

PROOF: By the remarks proceeding Lemma 4.12, it is clear that $\hat{S}\Gamma$ is a subdirect product if it is a semigroup; this will follow if we show that if $(\rho, \lambda), (\rho', \lambda') \in \hat{S}\Gamma$, $\rho\rho'$ is linked to $\lambda'\lambda$. By (4.16), there exist $(c, D), (c', D') \in v\mathcal{C} \times v\mathcal{D}$ such that $\rho \in \Gamma(c, D)$, $\lambda = \mathfrak{X}_\Gamma(c, D)(\rho)$ and $\rho' \in \Gamma(c', D')$, $\lambda' = \mathfrak{X}_\Gamma(c', D')(\rho')$. Then by Lemma 4.12, $\lambda'\lambda = \mathfrak{X}_\Gamma(c', D)(\rho\rho')$. Hence $(\rho\rho', \lambda'\lambda) \in \hat{S}\Gamma$.

Let $(\rho, \lambda) \in \hat{S}\Gamma$ so that for $(c, D) \in c\mathcal{C} \times v\mathcal{D}$, $\rho \in \Gamma(c, D)$, $\lambda = \mathfrak{X}_\Gamma(c, D)(\rho)$. By Proposition 4.10 (see the proof), if $\rho = \gamma(c', D)^* f^0$ when $c' \in M\Gamma(D)$ and $f\colon c' \to c$, there is $\gamma(c'', D')\omega\gamma(c', D)$ and an isomorphism $g\colon c'' \to c_1 \subseteq c$ such that $\rho = \gamma(c'', D')^* g$. By Lemma 4.11, $\mathfrak{X}_\Gamma(c, D)(\rho) = \mathfrak{X}_\Gamma(c_1, D')(\rho) = \lambda$. Hence if $g^*\colon D_1 \to D'$ is a transpose of $g\colon c'' \to c_1$ where $D_1 \in M\Gamma^*(c_1)$, then $\lambda = \gamma^*(c_1, D_1)^* g^*$ by (4.8). Now $g^{-1}\colon c_1 \to c''$ and the transpose of g^{-1} from D' to D_1 is $g^{*^{-1}}$. Hence $\mathfrak{X}_\Gamma(c'', D_1)(\gamma(c_1, D_1)^* g^{-1}) = \gamma^*(c'', D')^* g^{*^{-1}}$. Hence if

$\rho' = \gamma(c_1, D_1)^* g^{-1}$ and $\lambda' = \gamma^*(c'', D')^* g^{*^{-1}}$, then $(\rho', \lambda') \in \hat{S}\Gamma$ and it is easy to see (using (3.4)) that $(\rho, \lambda)(\rho', \lambda')(\rho, \lambda) = (\rho, \lambda)$ and $(\rho', \lambda')(\rho, \lambda)(\rho', \lambda') = (\rho', \lambda')$. Thus $\hat{S}\Gamma$ is regular.

Suppose that $\pi_1 : \hat{S}\Gamma \to U\Gamma$ is the projection. We first show that $U\Gamma$ is isomorphic to the regular representations $(\hat{S}\Gamma)_\rho$ of $\hat{S}\Gamma$. It is sufficient to show that the congruence *cong* π_1 induced by π_1 is the same as the congruence *cong* ρ on $\hat{S}\Gamma$ by the regular representation. Let $(\rho, \lambda_1), (\rho, \lambda_2) \in \hat{S}\Gamma$ (so that $(\rho, \lambda_1), (\rho, \lambda_2) \in$ *cong* π_1). Then there exists $(c_i, D_i) \in v\mathcal{C} \times v\mathcal{D}$, $i = 1,2$ such that $\rho = \gamma(c_i', D_i)^* f_i \in \Gamma(c_i, D_i)$ where f_i is an isomorphism and $\lambda_i = \mathfrak{X}_\Gamma(c_i, D_i)(\rho)$, $i = 1,2$. Then $\gamma(c_1', D_1) R \gamma(c_2', D_2)$ and so $\Gamma(D_1) = \Gamma(D_2)$. Hence $\gamma(c_k', D_1) = \gamma(c_k', D_2)$, $k = 1,2$. Hence $f_1 = \gamma(c_2', D_2)_{c_1'} f_2$. Now if $(\rho', \lambda') \in \hat{S}\Gamma$, where $\rho' = \gamma(c', D') \in \hat{S}\Gamma$, where $\rho' = \gamma(c', D')^* f' \in \Gamma(c'', D')$ with $f' : c' \to c''$ an isomorphism and $\lambda' = \mathfrak{X}_\Gamma(c'', D')(\rho')$, we have by Lemma 4.12, $\lambda_1 \lambda' = \mathfrak{X}_\Gamma(c, D')(\rho' \rho) = \lambda_2 \lambda'$ where $c = c_1 = c_2 = c_\rho$, the vertex of ρ. Hence $(\rho', \lambda')(\rho, \lambda) = (\rho' \rho, \lambda \lambda') = (\rho' \rho, \lambda_2 \lambda') = (\rho', \lambda')(\rho, \lambda_2)$. Hence $((\rho, \lambda_1), (\rho, \lambda_2) \in$ *cong* ρ. Thus *cong* $\pi_1 \subseteq$ *cong* ρ.

We now show that the congruence. *cong* ρ on $U\Gamma$ is the identity. It is sufficient to show that no two distinct idempotents of $U\Gamma$ are related by *cong* ρ, since $\mathcal{H} \cap$ *cong* ρ identity for some regular semigroup. If $(\gamma(c_1, D_1), \gamma(c_2, D_2) \in$ *cong* ρ, since *cong* $\rho \subseteq \mathcal{L}$, we have $c_1 = c_2$. Now $\rho\gamma(c_1, D_1) = \rho\gamma(c_1, D_2)$ for all $\rho \in U\Gamma$. Hence if $c' \in \gamma\mathcal{C}$ and if $\gamma(c', D')$ is any idempotent with vertex c', $\gamma(c', D')\gamma(c_1, D_1) = \gamma(c', D')\gamma(c_1, D_2)$. Hence $(\gamma(c', D')\gamma(c_1, D_1)) = \gamma(c_1, D_1)^0_{c'} = (\gamma(c', D')\gamma(c_1, D_2))'_{c'} = \gamma(c_1, D_2)^0_{c'}$. Thus $\gamma(c_1, D_1)_{c'} = \gamma(c_1, D_2)_{c'}$. This is true for all $c' \in v\mathcal{C}$ and so $\gamma(c_1, D_1) = \gamma(c_1, D_2)$. Now if $((\rho_1, \lambda_1), (\rho_2, \lambda_2))$ $\in$ *cong* ρ, then clearly $(\rho_1, \rho_2) \in$ *cong* ρ on $U\Gamma$ and so $\rho_1 = \rho_2$. Thus *cong* $\rho \subseteq$ *cong* π_1.

Let $(\rho_1, \lambda_1), (\rho_2, \lambda_2) \in \hat{S}\Gamma$. If $(\rho_1, \lambda_1)\mathcal{L}(\rho_2, \lambda_2)$, clearly $\rho_1 \mathcal{L} \rho_2$ in $U\Gamma$. Conversely if $\rho_2 = \rho' \rho_1$, then for any λ' linked to ρ', $\lambda_1 \lambda'$ is linked to $\rho' \rho_1 = \rho_2$ and so $(\rho_2, \lambda_1 \lambda') = (\rho', \lambda')(\rho_1, \lambda_1)$. Since $((\rho_2, \lambda_2), (\rho_2, \lambda_1 \lambda')) \in$ *cong* ρ, $(\rho_2, \lambda_2)\mathcal{L}(\rho_2, \lambda_1 \lambda')$ and so $(\rho_2, \lambda_2) \in \hat{S}\Gamma(\rho_1, \lambda_1)$. Hence if $\rho_1 \mathcal{L} \rho_2$ is $U\Gamma$, it follows that (ρ_1, λ_1) $\mathcal{L}(\rho_2, \lambda_2)$. Hence

$$vF'' : U\Gamma\rho \mapsto \hat{S}\Gamma(\rho, \lambda)$$

is a bijection of $vL(U\Gamma)$ to $vL(\hat{S}\Gamma)$, which by the above is inclusion preserving.

Also for $f: c \to c'$, and λ' linked to $\gamma(c,D)^* f^0$ with vertex D,

$$F''((\gamma(c,D),\gamma(c,U)^* f^0,\gamma(U,c'))$$
$$= ((\gamma(c,D)\gamma^*(c,D),(\gamma(c,D)^* f_1\lambda'),(\gamma(c',D'),\gamma^*(c',D'))$$

is well defined because the translation induced by any ρ in $(\gamma(c,D)^* f^0, \lambda')$ is equal to that induced by any other pair $(\gamma(c,D)^* f^0, \lambda'')$. It is easy to see that $F'': \mathbb{L}(U\Gamma) \to \mathbb{L}(\hat{S}\Gamma)$ is an isomorphism. Hence by Theorem 4.10, F_Γ defined by:

(4.17i) $\quad F_\Gamma(c) = \hat{S}\Gamma(\rho,\lambda)$, where ρ is a cone with vertex c; and

(4.17ii) $\quad F_\Gamma(f) = \rho((\gamma(c,D),\gamma^*(c,D)),(\gamma(c',D'),\gamma^*(c',D')))$, $\quad f: c \to c'$,

and λ is linked to $\rho = \gamma(c,D)^* f^0$ with vertex D is an isomorphism $F_\Gamma: \mathcal{C} \to \mathbb{L}(\hat{S}\Gamma)$.

Since $\Gamma^*: \mathcal{C} \to N^*\mathcal{D}$ is a cross-connection, by the above there is an isomorphism $\overline{F}: \mathcal{D} \to L(\hat{S}\Gamma^*)$. Now $\hat{S}\Gamma^* = \hat{S}\Gamma^{-1}$ and the map $(\lambda,\rho) \mapsto (\rho,\lambda)$ is an isomorphism of $\hat{S}\Gamma$. Clearly this induces an isomorphism $\overline{F}: \mathbb{L}(\hat{S}\Gamma^*) \to \mathbb{R}(\hat{S}\Gamma)$. Thus by (4.17) G_Γ defined by:

(4.17i*) $\quad G_\Gamma(D) = (\gamma(c,D),\gamma^*(c,D))\hat{S}\Gamma$

(4.17ii*) $\quad G_\Gamma(g) = \lambda((\gamma(c,D))\gamma^*(c,D)),(\rho,\lambda),(\gamma(c',D'),\gamma^*(c',D')))$

where $g: D' \to D$ and ρ is linked to $\lambda = \gamma^*(c',D')^* g^0$ with vertex c, is an isomorphism $G_\Gamma: \mathcal{D} \to \mathbb{R}(\hat{S}\Gamma)$. Note that $G_\Gamma = \overline{F} \circ \overline{\overline{F}}$. This completes the proof.

For convenience of later reference, we state the following as a corollary, and is a consequence of the proof of the above theorem above:

COROLLARY 4.14. *Let $\Gamma: \mathcal{D} \to N^*\mathcal{C}$ is a cross-connection. The following statements are equivalent:*

(1) *$\hat{S}\Gamma$ is right reductive.*

(2) *The projection $\pi_1: \hat{S}\Gamma \to U\Gamma$ is an isomorphism.*

(3) *$\Gamma: \mathcal{D} \to N^*\mathcal{C}$ is embedding (i.e. $v\Gamma$ is injective).*

PROOF: Equivalence of (1) and (2) follows from the equality $cong\ \rho = cong\ \pi_1$. If (2) holds and if $\Gamma(D_1) = \Gamma(D_2)$, for any $c \in M\Gamma(D_2)$, $\gamma(c,D_1) = \gamma(c,D_2)$ and so $((\gamma(c,D_1),\gamma^*(c,D_1)),(\gamma(c,D_2),\gamma^*(c,D_2)) \in cong\ \rho = cong\ \pi$ and so $\gamma^*(c,D_1) =$

$\gamma^*(c, D_2)$. Hence $D_1 = \text{vertex}\ \gamma^*(c, D_1) = \text{vertex}\ \gamma^*(c, D_2) = D_2$. Conversely if (3) holds and if $((\gamma(c, D), \lambda_1), (\gamma(c, D), \lambda_2)) \in cong\ \rho$, then by the proof above there is $D_1, D_2 \in M\Gamma^*(c)$ such that $\lambda_i = \gamma^*(c, D_i)$, $i = 1, 2$ and $\gamma(f, D) = \gamma(c, D_i)$. Then $\Gamma(D_1) = \Gamma(D_2)$ and so $D_1 = D_2$ by (3). Hence $\lambda_1 = \lambda_2$ and so (1) holds.

Let S be a regular semigroup and Γ, Δ and $\mathfrak{X}$ be defined by (4.1), (4.2*) and (4.3) respectively. By Theorem 4.1, Γ is a local isomorphism. Since $\Gamma(Sf) = H(\rho^f, -)$ and since ρ^f is a convex with vertex Sf (cf. (3.10)), we have $\gamma(Sf, fS) = \rho^f$. Hence Γ is a cross-connection. By (4.2) and Proposition 4.6 $\lambda(e, u, f) = \rho(f, u, e)^*$. Hence by (3.10) and (4.11), $\lambda^f_{eS} = \lambda(e, ef, f) = \rho(f, ef, e)^* = (\rho^e_{Sf})^*$. Therefore, $\gamma^*(Sf, fS)_{eS} = (\gamma(Se, eS)_{fS})^* = (\rho^{e*}_{Sf}) = \lambda^f_{eS}$. Thus $\gamma^*(Sf, fS) = \lambda^* = \Delta(fS)$. Similarly by Proposition 4.6 and (4.2*).

$$\begin{aligned}\Delta(\rho(e, u, f)) &= \eta_{\lambda^e} \cdot \mathbb{R}(S)(\lambda(f, u, e), -) \cdot \eta^{-1}_{\lambda^f} \\ &= \Gamma^*(\lambda(f, u, e)^*) = \Gamma^*(\rho(e, u, f)).\end{aligned}$$

Therefore $\Delta = \Gamma^*$. Further by (4.3) and (4.8$'$), for $Se \in vL(S)$ and $fS \in v\mathbb{R}(S)$. $\mathfrak{X}(Se, fS) = \mathfrak{X}_\Gamma(Se, fS)$. Thus $\mathfrak{X} = \mathfrak{X}_\Gamma$.

Now let $a \in S$. Then for any $e \in E(R_a)$,

$$\rho^a = \rho^e \rho^a = \rho^{e*} \rho(e, a, f) \quad \text{where} \quad f \in E(L_a).$$

Hence by Proposition 4.10, $\rho^a \in U\Gamma$. Conversely if $\gamma \in U\Gamma$, then $\gamma = \gamma(Se, fS)^* \rho$ where ρ is an isomorphism of Se onto Sh (say). Then $\rho = \rho(e, u, h)$ where $u \in R_e \cap L_h$. Also $Se \in M\Gamma(fS) = MH(\rho^f, -)$ and so by (3.11) there is $g \in L_e \cap R_f$. Hence $\gamma(Se, fS) = \gamma(Sg, gS) = \rho^g$ and $\gamma = \rho^{g*} \rho(g, gu, h) = \rho^{gu}$. Hence

$$U\Gamma = \{P\rho^a : u \in S\} \tag{4.18}$$

Let $a \in R_e \cap L_f$ where $e, f \in E(S)$. Then

$$\rho^a = \rho^{e*} \rho(e, a, f), \quad \lambda^a = \lambda^{f*} \lambda(f, a, e)$$

Hence by (4.14) and (4.3), λ^a is linked to ρ^a. Conversely let λ be linked to ρ^a. Then by (4.14), $\rho^a = \gamma(Se, fS)^* \rho^a_{Se}$, where ρ^a_{Se} is an isomorphism and

$\lambda = \mathcal{X}(Sh, Sf)(\rho^a)$ where $h\mathcal{L}a$. As before we can find $g \in E(S)$ such that $\gamma(Se, fS) = \rho^g$ and then

$$\rho^a = \rho^{g} \cdot \rho(g, ga, h) = \rho^{ga}$$

and by (4.3), $\lambda = \lambda^{h} \cdot \lambda(h, ga, g) = \lambda^{ga}$. Thus $(\rho^a, \lambda) = (\rho^{ga}, \lambda^{ga})$. Since S is weakly reductive, $a \mapsto (\rho^a, \lambda^a)$ is an isomorphism. Hence:

THEOREM 4.15. *Let S be a regular semigroup. Thus $\Gamma: \mathbb{R}(S) \to N^*\mathbb{L}(S)$ defined by (4.2) is a cross–connection such that $\Gamma^* = \Delta$ and $\mathcal{X}_\Gamma = \mathcal{X}$ where Δ and $\mathcal{X}$ are defined by (4.1*) and (4.3) respectively. Further*

$$\hat{S}\Gamma = \{(\rho^a, \lambda^a) : a \in S\}$$

and the mapping $\varphi_S: S \to \hat{S}\Gamma$ defined by

$$\varphi_S(a) = (\rho^a, \lambda^a) \tag{4.19}$$

is an isomorphism of S onto $\hat{S}\Gamma$.

