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ON MODULAR AND CANCELLABLE ELEMENTS OF THE LATTICE OF SEMIGROUP VARIETIES

D.V. SKOKOV, B.M. VERNIKOV

Abstract. We continue a study of modular and cancellable elements in the lattice **SEM** of all semigroup varieties. In 2007, the second author completely determined all commutative semigroup varieties that are modular elements in **SEM**. In 2018 the authors jointly with S.V.Gusev proved that, within the class of commutative varieties, the properties to be modular and cancellable elements in **SEM** are equivalent. The objective of this article is to verify that, within some slightly wider class of semigroup varieties, this equivalence is not the case. To achieve this goal, we completely classify semigroup varieties satisfying a permutational identity of length 3 that are modular elements in **SEM**. Further, we specify a variety with these properties that is not a cancellable element in **SEM**.

Keywords: semigroup, variety, lattice of varieties, permutational identity, modular element of a lattice, cancellable element of a lattice.

1. Introduction and summary

There are a number of articles devoted to an examination of special elements in the lattice **SEM** of all semigroup varieties (see surveys [8, Section 14] and [10]). Here we continue these considerations and examine special elements of two types, namely, modular and cancellable elements. Recall that an element x of a lattice

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$\langle L; \vee, \wedge \rangle$ is called

$$\text{\textit{modular} if} \qquad (\forall y, z \in L) \quad \bigl(y \le z \longrightarrow (x \vee y) \wedge z = (x \wedge z) \vee y\bigr),$$
$$\text{\textit{cancellable} if} \qquad (\forall y, z \in L) \quad (x \vee y = x \vee z \;\&\; x \wedge y = x \wedge z \longrightarrow y = z).$$

It is evident that every cancellable element of a lattice is modular. A valuable information about modular and cancellable elements in abstract lattices can be found in [5], for instance.

Several results about modular elements of the lattice **SEM** were provided in the papers [2, 7, 9]. In particular, commutative semigroup varieties that are modular elements in **SEM** are completely determined in [9, Theorem 3.1]. Further, it is verified in [1] that the properties of being modular and cancellable elements of **SEM** are equivalent in the class of commutative semigroup varieties. The objective of this article is to prove that this equivalence is not the case in slightly wider class, namely in the class of semigroup varieties satisfying a permutational identity of length 3.

A semigroup variety is called a *nil-variety* if it consists of nilsemigroups. Semigroup words unlike letters are written in bold. Two sides of identities we connect by the symbol $\approx$, while the symbol $=$ stands for the equality relation on the free semigroup. As usual, we write the pair of identities $x\mathbf{u} \approx \mathbf{u}x \approx \mathbf{u}$ where the letter x does not occur in the word $\mathbf{u}$ in the short form $\mathbf{u} \approx 0$ and refer to the expression $\mathbf{u} \approx 0$ as to a single identity. A *permutational* identity is an identity of the form

$$x_1 x_2 \cdots x_n \approx x_{1\pi} x_{2\pi} \cdots x_{n\pi} \tag{1}$$

where π is a non-trivial permutation on the set $\{1, 2, \dots, n\}$. The number n is called a *length* of the identity (1). We denote by $\mathbf{T}$ the trivial semigroup variety and by $\mathbf{SL}$ the variety of all semilattices.

The main result of this note is the following

Theorem 1.1. *A semigroup variety* $\mathbf{V}$ *satisfying a permutational identity of length* 3 *is a modular element in the lattice* **SEM** *if and only if* $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$ *or* $\mathbf{SL}$*, while the variety* $\mathbf{N}$ *satisfies one of the following identity systems:*

$$xyz \approx zyx,\ x^2y \approx 0; \tag{2}$$
$$xyz \approx yzx,\ x^2y \approx 0; \tag{3}$$
$$xyz \approx yxz,\ xyzt \approx xzty,\ xy^2 \approx 0; \tag{4}$$
$$xyz \approx xzy,\ xyzt \approx yzxt,\ x^2y \approx 0. \tag{5}$$

We denote by $\mathbb{S}_n$ the full permutation group on the set $\{1, 2, \dots, n\}$. The semigroup variety given by an identity system Σ is denoted by $\operatorname{var}\Sigma$. In [1, Question 3.2], the question is asked whether there is a semigroup variety that is a modular but not cancellable element of the lattice **SEM**. The following result gives the affirmative answer to this question.

Proposition 1.2. *Let* ρ *be a non-trivial permutation from* $\mathbb{S}_3$*. The variety*

$$\operatorname{var}\{xyzt \approx xyx \approx x^2 \approx 0,\ x_1x_2x_3 \approx x_{1\rho}x_{2\rho}x_{3\rho}\}$$

is a modular but not cancellable element of the lattice **SEM**.

The article consists of three sections. Section 2 contains auxiliary results. In Section 3 we verify Theorem 1.1 and Proposition 1.2.

2. Preliminaries

We denote by F the free semigroup over a countably infinite alphabet. If $\mathbf{u} \in F$ then $\mathrm{con}(\mathbf{u})$ denotes the set of all letters occurring in the word $\mathbf{u}$. The following claim is well known and easily verified.

Lemma 2.1. *An identity $\mathbf{u} \approx \mathbf{v}$ holds in the variety* **SL** *if and only if* $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v})$. □

An element x of a lattice $\langle L; \vee, \wedge\rangle$ is called *neutral* if, for any $y, z \in L$, the sublattice of L generated by x, y and z is distributive. It is well known that the variety **SL** is a neutral element of the lattice **SEM** [12, Proposition 4.1] and an atom of this lattice (see the survey [8], for instance). This fact together with [6, Corollary 2.1] immediately imply the following

Lemma 2.2. *For a semigroup variety* **V***, the following are equivalent:*
- (i) *the variety* **V** *is a modular element of the lattice* **SEM***;*
- (ii) *the variety* $\mathbf{SL} \vee \mathbf{V}$ *is a modular element of the lattice* **SEM***;*
- (iii) *the variety* $\mathbf{SL} \vee \mathbf{V}$ *is a modular element of the coideal* [**SL**) *of the lattice* **SEM**. □

The following claim gives a strong necessary condition for a semigroup variety to be a modular element in the lattice **SEM**.

Proposition 2.3. *If* **V** *is a modular element of the lattice* **SEM** *then either* **V** *coincides with the variety of all semigroups or* $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* **M** *is one of the varieties* **T** *or* **SL***, while* **N** *is a nil-variety.* □

This proposition was proved (in slightly weaker form and in some other terminology) in [2, Proposition 1.6]. It was formulated in the form given here in [9, Proposition 2.1]. A direct and transparent proof of Proposition 2.3 not depending on a technique from [2] is provided in [7].

As we have already mentioned, commutative varieties that are modular elements in **SEM** are completely classified in [9, Theorem 3.1]. We need the following consequence of this result that was verified in [11], in fact.

Proposition 2.4. *If a semigroup variety satisfies the identities $x^2y \approx 0$ and $xy \approx yx$ then it is a modular element in* **SEM**. □

We need also the following claim that is a part of [9, Theorem 4.5].

