CANCELLABLE ELEMENTS OF LATTICES OF SEMIGROUP AND EPIGROUP VARIETIES

Boris Vernikov

Ural Federal University

The joint work with Sergey Gusev and Dmitry Skokov

AAA94 + 5th NSAC

Novi Sad, June 15, 2017

An element x of a lattice L is called *neutral* if

$$\forall y, z \in L: \quad \text{the sublattice of all generated by } x, y \text{ and } z \text{ is distributive}$$

or, equivalently, if

$$\forall y, z \in L: \quad (x \vee y) \wedge (y \vee z) \wedge (z \vee x) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x).$$

If a is neutral in L then L is a subdirect product of $(a]$ and $[a)$,

L embeds in $(a] \times [a)$ by the rule

$$x \longmapsto (x \wedge a, x \vee a) \text{ for any } x \in L.$$

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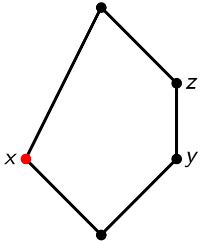
An element x of a lattice L is called *modular* if

$$\forall y, z: \quad y \leq z \to (x \vee y) \wedge z = (x \wedge z) \vee y.$$



This configuration is impossible

An element x of a lattice L is called *cancellable* if

$$\forall y, z: \quad x \wedge y = x \wedge z \;\&\; x \vee y = x \vee z \to y = z.$$

Every neutral element is cancellable.

Every cancellable element is modular.

Cancellable elements

An element x of a lattice L is called *cancellable* if

$$\forall y, z: \quad x \wedge y = x \wedge z \ \& \ x \vee y = x \vee z \rightarrow y = z.$$

Every neutral element is cancellable.

Every cancellable element is modular.

Let **SEM** be the lattice of all semigroup varieties.

Proposition 1 (Jezek and McKenzie, 1993; reproved in simpler way by Shaprynskiĭ, 2012)

If **V** *is a modular element of the lattice* **SEM** *then either* **V** *is the variety of all semigroups or* $\mathbf{V} \subseteq \mathbf{SL} \vee \mathbf{N}$ *where* **SL** *is the variety of semilattices, while* **N** *is a nilvariety.*

Proposition 2 (∼, 2007)

A commutative semigroup variety **V** *is a modular element of the lattice* **SEM** *if and only if* $\mathbf{V} \subseteq \mathbf{SL} \vee \mathbf{N}$ *where* **N** *satisfies the identities* $x^2y = 0$ *and* $xy = yx$.

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Theorem

For a commutative semigroup variety $\mathbf{V}$*, the following are equivalent:*

a) $\mathbf{V}$ *is a cancellable element of the lattice* $\mathbf{SEM}$*;*
b) $\mathbf{V}$ *is a modular element of the lattice* $\mathbf{SEM}$*;*
c) $\mathbf{V} \subseteq \mathbf{SL} \vee \mathbf{N}$ *where* $\mathbf{N}$ *satisfies the identities* $x^2y = 0$ *and* $xy = yx$*.*

($w = 0$ means $wx = xw = w$ where x does not occur in w)

Question 1

Does there exist a semigroup variety that is a modular but not cancellable element of the lattice **SEM**?

Proposition 1 completely reduces the problem of description of modular elements in **SEM** to the nil-case.

An important class of nilvarieties: a variety is called *0-reduced* if it is given by identities of the form $w = 0$.

Proposition 3 (∼ and Volkov, 1988; independently, Jezek and McKenzie, 1993)

A 0-reduced semigroup variety is a modular element of the lattice **SEM**.

Question 2

Is a 0-reduced semigroup variety a cancellable element of the lattice **SEM**?

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An *epigroup* is a semigroup S with the following property: for any $x \in S$ there is n such that x^n lies in some subgroup of S.

All periodic semigroups as well as all completely regular semigroups are epigroups.

Epigroups may be considered as *unary semigroups*, that is semigroups with an additional unary operation.

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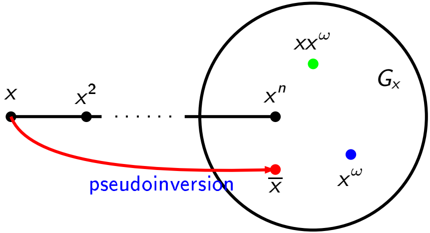
Let S be an epigroup, $x \in S$, G_x is the maximum subgroup of S containing x. Let x^ω be a unit element of G_x. Then $xx^\omega = x^\omega x \in G_x$. Put

$$\overline{x} = (xx^\omega)^{-1} \text{ in } G_x.$$

$\overline{x}$ is called *pseudoinverse* to x



Every periodic semigroup variety can be considered as a variety of epigroups.

If an epigroup variety **V** consists of periodic semigroups then the operation of pseudoinversion may be defined by multiplication. Namely, if **V** satisfies the identity $x^m = x^{m+n}$ then $\overline{x} = x^{(m+1)n-1}$. Thus a variety of periodic epigroups can be considered as epigroup variety.

Periodic varieties of epigroups may be identified with periodic varieties of semigroups.

Periodic case

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Periodic varieties of epigroups may be identified with periodic varieties of semigroups.

Let **EPI** be the lattice of all epigroup varieties.

Theorem

For a commutative epigroup variety **V***, the following are equivalent:*

a) **V** *is a cancellable element of the lattice* **EPI***;*
b) **V** *is a modular element of the lattice* **EPI***;*
c) $\mathbf{V} \subseteq \mathbf{SL} \vee \mathbf{N}$ *where* **N** *satisfies the identities* $x^2y = 0$ *and* $xy = yx$*.*

The equivalence of b) and c) was proved earlier by $\sim$, Skokov and Shaprynskii (2016).

Corollary

For a periodic commutative epigroup variety **V***, the following are equivalent:*

a) **V** *is a cancellable element of the lattice* **EPI***;*
b) **V** *is a cancellable element of the lattice* **SEM***.*

Cancellable elements in the lattice **EPI**

Let **EPI** be the lattice of all epigroup varieties.

Theorem

For a commutative epigroup variety $\mathbf{V}$, *the following are equivalent:*

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Corollary

For a periodic commutative epigroup variety $\mathbf{V}$, *the following are equivalent:*

a) $\mathbf{V}$ *is a cancellable element of the lattice* **EPI**;
b) $\mathbf{V}$ *is a cancellable element of the lattice* **SEM**.