REMARK 4.4. Theorem 4.13 constructs a regular semigroup $\hat{S}\Gamma$ from a cross–connection $\Gamma: \mathcal{D} \to N^*\mathcal{C}$ and Theorem 4.15 shows that, up to isomorphisms, every regular semigroup is of this form.

Now given two normal reductive categories $\mathcal{C}$ and $\mathcal{D}$ existence of a cross–connection between them, implies strong relations between $\mathcal{C}$ and $\mathcal{D}$. In fact, up to the vertex set $\mathcal{D}, \mathcal{D}$ is the same as a '*total*' ideal of $N^*\mathcal{C}$. Here an ideal I of $N^*\mathcal{C}$ is total if for each $c \in v\mathcal{C}$, there is $H(\epsilon, -) \in I$ such that $c \in MH(\epsilon, -)$. If a cross–connection $\Gamma: \mathcal{D} \to N^*\mathcal{C}$ exists, we can construct $\mathcal{D}$ from $v\mathcal{D}$ and Im Γ which is a total ideal of $N^*\mathcal{C}$. Now, clearly $v\mathcal{D}$ is a regular partially ordered set and by Proposition 4.6, $\mathcal{D}(D, D') = \{\eta_{\gamma(D',c)}\mathcal{C}(f, -) \cdot \eta_{\gamma(D',c')};\ f \in \mathcal{C}(c', c),\ c \in M\Gamma(D)$ and $c' \in M\Gamma(D')\}$. It is also that Γ is completely determined by $v\Gamma$. This suggests that a regular semigroup can be constructed from the following data:

(1) a normal reductive category $\mathcal{C}_0$,

(2) a regular partially ordered set I ; and

(3) A mapping $\theta: I \to vN^*\mathcal{C}$ such that

  (i) $\theta|I(x)$ is an isomorphism for each $x \in I$;

(ii) Im θ is total in $vN^*\mathcal{C}$ (is *teh* full subcategory generated by Im θ is a total ideal of $N^*\mathcal{C}$)

Given these, we construct the category $\mathcal{D}$ by setting:

$$(4.20) \qquad v\mathcal{D} = I, \quad \mathcal{D}(D, D') = \{\eta_{\gamma(D,c)} \cdot \mathcal{C}(f, -) \cdot \eta^{-1}_{\gamma(D',c')} : f \in \mathcal{C}(c', c).\}$$

where $c \in M\theta(D)$, $c' \in M\theta(c')$ and $\gamma(D, c)$ is the unique cone such that $\theta(D) = H(\gamma(D, c) : -)$. Γ is constructed by setting

$$(4.21) \qquad v\Gamma = \theta \quad \text{and} \quad \Gamma|\mathcal{D}(0, D') = 1_{\mathcal{D}(D,D')}.$$

It is easy to see that $\mathcal{D}$ is a normal reductive category and $\Gamma: \mathcal{D} \to N^*\mathcal{C}$ is a cross-connection.

By Corollary 4.14, the semigroup $\hat{S}\Gamma$ constructed as above is right reductive if and only if θ is an embedding of I as a total ideal of $vN^*\mathcal{C}$. Thus there is one-to-one correspondence between right reductive regular semimaps with $\mathbb{L}(S) \approx \mathcal{C}$ and total ideals of $N^*\mathcal{C}$.

5. Morphism of Cross-connections

In this section we wish to describe homomorphisms of regular semigroups in terms of cross-connections. We shall define a category **Cr** which will be called the category of cross-connections and show that **Cr** is naturally equivalent to the category **RS** of regular semigroups.

Let $\Gamma: \mathcal{D} \to N^*\mathcal{C}$ be a cross-connection. If $c \in M\Gamma(D)$, clearly, $(\gamma(c, D), \gamma^*(c, D))$ is an idempotent in $\hat{S}\Gamma$. Every idempotent of $\hat{S}\Gamma$ is of this form. For, if (ρ, λ) is an idempotent, then by (4.15) and (4.16), ρ is an idempotent in $U\Gamma$ and by Proposition 4.10, $\rho = \gamma(c, D)$ for $(c, D) \in v\mathcal{C} \times v\mathcal{D}$ such that $c \in M\Gamma(D)$. Now, by (4.14), there is $(c', D') \in v\mathcal{C} \times v\mathcal{D}$ and an isomorphism $f: c'' \to c$ with $c'' \in M\Gamma(D')$ (by Proposition 4.10) such that

$$\rho = \gamma(c'', D')^* f, \qquad \lambda = \mathcal{X}_\Gamma(c', D')(\rho)$$

Then $\gamma(c, D) = \gamma(c'', D')^* f$ and so $c' = c$ and by Lemma 3.5, $\gamma(c, D)\mathcal{R}\gamma(c'', D')$. Hence by Lemma 3.13, $\Gamma(D) = H(\gamma(c, D), -) = H(\gamma(c'', D'), -) = \Gamma(D')$. Hence $\gamma(c, D) = \gamma(c, D')$. It follows that $f = \gamma(c, D')_{c''}$ and so by (4.8')

$$\lambda = \mathcal{X}_\Gamma(c, D')(\gamma(c'', D')^* \gamma(c, D')_{c''}) = \mathcal{X}_\Gamma(c, D')(\gamma(c, D')) = \gamma^*(c, D').$$

Hence $(\rho, \lambda) = (\gamma(c, D'), \gamma^*(c, D'))$. It is clear that the mapping

$$(5.1) \quad \alpha: E\Gamma = \{(c, D): c \in M\Gamma(D)\} \to E(\hat{S}\Gamma), \quad \alpha(c, D) = (\gamma(c, D), \gamma^*(c, D))$$

is a bijection. Hence we may define a biorder structure on $E\Gamma$ which makes α biorder isomorphism.

Recall from §1 that a coimage of a morphism $f: c \to c'$ is the image of a retraction e such that $f = euj$ is a normal factorization of f. Thus the equation $c_1 = \text{coim } f$ means that there is a unique retraction $e: c \to c_1$ such that e is a factor of a normal factorization of f. Any two coimages of f are clearly isomorphic.

By routine calculation one can prove the following.

PROPOSITION 5.1. *Let $\Gamma: \mathcal{D} \to N^*\mathcal{C}$ be a cross-connection. Then $E\Gamma = \{(c, D): c \in M\Gamma(D)\}$ becomes a biordered set when we define quasiorders and basic products as follows:*

$$(5.2) \quad \begin{aligned} &(c', D')\omega^l(c, D) \Leftrightarrow c' \subseteq c; \\ &(c, D)(c', D') = (c', D''), \quad \text{where} \quad D'' = \text{Im}\,(\gamma^*(c, D)_{D'}). \end{aligned}$$

$$(5.2^*) \quad \begin{aligned} &(c', D')\omega^r(c, D) \Leftrightarrow D' \subseteq D; \\ &(c', D')(c, D) = (c'', D'), \quad \text{where} \quad c'' = \text{Im}\,(\gamma(c, D)_{c'}). \end{aligned}$$

If $(c, D), (c', D') \in E\Gamma$, the sandwich set $S((c, D), (c', D')) = \overline{S}(c, D')$ where

$$(5.3) \quad \overline{S}(c, D') = \{(c'', D''): c'' = \text{coim}\,(\gamma(c_1', D')_c), \quad D'' = \text{coim}\,(\gamma^*(c, D')_{D'})\}.$$

PROOF: We shall verify only (5.3). If e and f are idempotents of a regular semigroup S, it is well-known that $S(e, f) = V(ef) \cap M(e, f)$ (cf. [11]). Now let $(c_1, D'') \in S((c, D), (c', D'))$. Then clearly $\gamma(c_1, D'') \in S(\gamma(c, D), \gamma(c', D'))$ and so if $c'' =$ vertex of $\gamma(c, D)\gamma(c', D')$, $\gamma(c_1, D'') = \gamma(c'', D'')^*\gamma(c_1, D'')_{c''}$. Since $\gamma(c_1, D'') \in V(\gamma(c, D)\gamma(c', D'))$, there exists $D_1 \subseteq D$ such that $\gamma(c, D)\gamma(c', D') = \gamma(c_1, D_1)^*(\gamma(c_1, D'')_{c''})^{-1}$. Hence we have

$$\gamma(c_1, D_1) = \gamma(c, D)^*\gamma(c', D')^0_c(\gamma(c_1, D'')_{c''})$$

and $\gamma(c_1, D_1)\omega\gamma(c, D)$. Hence $h = \gamma(c', D')_{c''}: c \to c_1$ is a retraction such that $\gamma(c', D')^0_c = h(\gamma(c_1, D'')_{c''})^{-1}$ is a normal factorization of $\gamma(c', D')^0_c$. Thus $c_1 = \text{coim}\,(\gamma(c', D')^0_c) = \text{coim}\,(\gamma(c', D')_c)$. Similarly $D'' = \text{coim}\,(\gamma^*(c, D)_{D'})$ and so $(c_1, D'') \in \overline{S}(c, D')$. Conversely let $(c'', D'') \in \overline{S}(c, D')$. Since $c'' = \text{coim}\,\gamma(c', D')_c$ there is $D_1 \subseteq D$ such that $D_1 \in M\Gamma^*(c'')$ and

$$\gamma(c'', D_1)^* f = \gamma(c, D) \cdot \gamma(c', D') \quad \text{where} \quad f = \gamma(c', D')^0_{c''}$$

If $c_1 = \text{Im}\, f$, then there is $D'_1 \subseteq D'$ such that $c_1 \in M\Gamma(D_1)$. Let $h: c \to c''$ be the retraction such that $\gamma(c', D')_c = h\gamma(c', D')_{c''}$, $h' = (j^{c'}_{c'_1})^*: D' \to D'_1$ and $h'': D' \to D''$ be the retraction such that $\gamma^*(c, D)_{D'} = h''\gamma^*(c, D)_{D''}$. Now $\gamma(c', D')_c = hfj^{c'}_{c'_1}$ is a normal factorization of $\gamma(c', D')_c$. By Lemma 4.12,

$$\begin{aligned}\gamma^*(c', D')\gamma^*(c, D) &= \gamma^*(c', D')^*\gamma^*(c, D)^0_{D'} \\ &= \mathfrak{X}_\Gamma(c, D')(\gamma(c, D)\gamma(c', D')) \\ &= \mathfrak{X}_\Gamma(c, D')(\gamma(c, D)^*\gamma(c', D')^0_c) \\ &= \gamma^*(c', D')^*((\gamma(c', D')_c)^*)^0.\end{aligned}$$

Hence $\gamma(c', D')^*_c = \gamma^*(c, D)_{D'} = h'f^*j^D_{D_1}$. Hence $D'_1 = \text{coim}\,\gamma^*(c, D)_{D'}$. Since $D'' = \text{coim}\,\gamma^*(c, D)_{D'}$, $\gamma^*(c', D')^*h''$ is an idempotent in $U\Gamma^*$, $\mathcal{R}$ equivalent to $\gamma^*(c', D')^*h' = \gamma^*(c'_1, D'_1)$. Hence $D'' \in M\Gamma^*(c'_1)$ and $\gamma^*(c', D')^*h'' = \gamma(c'_1, D'')$. Now $\gamma(c'_1, D'')^*f^{-1}$ is an inverse of $\gamma(c, D)\gamma(c', D')$ in $M(\gamma(c, D), \gamma(c', D'))$ and so $\gamma(c'_1, D'')^*f^{-1} = \gamma(c'', D'')$ and $\gamma(c'', D'') \in S(\gamma(c, D), \gamma(c', D'))$. Similarly $\gamma^*(c'', D'') = \gamma^*(c'', D_1)^*f^{*^{-1}}$ and $\gamma^*(c'', D'') \in S(\gamma^*(c, D), \gamma^*(c, D))$. Thus $\alpha(c''.D'') = (\gamma(c'', D''), \gamma^*(c'', D''))$ belongs to $S(\alpha(c, D), \alpha(c', D'))$ and so $(c'', D'') \in S((c, D), (c', D'))$. This proves the equality (5.3). The remaining statements can be verified easily using Theorem 4.13.

Let $\Gamma: \mathcal{D} \to N^*\mathcal{C}$ be a cross-connection. In the following it will be convenient to use the following notations: if $e \in E\Gamma$, the pair of objects corresponding to e will be denoted by (c^e, D^e); that is, $e = (c^e, D^e)$.

DEFINITION 5.1. *Let $\Gamma: \mathcal{D} \to N^*\mathcal{C}$ and $\Gamma': \mathcal{D}' \to N\mathcal{C}'$ be cross-connections. A morphism $m: \Gamma \to \Gamma'$ is a pair $m = (F, G)$ of inclusion-preserving functors satisfying the following:*

I: *$(c, D) \in E\Gamma \Rightarrow (F(c), G(D)) \in E\Gamma'$ and $F(\gamma(c, D)_{c'}) = \gamma(F(c), G(D)_{F(c')})$ for all $c' \in v\mathcal{C}$.*

II: *If* $f^*: D' \to D$ *is a transpose of* $f: c \to c'$, *then* $G(f^*) = F(f)^*$.

It may be noted that statement I^* and II^* respectively dual to I and II (these obtained by interchanging $\mathcal{C}$ and $\mathcal{D}$, F and G etc.) are consequences of I and II. For if $g: D' \to D$ is given and if $g^* = c \to c'$ is a transpose, we have:

$$G(g)^* = G(g^{**})^* = F(g^*)^{**} = F(g^*)$$

by II. Thus II^* holds. To prove I^*, let $(c, D) \in E\Gamma$ and $D' \in v\mathcal{D}$. Choose $c' \in v\mathcal{C}$ such that $c' \in M\Gamma(D')$. Then by I, $(F(c'), G(D')) \in E\Gamma'$ and

$$\begin{aligned}\gamma^*(F(c), G(D))_{G(D')} &= (\gamma(F(c'), G(D'))_{F(c)})^* \quad \text{by (4.11)}\\ &= F(\gamma(c', D')_c)^* \quad \text{by I}\\ &= G(\gamma^*(c, D)_{D'}) \quad \text{by II and (4.11)}.\end{aligned}$$

Hence, in the following, we shall assume that I and II include their dual as well.

It is clear that if $m: \Gamma \to \Gamma'$ and $m': \Gamma' \to \Gamma''$ are morphisms

$$m \cdot m' = (F \circ F', G \circ G'): \Gamma \to \Gamma'', \quad m = (F, G), \quad m' = (F', u') \tag{5.4}$$

is a morphism. Also $(1_{\mathcal{C}}, 1_{\mathcal{D}}): \Gamma \to \Gamma$ is a morphism. Hence we have a category **Cr** whose objects are cross-connections and morphisms are these defined above. The category **Cr** will be called the *category of cross-connections*.

We shall now drive a useful alternate form of axiom II:

THEOREM 5.2. *Let* $\Gamma: \mathcal{D} \to N^*\mathcal{C}$, $\Gamma': \mathcal{D} \to N^*\mathcal{C}$ *be cross-connections and* $F: \mathcal{C} \to \mathcal{C}'$ *and* $G: D \to \mathcal{D}'$ *be inclusion-preserving functors satisfying axiom* I *(and its dual). Then* $(F, G): \Gamma \to \Gamma'$ *is a morphism if and only if it satisfies the following:*
II′ *There exists a natural transformation* $\zeta: \Gamma(-,-) \to (F \times G)\Gamma'(-,-)$ *such that for all* $(c, D) \in E\Gamma$, $\varphi(c, D)(\gamma(c, D)) = \gamma(F(c), G(D))$.

PROOF: First suppose that (F, G) satisfies II′. Let $\gamma = \gamma(\epsilon)^* f^0 \in \Gamma(c, D)$, where $e \in E\Gamma$ with $D^e = D$. Since $\zeta: \Gamma \to (F \times G)\Gamma'$ is natural, we have:

$$\begin{aligned}\zeta(c, D)(\gamma) = \gamma(c, D)(\gamma(e)^* f^0) &= \zeta(c, D)(\Gamma(f, D)(\gamma(e)))\\ &= \Gamma'(F(f), G(D))(\zeta(c^e, D)(\gamma(e)))\\ &= \Gamma'(F(f), G(D))(\gamma(F(c^e), G(D))) \quad \text{by II}'\\ &= \gamma(\theta(\epsilon))^* F(f)^0\end{aligned}$$

where $\theta: E\Gamma \to E\Gamma'$ is the map defined by $\theta(e) = (F(c^e), G(D^e))$. Hence for each $(c, D) \in v\mathcal{C} \times v\mathcal{D}$, the map $\zeta(c, D)$ is defined by:

$$\zeta(c,D)(\gamma) = \gamma(\theta(e))^* F(f)^0 \quad \text{if } \gamma = \gamma(e)^* f^0 \text{ with } e \in E\Gamma, \tag{5.5}$$
$$f: c^e \to c \text{ and } D^e = D.$$

To prove axiom II, consider $f: c \to c'$ and let $f^*: D' \to D$ be a transpose of f so that $e = (c, D)$, $e' = (c', D') \in E\Gamma$. By (4.12) we have $\Gamma(f, D)(\gamma(e)) = \Gamma(c', f^*)(\gamma(e'))$.

$$\begin{aligned} \text{Now} \quad \zeta(c',D)(\Gamma(c',f^*)(\gamma(e')) &= G(c',D)(\Gamma(f,D)(\gamma(e))) \\ &= \zeta(c',D)\gamma(e)^* f^0) = \gamma(\theta_e)^* F(f)^0 \quad \text{by (5.5)} \end{aligned}$$

$$\begin{aligned} \text{and} \quad \Gamma'(F(c'),G(f^*))(\zeta(c',D')(\gamma(e'))) &= \Gamma'(F(c'),G(f^*))(\gamma(\theta(e'))) \quad \text{by II}' \\ &= \Gamma'(G(f^*)^*, G(D))(\gamma(\theta(e))) \\ &= \gamma(\theta(e))^*(G(f^*)^*)^0 . \end{aligned}$$

Since $\zeta: \Gamma \to (F \times G)\Gamma'$ is a natural transformation, we have

$$\begin{aligned} \gamma(\theta(e))^* F(f)^0 = \zeta(c',D)(\Gamma(c',f^*)(\gamma(e'))) &= \Gamma'(F(c'),G(f^*))(\zeta(c',D')(\gamma(e'))) \\ &= \gamma(\theta(e))^*(G(f^*)^*)^0 . \end{aligned}$$

It follows that $F(f) = G(f^*)^*$. This proves axiom II.