Lemma 2.5. *Let* **V** *be a nil-variety of semigroups satisfying an identity of the form $x_1x_2x_3 \approx x_{1\pi}x_{2\pi}x_{3\pi}$ where π is a non-trivial permutation from $\mathbb{S}_3$. If* **V** *is a modular element of the lattice* **SEM** *then* **V** *satisfies also:*
- (i) *all permutational identities of length* 4*;*
- (ii) *the identity $xy^2 \approx 0$ whenever $\pi = (12)$;*
- (iii) *the identity $x^2y \approx 0$ whenever π is one of the permutations* (13)*,* (23) *or* (123). □

For a natural number n and a semigroup variety **V**, we denote by $\mathrm{Perm}_n(\mathbf{V})$ the set of all permutations $\pi \in \mathbb{S}_n$ such that **V** satisfies the identity (1). Clearly, $\mathrm{Perm}_n(\mathbf{V})$ is a subgroup in $\mathbb{S}_n$. If $1 \le i \le n$ then we denote by $\mathrm{Stab}_n(i)$ the set of all permutations $\pi \in \mathbb{S}_n$ with $i\pi = i$. Obviously, $\mathrm{Stab}_n(i)$ also is a subgroup in $\mathbb{S}_n$. Moreover, it is well known that $\mathrm{Stab}_n(i)$ is a maximal proper subgroup in $\mathbb{S}_n$. We need the following partial cases of results of the article [3].

Lemma 2.6. *Let* $\mathbf{V}$ *be a semigroup variety.*

1) *If* $\mathbf{V}$ *satisfies a non-trivial identity of the form* $x_1x_2x_3 \approx x_{1\pi}x_{2\pi}x_{3\pi}$ *and* $n \geq 4$ *then*
   (i) $\mathrm{Perm}_n(\mathbf{V}) \supseteq \mathrm{Stab}_n(n)$ *whenever* $\pi = (12)$*;*
   (ii) $\mathrm{Perm}_n(\mathbf{V}) \supseteq \mathrm{Stab}_n(1)$ *whenever* $\pi = (23)$*;*
   (iii) $\mathrm{Perm}_n(\mathbf{V}) = \mathbb{S}_n$ *otherwise.*
2) *If* $\mathbf{V}$ *satisfies the identity* $xyzt \approx xzty$ *and* $n \geq 5$ *then* $\mathrm{Perm}_n(\mathbf{V}) \supseteq \mathrm{Stab}_n(1)$. □

If $\mathbf{a} \in F$ then we denote by $\ell(\mathbf{a})$ the length of the word $\mathbf{a}$.

Lemma 2.7. *If a nil-variety of semigroups* $\mathbf{N}$ *satisfies a non-trivial identity of the form* $x_1x_2\cdots x_n \approx \mathbf{w}$ *then either this identity is permutational or* $\mathbf{N}$ *satisfies also the identity* $x_1x_2\cdots x_n \approx 0$.

Proof. Suppose that $\mathrm{con}(\mathbf{w}) \neq \{x_1, x_2, \ldots, x_n\}$. Then there is a letter x_i that occurs in exactly one of the words $x_1x_2\cdots x_n$ and $\mathbf{w}$. One can substitute 0 for x_i in the identity $x_1x_2\cdots x_n \approx \mathbf{w}$. We obtain that $x_1x_2\cdots x_n \approx 0$ holds in $\mathbf{N}$. Thus, we can assume that $\mathrm{con}(\mathbf{w}) = \{x_1, x_2, \ldots, x_n\}$. In particular, $\ell(\mathbf{w}) \geq n$. It readily follows from the proof of [4, Lemma 1] that if $\ell(\mathbf{w}) > n$ then $\mathbf{N}$ satisfies the identity $x_1x_2\cdots x_n \approx 0$. Finally, if $\ell(\mathbf{w}) = n$ then the identity $x_1x_2\cdots x_n \approx \mathbf{w}$ is permutational. □

If $\mathbf{a}, \mathbf{b} \in F$ and $\mathbf{b}$ may be obtained from $\mathbf{a}$ by renaming of letters then we say that the words $\mathbf{a}$ and $\mathbf{b}$ are *similar*. For brevity, we will denote the identity (1) by $p_n[\pi]$. If $\mathbf{a}$ is a word and π is a permutation on the set $\mathrm{con}(\mathbf{a})$ then we denote by $\pi[\mathbf{a}]$ the word that is obtained from $\mathbf{a}$ by the substitution $x \mapsto \pi(x)$ for every letter $x \in \mathrm{con}(\mathbf{a})$. A word $\mathbf{w}$ is called *linear* if every letter occurs in $\mathbf{w}$ at most once.

Lemma 2.8. *Let* $\mathbf{V}_1$ *and* $\mathbf{V}_2$ *be semigroup varieties and* n *be a natural number. Then:*

(i) $\mathrm{Perm}_n(\mathbf{V}_1 \vee \mathbf{V}_2) = \mathrm{Perm}_n(\mathbf{V}_1) \wedge \mathrm{Perm}_n(\mathbf{V}_2)$*;*
(ii) *if* $\mathbf{V}_1$ *and* $\mathbf{V}_2$ *are nil-varieties then*

$$\mathrm{Perm}_n(\mathbf{V}_1 \wedge \mathbf{V}_2) = \mathrm{Perm}_n(\mathbf{V}_1) \vee \mathrm{Perm}_n(\mathbf{V}_2).$$

Proof. (i) Let $\pi \in \mathrm{Perm}_n(\mathbf{V}_1 \vee \mathbf{V}_2)$. Then the variety $\mathbf{V}_1 \vee \mathbf{V}_2$ satisfies the identity $p_n[\pi]$. Hence this identity holds in both the varieties $\mathbf{V}_1$ and $\mathbf{V}_2$. Therefore, $\pi \in \mathrm{Perm}_n(\mathbf{V}_1)$ and $\pi \in \mathrm{Perm}_n(\mathbf{V}_2)$, whence $\pi \in \mathrm{Perm}_n(\mathbf{V}_1) \wedge \mathrm{Perm}_n(\mathbf{V}_2)$. Thus, $\mathrm{Perm}_n(\mathbf{V}_1 \vee \mathbf{V}_2) \subseteq \mathrm{Perm}_n(\mathbf{V}_1) \wedge \mathrm{Perm}_n(\mathbf{V}_2)$. Suppose now that $\pi \in \mathrm{Perm}_n(\mathbf{V}_1) \wedge \mathrm{Perm}_n(\mathbf{V}_2)$. Then the identity $p_n[\pi]$ holds in both the varieties $\mathbf{V}_1$ and $\mathbf{V}_2$. Therefore, it holds in $\mathbf{V}_1 \vee \mathbf{V}_2$. Thus, $\pi \in \mathrm{Perm}_n(\mathbf{V}_1 \vee \mathbf{V}_2)$ and $\mathrm{Perm}_n(\mathbf{V}_1) \wedge \mathrm{Perm}_n(\mathbf{V}_2) \subseteq \mathrm{Perm}_n(\mathbf{V}_1 \vee \mathbf{V}_2)$. This implies the required equality.

(ii) Let $\pi \in \mathrm{Perm}_n(\mathbf{V}_1) \vee \mathrm{Perm}_n(\mathbf{V}_2)$. Then there is a sequence of permutations $\pi_1, \pi_2, \ldots, \pi_m$ such that $\pi = \pi_1\pi_2\cdots\pi_m$ and, for each $i = 1, 2, \ldots, m$, the permutation π_i lies in one of the groups $\mathrm{Perm}_n(\mathbf{V}_1)$ or $\mathrm{Perm}_n(\mathbf{V}_2)$. Put $\mathbf{u}_0 = x_1x_2\cdots x_n$. For each $i = 1, 2, \ldots, m$, we define by induction the word $\mathbf{u}_i$ by the equality $\mathbf{u}_i = \pi_i[\mathbf{u}_{i-1}]$. It is clear that $\mathbf{u}_m = \pi[\mathbf{u}_0]$ and the sequence of words $\mathbf{u}_0$, $\mathbf{u}_1$, …, $\mathbf{u}_m$ is a deduction of the identity $p_n[\pi]$ from identities of the varieties $\mathbf{V}_1$ and $\mathbf{V}_2$. Then $\pi \in \mathrm{Perm}_n(\mathbf{V}_1 \wedge \mathbf{V}_2)$, whence $\mathrm{Perm}_n(\mathbf{V}_1) \vee \mathrm{Perm}_n(\mathbf{V}_2) \subseteq \mathrm{Perm}_n(\mathbf{V}_1 \wedge \mathbf{V}_2)$.