Suppose now that (F, G) is a morphism of Γ to Γ'. For each $(c, D) \in v\mathcal{C} \times v\mathcal{D}$, define the map $\zeta(c, D): \Gamma(c, D) \to \Gamma'(F(c), G(D))$ by (5.5). This is well defined. For if $\gamma = \gamma(e')^* f'^0$ is another representation of γ, thus $D^{e'}$ and so $e\mathcal{R}e'$ in $E\Gamma$ and $f' = \gamma(e)_{c^{e'}} f$ and so

$$\begin{aligned} \gamma(\theta e')^* F(f')^0 &= \gamma(\theta e')^* F(\gamma(e)_{c^{e'}}) F(f)^0, && \text{since } \gamma(e)_{c^{e'}}: c^{e'} \to e^e \text{ is an isomorphism,} \\ &= \gamma(\theta e')^* \gamma(\theta e)_{F(c^{e'})} F(f)^0 && \text{by I} \\ &= \gamma(\theta e)^* F(f)^0 && \text{since } e\mathcal{R}\theta e',\ F(c^{e'}) = c^{\theta e'} \end{aligned}$$

To show that the map $\zeta: (c, D) \mapsto \zeta(c, D)$ is a natural transformation, consider $f: c' \to c''$. If $\gamma = \gamma(e)^* \gamma_c \in \Gamma(c', D)$ (so that $e = (c, D)$), we have

$$\begin{aligned} (\Gamma(f,D)\zeta(c'',D))(\gamma) &= \zeta(c'',D)(\Gamma(f,D)(\gamma)) = G(c'',D)(\gamma^* f^0) \\ &= \zeta(c'',D)(\gamma(e)^* \gamma_c f^0) = \gamma(\theta e)^* F(\gamma_c) F(f^0)^\circ \end{aligned}$$

by (5.5), and

$$\begin{aligned}\zeta(c',D)\Gamma'(F(f),G(D))(\gamma) &= \Gamma'(F(f),G(D))(\gamma(\theta e)^*F(\gamma_c)) \\ &= \gamma(\theta e)^*F(\gamma_c)F(f)^0 .\end{aligned}$$

Hence $\Gamma(f,D)\zeta(c'',D) = \zeta(c',D)\Gamma'(F(f),G(D))$. Let $g: D \to D' \in \mathcal{D}$ and $g^*: c' \to c$ be a transpose of g. Then by (4.12), we have $\Gamma(c,g)(\gamma(e)) \oplus \Gamma(g^*,D')(\gamma(e'))$ where $e = (c,D)$ and $e' = (c',D') \in E\Gamma$. If $\gamma = (e)^*\gamma_c \in \Gamma(c'',D)$, then $\gamma = \Gamma(\gamma_c, D)(\gamma(e))$ and so,

$$\begin{aligned}\Gamma(c'',g)(\gamma) &= \Gamma(c,g)(\Gamma(\gamma_c,D)(\gamma(e))) \\ &= \Gamma(\gamma_c,D)(\Gamma(c,g)(\gamma(e))) \quad \text{since } \Gamma \text{ is a bifunctor,} \\ &= \Gamma(\gamma_c,D)(\Gamma(g^*,D')(\gamma(e'))) = \gamma(e')^*(g(\gamma_c)^0)\end{aligned}$$

Hence $(\Gamma(c'',g)\zeta(c'',D'))(\gamma) = \gamma(\theta e')^*F(g^*\gamma_c)^0$, by (5.5). Similarly

$$\begin{aligned}(\zeta(c'',D)\Gamma'(F(c''),G(g)))(\gamma) &= \Gamma'(F(c''),G(g))(\gamma(\theta e)^*F(\gamma_c)) \\ &= \gamma(\theta e')^*(G(g)^*F(\gamma_c))^0 = \gamma(\theta e')^*F(g^*\gamma_c)^0\end{aligned}$$

since by II, $G(g)^* = F(g^*)$. Thus $\Gamma(c'',g)\zeta(c'',D') = \zeta(c'',D)\Gamma'(F(c''),G(g))$. It now follows from bifunctor criterion that ζ is a natural transformation.

REMARK 5.1. Let Γ, Γ', F and G be as in the statement of Theorem 5.2. If there exists $\zeta: \Gamma \to (F \times G)\Gamma'$ satisfying II$'$, then φ is uniquely determined by F and G (in view of (5.5)).

REMARK 5.2. In view of the symmetry between Γ and Γ^*, F and G, it follows that the dual of Theorem 5.2 also holds. Thus, given (F,G) satisfying axiom I, axiom II holds if and only if:

II$^{1^*}$ There is a natural transformation $\zeta^*: \Gamma^* \to (F \times G)\Gamma'^*$ such that for each $(c,D) \in E\Gamma$, $\zeta^*(c,D)(\gamma^*(c,D)) = \gamma^*(F(c),G(D))$. When ζ^* exists, it is defined by the dual (5.5^*) of Equation (5.5). Thus given a pair of functors (F,G) satisfying axiom I, whenever (F,G) satisfies one of II, II*, II$'$, II$'^*$, then it satisfies all others and then (F,G) is a morphism of cross-connections.

PROPOSITION 5.3. *Let $(F,G): \Gamma \to \Gamma'$ be a morphism of cross-connections. Then the following diagram of functor and natural transformation commute:*

$$\begin{array}{ccc}
\Gamma & \xrightarrow{\mathfrak{X}_\Gamma} & \Gamma^* \\
\downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \zeta^*} \\
(F \times G)\Gamma' & \xrightarrow{(F\times G)\mathfrak{X}_\Gamma} & (F \times G)\Gamma'^*
\end{array}$$

Here $(F \times G)\mathfrak{X}_{\Gamma'}$ *is the natural transformations* $(c, D) \mapsto \mathfrak{X}_{\Gamma'}(F(c), G(D))$.

PROOF: Let $\gamma = \gamma(e)^* f \in \Gamma(c, D)$ where $e \in E\Gamma$, $D^e = D$. Then $f: c^e \to c$ is an isomorphism and by $(4.8^{'})$ and (5.5^{*}), we have

$$\begin{aligned}
(\mathfrak{X}_\Gamma(c, D)\zeta^*(c, D))(\gamma) &= \zeta^*(c, D)(\mathfrak{X}_\Gamma(c, D)(\gamma(e)^* f)) \\
&= \zeta^*(c, D)(\gamma^*(')^* f^*) \\
&= \gamma^*(\theta e')^* G(f^*),
\end{aligned}$$

where $e' \in E\Gamma$ with $c^{e'} = c$. Similarly,

$$\begin{aligned}
\zeta(c, D)(F \times G)\mathfrak{X}_{\Gamma'}(c, D)(\zeta) &= \mathfrak{X}_{\Gamma'}(F(c), G(D))(\zeta(c, D)(\gamma(e)^* f)) \\
&= \mathfrak{X}_{\Gamma'}(F(c), G(D))(\gamma(\theta e)^* F(f)) \quad \text{by } (5.5) \\
&= \gamma^*(\theta e')^* F(f)^* \quad \text{by } (4.8^{'})
\end{aligned}$$

Hence by II, $\mathfrak{X}_\Gamma(c, D)\zeta^*(c, D) = \zeta(c, D)(F \times G)\mathfrak{X}_{\Gamma'}(c, D)$. For all $(c, D) \in v\mathcal{C} \times v\mathcal{D}$. This completes the proof.

We have seen that a cross-connection $\Gamma: \mathcal{D} \to N^*\mathcal{C}$ is completely determined by $\mathcal{C}$ and the vertex map $v\Gamma$ of Γ (cf. Remark 4.1). Analogously, we proceed to show that a morphism of cross-connection (F, G) is completely determined by one of the functors and the vertex map of the other.

THEOREM 5.4. *Let* $\Gamma: \mathcal{D} \to N^*\mathcal{C}$, $\Gamma': \mathcal{D}' \to N^*\mathcal{C}'$ *be cross-connections. Suppose that* $F: \mathcal{C} \to \mathcal{C}'$ *is an inclusion-preserving functor and* $\theta: E\Gamma \to E\Gamma'$ *is a regular bimorphism such that for all* $e, e' \in E\Gamma$,

Γ'. $F(\gamma(e)_{c^{e'}}) = \gamma(\theta e)_{c^{\theta e'}}$

Then there exists a unique functor $G: \mathcal{D} \to \mathcal{D}'$ *such that* $(F, G): \Gamma \to \Gamma'$ *is a morphism of cross-connections such that for all* $e \in E\Gamma$,

$$\theta e = (F(c^e), G(D^e))$$

Conversely if $(F,G)\colon \Gamma \to \Gamma'$ is a morphism, then the map θ defined by (5.6) is a regular bimorphism such that the pair (θ, F) satisfies I' (and (θ, G) satisfies I'^{*}).

PROOF: Let $c \in v\mathcal{C}$. We first show that $F(c) = c^{\theta e}$ where $e \in E\Gamma$ with $c^e = c$. For, by I', $F(1_c) = F(\gamma(e)_c) - \gamma(\theta e)_{c^{\theta e}} = 1_{F(c)}$. This proves the required equality. Next, for each $D \in v\mathcal{D}$, define

$$G(D) = D^{\theta e}, \quad \text{where } e \in E\Gamma \text{ with } D^e = D. \tag{5.7i}$$

If $e' \in E\Gamma$ with $D^{e'} = D$, then $e\mathcal{R}e'$ and since θ is a bimorphism, $\theta e\mathcal{R}\theta e'$. Hence $D^{\theta e} = D^{\theta e'}$ and so (5.7i) defines a single-valued map. If $g\colon D \to D'$, and if $g^*\colon c' \to c$ is a transpose of g, then $e = (c, D)$, $e' = (c', D') \in E\Gamma$. So $\theta e = (F(c), G(D))$, $\theta e' = (F(c'), G(D')) \in E\Gamma'$. Hence $F(g^*)\colon F(c') \to F(c)$ has a transpose $F(g^*)^*\colon G(D) \to G(D')$; we define

$$G(g) = F(g^*)^*\colon G(D) \to G(D'). \tag{5.7ii}$$

To see that this is well-defined, consider another transpose $f\colon c_1' \to c_1$ of g. Then $e_1 = (c_1, D)$, $e_1' = (c_1', D) \in E\Gamma$, $e_1\mathcal{R}e'$ and $e_1'\mathcal{R}e'$. Also, by (4.11), $(\gamma(e_1)_c)^* = \gamma^*(e)_D = 1_D$. Similarly, $(\gamma(e')_{c_1'})^* = 1_{D'}$, $(\gamma(\theta e_1)_{F(c)})^* = 1_{G(D)}$ and $(\gamma(\theta e')_{F(c_1')})^* = 1_{G(D')}$. Hence by Corollary 4.7, $(\gamma(e')_{c_1'} g^*(\gamma(e_1)_c)^* = g$ and so by Proposition 4.6, $f = \gamma(e')_{c_1'} g^*(e_1)_c$. Hence

$$\begin{aligned} F(f)^* &= (F(\gamma(e')_{c_1'})F(g^*)F(\gamma(e_1)_c))^* \\ &= (\gamma(\theta e_1)_{F(c)})^* F(g^*)^* (\gamma(\theta e')_{F(c_1')})^* \\ &= F(g^*)^*. \end{aligned}$$

Clearly $G(1_D) = 1_{G(D)}$. If $g_1\colon D_1 \to D_2$, $g_2\colon D_2 \to D_3$ are morphisms of $\mathcal{D}$, $G(g_1g_2) = F((g_1g_2)^*)^* = F(g_2^*g_1^*)^* = (F(g_2^*)F(g_1^*))^* = F(g_1^*)^*F(g_2^*)^* = G(g_1)G(g_2)$. Hence $G\colon \mathcal{D} \to \mathcal{D}'$ is a functor. If $g\colon D \subseteq D'$, we can find a transpose g^* of g which is a retraction and then $F(g^*)$ is a retraction of $\mathcal{D}$. Since θ is a bimorphism, the map defined by (5.7i) is inclusion-preserving. So $G(D) \subseteq G(D')$. Hence, the transpose of the retraction $F(g^*)$ from $G(D)$ to $G(D')$ is the inclusion. Hence G is an inclusion-preserving functor. From (5.7ii), we see that axiom II holds and so $(F,G)\colon \Gamma \to \Gamma'$ is a morphism. Since $F(c^e) = c^{\theta e}$, $G(D^e) = D^{\theta e}$, Equation (5.6) holds. Uniqueness of G also follows readily.

Conversely, assume that $(F,G)\colon \Gamma \to \Gamma'$ is a morphism and define θ by (5.6). By axiom I, θ is single-valued and since F and G are inclusion-preserving, θ preserves ω^r and ω^l by (5.2) and (5.2*). Now for any $f\colon c \to c'$ it is clear that $\operatorname{Im} F(f) = F(\operatorname{Im} F)$ and if $c_1 = \operatorname{coim} f$, then $F(c_1) = \operatorname{coim} F(f)$. Hence it follows from (5.2), (5.2*) and (5.3) that θ preserves basic products and sandwich sets. Therefore θ is a regular bimorphism. It is immediate from axiom I that the pair (θ, F) satisfies axiom I$'$ and (θ, G) satisfies axiom I$'^*$. This completes the proof.

Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor. F is said to be *injective* if vF is injective and F is faithful. F is [*weakly*] *surjective* if [the morphism map $Mor\ F = f \mapsto F(f)$ of F is surjective] F is full and vF is surjective. If $(F,G)\colon \Gamma \to \Gamma'$ is a morphism of cross-connections, then (F,G) is *injective* if both F and G are injective and (F,G) is *surjective* if both F and G are weakly surjective. Note that by Theorem 5.4 above (F,G) is injective [surjective] if and only if both θ and F (or both θ and G) are injective [θ is surjective and F (or G) is weakly surjective].

We introduce the following notations for convenience: For $(c,D) \in \mathcal{C} \times \mathcal{D}$

(5.8)
$$U\Gamma(c,D) = \{\gamma(e)^* f \colon e \in E\Gamma \text{ with } D^e = D \text{ and } f\colon c^e \to c \text{ is an isomorphism}\}$$

(5.8*)
$$U\Gamma^*(c,D) = \{\gamma^*(e)^* g \colon e \in E\Gamma \text{ with } c^e = c \text{ and } g\colon D^e \to D \text{ is an isomorphism.}\}$$

By Proposition 4.10, $\bigcup_{(c,D)\in v\mathcal{C}\times v\mathcal{D}} U\Gamma(c,D) = U\Gamma$ and $\bigcup_{(c,D)\in v\mathcal{C}\times v\mathcal{D}} U\Gamma^*(c,D) = U\Gamma^*$.

Let $m = (F,G)\colon \Gamma \to \Gamma'$ be a morphism and for each $\rho \in U\Gamma(c,D)$, define

$$\zeta_m(\rho) = \zeta_m(c,D)(\rho) \tag{5.9}$$

where ζ_m is the natural transformation satisfying II$'$ of Theorem 5.2. Dually we define ζ_m^* by (5.9*).