It remains to verify that $\mathrm{Perm}_n(\mathbf{V}_1 \wedge \mathbf{V}_2) \subseteq \mathrm{Perm}_n(\mathbf{V}_1) \vee \mathrm{Perm}_n(\mathbf{V}_2)$. Let $\pi \in \mathrm{Perm}_n(\mathbf{V}_1 \wedge \mathbf{V}_2)$. Then the identity $p_n[\pi]$ holds in the variety $\mathbf{V}_1 \wedge \mathbf{V}_2$. Let

$\mathbf{u}_0$, $\mathbf{u}_1$, ..., $\mathbf{u}_m$ be a deduction of this identity from identities of the varieties $\mathbf{V}_1$ and $\mathbf{V}_2$. In other words, $\mathbf{u}_0 = x_1x_2\cdots x_n$, $\mathbf{u}_m = x_{1\pi}x_{2\pi}\cdots x_{n\pi}$ and, for each $i = 0, 1, \dots, m-1$, the identity $\mathbf{u}_i \approx \mathbf{u}_{i+1}$ holds in either $\mathbf{V}_1$ or $\mathbf{V}_2$. We will assume that $\mathbf{u}_0$, $\mathbf{u}_1$, ..., $\mathbf{u}_m$ is the shortest sequence of words with the mentioned properties. Suppose that there is an index i such that either $\mathbf{u}_i$ is non-linear or $\mathrm{con}(\mathbf{u}_i) \ne \{x_1, x_2, \dots, x_n\}$. Let i be the least index with such a property. Clearly, $0 < i < m$. The word $\mathbf{u}_{i-1}$ is similar to $x_1x_2\cdots x_n$. The identity $\mathbf{u}_{i-1} \approx \mathbf{u}_i$ holds in one of the varieties $\mathbf{V}_1$ or $\mathbf{V}_2$, say, in $\mathbf{V}_1$. This identity is non-permutational. Lemma 2.7 implies then that $\mathbf{V}_1$ satisfies the identity $x_1x_2\cdots x_n \approx 0$ and therefore, the identity $p_n[\pi]$. But then the sequence of words $\mathbf{u}_0$, $\mathbf{u}_1$, ..., $\mathbf{u}_{i-1}$, $\mathbf{u}_m$ is a deduction of the identity $p_n[\pi]$ from identities of the varieties $\mathbf{V}_1$ and $\mathbf{V}_2$ shorter than the deduction $\mathbf{u}_0$, $\mathbf{u}_1$, ..., $\mathbf{u}_m$.

Therefore, the words $\mathbf{u}_0$, $\mathbf{u}_1$, ..., $\mathbf{u}_m$ are linear and $\mathrm{con}(\mathbf{u}_i) = \{x_1, x_2, \dots, x_n\}$ for all $i = 0, 1, \dots, m$. Hence there are permutations $\pi_1, \pi_2, \dots, \pi_m \in \mathbb{S}_n$ such that $\mathbf{u}_i = \pi_i[\mathbf{u}_{i-1}]$ for all $i = 1, 2, \dots, m$ and each of the permutations $\pi_1, \pi_2, \dots, \pi_m$ lies in either $\mathrm{Perm}_n(\mathbf{V}_1)$ or $\mathrm{Perm}_n(\mathbf{V}_2)$. Clearly, $\pi = \pi_1\pi_2\cdots\pi_m$, whence $\pi \in \mathrm{Perm}_n(\mathbf{V}_1) \vee \mathrm{Perm}_n(\mathbf{V}_2)$. □

To verify Proposition 1.2, we need the information about the structure of the subgroup lattice of the group $\mathbb{S}_3$. It is generally known and easy to check that this lattice has the form shown in Fig. 1. Here $\mathbb{T}$ is the trivial group and $\mathrm{gr}\{\pi\}$ denotes the subgroup of $\mathbb{S}_3$ generated by the permutation π.

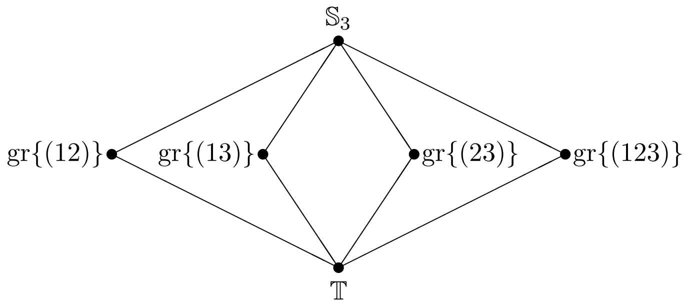

Рис. 1. The subgroup lattice of the group $\mathbb{S}_3$

3. The proof of the main results

Proof of Theorem 1.1. Necessity immediately follows from Proposition 2.3 and Lemma 2.5.

Sufficiency. Let $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ where $\mathbf{M}$ is one of the varieties $\mathbf{T}$ or $\mathbf{SL}$, while the variety $\mathbf{N}$ satisfies one of the identity systems (2)–(5). We need to verify that the variety $\mathbf{V}$ is a modular element in the lattice $\mathbf{SEM}$. In view of Lemma 2.2, we may assume that $\mathbf{M} = \mathbf{SL}$, so $\mathbf{V} = \mathbf{SL} \vee \mathbf{N}$. The same lemma shows that it suffices to prove that $\mathbf{V}$ is a modular element of the coideal $[\mathbf{SL})$ of the lattice $\mathbf{SEM}$. In other words, we have to check that

$$(\mathbf{V} \vee \mathbf{Y}) \wedge \mathbf{Z} = (\mathbf{V} \wedge \mathbf{Z}) \vee \mathbf{Y}$$

for arbitrary varieties $\mathbf{Y}$ and $\mathbf{Z}$ with $\mathbf{SL} \subseteq \mathbf{Y} \subseteq \mathbf{Z}$. Moreover, it suffices to verify that

$$(\mathbf{V} \vee \mathbf{Y}) \wedge \mathbf{Z} \subseteq (\mathbf{V} \wedge \mathbf{Z}) \vee \mathbf{Y}$$

because the opposite inclusion is evident. Thus, we need to prove that if a non-trivial identity $\mathbf{u} \approx \mathbf{v}$ holds in $(\mathbf{V} \wedge \mathbf{Z}) \vee \mathbf{Y}$ then it holds in $(\mathbf{V} \vee \mathbf{Y}) \wedge \mathbf{Z}$ too.

So, let $\mathbf{SL} \subseteq \mathbf{Y} \subseteq \mathbf{Z}$ and $\mathbf{u} \approx \mathbf{v}$ be a non-trivial identity that holds in the variety $(\mathbf{V} \wedge \mathbf{Z}) \vee \mathbf{Y}$. Then $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{Y}$ and there is a deduction of this identity from identities of the varieties $\mathbf{V}$ and $\mathbf{Z}$, that is, a sequence of words

$$\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_m \tag{6}$$

such that $\mathbf{w}_0 = \mathbf{u}$, $\mathbf{w}_m = \mathbf{v}$ and, for each $i = 0, 1, \dots, m-1$, the identity $\mathbf{w_i} \approx \mathbf{w_{i+1}}$ holds in one of the varieties $\mathbf{V}$ or $\mathbf{Z}$. We will assume that (6) is the shortest deduction of $\mathbf{u} \approx \mathbf{v}$ from identities of $\mathbf{V}$ and $\mathbf{Z}$. In particular, this means that the words $\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_m$ are pairwise distinct, there is no $i \in \{0, 1, \dots, m-2\}$ such that $\mathbf{w}_i \approx \mathbf{w}_{i+1} \approx \mathbf{w}_{i+2}$ hold in one of the varieties $\mathbf{V}$ or $\mathbf{Z}$, and there is no $i \in \{0, 1, \dots, m-1\}$ such that $\mathbf{w}_i \approx \mathbf{w}_{i+1}$ holds in both of these two varieties. Since each of the varieties $\mathbf{V}$ and $\mathbf{Z}$ contains $\mathbf{SL}$, Lemma 2.1 implies that

$$\mathrm{con}(\mathbf{w}_0) = \mathrm{con}(\mathbf{w}_1) = \cdots = \mathrm{con}(\mathbf{w}_m). \tag{7}$$

This fact and Lemma 2.1 imply that if, for some $i \in \{0, 1, \dots, m-1\}$, the identity $\mathbf{w}_i \approx \mathbf{w}_{i+1}$ holds in $\mathbf{N}$ then it holds in $\mathbf{V}$ too. All these observations will be used below without references.

The case $m = 1$ is evident. Indeed, if $m = 1$ then the identity $\mathbf{u} \approx \mathbf{v}$ holds in one of the varieties $\mathbf{V}$ or $\mathbf{Z}$. This identity holds also in $\mathbf{Y}$, whence it holds in one of the varieties $\mathbf{V} \vee \mathbf{Y}$ or $\mathbf{Z}$, and moreover in $(\mathbf{V} \vee \mathbf{Y}) \wedge \mathbf{Z}$.

Suppose that $m = 2$. By symmetry, we may assume that $\mathbf{u} \approx \mathbf{w}_1$ holds in $\mathbf{V}$ and $\mathbf{w}_1 \approx \mathbf{v}$ holds in $\mathbf{Z}$. Then $\mathbf{w}_1 \approx \mathbf{v} \approx \mathbf{u}$ hold in $\mathbf{Y}$. We see that the identities $\mathbf{u} \approx \mathbf{w}_1$ and $\mathbf{w}_1 \approx \mathbf{v}$ hold in $\mathbf{V} \vee \mathbf{Y}$ and $\mathbf{Z}$ respectively, whence $\mathbf{u} \approx \mathbf{v}$ holds in $(\mathbf{V} \vee \mathbf{Y}) \wedge \mathbf{Z}$.

At the rest part of the proof we suppose that $m \geq 3$.

First of all, we consider one very special but important partial case.

Lemma 3.1. *If $m = 3$, the identities $\mathbf{w}_0 \approx \mathbf{w}_1$ and $\mathbf{w}_2 \approx \mathbf{w}_3$ hold in the variety $\mathbf{Z}$, and the identity $\mathbf{w}_1 \approx \mathbf{w}_2$ holds in the variety $\mathbf{V}$ then the identity $\mathbf{u} \approx \mathbf{v}$ holds in the variety $(\mathbf{V} \vee \mathbf{Y}) \wedge \mathbf{Z}$.*

Proof. Recall that the identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{Y}$, $\mathbf{u} = \mathbf{w}_0$ and $\mathbf{v} = \mathbf{w}_3$. Since $\mathbf{Y} \subseteq \mathbf{Z}$, we have that $\mathbf{w}_1 \approx \mathbf{w}_0 \approx \mathbf{w}_3 \approx \mathbf{w}_2$ hold in $\mathbf{Y}$. Therefore, the identity $\mathbf{w}_1 \approx \mathbf{w}_2$ holds in the variety $\mathbf{V} \vee \mathbf{Y}$. It remains to take into account that the identities $\mathbf{w}_0 \approx \mathbf{w}_1$ and $\mathbf{w}_2 \approx \mathbf{w}_3$ hold in $\mathbf{Z}$. □

Put

$$Z = \{\mathbf{w} \in F \mid \text{ the identity } \mathbf{w} \approx 0 \text{ holds in the variety } \mathbf{N}\},$$
$$L = \{\mathbf{w} \in F \mid \text{ the word } \mathbf{w} \text{ is linear and } \mathbf{w} \notin Z\},$$
$$S = \{\mathbf{w} \in F \mid \mathbf{w} \notin Z \cup L\}.$$

Lemma 2.5 readily implies that if $S \neq \varnothing$ then any word from S is similar to:

- one of the words x^2 or xyx whenever $\mathbf{V}$ satisfies (2);
- the word x^2 whenever $\mathbf{V}$ satisfies (3);
- one of the words x^2 or x^2y whenever $\mathbf{V}$ satisfies (4);
- one of the words x^2 or xy^2 whenever $\mathbf{V}$ satisfies (5).

Let S_k be the set of all words from S depending on k letters. We have the following

Lemma 3.2. *$S = S_1 \cup S_2$; if $\mathbf{u} \in S_1$ then $\mathbf{u}$ is similar to x^2; if $\mathbf{u}, \mathbf{v} \in S_2$ then $\mathbf{u}$ and $\mathbf{v}$ are similar.* □

It is evident that each of the words $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_m$ lies in exactly one of the sets Z, L or S. It is clear also that if $\mathbf{w}', \mathbf{w}'' \in Z$ then $\mathbf{w}' \approx \mathbf{w}''$ holds in $\mathbf{N}$; if, besides that, $\mathrm{con}(\mathbf{w}') = \mathrm{con}(\mathbf{w}'')$ then $\mathbf{w}' \approx \mathbf{w}''$ holds in $\mathbf{V}$. In particular, if $\mathbf{w}_i, \mathbf{w}_j \in Z$ for some $i, j \in \{0, 1, \ldots, m\}$ then $\mathbf{w}_i \approx \mathbf{w}_j$ holds in $\mathbf{V}$. All these observations will be used below without references.

One can provide some properties of the sequence (6).

Lemma 3.3. *If $\mathbf{w}_i, \mathbf{w}_j \in Z$ for some $0 \le i < j \le m$ then $j = i + 1$. In particular, the sequence* (6) *contains at most two words from the set Z.*

Proof. The identity $\mathbf{w}_i \approx \mathbf{w}_j$ holds in the variety $\mathbf{V}$. If $j > i + 1$ then the sequence of words

$$\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_i, \mathbf{w}_j, \ldots, \mathbf{w}_m$$

is a deduction of the identity $\mathbf{u} \approx \mathbf{v}$ from identities of the varieties $\mathbf{V}$ and $\mathbf{Z}$ shorter than the sequence (6). □

Lemma 3.4. *If $\mathbf{w}_i \in S_1$ for some $i \in \{0, 1, \ldots, m\}$ then:*

(i) *either $i = 0$ or $i = m$;*

(ii) *if $i = 0$ [respectively $i = m$] then the identity $\mathbf{w}_0 \approx \mathbf{w}_1$ [respectively $\mathbf{w}_{m-1} \approx \mathbf{w}_m$] holds in the variety $\mathbf{Z}$.*

Proof. In view of Lemma 3.2, we may assume without loss of generality that $\mathbf{w}_i = x^2$. Suppose that $0 < i < m$. Then $\mathbf{w}_i \approx \mathbf{w}_j$ holds in $\mathbf{V}$ for some $j \in \{i-1, i+1\}$. In particular, $\mathbf{w}_i \approx \mathbf{w}_j$ in $\mathbf{N}$. Since $\mathrm{con}(\mathbf{w}_j) = \mathrm{con}(\mathbf{w}_i) = \{x\}$ and $\mathbf{w}_j \ne \mathbf{w}_i$, we have that $\mathbf{w}_j = x^k$ for some $k \ne 2$. It is evident that $x^k \approx x^2$ implies $x^2 \approx 0$ in any nil-variety. Therefore, $\mathbf{w}_i \approx 0$ holds in $\mathbf{N}$ contradicting the claim $\mathbf{w}_i \in S$. The claim (i) is proved. Analogous arguments show that if $i = 0$ [respectively $i = m$] then the identity $\mathbf{w}_0 \approx \mathbf{w}_1$ [respectively $\mathbf{w}_{m-1} \approx \mathbf{w}_m$] fails in $\mathbf{V}$ and therefore, holds in $\mathbf{Z}$. The claim (ii) is proved as well. □