PROPOSITION 5.5. *Let $m = (F,G)\colon \Gamma \to \Gamma'$ be a morphism of cross-connections. Then (5.9) and (5.9*) defines homomorphisms $\zeta_m\colon U\Gamma \to U\Gamma'$ and $\zeta_m^*\colon U\Gamma^* \to U\Gamma'^*$ such that ζ_m and ζ_m^* are both injective [surjective] if and only if (F,G) is injective [surjective].*

PROOF: We first show that ζ_m is well-defined by (5.9). Suppose that $\rho \in \Gamma(c,D)$ and $\rho \in \Gamma(c',D')$. Then vertex $\rho = c = c'$ and if $e, e' \in E\Gamma$ with $D^e =$

D, $D^{e'} = D'$, then $\gamma(e)\mathcal{R}\rho\mathcal{R}\gamma(e')$. Hence $\gamma(e)\mathcal{R}\gamma(e')$ and so $\Gamma(D) = \Gamma(D')$. If $c_1^* \in M\Gamma(D)$, we have $e_1 = (c_1, D)$, $e_1' = (c_1, D') \in E\Gamma$ and $\gamma(e_1^* f) = \rho = \gamma(c_1')^* f$ for some isomorphism $f: c_1 \to c$. Hence $\zeta = (c, D)(\rho) = \gamma(e_1)^* F(f) = \gamma(\theta e_1')^* F(f) = \zeta(c', D')(\rho)$ by (5.5). This proves that ζ_m is a mapping. We further observe that

$$\zeta_m(\rho) = \zeta(c, D)(\rho) \tag{5.9$'$}$$

for any $(c, D) \in v\mathcal{C} \times v\mathcal{D}$ such that $\rho \in \Gamma(c, D)$. For if $\rho = \gamma(e)^* f^0 \in \Gamma(c, D)$, where $f \in \mathcal{C}(c^e, c)$, by Proposition 4.10, $\rho = \gamma(e')^* g$ where $e'we$ and g is the isomorphism factor of f^0; i.e., $hg = f^0$ where $h: c^e \to c^{e'}$ is the retraction. Then $\rho \in U\Gamma(c', D^{e'})$ where $c' = \operatorname{Im} f$. By Lemma 4.11, $\zeta(c, D)(\rho) = \zeta(c', D^{e'})(\rho) = \zeta_m(\rho)$. In a similar way, we see that (5.9$'^*$) holds for ζ_m^* and ζ_m^* is a mapping.

Let $\rho_i = \gamma(e_i)^* f_i \in U\Gamma(c_i, D_i)$, $i = 1, 2$. Then $\rho_1\rho_2 \in \Gamma(c_2, D_1)$ and so,

$$\begin{aligned}
\zeta_m(\rho_1\rho_2) &= \zeta(c_2, D_1)(\gamma(e_1)^* f_1(\gamma(e_2)_{c_1}) f_2)^0) && \text{by (5.9$'$)} \\
&= \zeta(c_2, D_1)(\gamma(e_1)^*(f_1\gamma(e_2)_{c_1} f_2)^0) \\
&= \gamma(\theta e_1)^* F(f_1\gamma(e_2)_{c_1} f_2)^0 && \text{by (5.5)} \\
&= \gamma(\theta e_1)^*(F(f_1)\gamma(\theta e_2)_{F(G)} F(f_2))^0 && \text{by I}' \\
&= (\gamma(\theta e_1)^* F(f_1))(\gamma(\theta e_2)^* F(f_2)) = \zeta_m(\rho_1)\zeta_m(\rho_2).
\end{aligned}$$

Hence $\zeta_m: U\Gamma \to U\Gamma'$ is a homomorphism. Similarly $\zeta_m^*: U\Gamma^* \to U\Gamma'^*$ is also a homomorphism.

Suppose that (F, G) be injective. Then, clearly, the map θ defined by (5.6) is injective. Let $\rho_i = \gamma(e_i)^* f_i$ and $\zeta_m(\rho_1) = \zeta_m(\rho_2)$. By (5.5), $\gamma(\theta e_1)^* F(f_1) = \gamma(\theta e_2)^* F(f_2)$ and so $\gamma(\theta e_1)\mathcal{R}\gamma(\theta e_2)$. Hence $e \in \Gamma'(D^{\theta e_2}) = \Gamma'(D^{\theta e_1})$. Therefore $\bar{e} = (c^{\theta e_1}, D^{\theta e_2}) \in E\Gamma'$, and $\theta e_1 \mathcal{L}\bar{e}\mathcal{R}\theta e_2$. Let $e \in S(e_1, e_2)$. Then $\theta e \in S(\theta e_1, \theta e_2) = \{\bar{e}\}$. Moreover since θ is injective, $e_1 e \omega e_1$ and $\theta(ee_1) = \theta e_1$ implies $e_1 e = e_1$. Hence $e\mathcal{L}e_1$. Similarly $e\mathcal{R}e_2$. Then $\zeta_m(\gamma(e)^* f_1) = \gamma(\theta e)^* F(f_1) = \gamma(\theta e_1)^* F(f_1) = \zeta_m(\rho_1) = \zeta_m(\rho_2)$. Since $e\mathcal{R}e_2$, there is $f: c^e \to \operatorname{Im} f_2$, such that $\gamma(e)^* f = \gamma(e_2)^* f_2$. So $\gamma(\theta e)^* F(f) = \gamma(\theta e)^* F(f_1)$. Hence $F(f) = F(f_1)$ and so $f = f_1$. Now for any $c \in v\mathcal{C}$, $F(\gamma(e_1)_c) = \gamma(\theta e_1)_{F(c)} = F(\gamma(e)_c)$ and since F is faithful, we have $\gamma(e_1) = \gamma(e)$. Hence $\rho_2 = \gamma(e)^* f_1 = \gamma(e_1)^* f_1 = \rho_1$. Thus ζ_m is one-to-one. Similarly ζ_m^* is also one-to-one.

Now suppose that both ζ_m and ζ_m^* are one-to-one. If $\theta e_1 = \theta e_2$ then $\gamma(\theta e_1) = \zeta_m(\gamma(e_1)) = \gamma(\theta e_2) = \zeta_m(\gamma(e_2))$ and so $\gamma(e_1) = \gamma(e_2)$. Similarly

$\gamma^*(e_1) = \gamma^*(e_2)$. Thus $\alpha(e_1) = (\gamma(e_1), \gamma^*(e_1)) = \alpha(e_2)$. Since α is an isomorphism by Proposition 5.1, we have $e_1 = e_2$. Thus θ is one-to-one.

Let $F(c_1) = F(c_2)$. Then there exists $e_1, e_2 \in E\Gamma$ with $c^{e_1} = c_1$ and $c^{e_2} = c_2$. Then by (5.6), $\theta e_2 \mathcal{L} \theta e_2$. Now if $h \in S(e_1, e_2), \theta h \in S(\theta e_1, \theta e_2) = \{\theta e_2\}$ and so $h = e_2$ since θ is injective. It follows that $e_1 \mathcal{L} e_2$ and so $c_1 = c_2$. Thus vF is injective. Let $f, f' \in \mathcal{C}(c, c')$ and $F(f) = F(f')$. If $e \in E\Gamma$ with $c^e = c$, then $\zeta_m(\gamma(e)^* f^0) = \gamma(\theta e)^* F(f)^0 = \gamma(\theta e)^* F(f')^0 = \zeta_m(\gamma(e)^* f'^0)$. Since ζ_m is injective we have $\gamma(e)^* f^0 = \gamma(e)^* f'^0$ and so $f^0 = f'^0$. Hence $f = f'$. Hence F is faithful and so F is injective. Similarly G is also injective.

Let (F, G) be surjective. If $e' \in E\Gamma'$, there is $(c, D) \in v\mathcal{C} \times v\mathcal{D}$ such that $F(c) = c^{e'}$, $G(D) = D^{e'}$. Let $e \in \overline{S}(c, D)$. Then $\theta e = \overline{S}(F(c), G(D)) = \overline{S}(c^{e'}, D^{e'}) = \{e'\}$. Hence θ is surjective. Let $\rho' = \gamma(e')^* f' \epsilon U\Gamma'$ where $f' : c^{e'} \to c'$ is an isomorphism. Since θ is surjective there is $e \in E\Gamma$ with $\theta e = e'$. Since F is weakly surjective, there is $f : \bar{c} \to c$ such that $F(f) = f'$. Since F is inclusion-preserving, F preserves inclusions and retractions and so, we may assume that f is an isomorphism. If $\bar{e} \in E\Gamma$, with $c^{\bar{e}} = \bar{c}$, $c^{\theta \bar{e}} = c^{e'}$ and so $\theta \bar{e} \mathcal{L} e' = \theta e$. Hence if $e'' \in S(\bar{e}, e)$, $\theta e'' = e$ and $c^{e''} \subseteq \bar{c}$. If $\bar{f} = (j^{\bar{c}}_{c^{e''}} f)^0$, then $\bar{f}$ is an isomorphism. Since $F(c^{e''}) = F(\bar{c})$, we have $F(\bar{f}) = F(f) = f'$. Hence $\zeta_m(\gamma(e'')^* \bar{f}) = \gamma(\theta e'')^* F(\bar{f}) = \gamma(e')^* f'$. Hence ζ_m is surjective. Similarly ζ_m^* is also surjective.

Conversely, assume that both ζ_m and ζ_m^* be surjective. If $e' \in E\Gamma'$, there exists $e_1, e_2 \in E\Gamma$ such that $\gamma(\theta e_1) = \gamma(e')$ and $\gamma^*(\theta e_2) = \gamma^*(e')$ since ζ_m and ζ_m^* are surjective. If $e \in S(e_1, e_2)$, then $\theta e \in S(\theta e_1, \theta e_2) = \{e'\}$. Hence θ is surjective. Now let $f' : c' \to c''$ be any morphism and let $f'^0 : c' \to c''_1 \subseteq c''$. Since θ is surjective there is $e \in E\Gamma$ with $c^{\theta e} = c'$. If $\rho' = \gamma(\theta e)^* f'^0$, there is $\rho = \gamma(e_1)^* g \in U\Gamma$, with $\zeta_m(\rho) = \gamma(\theta e_1)^* F(g) = \gamma(\theta e)^* f'^0$, where g is an epimorphism. If $e' \in S(e, e_1)$, since $\theta e \mathcal{R} \theta e_1$, $\theta e' = \theta e$. From $\gamma(\theta e_1)^* F(g) = \gamma(\theta e)^* f'^0$, we get $f'^0 = \gamma(\theta e_1)_{c^{\theta e}} F(g) = F(g')$, where $g' = \gamma(e_1)_{c^e} g$. Now, let $\bar{e}_1, \bar{e}_2 \in E\Gamma$ such that $c^{\theta \bar{e}_1} = c''$, $c^{\theta \bar{e}_2} = c''_1$. Since θ is surjective, there is $\bar{e} \in M(\bar{e}_1, \bar{e}_2)$ such that $\theta \bar{e} = \theta \bar{e}_2$. Then $c^{\bar{e}} \subseteq c^{\bar{e}_1}$ and $c^{\theta \bar{e}} = c^{\theta \bar{e}_2} = c''_1$. Hence $F(\gamma(\bar{e})_{c^{\bar{e}_2}}) = \gamma(\theta \bar{e})_{\theta \bar{e}_2} = 1_{c''_1}$. Hence if $g'' = g' \gamma(\bar{e})_{c^{\bar{e}_2}}$, then $F(g'') = f'^0$. Hence if $f = g'' j^c_{\mathrm{Im}\, g''}$, then $F(f) = f'^0 j^{c''}_{c''_1} = f'$, since $\mathrm{Im}\, g'' = c^{\bar{e}}$. Hence F is weakly surjective, since vF is obviously surjective. Similarly G is also weakly surjective. Hence (F, G) is surjective.

THEOREM 5.6. *Let $m = (F, G): \Gamma \to \Gamma'$ be a morphism of cross-connections. Then*

$$\hat{S}m(\rho, \lambda) = (\zeta_m(\rho), \zeta_m^*(\lambda)), \quad \text{for all } (\rho, \lambda) \in \hat{S}\Gamma \tag{5.10}$$

defines a homomorphism $\hat{S}m: \hat{S}\Gamma \to \hat{S}\Gamma'$ such that $\hat{S}m$ is injective [surjective] if and only if m is injective [surjective]. Further,

$$\hat{S}1_\Gamma = 1_{\hat{S}\Gamma} \quad \text{and} \quad \hat{S}(mm') = \hat{S}m\hat{S}m'$$

where $m: \Gamma \to \Gamma'$ and $m': \Gamma' \to \Gamma''$ are morphisms. Thus $\hat{S}: \mathbf{Cr} \to \mathbf{RS}$ defined by:

$$\Gamma \mapsto \hat{S}\Gamma, \quad m: \Gamma \to \Gamma' \mapsto \hat{S}m: \hat{S}\Gamma \mapsto \hat{S}p'$$

is a functor.

PROOF: Let $(\rho, \lambda) \in \hat{S}\Gamma$. Let $\rho \in U\Gamma(c, D)$ and $\lambda \in U\Gamma^*(c, D)$. Then

$$\begin{aligned}\chi_{\Gamma'}(F(c), G(D))(\zeta_m(\rho)) &= \chi_{\Gamma'}(F(c), G(D))(\zeta(c, D)(\rho))\zeta^*(c, D)(\chi_\Gamma(c, D)(\rho)) \\ &= \zeta^*(c, D)(\lambda) = \zeta_m^*(\lambda)\end{aligned}$$

by Proposition 5.3. Hence $(\zeta_m(\rho), \zeta_m^*(\lambda)) \in \hat{S}\Gamma'$ and so (5.10) defines a mapping of $\hat{S}\Gamma$ into $\hat{S}\Gamma$. By Proposition 5.5, this is a homomorphism such that it is injective [surjective] if and only if m is injective [surjective]. Clearly $1_\Gamma = (1_{\mathcal{C}}, 1_{\mathcal{D}}): \Gamma \to \Gamma$ is a morphism and $\hat{S}1_\Gamma = 1_{\hat{S}\Gamma}$. Let $m = (F, G): \Gamma \to \Gamma'$ and $m' = (F', G'): \Gamma' \to \Gamma''$ be morphisms. From (5.5) and (5.9) (also (5.9$'$)) and their duals we see that $\zeta_{mm'} = \zeta_m\zeta_{m'}$, $\zeta_{mm'}^* = \zeta_m^*\zeta_{m'}^*$. Hence $\hat{S}(mm') = (\zeta_{mm'}, \zeta_{mm'}^*) = (\zeta_m\zeta_{m'}, \zeta_m^*\zeta_{m'}^*) = (\zeta_{m'}, \zeta_m^*)(\zeta_{m'}, \zeta_{m'}^*) = \hat{S}m\hat{S}m'$. Hence $\hat{S}: \mathbf{Cr} \to \mathbf{RS}$ is a functor.

We proceed to show that $\hat{S}$ is an adjoint equivalence of the categories $\mathbf{Cr}$ and $\mathbf{RS}$. Next theorem constructs the inverse $\hat{\Gamma}: \mathbf{RS} \to \mathbf{Cr}$ of $\hat{S}$.

THEOREM 5.7. *Let $h: S \to S'$ be a homomorphism of regular semigroups. Define*

$$F_h(Sx) = S'(hx), \quad F_h(\rho(e, u, f)) = \rho(he, hu, hf); \tag{5.11}$$

$$G_h(xS) = (hx)S', \quad G_h(\lambda(e, u, f)) = \lambda(he, hu, hf) \tag{5.11*}$$

Thus $F_h:\mathbb{L}(S) \to \mathbb{L}(S')$, $G_h:\mathbb{R}(S) \to \mathbb{R}(S')$ *are inclusion-preserving functors such that*

$$\hat{\Gamma}h = (F_h, G_h): \Gamma_S \to \Gamma_{S'}$$

is a morphism of cross-connections. Further, the assignments

$$S \mapsto \Gamma_S = \hat{\Gamma}S, \qquad h \mapsto \hat{\Gamma}h$$

is a functor $\hat{\Gamma}: \mathbf{RS} \to \mathbf{Cr}$ *and if* $\varphi_S: S \to \hat{S}\Gamma_S$ *is the isomorphism defined by (4.19), the mapping*

$$S \mapsto \varphi_S: 1_{RS} \to \hat{\Gamma}\hat{S}$$

is a natural isomorphism.

Proof: From (3.6) and (3.8), it is immediate that F_h is an inclusion-preserving functor. Similarly from (3.6^*) and (3.8^*) we obtain that G_h is also an inclusion-preserving functor. Now $(Se, fS) \in E\Gamma_S$ if and only if $\gamma(Se, fS) = \gamma(Sg, gS) = \rho^g$ for some $g \in E(S)$. Hence it is clear from the definitions of F_h and G_h that if $(Se, fS) \in E\Gamma_S$ then $(F_h(Se), G_h(fS)) = (S'hg, hgS') \in E\Gamma_{S'}$. Also by (3.10), $\gamma(Se, eS)_{Sf} = \rho^e_{Sf} = \rho(f, fe, e)$ and so $F(\gamma(Se, eS)_{Sf}) = \rho(hf, hfhe, he) = \gamma(S'he, heS')_{S'hf}$. Hence the pair (F_h, G_h) satisfies axiom I. Now by Theorem 4.15, (4.3), $(4.8')$, we have $\rho(e, u, f)^* = \lambda(f, u, e)$. Hence $G(\rho(e, u, f)^*) = \lambda(hf, hu, he) = \rho(he, hu, hf)^* = F(\rho(e, u, f))^*$. This proves that $(F_h, G_h): \Gamma_S \to \Gamma_{S'}$ is a morphism of cross-connections.

Clearly $(F_{1_S}, G_{1_S}) = 1_{\Gamma_S}$. If $h: S \to S'$ and $h': S' \to S''$ are homomorphisms by (5.11) and (5.11^*), $F_{hh'} = F_h F_{h'}$, $G_{hh'} = G_h G_{h'}$. Hence $\hat{\Gamma}hh' = \hat{\Gamma}h\hat{\Gamma}h'$. Thus $\hat{\Gamma}: \mathbf{RS} \to \mathbf{Cr}$ is a functor.

Finally let $h: S \to S'$ be a homomorphism and let $\varphi_S: S \to \hat{S}\Gamma_S$ be the isomorphism defined by (4.19). We prove that the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\varphi_S} & \hat{S}\hat{\Gamma}_S = \hat{S}\Gamma_S \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \hat{S}\hat{\Gamma}h} \\ S' & \xrightarrow{\varphi_{S'}} & \hat{S}\hat{\Gamma}_{S'} = \hat{S}\Gamma_{S'} \end{array}$$

Let $x \in S$. Then

$$\varphi_{S'}(hx) = (\rho^{hx}, \lambda^{hx})$$

and $\hat{S}\hat{\Gamma}h(\varphi_S(x)) = \hat{S}\hat{\Gamma}g(\rho^x, \lambda^x) = (\zeta(\rho^x), \zeta^*(\lambda^*))$, by (5.10) and the definition of $\hat{\Gamma}h$ above, when ζ is the natural transformation satisfying II′ (associated with $\hat{\Gamma}h$). By (5.5) and (5.11), using (3.8) and (3.10), we obtain

$$\zeta(\rho^x) = \gamma(he)^*\rho(he, hx, hf) = \rho^{hx}, \quad \text{when } x \in R_e \cap L_f.$$

$$\begin{aligned} \text{and} \quad \zeta(\lambda^x) &= \zeta^*(\mathfrak{X}_S(\rho^x)) && \text{by Theorem 4.15} \\ &= \mathfrak{X}_{S'}(\zeta(\rho^*)) && \text{by Theorem 5.3} \\ &= \mathfrak{X}_{S'}(\rho^{hx}) = \lambda^{hx} && \text{by Theorem 4.15} \end{aligned}$$

Hence $h\varphi_{S'}(x) = \varphi_S(\hat{S}\hat{\Gamma}h)(x)$. This proves the given diagram is commutative. Therefore $\varphi: S \mapsto \varphi_S$ is a natural isomorphism.

Corollary 5.8. *Let h be a homomorphism of regular semigroup. Then the morphism $\hat{\Gamma}h: \Gamma_S \to \Gamma_{S'}$ is injective [surjective] if and only if $h: S \to S'$ is injective [surjective].*

Proof: If h is injective [surjective], then $\hat{S}(\hat{\Gamma}h)$ is injective [surjective] since the diagram above commutes and $\varphi_S, \varphi_{S'}$ are isomorphisms. Then by Theorem 5.6, $\hat{\Gamma}h$ has this property. Similarly if $\hat{\Gamma}h$ is injective [surjective], then $\hat{S}(\hat{\Gamma}h)$ has the property by Theorem 5.6 and so, as above, h has this property.

Theorem 5.9. *For each cross-connection $\Gamma: \mathcal{D} \to N^*\mathcal{C}$, there is an isomorphism $\psi_\Gamma: \Gamma \to \Gamma_{\hat{S}\Gamma} = \hat{\Gamma}(\hat{S}\Gamma)$ such that the assignment*

$$\psi: \Gamma \mapsto \psi_\Gamma$$

is a natural isomorphism $\psi: 1_{\mathbf{Cr}} \to \hat{S}\hat{\Gamma}$.

Proof: Let F_Γ and G_Γ be isomorphisms defined by (4.17) and (4.17*) respectively. We first show that

$$\psi_\Gamma = (F_\Gamma, G_\Gamma) \tag{5.12}$$

satisfies axioms I and II so that $\psi_\Gamma: \Gamma \to \Gamma_S$ is a morphism, and since F_Γ and G_Γ are isomorphisms, ψ_Γ is an isomophism. Here $S = \hat{S}\Gamma$.

Let $(c, D) \in E\Gamma$. Then by (4.17i) and (4.17i*)

$$F_\Gamma(c) = S\alpha(c, D), \qquad G_\Gamma(c) = \alpha(c, D)S$$

where $\alpha: E\Gamma \to E(S)$ is the isomorphism defined by (5.1). Now $\gamma(S\alpha(c, D), \alpha(c, D)S) = \rho^{\alpha(c,D)}$ and so $(S\alpha(c, D), \alpha(c, D)S) \in E\Gamma_S$. Further by (4.17ii), for an $f: c \to c'$ and $D \in M\Gamma^*(c)$, $D' \in M\Gamma^*(c')$,

$$F_\Gamma(F) = \rho(\alpha(c, D)(\gamma(c, D)^* f^0, \gamma^*(c', D')^*(f^*)^0), (\alpha(c', D'))$$

Hence

$$\begin{aligned}
F_\Gamma(\gamma(c, D)_{c'}) &= \rho(\alpha(c', D'), (\gamma(c', D')^* \gamma(c, D)_{c'}, \gamma^*(c, D)^* \gamma^*(c', D')_D), \alpha(c, D)) \\
&= \rho(\alpha(c', D'), (\gamma(c', D')\gamma(c, D), \gamma^*(c, D)\gamma^*(c', D')), \alpha(c, D)) \\
&= \rho(\alpha(c', D'), \alpha(c', D')\alpha(c, D), \alpha(c, D)) \\
&= \rho^{\alpha(c,D)}_{S\alpha(c',D')} = \gamma(S\alpha(c, D), \alpha(c, D)S)_{S\alpha(c',D')} \\
&= \gamma(F_\Gamma(c), G_\Gamma(D))_{F_\Gamma(c')}
\end{aligned}$$

This proves that ψ_Γ satisfies axiom I of Definition 5.1. Let $f: c \to c'$, $c \in M\Gamma(D)$ and $c' \in M\Gamma(D')$. Then by (4.17ii) and (4.3),

$$\begin{aligned}
F_\Gamma(f)^* &= \lambda(\alpha(c', D'), (\gamma(c, D)^* f^0, \gamma^*(c', D')^*(f^*)^0), \alpha(c, D)) \\
&= G_\Gamma(f^*) \quad \text{by (4.17}^*\text{), where } f^*: D' \to D \text{ is the transpose of } f.
\end{aligned}$$

Hence ψ_Γ satisfies axiom II and hence $\psi_\Gamma: \Gamma \to \Gamma_S$ is an isomorphism.

To prove that $\Gamma \mapsto \psi_\Gamma$ is a natural isomorphism, let $m = (F, G)$ be a morphism $m: \Gamma \to \Gamma'$. We show that $\psi_\Gamma \psi_{\bar m} = m\psi_{\Gamma'}$ where $\bar m = \hat\Gamma(\hat S m)$. By (5.4), this is equivalent to the commutativity of the following diagrams:

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F_\Gamma} & \mathbb{L}(S) \\
\downarrow F & & \downarrow F_h \\
\mathcal{C}' & \xrightarrow{F_{\Gamma'}} & \mathbb{L}(S')
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{G_\Gamma} & \mathbb{R}(S) \\
\downarrow G & & \downarrow G_h \\
\mathcal{D}' & \xrightarrow{G_{\Gamma'}} & \mathbb{R}(S')
\end{array}$$

where $h = \hat S m: \hat S\Gamma = S \to S' = \hat S\Gamma'$ is defined by (5.10). Let $C \in v\mathcal{C}$. Then $F_\Gamma F_h(C) = F_h(F_\Gamma(C)) = F_h(S\alpha(C, D)) = S'h\alpha(C, D)$, by (5.11), where $D \in M\Gamma^*(C)$. Now by (5.9), (5.9*) and (5.10), $h\zeta(C, D) = (\zeta_m(\gamma(C, D)), \zeta^*_m(\gamma^*(C, D)))$

$= \alpha(F(C), G(D))$. Also $F\ F_\Gamma(C) = F_{\Gamma'}(F(C)) = S'\alpha(F(C), G(D))$. Thus for each $C \in v\mathcal{C}$, $F_\Gamma F_h(C) = F\ F_{\Gamma'}(C)$. Let $f: C \to C'$ be any morphism and $f^*: D' \to D$ be a transpose of f. Then by (4.17ii), $F_\Gamma(f) = \rho(\alpha(C, D), (\rho, \lambda), \alpha'(C', D'))$ where $\rho = \gamma(C, D) * f^\circ$ and $\lambda = \mathfrak{X}_\Gamma(C', D)(\rho) = \gamma^*(C, D) * (f^*)^\circ$ by (4.8). Hence $F_h(F_\Gamma(f)) = \rho(\alpha(F(C), G(D)), (\zeta_m(\rho), \zeta^*_m(\lambda)), \alpha(F(C'), G(D')))$. By (5.5), $\zeta_m(\rho) = \gamma(F(C), G(D) * F(f)^\circ)$, and by (5.5*),

$$\begin{aligned} \zeta^*_m(\lambda) &= \gamma^*(F(C'), G(D') * G(f^*)^\circ) = \gamma^*(F(C'), G(D')) * (F(f)^*)^\circ \quad \text{by II.} \\ &= \psi_{\Gamma'}(F(C), G(D'))(\zeta_m(\rho)). \end{aligned}$$

Hence by (4.17), $F_h(F_\Gamma(f)) = F_{\Gamma'}(F(f))$. This proves the commutativity of the first diagram above. Commutativity of the second can be proved similarly. Hence $\psi: \Gamma \to \psi_\Gamma: 1_{\mathbf{Cr}} \to \hat{S}\hat{\Gamma}$ is a natural isomorphism.

From Theorems 5.7 ad 5.9, we obtain:

THEOREM 5.10. *Let $\hat{S}: \mathbf{Cr} \to \mathbf{RS}$ be the functor defined in Theorem 5.6. Then $\hat{S}$ is an adjoint equivalence of the categories $\mathbf{Cr}$ and $\mathbf{RS}$ with inverse $\hat{\Gamma}: \mathbf{RS} \to \mathbf{Cr}$ defined in Theorem 5.7.*

6. The Cross–connection of a Fundamental Regular Semigroup

In [9] Grillet had introduced the concept of cross–connections between regular partially ordered sets in order to characterize fundamental regular semigroups. In this section we wish to show how Grillet's theory would be deduced from the more general theory given in §4 and §5.

First we shall obtain a characterization of the normal categories $\mathbb{L}(S)$ and $\mathbb{R}(S)$ where S is fundamental. We need the following:

DEFINITION 6.1. *Let $\mathfrak{b}$ be a normal reductive category. We shall say that an object $C \in v\mathfrak{b}$ is reduced in $\mathfrak{b}$ if for $f, g \in \mathfrak{b}(C, C)$. $f(C') = Im\,(j^c_{c'} f) = g(C')$ for all $C' \subseteq C$, implies $f = g$. If C is reduced and $C' \subseteq C$, then C' is reduced. For if $f, g \in \mathfrak{b}(C', C')$ satisfy the condition $f(C'_1) = g(C'_1)$ for all $C'_1 \subseteq C'$, then $f'(C_1) = g'(C_1)$ for all $C_1 \subseteq C$, where $f' = hfj^C_{C'}$ and $g' = hgj^C_{C'}$, with $h: C \to C'$ a retraction. Hence $f' = g'$ and so $f = g$. Thus the set of reduced objects in $\mathfrak{b}$ is a complex in $\mathfrak{b}$ and the ideal $\mathfrak{b}'$ generated by this complex is called* *the reduced ideal of* $\mathfrak{b}$. $\mathfrak{b}$ *is said to be reduced if* $\mathfrak{b}' = \mathfrak{b}$ *; i.e., every object of* $\mathfrak{b}$ *is reduced.*

Let I be a regular partially ordered set. Define the category $\mathfrak{b}(I)$ by:

$$v\mathfrak{b}(I) = \{I(x) : x \in I\},$$
$$\mathfrak{b}(I)\,(I(x), I(y)) = \{f : I(x) \to I(y)\,; \quad f \text{ is normal}\}.$$

PROPOSITION 6.1. *For each regular partially ordered set* I, $\mathfrak{b}(I)$ *defined by (6.1) is a normal reductive category.*

PROOF: Since composition of normal maps are normal $\mathfrak{b}(I)$ is a category (identity map on $I(x)$ is evidently normal). Also if $I(x) \subseteq I(y)$, then the inclusion map $j^{I(y)}_{I(x)}$ is normal and so $\mathfrak{b}(I)$ has subobjects. Also, when $I(x) \subseteq I(y)$, the fact that $I(y)$ is a regular partially ordered set implies that there is a normal retraction of $I(y)$ onto $I(x)$. This retraction is clearly a right inverse of $j^{I(y)}_{I(x)}$. Hence every inclusion is a fiberation. If $f : I(x) \to I(y)$ is any morphism of $\mathfrak{b}(I)$, any normal factorization of f as a normal map (cf. [9]) provides a normal factorization of f in the category $\mathfrak{b}(I)$. Hence $\mathfrak{b}(I)$ is normal. To prove that it is reductive consider $I(x) \in v\mathfrak{b}(I)$. Since I is regular, there is a normal retraction $\epsilon : I \to I(x)$. Now define $\bar{\epsilon} : v\mathfrak{b}(I) \to Mor\ (\mathfrak{b}(I))$ by:

$$\bar{\epsilon}_{I(y)} = \epsilon | I(y) \tag{6.2}$$

Clearly $\bar{\epsilon}$ is a cone from the base $v\mathfrak{b}(I)$ to the vertex $I(x)$. Since $\bar{\epsilon}_{I(x)} = 1_{I(x)}$, it follows that $\bar{\epsilon}$ is a normal cone which is idempotent. Hence $\mathfrak{b}(I)$ is reductive.

Observe that the mapping $x \to I(x)$ is an order isomorphism of I onto the partially ordered set $v\mathfrak{b}(I)$. We shall therefore identify I with $v\mathfrak{b}(I)$ by this isomorphism and regard $\mathfrak{b}(I)$ as the category with $v\mathfrak{b}(I) = I$.

The proof above shows that we can associate with every normal retraction ϵ, a normal idempotent cone $\bar{\epsilon}$ in $\mathfrak{b}(I)$. More generally if $f \in S(I)$ is any normal mapping with $\operatorname{Im} f = I(x)$, define $\bar{f}$ by:

$$\bar{f}_y = f | I(y) \quad \text{for all } y \in I. \tag{6.2'}$$

It is clear that $y \to \bar{f}_y$ defines a normal cone with vertex $I(x)$ such that the M-set Mf of f as a normal map (cf. [9]) is the same as $M\bar{f}$. Conversely, given $\sigma \in T\mathfrak{b}(I)$, define f by:

$$f(y) = \sigma_y(y) \quad \text{for all } y \in I. \tag{6.3}$$

Thus $f(y)$ is the generator of the principal ideal $\operatorname{Im}\sigma_y$. If $y_1 \le y$, then $\operatorname{Im}\sigma_{y_1} = \sigma_y(I(y_1)) = I(\sigma_y(y_1)) = I(f(y_1))$ by the definition. Hence $f(y_1) = \sigma_y(y_1)$ and so f is a well-defined map of I into I such that

$$\sigma_y = f|I(y) \quad \text{for all } y \in I.$$

Since σ_y is normal for each $y \in I$, it follows that f is normal and that $\operatorname{Im} f$ is the vertex of σ. Moreover, we have

$$\bar{f} = \sigma$$

Thus the map

$$f \to \bar{f}$$

is a bijection of $S(I)$ onto $T\mathfrak{b}(I)$. Moreover, if $f, g \in S(I)$, we have

$$fg = f(g|I(x)), \quad I(x) = \operatorname{Im} f.$$

Hence, by the definition of product in $T\mathfrak{b}(I)$, we have

$$(\overline{fg})_y = fg|I(y) = (f|I(y))(g|I(x)) = \bar{f}_y\bar{g}_x = (\bar{f},\bar{g})_y$$

for every $y \in I$. Hence $\overline{fg} = \bar{f}\bar{g}$. We thus have the following:

Proposition 6.2. *Let I be a regular partially ordered set. Then the map*

$$f \to \bar{f}$$

is a natural isomorphism from $S(I)$ onto $T\mathfrak{b}(I)$.

Suppose that I and Λ are regular partially ordered sets and $\Gamma: I \to \Lambda^\circ$, $\Delta: \Lambda \to I^\circ$ be maps satisfying conditions (Cr 1) and (Cr 2) of Theorem 2 of [26]. Here we shall refer to the pair (Γ, Δ) as a fundamental cross-connection between I and Λ (or more precisely, (Γ, Δ) is a fundamental cross-connection for (Λ, I)). Recall that the set $U = U(I, \Lambda; \Gamma, \Delta)$ consists of all pairs $(f, g) \in S^{op}(I) \times S(\Lambda)$ satisfying conditions

C1 $\operatorname{Im} f = I(x)$, $\operatorname{Im} g = \Lambda(y) \Rightarrow \ker f = \Delta(y)$, $\ker g = \Gamma(x)$;

C2 The following diagrams commute:

$$\begin{array}{ccccccc} I & \xrightarrow{\Gamma} & \Lambda^\circ & \quad & \Lambda & \xrightarrow{\Delta} & I^\circ \\ \downarrow f & & \downarrow g^\circ & & \downarrow g & & \downarrow f^\circ \\ I & \xrightarrow{\Gamma} & \Lambda^\circ & & \Lambda & \xrightarrow{\Delta} & I^\circ \end{array}$$

Here f° and g° are normal maps of I° and Λ° respectively defined in Proposition 1 of [26].