Lemma 3.5. *If the sequence* (6) *does not contain a word from the set S_2 and $\mathbf{w}_i \in L$ for some $i \in \{1, \ldots, m-1\}$ then either $\mathbf{w}_{i-1} \in L$ or $\mathbf{w}_{i+1} \in L$.*

Proof. Arguing by contradiction, suppose that $\mathbf{w}_{i-1}, \mathbf{w}_{i+1} \in Z \cup S$. If $\mathbf{w}_{i-1} \in S$ then $\mathbf{w}_{i-1} \in S_1$ and Lemma 3.4 applies with the conclusion that the identity $\mathbf{w}_{i-1} \approx \mathbf{w}_i$ holds in $\mathbf{Z}$. Let now $\mathbf{w}_{i-1} \in Z$. Then the variety $\mathbf{N}$ satisfies the identity $\mathbf{w}_{i-1} \approx 0$. On the other hand, the identity $\mathbf{w}_i \approx 0$ fails in $\mathbf{N}$ because $\mathbf{w}_i \in L$. Therefore, the identity $\mathbf{w}_{i-1} \approx \mathbf{w}_i$ fails in $\mathbf{V}$, whence it holds in $\mathbf{Z}$. We see that $\mathbf{w}_{i-1} \approx \mathbf{w}_i$ holds in $\mathbf{Z}$ in any case. Analogous arguments show that $\mathbf{w}_i \approx \mathbf{w}_{i+1}$ holds in $\mathbf{Z}$. Thus, $\mathbf{w}_{i-1} \approx \mathbf{w}_i \approx \mathbf{w}_{i+1}$ hold in $\mathbf{Z}$ that is impossible. □

Lemma 3.6. *If the sequence* (6) *does not contain a word from the set S_2 and $\mathbf{w}_i, \mathbf{w}_{i+1} \in L$ for some $i \in \{0, 1, \ldots, m-1\}$ then the sequence* (6) *does not contain words from the set Z.*

Proof. Suppose that $i + 1 < m$ and $\mathbf{w}_{i+2} \in Z$. Then the variety $\mathbf{N}$ satisfies the identity $\mathbf{w}_{i+2} \approx 0$ but does not satisfy the identity $\mathbf{w}_{i+1} \approx 0$. Hence the identity $\mathbf{w}_{i+1} \approx \mathbf{w}_{i+2}$ is false in $\mathbf{N}$, and moreover in $\mathbf{V}$. Therefore, $\mathbf{w}_{i+1} \approx \mathbf{w}_{i+2}$ holds in

$\mathbf{Z}$, whence $\mathbf{w}_i \approx \mathbf{w}_{i+1}$ holds in $\mathbf{V}$. Clearly, $\mathbf{w}_i = \sigma[\mathbf{w}_{i+1}]$ for some permutation σ on the set $\mathrm{con}(\mathbf{w}_{i+1})$. Then $\mathbf{w}_i = \sigma(\mathbf{w}_{i+1}) \approx \sigma(\mathbf{w}_{i+2})$ holds in $\mathbf{Z}$. Since $\mathbf{w}_{i+2} \approx 0$ holds in $\mathbf{N}$, we have that $\sigma[\mathbf{w}_{i+2}] \approx 0$ holds in $\mathbf{N}$ too. Thus $\sigma[\mathbf{w}_{i+2}] \approx \mathbf{w}_{i+2}$ holds in $\mathbf{N}$ and therefore, in $\mathbf{V}$. Recall that $m \geq 3$. Hence either $i > 0$ or $i + 2 < m$. If $i > 0$ then $\mathbf{w}_{i-1} \approx \mathbf{w}_i \approx \sigma[\mathbf{w}_{i+2}]$ hold in $\mathbf{Z}$ and the sequence of words

$$\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_{i-1}, \sigma[\mathbf{w}_{i+2}], \mathbf{w}_{i+2}, \ldots, \mathbf{w}_m$$

is a deduction of the identity $\mathbf{u} \approx \mathbf{v}$ from identities of the varieties $\mathbf{V}$ and $\mathbf{Z}$ shorter than the sequence (6). Further, if $i+2 < m$ then $\sigma[\mathbf{w}_{i+2}] \approx \mathbf{w}_{i+2} \approx \mathbf{w}_{i+3}$ hold in $\mathbf{V}$ and we also have a deduction of the identity $\mathbf{u} \approx \mathbf{v}$ from identities of the varieties $\mathbf{V}$ and $\mathbf{Z}$ shorter than the sequence (6), namely the sequence of words

$$\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_i, \sigma[\mathbf{w}_{i+2}], \mathbf{w}_{i+3}, \ldots, \mathbf{w}_m.$$

Both these cases are impossible. We prove that if $i + 1 < m$ then $\mathbf{w}_{i+2} \notin Z$. By symmetry, if $i > 0$ then $\mathbf{w}_{i-1} \notin Z$.

Let now $\mathbf{w}_j, \mathbf{w}_{j+1}, \ldots, \mathbf{w}_{j+k}$ (where $0 \leq j < j + k \leq m$) be the maximal subsequence of the sequence (6) consisting of words from the set L. In other words, $\mathbf{w}_j, \mathbf{w}_{j+1}, \ldots, \mathbf{w}_{j+k} \in L$, $\mathbf{w}_{j+k+1} \notin L$ whenever $j+k < m$, and $\mathbf{w}_{j-1} \notin L$ whenever $j > 0$. Suppose that $j + k < m$. As we have seen in the previous paragraph, this implies that $\mathbf{w}_{j+k+1} \notin Z$, whence $\mathbf{w}_{j+k+1} \in S$. Now Lemma 3.4(i) applies, and we conclude that $j + k + 1 = m$. Analogously, if $j > 0$ then $j = 1$ and $\mathbf{w}_0 \in S$. We see that words from the set Z are absent in the sequence (6). $\square$

Further considerations are divided into two cases.

Case 1: the sequence (6) does not contain a word from the set S_2. Suppose at first that the sequence (6) does not contain adjacent words from the set L. Lemmas 3.4(i) and 3.5 imply that $\mathbf{w}_1, \ldots, \mathbf{w}_{m-1} \in Z$ in this case, whence $\mathbf{w}_1 \approx \cdots \approx \mathbf{w}_{m-1}$ in $\mathbf{V}$. Therefore, $m - 1 = 2$, that is, $m = 3$. Since the identity $\mathbf{w}_1 \approx \mathbf{w}_2$ holds in $\mathbf{V}$, the identities $\mathbf{w}_0 \approx \mathbf{w}_1$ and $\mathbf{w}_2 \approx \mathbf{w}_3$ hold in $\mathbf{Z}$. Now Lemma 3.1 applies and we are done.

Suppose now that the sequence (6) contains adjacent words from the set L but does not contain three words in row from this set. By Lemma 3.6 the sequence (6) does not contain words from the set Z. Then Lemma 3.4(i) implies that $m = 3$, $\mathbf{w}_0, \mathbf{w}_3 \in S$ and $\mathbf{w}_1, \mathbf{w}_2 \in L$. By Lemma 3.4(ii) the identities $\mathbf{w}_0 \approx \mathbf{w}_1$ and $\mathbf{w}_2 \approx \mathbf{w}_3$ hold in $\mathbf{Z}$. Therefore, $\mathbf{w}_1 \approx \mathbf{w}_2$ holds in $\mathbf{V}$ and Lemma 3.1 applies again.