PROPOSITION 6.3. *Let ρ denotes the right regular representation of U and let $\Pi_1 : (f,g) \to g$ be the projection homomorphism of U into $S(I)$. Then*

$$ker\ \rho = ker\ \Pi_1$$

Dually if λ denotes regular anti-representation and if Π_2 is the projection of U into $S^{op}(I)$, then $ker\ \lambda = ker\ \Pi_2$.

PROOF: Let $u = (f_u, g_u)$, $v \in (f_v, g_v) \in U$. We show that

$$\rho_u = \rho_v \Longleftrightarrow g_u = g_v .$$

If $\rho_u = \rho_v$ then for all $w = (f_w, g_w) \in U$, $wu = wv$; is

$$f_u f_w = f_v f_w \quad \text{and} \quad g_w g_u = g_w g_v .$$

Let $y \in \Lambda$ and let $e = (f_e, g_e)$ be an idempotent in U such that g_e is a normal retraction of $\Lambda(y)$. Note that any idempotent in the class $\mathcal{L}$-class of U corresponding to $y \in \Lambda$ will satisfy this condition. Thus $g_e g_u = g_e g_v$ and so, since $g_e(y) = y$,

$$g_u(y) = g_u(g_e(y)) = g_e g_u(y) = g_e g_v(y) = g_v(g_e(y)) = g_v(y)$$

Hence $g_u = g_v$.

Conversely assume that $g_u = g_v$. We wish to show that for all $w \in U$

$$f_u f_w = f_v f_w, \quad g_w g_u = g_w g_v .$$

The second equation clearly holds and holds and so it is sufficient to prove the first. Now from $g_u = g_v$, we get by condition (C1) above:

$$ker\ f_u = \Delta(b) = ker\ f_v, \quad \Gamma(a_u) = ker\ g_u = \Gamma(a_v)$$

where Im $g_u = \Lambda(b)$, Im $f_u = I(a_u)$ and Im $f_v = I(a_v)$. Now if Im $g_w = \Lambda(\bar{b})$, then by (C1), $ker\ f_w = \Delta(\bar{b})$ and by (C2)

$$\begin{aligned} ker\ f_u f_w = f_u^\circ(ker\ f_w) = f_u^\circ(\Delta(\bar{b})) = \Delta(g_u(\bar{b})) = \Delta(g_v(\bar{b})) \\ = f_v^\circ(\Delta(\bar{b})) = ker\ f_v f_w\,, \end{aligned}$$

Let $a \in M\Delta(g_u(\bar{b}))$. Then $e = (\epsilon(\Delta(g_u(\bar{b})), a), \epsilon(\Gamma(a), g_u(\bar{b}))$ is an idempotent of U in the $\mathcal{L}$-class $L_{wu} = L_{wv}$. Moreover both $f_u f_w$ and $f_v f_w$ have normal factorizations of the form $\epsilon(\Delta(g_u(\bar{b})), a)\alpha = f_u f_w$ and $\epsilon(\Delta(g_u(\bar{b}), a)\alpha' = f_u f_w$. Hence $f_u f_w = f_v f_w$ if $f_u f_w(x) = f_v f_w(x)$ for all $x \leq a$.
Let $e' = (\epsilon(\Delta(y), x), \epsilon(\Gamma(x), y))$ be an idempotent in the $\mathcal{R}$-class corresponding to x such that $e'we$. This implies that $y \leq g_u(b)$. Since g_u is normal, there is $\bar{y} \leq \bar{b}$ such that $g_u|\Lambda(\bar{y})$ is an isomorphism onto $\Lambda(g_u(\bar{y})) = \Lambda(y)$. Also Im $f_u e' =$ Im $(\epsilon(\Delta(y), x) f_u) = I(f_u(x))$, $ker\ fue' = \Delta(y)$. Hence by (C1),

$$\text{Im } (g_u \epsilon(\Gamma(x), y)) = \text{Im } g_{ue'} = \Lambda(y) = \Lambda(g_u(\bar{y}))\,,$$

$$ker\ g_{ue'} = \Gamma(f_u(x))\,.$$

Now $g_{ue'}|\Lambda(\bar{y}) = g_u|\Lambda(\bar{y})$ and since the latter is an isomorphism onto Im $g_{ue'}$, it follows that $\bar{y} \in M\Gamma(f_u(x))$. Consider

$$e_1 = (\epsilon(\Delta(\bar{y}), f_u(x), \epsilon(\Gamma(f_u(x), \bar{y}))$$

Then $f_{we_1} = \epsilon(\Delta(\bar{y}), f_u(x)) f_w$ and since $\bar{y} \leq \bar{b}$, $ker\ f_{we_1} = \Delta(\bar{y})$. Now $\Gamma(f_u(x)) = g_u^\circ(\Gamma(x)) = g_v^\circ(\Gamma(x)) = \Gamma(f_v(x))$ and since $\bar{y} \in M\Gamma(f_u(x))$, we have, $f_u(x)$, $f_v(x) \in M(\Delta(\bar{y})) = M\, ker\ f_{we_1}$. Hence $f_{we_1}(f_u(x)) = f_{we_1}(f_v(x))$ is the generator of the image of f_{we_1}. But

$$f_{we_1}(f_u(x)) = f_w(\epsilon(\Delta(\bar{y}), f_u(x))(f_u(x))) = f_w(f_u(x))\,.$$

Similarly $f_{we} = \epsilon(\Delta(\bar{y}), f_v(x))(f_w|I(f_v(x)))$ is a normal factorization of f_{we_1}, and so

$$f_{we_1}(f_v(x)) = f_w(\epsilon(\Delta(\bar{y}), f_v(x))(f_v(x))) = f_w(f_v(x))\,.$$

Consequently we have $f_u f_w(x) = f_v f_w(x)$ for all $x \leq a$. Hence $f_u f_w = f_v f_w$. This completes the proof that $ker\ \rho = ker\ \pi_1$.

To prove the dual statement, note that if $U = U(I, \Lambda; \Gamma\Delta)$, then $U^{op} = U(\Lambda, I; \Delta, \Gamma)$ and that the kernel of the regular anti-representation of U is the same as the kernel of the regular representation of U^{op}. Hence the dual statement follows from the part proved above. This completes the proof of the proposition.

PROPOSITION 6.4. *Let $U = U(I, \Lambda; \Gamma\Delta)$ where as above, (Γ, Δ) is a fundamental cross-connection between I and Λ. Then there exist natural embeddings,*

$$J: \mathbb{L}(U) \to \mathfrak{b}(\Lambda) \quad and \quad J^*: \mathbb{R}(U) \to \mathfrak{b}(I).$$

PROOF: Define J by:

(6.4i) $J(Ue) = b_e$ where $e \in E(U)$ and $\mathrm{Im}\ g_e = \Lambda(b_e)$; and if $\rho(e,u,f)$: $Ue \to Uf$ in $\mathbb{L}(U)$, with $u = (f_u, g_u) \in eUf$,

(6.4ii) $J(\rho(e,u,f)) = g_u|\mathrm{Im}\ g'_e$.

Clearly, J defined on $v\mathbb{L}(U)$ is the same as the isomorphism of $U/\mathcal{L}$ onto Λ. Further, the morphism map defined by (6.4ii) is well defined. For if $\rho(e,u,f) = \rho(e',v,f')$, then $f\mathcal{L}f'$ and $e\mathcal{L}e'$. Also, by (3.7), $v = e'u$. Since $\mathrm{Im}\ g_e = \mathrm{Im}\ g_{e'}$

$$g_u = g_{eu} = g_e g_u = g_e(g_u|\mathrm{Im}\ g_e)$$

and $g_v = g_{e'u} = g_{e'}g_u = g_{e'}(g_u|\mathrm{Im}\ g_{e'})$ it follows that $g_u|\mathrm{Im}\ g_e = g_v|\mathrm{Im}\ g_e$. Since $g_{uv} = g_u g_v$, it follows from (3.8) that J is a functor. To prove that J is faithful, consider $\rho(e,u,f)$, $\rho(e,v,f)$: $Ue \to Uf$ such that $g_u|\mathrm{Im}\ g_e = g_v|\mathrm{Im}\ g$. Then

$$g_e g_u|\mathrm{Im}\ g_e = g_e g_u = g_{eu} = g_u = g_e(g_v|\mathrm{Im}\ g_e) = gv.$$

By Proposition 6.3, $g_u = g_v$ implies $\rho u = \rho v$ and so $\rho(e,u,f) = \rho u|ue = \rho v|ue = \rho(e,v,f)$. Hence $J: \mathbb{L}(U) \to \mathfrak{b}(\Lambda)$ is an embedding.

Now $\mathbb{R}(u) = \mathbb{L}(U^{op})$. So, the result proved above shows that there is an embedding $J^*: \mathbb{R}(U) = \mathbb{L}(U^{op}) \to \mathfrak{b}(I)$.

In view of the result proved above, we shall assume in what follows that $\mathbb{L}(U) \subseteq \mathfrak{b}(\Lambda)$ and $\mathbb{R}(U) \subseteq \mathfrak{b}(I)$, where $U = U(I, \Lambda; \Gamma, \Delta)$.

Let S be any regular semigroup such that $S/\mathcal{L} = \Lambda$ and $S/\mathcal{R} = I$. If $(\Gamma^\circ, \Delta^\circ)$ is the fundamental cross-connection induced by S, then it is well-known that there is a natural homomorphism $\bar{a}: S \to U$ such that $\mu(S) = ker\ \bar{a}$ is the maximum idempotent separating congruence on S. By Theorem 5.7 there is pair $(F_{\bar{a}}, G_{\bar{a}})$ of functors, which induces a morphism of the cross-connections $\Gamma_S: \mathbb{R}(S) \to N^*\mathbb{L}(S)$ and $\Gamma_u: \mathbb{R}(U) \to N^*\mathbb{L}(U)$. Then $F_\mu = F_{\bar{a}}J$, $G_\mu = G_{\bar{a}}J^*$ where J and J^* are functors defined by (6.4) and its dual, are functors from $\mathbb{L}(S)$ into $\mathfrak{b}(\Lambda)$ and $\mathbb{R}(S)$ into $\mathfrak{b}(I)$ respectively. By Corollary 5.8, $F_{\bar{a}}$ and $G_{\bar{a}}$ are injective if and only if $\bar{a}$ is injective or equivalently if and only if S is fundamental. We have

PROPOSITION 6.5. *Let S be a regular semigroup such that $S/\mathcal{L} = \Lambda$, $S/\mathcal{R} = I$. Let $U(I, \Lambda; \Gamma^\circ, \Delta^\circ)$ where $(\Gamma^\circ, \Delta^\circ)$ is the fundamental cross-connection induced by S. If $F_\mu(S)$ and $G_\mu(S)$ are natural functors defined by (6.5), they are embeddings if and only if S is fundamental.*

Let I be any regular partially ordered set. By Proposition 6.2, $S(I)$ is isomorphic to $T\mathfrak{b}(I)$. Now by Theorem 3.10 of [25], $S(I)$ is fundamental and so by Theorem 3.14 and the result abvoe $N^*\mathfrak{b}(I) \approx \mathbb{R}(T\mathfrak{b}(I))$ is isomorphic to a subcategory of $\mathfrak{b}(I^\circ)$ since $I^\circ \approx T\mathfrak{b}(I)/\mathcal{R}$. Hence $N^*\mathfrak{b}(I)$ may be identified with a subcategory of $\mathfrak{b}(I^\circ)$. Moreover, if $\mathfrak{b} \subseteq \mathfrak{b}(I)$ is a reductive subcategory of $\mathfrak{b}(I)$ such that $v\mathfrak{b} = I$, then clearly, $T\mathfrak{b}$ is a subsemigroup of $T\mathfrak{b}(I)$ which intersects every $\mathcal{L}$-class of $T\mathfrak{b}(I)$. Then by Theorem 3.10 of [25], $T\mathfrak{b}$ is fundamental. Hence by Theorem 3.14, it follows that $N^*\mathfrak{b}$ is isomorphic to a subcategory of $\mathfrak{b}(I^\circ)$. Thus:

COROLLARY 6.6. *Let I be a regular partially ordered set and let $\mathfrak{b} \subseteq \mathfrak{b}(I)$ be a normal, reductive subcategory such that $v\mathfrak{b} = I$. Then $N^*\mathfrak{b}$ is isomorphic to a subcategory of $\mathfrak{b}(I^\circ)$.*

Recall that a normal category reductive category $\mathfrak{b}$ is reduced if for all $f, g \in \mathfrak{b}(C, C')$, $f(C') = g(C')$ for all $C' \in C$ implies $f = g$ (cf. Definition 6.1). If we let $I = v\mathfrak{b}$, it is easy to see that this condition is equivalent to the fact that the natural functor $F_\mu : \mathbb{L}(T\mathfrak{b}) \approx \mathfrak{b} \to \mathfrak{b}(I)$ is an embedding. Consequently, by Proposition 6.5. S is fundamental if and only if both $\mathbb{L}(S)$ and $\mathbb{R}(S)$ are reduced and so, if $\mathfrak{b}$ is reduced, so is $N^*\mathfrak{b}$. In the following, we shall assume that, whenever $\mathfrak{b}$ is reduced, $\mathfrak{b}$ has been identified with subcategory of $\mathfrak{b}(I)$ by the natural functor F_μ (where $I = v\mathfrak{b}$). Also we shall assume that, in this case, $N^*\mathfrak{b} \subseteq \mathfrak{b}(I^\circ)$. Observe that when we make this identification, objects of $N^*\mathfrak{b}$, which are functors of the form $H(\epsilon, -)$ where ϵ denotes an idempotent of $T\mathfrak{b}$, gets identified with $ker\ \epsilon$, the normal equivalence relation determined by ϵ (which may be identified with a normal retraction of I by Propositions 6.2 and 6.5). In particular, if $U = U(I, \Lambda; \Gamma^\circ, \Delta^\circ)$, then by Equation 4.2,

$$\Gamma_U(a) = H(\rho^e, -)$$

where e is an idempotent of U in the $\mathcal{R}$-class represented by a. By Proposition 6.3, we may identify ρ^e with $\epsilon(\Gamma^\circ(a), \mathfrak{b})$ for some $b \in M\Gamma(a)$. Hence by the remarks above, identification of $N^*\mathbb{L}(U)$ with a subcategory of $\mathfrak{b}(\Lambda^\circ)$ identifies $\Gamma_U(a)$ with $\Gamma^\circ(a)$. Hence it follows that

$$v\Gamma_U(a) = \Gamma^\circ, \qquad v\Gamma_U^* = \Delta^\circ \tag{6.6}$$

More generally if S is any regular semigroup and if $I = S/\mathcal{R}$, $\Lambda = S/\mathcal{L}$ then the identification of $S/\mathcal{R}$ and $U/\mathcal{R}$ with I (by means of the natural homomorphism of S into U) lead to the identification of the vertex maps of Γ_S with that of Γ_U and $v\Gamma_S^*$ with $v\Gamma_U^*$. Hence from (6.6) we obtain the following which describes the relation between fundamental cross-connection and those defined by Definition 4.2.

THEOREM 6.7. *Let S be a regular semigroup and let Γ_S and Γ_S^* be the functors defined by (4.2) and (4.2*) respectively. Then $(v\Gamma_S, v\Gamma_S^*)$ is a fundamental cross-connection between $I = S/\mathcal{R}$ and $\Lambda = S/\mathcal{L}$. Every fundamental cross-connection arises in this way.*

7. Remarks and Examples

In this section we wish to discuss some examples to illustrate the various results of the paper.

We first consider the class of strongly regular Baer semigroups. Recall that a semigroup S is strongly regular Baer semigroup if the left [right] annihilator of every element is generated by an idempotent and every principal left [right] ideal arises that way (cf. [30], [26]). Also recall that the kernel of a morphism F in a category $\mathfrak{b}$ is a morphism $\alpha: K \to C = \operatorname{dom} f$ such that $\alpha f = 0$ and if $\beta: K' \to C$ is another monomorphism with $\beta f = 0$, then there is a unique morphism $\gamma: K' \to K$ such that $\beta = \gamma\alpha$ (cf. [16], [32]). Cokernels are defined dually. Now if $\mathfrak{b}$ has subobjects, then clearly kernels may be taken as inclusions and so, may be identified with their domains. Thus, in normal categories, when we talk of kernels, we shall mean a unique subobjects of the domain. Notice that in normal categories, cokernels need not be unique (only unique upto isomorphism). We first characterize those regular semigroups S for which categories $\mathbb{L}(S)$ and $\mathbb{R}(S)$ have kernels and cokernels.

Notice that $\mathbb{L}(S)$ has kernels implies that $\mathbb{L}(S)$ has zero which is a minimum left ideal. This must also be a $\mathcal{D}$-class. Now in order that $\mathbb{R}(S)$ may have kernel or cokernel, $\mathbb{R}(S)$ must also have zero. This must therefore represent an $\mathcal{H}$-class of S. Thus, the quotient of S with respect to this $\mathcal{D}$-class is a regular semigroup with zero. Consequently, we shall assume in the following that S is a regular semigroup with zero so that $\mathbb{L}(S)$ and $\mathbb{R}(S)$ are categories with zero.

LEMMA 7.1. *$\mathbb{L}(S)$ has kernels [cokernels] if and only if $\mathbb{R}(S)$ has cokernels [kernels].*

PROOF: Recall from Theorem 4.1, Proposition 4.6 and Theorem 4.8, that if $\rho(e,u,f): Se \to Sf$ is any morphism in $\mathbb{L}(S)$, then its transpose $\rho^*(e,u,f): fS \to eS$ is $\lambda(f,u,e)$. Also notice that if 0 is the zero morphism of $\mathbb{L}(S)$, its transpose is the zero of $\mathbb{R}(S)$. Now assume that $\mathbb{L}(S)$ has kernels and let $\lambda = \lambda(f,u,e) \in \mathbb{R}(S)$. Then $\lambda^* = \rho(e,u,f): Se \to Sf$. Let Sg be the kernel of λ^*, where $g\omega e$. If $j = \rho(g,g,e): Sg \subseteq Se$, then by the remarks above and Corollary 4.7, $\lambda j^* = (j\lambda^*)^* = 0^* = 0$. Now suppose that $\lambda k = 0$. We may assume without loss of generality that k is a retraction. Let $k = \lambda(e,g',g')$, $g'\omega e$. Then $k^* = \rho(g',g',e)$ is the inclusion of Sg' in Se and $k^*\lambda^* = (\lambda k)^* = 0^* = 0$. Since Sg is the kernel of λ^*, we have $Sg' \subseteq Sg$. Hence $g'\omega^l g$. Therefore by Equation (3.8*), $\lambda(e,g,g)\lambda(g,g',g') = \lambda(e,g',g')$. This proves that gS (i.e., the retraction $\lambda(e,g,g)$) is a cokernel of λ.

Now assume that $\mathbb{L}(S)$ has cokernel and let Sg be a cokernel of $\lambda^* = \rho(f,u,e)$ where $\lambda = \lambda(e,u,f)$. Assume that $g\omega e$. Thus Sg is the image of retraction α such that $\lambda^*\alpha = 0$. Clearly, $\alpha = \rho(e,g,g)$. Then $\alpha^*\lambda = (\lambda^*\alpha)^* = 0$. Now if $k = \lambda(h,h,e)$, $h\omega e$ is an inclusion such that $k\lambda = 0$, then $k^* = \rho(e,h,h)$ is a retraction with $\lambda^*k^* = 0$. Since Sg is a cokernel, there is a morphism $\rho' = \rho(g,v,h)$ where $v \in gSh$ such that $\alpha\rho' = \rho(e,g,g)\rho(g,v,h) = \rho(e,gv,h) = \rho(e,h,h)$. Hence $gv = v = h$. This proves that $h\omega^r g$ and hence $hS \subseteq gS$. Therefore gS is the kernel of λ. Since $\mathbb{L}(S^\circ) = \mathbb{R}(S)$, where S° is the left-right dual of S, this completes the proof of the lemma.

THEOREM 7.2. *Let S be a regular semigroup with zero. Then the category $\mathbb{L}(S)$ has kernels if and only if for every $e \in E(S)$, the left annihilator of every $x \in eSe$ is a principal ideal in eSe.*

PROOF: Assume that S satisfies the condition that for every $e \in E(S)$, the left annihilator of $x \in eSe$ in eSe is a principal ideal in eSe. Since every morphism in $\mathbb{L}(S)$ has a normal factorization it is sufficient to show that every retraction in

$\mathbb{L}(S)$ has kernel. Let $\rho = \rho(e,g,g)$, $g\omega e$, be a retraction. Then the annihilator of $g \in S' = eSe$ is a principal ideal, say $S'h$. Since $h\omega e$, $j = \rho(h,h,e): Sh \subseteq Se$ is an inclusion and $j\rho = \rho(h,hg,e) = \rho(h,0,e) = 0$ by Equation (3.8). Now let $Sk \subseteq Se$ (with $k\omega e$) and $j'\rho = 0$, where $j' = \rho(k,k,e): Sk \subseteq Se$. Then $\rho(k,kg,e) = 0$ and this implies $kg = 0$. Hence $k \in S'h$ and so $k\omega^l h$. Hence $Sk \subseteq Sh$ and this proves that Sh is the kernel of $\rho(e,g,g)$. Therefore every morphism in $\mathbb{L}(S)$ has the kernel.

Conversely, assume that $\mathbb{L}(S)$ has kernels. Let $e \in E(S)$ and $x \in S' = eSe$. Also, let Sh be the kernel of $\rho = \rho(e,x,e)$, where $h\omega e$. Then $j\rho = 0$, where $j = \rho(h,h,e): Sh \subseteq Se$. If $u \in S'$ and $ux = 0$, then $\rho(e,u,e)\rho = \rho(e,ux,e) = 0$. Hence $ux = 0$. It follows that $\operatorname{Im} \rho(e,u,e) = Su \subseteq Sh$; i.e., $u \in Sh$. Thus $u \in eSe \cap Sh = S'h$. Therefore $S'h$ is the annihilator of x in S'. This completes the proof.

Lemma 7.1 and Theorem 7.2 lead to the following characterization of those regular semigroups S for which $\mathbb{L}(S)$ has both kernels and cokernels.

THEOREM 7.3. *For a regular semigroup S with zero, the following statements are equivalent.*

(1) *For each $e \in E(S)$, eSe is a strongly regular Baer semigroup.*

(2) *$\mathbb{L}(S)$ and $\mathbb{R}(S)$ have kernels.*

(3) *$\mathbb{L}(S)$ has both kernels and cokernels.*

PROOF: Equivalence of (2) and (3) follows from Lemma 7.1. Statement (1) implies Statement (2) by Theorem 7.2.

Assume that (2) holds and let $e \in E(S)$. Then every element of $S' = eSe$ has both left and right annihilators and these are principal ideals in S'. Now if $S'g$ is any principal left ideal of S' and if $\rho = \rho(e,g,g)$ is the retraction of S' onto $S'g$, then $S'g$ is the kernel of the retraction onto a cokernel, say, $S'h$ of ρ. If $\rho' = \rho(e,h,h)$ is this retraction, then it can be seen that $S'g$ is the left annihilator of h. Similarly every right ideal of S' is the kernel of some retraction of $\mathbb{R}(S)$ and so, is the right annihilator of some elements of S'. It follows that S' is a strongly regular Baer semigroup. This completes the proof.

The following corollary is immediate. For the definitions of lattice theoretic concepts, we refer the reader to [5]. The equivalence of statements (1) and (2) below is well-known (see [26] and [30]).

Corollary 7.4. *For a regular monoid S with o, the following statements are equivalent.*

(1) *S is strongly regular Baer semigroup.*

(2) *$S/\mathcal{L}$ and $S/\mathcal{R}$ are dually isomorphic, complemented modular lattices.*

(3) *Categories $\mathbb{L}(S)$ and $\mathbb{R}(S)$ have kernels.*

(4) *The category $\mathbb{L}(S)$ has both kernels and cokernels.*

Recall from [26] that if L is a complemented modular lattice, then the set $B(L)$ of all binormal mappings of L into itself is the maximum fundamental strongly regular Baer semigroup. It follows from Theorem 6.6 and the Corollary above that $\mathbb{L}(B(L))$ is isomorphic to a subcategory of $\mathfrak{b}(L)$, the category of all normal mappings on L. Also, $\mathbb{L}(B(L))$ is clearly the maximum subcategory of $\mathfrak{b}(L)$ having kernel and cokernel. Hence we have:

Corollary 7.5. *Let L be a regular partially ordered set. Then L is a complemented modular lattice if and only if $\mathfrak{b}(L)$ contains a subcategory $\bar{\mathfrak{b}}(L)$ with kernels and cokernels such that $v\bar{\mathfrak{b}}(L) = L$. Moreover, if L satisfies this condition and if $\bar{\mathfrak{b}}(L)$ is the maximum subcategory with this property, then $T\bar{\mathfrak{b}}(L)$ is isomorphic to $B(L)$.*

Recall from [32] that a category $\mathfrak{b}$ is pre-additive if every home-set $\mathfrak{b}(c,d)$ of $\mathfrak{b}$ carries the structure of an Abelian group in which the addition is bilinear with respect to the composition; i.e. if $f,g \in \mathfrak{b}(c,d)$, $h \in (c',c)$ and $k \in \mathfrak{b}(d,d')$, then

$$h(f+g) = hf + hg, \qquad (f+g)k = fk + gk\,.$$

Also, if $\mathfrak{b}$ has subobjects, we shall say that $\mathfrak{b}$ is *bounded* if $\mathfrak{b}$ has 0 and a largest object, say 1. By Remark 3.3, if $\mathfrak{b}$ is normal, then the semigroup $T\mathfrak{b}$ is isomorphic to $\mathfrak{b}(1,1)$ and if $\mathfrak{b}$ is pre-additive, then $\mathfrak{b}(1,1)$ is a regular ring (cf. [10]). Conversely, if R is a regular ring in the sense of [10], then it is clear that $\mathbb{L}(R)$ is bounded, pre-additive, normal category. Thus we have the following theorem.

Theorem 7.6. *A bounded normal category $\mathfrak{b}$ is pre-additive if and only if $\mathfrak{b}$ is isomorphic to $\mathbb{L}(R)$ for some regular ring R.*

Remark 7.1. If R is a regular ring, it is well-known that $R/\mathcal{L} = L$ is a complemented modular lattice (see [10], [30]) and the multiplicative semigroup of R

is a strongly regular Baer semigroup. Hence the fundamental representation of R is a subsemigroup $B(L)$. Consequently, the functor F_μ defined by (6.5) is a quotient functor $F_\mu : \mathbb{L}(R) \to \hat{\mathfrak{b}}(L)$, where $\hat{\mathfrak{b}}(L)$ denote the subcategory of $\mathfrak{b}(L)$ whose morphisms have kernels and cokernels. F_μ is not, in general, surjective. But all retractions and inclusions of $\hat{\mathfrak{b}}(L)$ belong to the image of F_μ and so $\hat{\hat{\mathfrak{b}}}(L) = Cor\ \hat{\mathfrak{b}}(b)$ belongs to Im F_μ (see §2). Hence there is a regular subring $R' \subseteq R$ such that $R/\mathcal{L} = R'/\mathcal{L}$ and F_μ is surjective from $\mathbb{L}(R')$ onto $\hat{\mathfrak{b}}(L)$. Therefore, given a complemented modular lattice L, the problem of constructing a coordinatizing ring R for L, is equivalent to constructing a pre-additive normal category $\mathfrak{b}$ having $\hat{\mathfrak{b}}(L)$ as a quotient. Notice that given the category $\hat{\mathfrak{b}}(L)$, we can always construct a universal pre-additive category Add $(\hat{\mathfrak{b}}(L))$ (cf. [32], p. 22). However, this category need not be normal. Hence L co-ordinatizes a regular ring if and only if Add $(\hat{\mathfrak{b}}(L))$ has a quotient which is normal and pre-additive. When L has a homogeneous basis of order > 4 or if it has a homogeneous basis of order 3 and satisfies certain additional conditions, one can extract the construction of a pre-additive normal category from the construction given in [8].

Another important class of examples for normal categories and cross-connections arise in the study of semisimple objects of various types (see [32]) for an abstract definition of semisimple objects in a category). It is easy to see that given any semisimple object A, the category $\mathfrak{b}(A)$ of subobjects of A is normal. Here, we shall consider the category that arises as subobjects category of group-algebras (cf. [29]).

Let G be a group and K be a field such that $K[G]$ is a semisimple. This is true if either K is an uncountable field and G is a p'-group if characteristic of K is p (i.e., G does not contain any element of order p), or G is a finite p'-group (and K is any field of characteristic p) (see [29]). Thus if ch $K = 0$, for any group G, $K[G]$ is semisimple if K is non-denumerable and if G is finite, then $K[G]$ is always semisimple. In the following we shall always assume that K and G are such that $K[G]$ is semisimple. In this case, any G-module V (that is, a vector space over K which is a $K[G]$ module) is semisimple. Note that a vector space V over K is a G-module if there is an action of G on V; that is, a homomorphism of G into the group of automorphisms of V. We shall write this action on the right and call V, a right G-module. If V is a right G-module, it is easy to see that $V^* = \text{Hom}\,(V, K)$ is a left G-module under the action defined by

$$(v)(gf) = (vg)f \tag{7.1}$$

for all $v \in V$, $g \in G$ and $f \in V^*$. Further if $\varphi: V \to V'$ is a right G-module morphism (i.e., a linear map of vector spaces such that $\varphi(vg) = \varphi(v)g$ for all $v \in V$, $g \in G$, or equivalently, a $K(G)$-linear map), then we define $\varphi^*: V'^* \to V^*$ by:

$$\varphi^*(f) = \varphi f \tag{7.2}$$

for all $f \in V'^*$. It is easy to verify that φ^* is a (left) G-module morphism and that the assignments

$$D_r: V \to V^*, \qquad \varphi \to \varphi^* \tag{7.3}$$

is a contravariant functor of the category of all right G-modules into the category of left G-module. Clearly, there is a dual D_l from left G-modules into right G-modules. In particular, if $S(V) = End\ V$ (the endomorphism semigroup of the G-module), then the mapping $\varphi \to \varphi^*$ is an antihomomorphism of $S(V)$ into $S(V^*)^\circ$ which is injective. This homomorphism of $S(V)$ into $S(V^*)$ (the left-right dual of $S(V^*)$) will be called the dual homomorphism. We have the following:

PROPOSITION 7.7. *Let V be a G-module. Let $M(V)$ denote the category whose objects are submodules of V and morphisms are G-morphisms. Then $S(V)$ is a regular semigroup such that the following holds.*
(1) *$\mathbb{L}(S(V))$ is isomorphic to $M(V)$.*
(2) *$\mathbb{R}(S(V))$ is isomorphic to $N^*M(V)$.*

PROOF: Regularity of $S(V)$ is an immediate consequence of semisimplicity. Indeed, if $\alpha \in S(V)$, and if p is a projection onto a complement of $ker\ \alpha$, then the corresponding idempotent of $S(V)$ is $\mathcal{R}$-related to α in $S(V)$. Thus α is regular.

The isomophism (1) may be defined as follows:

$$F(Se) = \text{Im } e, \qquad F(\rho(e, u, f)) = u|\text{Im } e.$$

Isomorphism (2) is a consequence of Theorem 3.14. Notice that the map $\sigma \to \sigma_v$, $\sigma \in TM(V)$ is an isomorphism of $TM(V)$ onto $S(V)$.

Now for $H(e,-) \in N^*M(V)$, define (where $e \in E(S(V))$),

$$(7.4\text{i}) \qquad F_*(H(e,-)) = \hat{ker}\ E$$

where for $W \subseteq V$, $\hat{W}$ denotes the annihilator of W. If $\eta: H(e,-) \to H(e',-)$ is a morphism in $N^*M(V)$, η induces a unique $\alpha: \text{Im } e' \to \text{Im } e$. Identifying $(\text{Im } e)^*$ with $\text{Im } e^*$, we see that there is the unique morphism $\alpha^*: \text{Im } e^* \to \text{Im } e'^*$. Since $\text{Im } e^* = \hat{ker}\ e$, α^* is a morphism from $\hat{ker}\ e$ to $\hat{ker}\ e'$. We set

$$(7.4\text{ii}) \qquad F_*(\eta) = \alpha^*$$

Dually for $W \in VM(V)$, let

$$(7.5\text{i}) \qquad G_*(W) = H(e^*,-)$$

where e is an idempotent in $S(V)$ with $\text{Im } e = W$. This is well-defined since if $\text{Im } e = \text{Im } e'$, then $e\mathcal{L}e'$ in $S(V)$ and so $e^*\mathcal{R}e'^*$ in $S(V^*)$. Hence by Lemma 3.13, $H(e^*,-) = H(e'^*,-)$. Further, if $\alpha: W_1 \to W_2$ is a morphism and if $e_i \in E(S(V))$ is such that $\text{Im } e_i = W_i$, then α induces a morphism $\alpha^*: \text{Im } e_2^* \to \text{Im } e_1^*$. Since $H(e_i^*,-)$ is representable by $\text{Im } e_i^*$, this induces a unique morphism $\eta: H(e_1^*,-) \to H(e_2^*,-)$. In addition, η is independent of the choice of the idempotents e_1 and e_2. Hence

$$(7.5\text{ii}) \qquad G_*(\alpha) = \eta$$

is well-defined.

PROPOSITION 7.8. *$F_*: N^*M(V) \to M(V^*)$, $G_*: M(V) \to N^*M(V^*)$ are injective functors such that (F_*, G_*) is the morphisms of cross-connections inducing the dual homomorphism.*

PROOF: We first show that the map defined by (7.4i) is injective. We have, by (7.4i),

$$F_*(H(e,-)) = F_*(H(e',-)) \Leftrightarrow \hat{ker}\ e = \hat{ker}\ e' \Leftrightarrow ker\ e = ker\ e'$$

This implies that $e\mathcal{R}e'$ in $S(V)$ and so by Lemma 3.13, $H(e,-) = H(e',-)$.