Finally, suppose that the sequence (6) contains three words in row from the set L. Let $\mathbf{w}_i, \mathbf{w}_{i+1}, \mathbf{w}_{i+2} \in L$ for some $i \in \{0, 1, \ldots, m-2\}$. Put $k = \ell(\mathbf{w}_i)$. In view of (7), we have $\ell(\mathbf{w}_{i+1}) = \ell(\mathbf{w}_{i+2}) = k$. One of the identities $\mathbf{w}_i \approx \mathbf{w}_{i+1}$ or $\mathbf{w}_{i+1} \approx \mathbf{w}_{i+2}$ holds in the variety $\mathbf{V}$, whence it holds in $\mathbf{N}$.

Suppose that $k = 2$. The variety $\mathbf{N}$ is commutative in this case. Then $\mathbf{N}$ is a modular element of **SEM** by Proposition 2.4 and therefore, $\mathbf{V}$ has the same property by Lemma 2.2.

Let now $k \geq 3$. We may assume without loss of generality that $\mathrm{con}(\mathbf{w}_j) = \{x_1, x_2, \ldots, x_k\}$ for all $j = 0, 1, \ldots, m$. According to Lemma 3.6 the sequence (6) does not contain words from the set Z. Further, by Lemma 3.2 if $\mathbf{w} \in S$ then $\mathbf{w}$ is similar to x^2, whence $\mathrm{con}(\mathbf{w}) \neq \{x_1, x_2, \ldots, x_k\}$. Hence (6) does not contain words from the set S. Therefore, $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_m \in L$.

Suppose that $k = 3$. Since $\mathbf{w}_i \approx \mathbf{w}_{i+1}$ holds in $\mathbf{Z}$ for some $i \in \{0, 1, \ldots, m-1\}$, the variety $\mathbf{Z}$ satisfies a permutational identity of length 3. In other words, the group $\mathrm{Perm}_3(\mathbf{Z})$ contains some non-trivial permutation σ. It is clear that $\mathbf{w}_m = \tau[\mathbf{w}_0]$ for some permutation $\tau \in \mathbb{S}_3$. The permutation τ is non-trivial because the identity $\mathbf{u} \approx \mathbf{v}$ is non-trivial. If $\tau \in \mathrm{Perm}_3(\mathbf{Z})$ then the identity $\mathbf{w}_0 \approx \mathbf{w}_m$ (that is, $\mathbf{u} \approx \mathbf{v}$) holds in the variety $\mathbf{Z}$ and therefore, in $(\mathbf{V} \vee \mathbf{Y}) \wedge \mathbf{Z}$. Suppose now that $\tau \notin \mathrm{Perm}_3(\mathbf{Z})$. The identity $\mathbf{u} \approx \mathbf{v}$ has the form $\mathbf{u} \approx \tau[\mathbf{u}]$. Recall that this identity holds in the variety $\mathbf{Y}$, whence $\tau \in \mathrm{Perm}_3(\mathbf{Y})$. Besides that, $\sigma \in \mathrm{Perm}_3(\mathbf{Y})$ because $\mathbf{Y} \subseteq \mathbf{Z}$. Sinse $\tau \notin \mathrm{Perm}_3(\mathbf{Z})$, we have that σ and τ generate distinct non-trivial subgroups of $\mathbb{S}_3$. Therefore, the subgroup of $\mathbb{S}_3$ generated by these two permutations coincides with $\mathbb{S}_3$ (see Fig. 1), that is, $\mathrm{Perm}_3(\mathbf{Y}) = \mathbb{S}_3$. In other words, $\mathbf{Y}$ satisfies all permutational identities of length 3. This means that, for each $i = 0, 1, \dots, m-1$, the identity $\mathbf{w}_i \approx \mathbf{w}_{i+1}$ holds in one of the varieties $\mathbf{V} \vee \mathbf{Y}$ or $\mathbf{Z}$, whence $\mathbf{u} \approx \mathbf{v}$ holds in $(\mathbf{V} \vee \mathbf{Y}) \wedge \mathbf{Z}$.

Finally, let $k \geq 4$. The identity $\mathbf{u} \approx \mathbf{v}$, that is, $\mathbf{w}_0 \approx \mathbf{w}_m$ is a permutational identity of length k. In other words, there is a permutation $\xi \in \mathbb{S}_k$ with $\mathbf{v} = \xi[\mathbf{u}]$. If $\mathbf{N}$ satisfies one of the identity systems (2) or (3) then the claim 1(iii) of Lemma 2.6 applies with the conclusion that $\xi \in \mathrm{Perm}_k(\mathbf{N})$. This means that the identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{N}$ and therefore, in $\mathbf{V}$. In this case $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{V} \vee \mathbf{Y}$, and moreover in $(\mathbf{V} \vee \mathbf{Y}) \wedge \mathbf{Z}$. By symmetry, it remains to consider the case when $\mathbf{N}$ satisfies the identity system (4).

Suppose that $k = 4$. In view the claim 1(i) of Lemma 2.6, $\mathrm{Perm}_4(\mathbf{N}) \supseteq \mathrm{Stab}_4(4)$. Besides that, $\mathbf{N}$ satisfies the identity $xyzt \approx xzty$, whence the group $\mathrm{Perm}_4(\mathbf{N})$ contains the permutation (234). Since this permutation does not lie in $\mathrm{Stab}_4(4)$ and $\mathrm{Stab}_4(4)$ is a maximal proper subgroup in $\mathbb{S}_4$, we have that $\mathrm{Perm}_4(\mathbf{N}) = \mathbb{S}_4$. Further, let $k \geq 5$. Then the claims 1(i) and 2 of Lemma 2.6 imply that the group $\mathrm{Perm}_k(\mathbf{N})$ contains both the groups $\mathrm{Stab}_k(1)$ and $\mathrm{Stab}_k(k)$, whence $\mathrm{Perm}_k(\mathbf{N}) = \mathbb{S}_k$. Thus, the last equality holds for any $k \geq 4$. In particular, $\xi \in \mathrm{Perm}_k(\mathbf{N})$ in this case. As we have already seen in the previous paragraph, this implies the requirement conclusion.

Case 2: the sequence (6) contains a word from the set S_2. All words from S_2 depend on two letters. In view of (7), we may assume that $\mathrm{con}(\mathbf{w}_i) = \{x, y\}$ for each $i = 0, 1, \dots, m$. In particular, the sequence (6) does not contain words from the set S_1. Besides that, Lemma 3.2 shows that if $\mathbf{w}_i, \mathbf{w}_j \in S_2$ for some $0 \leq i < j \leq m$ then the words $\mathbf{w}_i$ and $\mathbf{w}_j$ are similar. Since words from S_2 depend on two letters, this implies that the sequence (6) contains at most two words from the set S_2. All these observations will be used below without references. We need two additional lemmas.

Lemma 3.7. *If the sequence* (6) *contains a word from the set* S_2*,* $\mathbf{w}_i \in L$ *for some* $i \in \{0, 1, \dots, m\}$ *and the variety* $\mathbf{V}$ *satisfies a non-trivial identity of the form* $\mathbf{w}_i \approx \mathbf{w}$ *then the variety* $\mathbf{V}$ *is a modular element in* $\mathbf{SEM}$.

Proof. The word $\mathbf{w}_i$ is linear and $\mathrm{con}(\mathbf{w}_i) = \{x, y\}$. Therefore, $\mathbf{w}_i \in \{xy, yx\}$. We may assume without loss of generality that $\mathbf{w}_i = xy$. The identity $xy \approx \mathbf{w}$ holds in the variety $\mathbf{N}$. If $\mathbf{w} = yx$ then the variety $\mathbf{N}$ is a modular element in $\mathbf{SEM}$ by Proposition 2.4, whence the variety $\mathbf{V}$ has the same property by Lemma 2.2. Suppose now that $\mathbf{w} \neq yx$. Then $\ell(\mathbf{w}) \neq 2$. By Lemma 2.7 $xy \approx 0$ holds in $\mathbf{N}$. Then the variety $\mathbf{V} = \mathbf{SL} \vee \mathbf{N}$ is a neutral element of the lattice $\mathbf{SEM}$ by [12, Proposition 4.1]. Therefore, the variety $\mathbf{V}$ is a modular element in $\mathbf{SEM}$. □

Lemma 3.8. *If the variety* $\mathbf{V}$ *satisfies the identity* $\mathbf{w}_i \approx \mathbf{w}_{i+1}$ *for some* $i \in \{0, 1, \dots, m-1\}$ *and one of the words* $\mathbf{w}_i$ *or* $\mathbf{w}_{i+1}$ *lies in* S_2 *then either the variety* $\mathbf{V}$ *is a modular element in* **SEM** *or both the words* $\mathbf{w}_i$ *and* $\mathbf{w}_{i+1}$ *lie in* S_2.

Proof. We may assume without loss of generality that $\mathbf{w}_i \in S_2$. If $\mathbf{w}_{i+1} \in Z$ then $\mathbf{w}_i \approx \mathbf{w}_{i+1} \approx 0$ in $\mathbf{N}$ contradicting the claim $\mathbf{w}_i \in S$. Further, if $\mathbf{w}_{i+1} \in L$ then Lemma 3.7 applies with the conclusion that the variety $\mathbf{V}$ is a modular element in **SEM**. Finally, if $\mathbf{w}_{i+1} \notin Z \cup L$ then $\mathbf{w}_{i+1} \in S$. Since the sequence (6) does not contain words from the set S_1, we have that $\mathbf{w}_{i+1} \in S_2$. □

Suppose now that the sequence (6) contains only one word from the set S_2. Namely, let $\mathbf{w}_i \in S_2$. Suppose that $i \in \{1, \dots, m-1\}$. Then the variety $\mathbf{V}$ satisfies one of the identities $\mathbf{w}_{i-1} \approx \mathbf{w}_i$ or $\mathbf{w}_i \approx \mathbf{w}_{i+1}$. Since $\mathbf{w}_{i-1}, \mathbf{w}_{i+1} \notin S_2$, Lemma 3.8 implies that the variety $\mathbf{V}$ is a modular element in **SEM**. It remains to consider the case when either $i = 0$ or $i = m$. By symmetry, we may suppose that $i = 0$. Then $\mathbf{w}_j \notin S$ for all $j = 1, 2, \dots, m$. If the identity $\mathbf{w}_0 \approx \mathbf{w}_1$ holds in $\mathbf{V}$ then we may apply Lemma 3.8 and conclude that the variety $\mathbf{V}$ is a modular element in **SEM**. Suppose now that the identity $\mathbf{w}_0 \approx \mathbf{w}_1$ holds in $\mathbf{Z}$. Then $\mathbf{w}_1 \approx \mathbf{w}_2$ in $\mathbf{V}$. In view of Lemma 3.7, we may assume that $\mathbf{w}_2 \notin L$. Therefore, $\mathbf{w}_2 \in Z$. Further, $\mathbf{w}_2 \approx \mathbf{w}_3$ in $\mathbf{Z}$. If $m = 3$ then Lemma 3.1 applies with the required conclusion. Otherwise, $\mathbf{w}_3 \approx \mathbf{w}_4$ in $\mathbf{V}$. Lemma 3.7 allows us to assume that $\mathbf{w}_4 \notin L$, whence $\mathbf{w}_4 \in Z$. Therefore, $\mathbf{w}_2 \approx \mathbf{w}_4$ in $\mathbf{V}$. This means that the sequence of words

$$\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4, \dots, \mathbf{w}_m$$

is a deduction of the identity $\mathbf{u} \approx \mathbf{v}$ from identities of the varieties $\mathbf{V}$ and $\mathbf{Z}$ shorter than (6).

Finally, suppose that the sequence (6) contains two words from the set S_2. Suppose at first that $\mathbf{w}_i \in S_2$ for some $i \in \{1, \dots, m-1\}$. Then the variety $\mathbf{V}$ satisfies the identity $\mathbf{w}_i \approx \mathbf{w}_j$ for some $j \in \{i-1, i+1\}$. In view of Lemma 3.8, we may assume that $\mathbf{w}_j \in S_2$ in this case. Suppose now that $\mathbf{w}_i \notin S_2$ whenever $i \in \{1, \dots, m-1\}$. Then $\mathbf{w}_0, \mathbf{w}_m \in S_2$. If, besides that, at least one of the identities $\mathbf{w}_0 \approx \mathbf{w}_1$ or $\mathbf{w}_{m-1} \approx \mathbf{w}_m$ holds in the variety $\mathbf{V}$ then Lemma 3.8 applies and we conclude that $\mathbf{V}$ is a modular element in **SEM**. We see that either $\mathbf{w}_0, \mathbf{w}_m \in S_2$ and the identities $\mathbf{w}_0 \approx \mathbf{w}_1$ and $\mathbf{w}_{m-1} \approx \mathbf{w}_m$ hold in $\mathbf{Z}$ or $\mathbf{w}_i, \mathbf{w}_{i+1} \in S_2$ for some $i \in \{0, 1, \dots, m-1\}$ and $\mathbf{w}_i \approx \mathbf{w}_{i+1}$ holds in $\mathbf{V}$.

Suppose that $\mathbf{w}_0, \mathbf{w}_m \in S_2$ and the identities $\mathbf{w}_0 \approx \mathbf{w}_1$ and $\mathbf{w}_{m-1} \approx \mathbf{w}_m$ hold in $\mathbf{Z}$. Then $\mathbf{w}_1, \mathbf{w}_{m-1} \notin S$. The identities $\mathbf{w}_1 \approx \mathbf{w}_2$ and $\mathbf{w}_{m-2} \approx \mathbf{w}_{m-1}$ hold in the variety $\mathbf{V}$. Lemma 3.7 permits to assume that $\mathbf{w}_1, \mathbf{w}_{m-1} \notin L$. Since $\mathbf{w}_1, \mathbf{w}_{m-1} \notin S$, we have $\mathbf{w}_1, \mathbf{w}_{m-1} \in Z$. Then $\mathbf{w}_1 \approx \mathbf{w}_{m-1}$ holds in $\mathbf{V}$. This means that the sequence of words

$$\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_{m-1}, \mathbf{w}_m$$

is a deduction of the identity $\mathbf{u} \approx \mathbf{v}$ from identities of the varieties $\mathbf{V}$ and $\mathbf{Z}$. If $m > 3$ then this deduction shorter than (6), while if $m = 3$ then Lemma 3.1 applies.