We next show that F_* is a functor. Consider morphisms $\eta: H(e,-) \to H(e',-)$ and $\eta': H(e',-) \to H(e'',-)$. Recall that if $\alpha: \mathrm{Im}\, e' \to \mathrm{Im}\, e$ is the morphism in $M(V)$ induced by η, then

$$\eta = \eta_e M(V)(\alpha,-)\eta_{e'}^{-1}$$

where $\eta_e: H(e,-) \to M(V)(\mathrm{Im}\, e,-)$ is the canonical representation of $H(e,-)$ (see Proof of Lemma 3.13). Thus we have

$$\eta\eta' = \eta_e M(V)(\alpha'\alpha,-)\eta_{e''}^{-1}$$

where $\alpha': \mathrm{Im}\, e'' \to \mathrm{Im}\, e'$ is the morphism induced by η'. It follows by (7.4ii) that:

$$F_*(\eta\eta') = (\alpha'\alpha)^* = \alpha^*\alpha'^* = F_*(\eta)F_*(\eta')$$

Suppose that $F_*(\eta) = F_*(\eta')$ where $\eta, \eta': H(e,-) \to H(e',-)$. By (7.4ii) $\alpha^* = \alpha'^*$. Now, $\mathrm{Im}\, e^* = ker\, e$ and for all $f \in \mathrm{Im}\, e^*$,

$$\alpha^*(f) = \alpha f = \alpha'^*(f) = \alpha' f$$

If $\alpha \neq \alpha'$, we can find $x \in \mathrm{Im}\, e'$ such that $\alpha(x) \neq \alpha'(x)$. This implies that there is $f \in \mathrm{Im}\, e^*$ such that $(\alpha(x))f = 0$ and $(\alpha'(x))f \neq 0$. Thus $\alpha f \neq \alpha' f$ and this contradicts the equality $\alpha^* = \alpha'^*$. Hence $\alpha = \alpha'$, i.e., $\eta = \eta'$. Therefore F_* is faithful. This proves that F_* is injective.

In a similar way, if $w_1, w_2 \in VM(v)$,

$$G_*(W_1) = G_*(W_2) \Leftrightarrow H(e_1^*,-) = H(e_2^*,-),$$

by (7.5i), where $e_1, e_2 \in E(S(v))$ with $\mathrm{Im}\, e_i = W_i$, $i = 1,2$. By Lemma 3.13, this is true if and only if $e_1^*\mathcal{R}e_2^*$ in $S(v^*)$ and so $e_1\mathcal{L}e_2$ in $S(V)$. Thus $w_1 = \mathrm{Im}\, e_1 = \mathrm{Im}\, e_2 = w_2$. Hence the map defined by (7.5i) is injective. Now if $\alpha: w_1 \to w_2$ is a morphism in $M(V)$ and if $\alpha^*: \mathrm{Im}\, e_2^* \to \mathrm{Im}\, e_1^*$ is the transpose of α (where $e_i \in E(S(v))$, with $\mathrm{Im}\, e_i = W_i$), then the unique morphism $\eta: H(e_1^*,-) \to H(e_2^*,-)$ induced by α^* is

$$\eta = \eta_{e_1^*} M(V)(\alpha^*,-)\eta_{e_2^*}^{-1}$$

Thus

$$G_*(\alpha) = \eta_{e_1^*} M(V)(\alpha^*, -)\eta_{e_2^*}^{-1}$$

It follows immediately (as in the proof for F_*) that G_* is a functor and that G_* is faithful. This proves that G_* is also injective.

Proposition 7.7 shows that $S(V)$ induces a cross-connection, say Γ between $M(V)$ and $N^*M(V)$ (which is isomorphic to the identity functor on $N^*M(V)$ since $S(V) \approx TM(V)$). Let $E \in \Gamma$ denote the biordered set of Γ (cf. Proposition 5.1). Then $(W, H(e_1, -)) \in E\Gamma$ if and only if $W \in MH(e, -)$ and so there is an idempotent $e' \in S(V)$ where $\operatorname{Im} e' = W$ and $e'\mathcal{R}e$. Then by (7.4i) and (7.5i), $G_*(W), F_*(H(e', -))) \in E\Gamma^*$ if Γ^* is the cross-connection induced by $S(V^*)$. This proves that the pair (G_*, F_*) satisfies axiom I of Definition 5.1. Axiom II follows from the fact that the transpose of $\alpha: W_1 \to W_2$ from $H(e_2, -)$ to $H(e_1, -)$, where $\operatorname{Im} e_i = W_i$, is $\eta_{e_2} M(v)(\alpha_1, -)\eta_{e_1}^{-1}$. A little computation using (5.5), (5.5*) and (5.10) shows that $\hat{S}(G_*, F_*)$ is the same as the dual homomorphism.

REMARK 7.2. Identifying $N^*M(V)$ with the image of F_*, we can say that there exists a cross-connection between $M(V)^* = \operatorname{Im} F_*$ and $M(V)$. This identification, identifies the transpose of $\alpha: W_1 \to W_2$ with the usual transpose α^* of the classical linear algebra. This identification is particularly is meaningful in the case in which V is a reflexive G-module.

If V is a G-module, it is easy to verify that the natural map of V into V^{**} defined by:

$$x \to \hat{x}, \quad \hat{x}(f) = (x)f \quad \text{for all } f \in V^* \tag{7.6}$$

is a G-module morphism. For, if $x \in V$, $g \in G$, then

$$(x\hat{g})(f) = (xg)f = (x)(gf) = \hat{x}(gf) = (\hat{x}g)(f),$$

for all $f \in V^*$. If V is reflexive, i.e. if $x \to \hat{x}$ is an isomorphism it is easy to see that every subspace of V^* is the annihilator of some subspace of V and every morphism of subspaces of V^* is the transpose of some morphism of subspaces of V. Thus, in this case, the functor F_* defined by (7.4) is an isomorphism. Hence:

THEOREM 7.9. *Let* V *be a reflexive* G*-module (over* K*). Then there exists a cross-connection* $\Gamma: M(V^*) \to M(V)$ *such that:*
(1) $\hat{S}\Gamma$ *is isomorphic to* $S(V)$;
(2) *The transpose of* $\alpha: W_1 \to W_2$ *relative to* Γ *is the same as the usual transpose* $\alpha^*: W_2 \to W_1$ *(where* W_i^* *is identified with a subspace of* V^**).*

Let V be a G-module. If $\mathcal{L}T(V)$ denote the semigroup of all linear endmorphisms of V as a vector space, then it is well-known that the maximum idempotent separating congruence on $\mathcal{L}T(V)$ is given by (see [26], Example 19):

$$\varphi\mu\varphi' \Leftrightarrow \varphi' = \alpha\varphi \quad \text{for} \quad \alpha \in K \tag{7.7}$$

If $L = \mathcal{P}(V)$, the lattice of all subspaces of V, then L is a complemented modular lattice and $\mathcal{L}T(V)/\mu$ can be identified with a subsemigroup of $B(L)$. This identifies $S(V)/\mu$ also as a semigroup of $B(L)$. Moreover, the action of G on V induces an action on L by collineations and the mapping belonging to $S(V)/\mu$ are those that preserve this action. Thus $S(L) = S(V)/\mu$ gives rise to a normal subcategory of the category $\hat{\mathfrak{b}}(L)$, the maximum subcategory of $\mathfrak{b}(L)$ having kernels and cokernels.

More generally, by a G-lattice (where G is a group) we shall mean a complemented modular lattice L on which G acts by collineation (see [1]). In the following, we shall consider only those lattices L which are isomorphic to the subspace lattices of a vector space over a field. Structure of such lattices are quite well understood (see [5]). We have seen that G-lattices (of this type) naturally arise in the study of G-modules (or representations of groups). Perhaps they are even more relevant in the study of projective representations of groups. Recall that a projective representation of a group G is a mapping $\rho: G \to GL(V)$ where V is a vector space of finite dimension and $GL(V)$ denote the group of all linear isomorphisms of V (called the general linear group) such that for each pair $g, h \in G$, we have

$$\rho(g)\rho(h) = \alpha\rho(gh) \tag{7.8}$$

for some $\alpha = \alpha(g,h) \in K$ (cf. [14]). It is clear that ρ induces a homomorphism $\bar{\rho}: G \to GL(V)/k^*$ where k^* is the multiplicative group of non-zero elements of k. It is well-known that $GL(V)/k^*$ may be identified with the group of projective transformations of the projective geometry $L = \mathcal{P}(V)$ which is a subgroup of the group of units of $B(L)$. Notice that the group of units of $B(L)$ are precisely the group of collineations of L and so any projective representation induces a unique G–lattice structure on L.

Let L be a G–lattice. An ideal $L' = \{a' : a' \leq a,\ a \in L\}$ is called a G–sublattice of L if L' is stable under the action of G, this is equivalent to the fact that the largest element a of L' is a fixed point for the action of G. We shall say that L is reducible if for every G–sublattice L', there is a saski projection (i.e., an idempotent of $B(L)$ see [26] and [30]) ϵ onto L' such that $\epsilon(x)g = \epsilon(xg)$ for all $x \in L$ and $g \in G$. From [26], we see that this is equivalent to requiring that whenever a is a fixed point of L for the action of G, then there is a complement a' of a which is also fixed by G.

The following is the analogue of Maschke's theorem for G–lattices.

Theorem 7.10. *Let K be a field and G be a group. Assume that L is a G–lattice, where $L = P(V)$ for some vector space V over K of dimension ≥ 3. Then L is reducible in the following cases:*

(1) *K is uncountable and $ch\ K = 0$.*

(2) *K is uncountable, $ch\ K = p$ and no element of G is of order p.*

(3) *G is finite and $ch\ K$ does not divide the order of G.*

Proof: Suppose that L is a G–lattice with homomorphism $\rho : G \to \mathrm{Aut}(L)$, where $\mathrm{Aut}(L)$ is a group of collineation of L (or the group of units of $B(L)$). Now for each $g \in G$, $\rho(g)$ is a collineation of L and since $\dim L \geq 3$ (since $\dim V \geq 3$ by hypothesis) by fundamental theorem of projective geometry (see [1], and Remark 7.3 below), there is a semilinear isomorphism $\bar{\rho}(g)$ of V such that for all subspace $W \subseteq V$, we have

$$\rho(g)[W] = \bar{\rho}(g)(W) = Wg \quad \text{(say)}$$

Recall (cf. [1]), that a semilinear map of V, is an additive homomorphism f such that there is an automorphism μ of K satisfying $f(\alpha v) = \mu(\alpha) f(v)$ for all $\alpha \in K$ and $v \in V$. For each $g \in G$ choose $\bar{\rho}(g)$ as above. For brevity, the automorphism associated with $\bar{\rho}(q)$ will be indicated by the assignment

$$\alpha \to \alpha^g, \qquad \alpha \in K.$$

Also for $v \in V$, $v\bar{\rho}(g)$ will be written as $v\bar{g}$.

Suppose that $g, h \in G$. Then for any $v \in V$, we have

$$< v\bar{g}\bar{h} > = < v\bar{g} > h = < v > gh = < v\overline{gh} >$$

where $< v >$ denote the subspace of V generated by v. So there is $\alpha \in K$ such that

$$v\bar{g}\bar{h} = \alpha(v\overline{gh})$$

Now if $w \in V$ is another vector, then we must have $w\bar{g}\bar{h} = \beta(w\overline{gh})$ and $(v+w)\bar{g}\bar{h} = \gamma(v+w)\overline{gh}$ for $\beta, \gamma \in K$. Hence

$$\alpha w\bar{g}\bar{h} + \beta w\overline{gh} = (v+w)\bar{g}\bar{h} = \gamma v\overline{gh} + \gamma w\overline{gh}$$

If v and w are linearly independent, we must have $\alpha = \gamma = \beta$. Hence for each pair $g, h \in G$, there is a unique constant $\alpha[g,h] \in K$ such that (cf. Equation 7.8):

$$v\bar{g}\bar{h} = \alpha[g,h](v\overline{gh})$$

for all $v \in V$. We may choose $\bar{1} = 1_v$ where 1 is the identity of G. From associativity of action of G on L, we obtain

$$\alpha[g, hk]\alpha[h,k] = \alpha[gh,k]\alpha[g,h]^k \tag{7.9}$$

for all $g, h, k \in G$. This shows that the map $\alpha[\ ,\ ]: G \times G \to K^*$ is a factor system for an extension of G by K^*. Let $H = \{\bar{g}\alpha : (g,\alpha) \in G \times K^*\}$, where $\bar{g}\alpha$ denote the map $v \to (v\bar{g})\alpha$. Define multiplication H by:

$$\bar{g}\alpha\bar{h}\beta = \overline{gh}\alpha[g,h]\alpha^h\beta \tag{7.10}$$

It is routine to check that H becomes a group under this multiplication, which is an extension of G by K^*. Also H naturally acts on V and hence V is an H-module.

Now by Lemma 7.1.6 of [29], $K[H]$ is semisimple in cases (1) and (2). In case (3), semisimplicity follows from Theorem 7.2.1. Since V is an H-module, semisimplicity of $K(H)$ implies semisimplicity of V (see [32]). If $W \subseteq V$ is an H-submodule, clearly G fixes W and so $L(W)$ is G-sublattice. Moreover there is a projection $p: V \to W$ which is H-linear. Hence the map $[p]: V' \to V'p$ is a projection of L onto $L(W)$. Conversely $W \in L$ is fixed by G, then for any $w \in W$, $\bar{g}\alpha \in H$, $w\bar{g}\alpha = (w\bar{g})\alpha \in Wg = W$. Hence by the semisimplicity of V, there is a projection $p: V \to W$ that is H-linear. This completes the proof.

REMARK 7.3. The fundamental theorem of projective geometry applied above states that given a collineation $\varphi: B \to B$ of a finite dimensional projective geometry over a fixed K, there is a automorphism μ of K and a transformation φ', semilinear with respect to μ such that for all subspace $W \subseteq V$, (where V coordinatizes B) we have $\varphi(W) = W\varphi'$ (see [1]). This result can be extended to an infinite dimensional lattice $L = P(V)$ by an application of Zon's Lemma. Using results of [29] it is possible to extend the statement (3) above to some wider class of groups – for example, the class of groups having an Abelian subgroup of finite index.

Suppose that L is a reducible G–lattice. It is easy to see that there is a subcategory $\mathfrak{b}(G, L)$ whose objects are G–sublattices of L and morphisms are morphisms of $\tilde{\mathfrak{b}}(L)$ (the maximum subcategory of $\mathfrak{b}(L)$ having kernels and cokernel– see Corollary 7.5) which are compatible with the action of G on L; i.e., $f: L(a) \to L(a')$ is a morphism in $\mathfrak{b}(G, L)$ if $f(xg) = f(x)g$ for all $x \in L(x)$ and $g \in G$. By Corollary 7.4 $T\mathfrak{b}(G, L) = S(G, L)$ (say) is a strongly regular Baer semigroup. $S(G, L)$ is a regular subsemigroup of $B(L)$. We have:

THEOREM 7.11. *Let G be a group and L be a reducible G–lattice. Then the category $\mathfrak{b}(G, L)$ of all G–sublattices of L is normal and the semigroup $T\mathfrak{b}(G, L) = S(G, L)$ is a strongly regular Baer semigroup.*

REMARK 7.4. Given a G–lattice L, we may study the representation associated with L by studying the fine structure of the semigroup $S(G, L)$ in the same way as we study linear representations using the associated modules. In particular, this is particularly suitable in studying projective representation, the action of G on the associated lattice L (or projective geometry) is by projective isomorphisms; i.e., collineation that preserve cross–ratios. The classical procedure in this case is to study the twisted group algebra determined by the factor system associated with the representation (see [14] and [29]). This factor system (which is a co–chain in this case) is completely determined by the associated G–lattice and any information that can be obtained from the twisted group algebra could also be deduced the associated G–lattice of subspaces of the algebra in a co–ordinate free way. In the more general case also, the factor system (see Equation (7.8)) could be used to define a 'twisted algebra' associated with the representation. We also remark that, given a G–lattice L, the semigroup $S(G, L)$ is a subsemigroup of the fundamental semigroup $B(L)$; but $S(G, L)$ is not in general, fundamental. For, even when the representation is linear, the μ–class containing identity of $S(G, L)$, is in one–to–one correspondence with a projective geometry of dimension $h-1$, where h is the dimension of the centre of commutant algebra of the algebra of linear transformations generated by the representation (see [35] for details).

We end this paper by indicating some further classes of examples of normal categories. One such example arises in connection with unit regular semigroups (see [37]). This arises as category of G–subsets of a G set (i.e., a group G and a set on which G acts by permutations).

Another interesting class of examples arise as categories of finite Boolean algebras with morphisms as Boolean algebra homomorphism which are also normal in the sense of Grillet [8]. This includes complexes (see [33]), buildings of spherical type [36], etc. as examples. Results from [33] and [36] indicate that study of these normal categories as well as semigroups associated with them will be of considerable interest. Other sources include categories whose objects are closed subspaces of a Hilbert or Banach space (see [13]), categories of finitely generated projective modules over regular rings [10] etc.

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LIST OF SYMBOLS

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INDEX

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