It remains to consider the case when $\mathbf{w}_i, \mathbf{w}_{i+1} \in S_2$ for some $i \in \{0, 1, \dots, m-1\}$ and $\mathbf{w}_i \approx \mathbf{w}_{i+1}$ holds in $\mathbf{V}$. Suppose at first that $m - 3 < i < 2$. Since $m \geq 3$, we have $m = 3$ and $i = 1$. Then the identity $\mathbf{w}_1 \approx \mathbf{w}_2$ holds in $\mathbf{V}$, whence the identities $\mathbf{w}_0 \approx \mathbf{w}_1$ and $\mathbf{w}_2 \approx \mathbf{w}_3$ hold in $\mathbf{Z}$. Now Lemma 3.1 applies. So, we may assume that either $i \geq 2$ or $i \leq m - 3$. By symmetry, it suffices to consider the former case. So, let $i \geq 2$. Then the identity $\mathbf{w}_{i-1} \approx \mathbf{w}_i$ holds in $\mathbf{Z}$, while the identity $\mathbf{w}_{i-2} \approx \mathbf{w}_{i-1}$ holds in $\mathbf{V}$. Clearly, $\mathbf{w}_{i-1} \notin S$. Besides that, Lemma 3.7 allows us to assume that $\mathbf{w}_{i-1} \notin L$. Therefore, $\mathbf{w}_{i-1} \in Z$. Since $\mathbf{w}_{i-2} \approx \mathbf{w}_{i-1}$ holds in $\mathbf{V}$, this identity holds also in $\mathbf{N}$. Hence $\mathbf{w}_{i-2} \approx \mathbf{w}_{i-1} \approx 0$ hold in $\mathbf{N}$, that is $\mathbf{w}_{i-2} \in Z$. Let π be a unique non-trivial permutation on the set $\{x, y\}$. Lemma 3.2 implies that $\mathbf{w}_{i+1} = \pi[\mathbf{w}_i]$. Since the variety $\mathbf{Z}$ satisfies the identity $\mathbf{w}_{i-1} \approx \mathbf{w}_i$, it satisfies also the identity $\pi[\mathbf{w}_{i-1}] \approx \pi[\mathbf{w}_i]$, that is $\pi[\mathbf{w}_{i-1}] \approx \mathbf{w}_{i+1}$. Further, the variety $\mathbf{N}$ satisfies the identity $\pi[\mathbf{w}_{i-1}] \approx 0$ because it satisfies $\mathbf{w}_{i-1} \approx 0$. Thus, the identity $\mathbf{w}_{i-2} \approx \pi[\mathbf{w}_{i-1}]$ holds in $\mathbf{N}$ and therefore, in $\mathbf{V}$. We see that the sequence of words

$$\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_{i-2}, \pi[\mathbf{w}_{i-1}], \mathbf{w}_{i+1}, \ldots, \mathbf{w}_m$$

is a deduction of the identity $\mathbf{u} \approx \mathbf{v}$ from identities of the varieties $\mathbf{V}$ and $\mathbf{Z}$ shorter than (6).

We complete the proof of Theorem 1.1. □

Proof of Proposition 1.2. Put

$$\mathbf{V} = \mathrm{var}\{xyzt \approx xyx \approx x^2 \approx 0,\ x_1x_2x_3 \approx x_{1\rho}x_{2\rho}x_{3\rho}\}.$$

The variety $\mathbf{V}$ satisfies one of the identity systems (2)–(5). According to Theorem 1.1 $\mathbf{V}$ is a modular element in the lattice **SEM**. It remains to check that $\mathbf{V}$ is not a cancellable element of this lattice. Let σ and τ be non-trivial permutations from $\mathbb{S}_3$ such that the groups $\mathrm{gr}\{\rho\}$, $\mathrm{gr}\{\sigma\}$ and $\mathrm{gr}\{\tau\}$ are pairwise distinct. Put

$$\mathbf{X} = \mathrm{var}\{xyzt \approx xyx \approx x^2 \approx 0,\ x_1x_2x_3 \approx x_{1\sigma}x_{2\sigma}x_{3\sigma}\},$$

$$\mathbf{Y} = \mathrm{var}\{xyzt \approx xyx \approx x^2 \approx 0,\ x_1x_2x_3 \approx x_{1\tau}x_{2\tau}x_{3\tau}\}.$$

Clearly, $\mathbf{X} \neq \mathbf{Y}$. It is evident that $\mathrm{Perm}_3(\mathbf{V}) = \mathrm{gr}\{\rho\}$, $\mathrm{Perm}_3(\mathbf{X}) = \mathrm{gr}\{\sigma\}$ and $\mathrm{Perm}_3(\mathbf{Y}) = \mathrm{gr}\{\tau\}$. Lemma 2.8(ii) and Fig. 1 imply that

$$\mathrm{Perm}_3(\mathbf{V} \wedge \mathbf{X}) = \mathrm{Perm}_3(\mathbf{V}) \vee \mathrm{Perm}_3(\mathbf{X}) = \mathbb{S}_3.$$

This means that the variety $\mathbf{V} \wedge \mathbf{X}$ satisfies all permutational identities of length 3. Hence $\mathbf{V} \wedge \mathbf{X} \subseteq \mathbf{Y}$ and therefore, $\mathbf{V} \wedge \mathbf{X} \subseteq \mathbf{V} \wedge \mathbf{Y}$. The opposite inclusion is verified analogously, whence $\mathbf{V} \wedge \mathbf{X} = \mathbf{V} \wedge \mathbf{Y}$.

It remains to verify that $\mathbf{V} \vee \mathbf{X} = \mathbf{V} \vee \mathbf{Y}$. Suppose that an identity $\mathbf{u} \approx \mathbf{v}$ holds in the variety $\mathbf{V} \vee \mathbf{X}$. We are going to check that then this identity holds in $\mathbf{Y}$. We can assume that the identity $\mathbf{u} \approx \mathbf{v}$ is non-trivial because the required conclusion is evident in the contrary case. If each of the words $\mathbf{u}$ and $\mathbf{v}$ either is non-linear or has the length > 3 then $\mathbf{u} \approx 0 \approx \mathbf{v}$ hold in $\mathbf{Y}$. Let now $\mathbf{u}$ and $\mathbf{v}$ be linear words of length ≤ 3. If $\mathrm{con}(\mathbf{u}) \neq \mathrm{con}(\mathbf{v})$ then $\mathbf{u} \approx 0 \approx \mathbf{v}$ hold in $\mathbf{Y}$ again because $\mathbf{Y}$ is a nil-variety. Therefore, we can assume that $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v})$. In particular, this means that $\ell(\mathbf{u}) = \ell(\mathbf{v})$. Clearly, $\ell(\mathbf{u}) > 1$ because the identity $\mathbf{u} \approx \mathbf{v}$ is trivial otherwise. If $\ell(\mathbf{u}) = 2$ then $\mathbf{u} \approx \mathbf{v}$ is the commutative law. But then $\mathbf{u} \approx \mathbf{v}$ is false in $\mathbf{V}$, and moreover in $\mathbf{V} \vee \mathbf{X}$. Finally, suppose that $\ell(\mathbf{u}) = 3$. Then $\mathbf{u} \approx \mathbf{v}$ is a permutational identity of length 3. Lemma 2.8(i) and Fig. 1 imply that

$$\mathrm{Perm}_3(\mathbf{V} \vee \mathbf{X}) = \mathrm{Perm}_3(\mathbf{V}) \wedge \mathrm{Perm}_3(\mathbf{X}) = \mathbb{T}.$$

Therefore, the identity $\mathbf{u} \approx \mathbf{v}$ is trivial. We have a contradiction. Thus, $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{Y}$. Hence $\mathbf{V} \vee \mathbf{X} \supseteq \mathbf{Y}$ and therefore, $\mathbf{V} \vee \mathbf{X} \supseteq \mathbf{V} \vee \mathbf{Y}$. The opposite inclusion is verified analogously, whence $\mathbf{V} \vee \mathbf{X} = \mathbf{V} \vee \mathbf{Y}$ and we are done. □

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Dmitry Vyacheslavovich Skokov
Ural Federal University, Institute of Natural Sciences and Mathematics,
51, Lenina str.,
Ekaterinburg, 620000, Russia
E-mail address: dmitry.skokov@gmail.com

Boris Munevich Vernikov
Ural Federal University, Institute of Natural Sciences and Mathematics,
51, Lenina str.,
Ekaterinburg, 620000, Russia
E-mail address: bvernikov@gmail.com