LATTICES OF SEMIGROUP VARIETIES

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Dedicated to the 50th anniversary of the journal

Abstract. We survey a great number of results obtained during four decades of investigations on lattices of semigroup varieties and formulate several open problems.

CONTENTS

0. Introduction

0.1. Introductory remarks. The theory of semigroup varieties has been intensively studied for more than four and half decades, and a huge and very diverse material has been accumulated here. A systematic overview of this material was due already long ago. This has led the first author to the idea

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to write a series of surveys concerning the area. The first of them [94] is devoted to equational aspects of the theory of semigroup varieties[1]. The subject of the second article [93] was related to consideration of structural properties of semigroups in varieties such as local finiteness and residual finiteness, decompositions into bands and embeddings. The present article is the third in the planned series[2]. Unfortunately, the preparation of this article was significantly delayed, but the delay has created an opportunity to present here also achievements of the two last decades.

More than 200 papers devoted (completely or partially) to lattices of semigroup varieties have been published so far. In choosing material to be reviewed in the article we have been somewhat restricted by space limitation. The choice has been directed by our intention to present the main developments in the area under review and to outline the frontier between advance achieved so far and problems that still remain open. We reproduce some most essential facts about lattices of semigroup varieties from the earlier surveys [15] and [4] but an overwhelming majority of the content of the present article is based upon results obtained during last 30 years, that is, after the second of these surveys had appeared. The list of references includes only sources cited in the text, and as a rule we do not mention publications whose results were superseded by later papers.

In investigations on the theory of varieties, much attention has been paid to *unary semigroups*, that is, semigroups with an additional unary operation. Two most important types of unary semigroups are inverse semigroups with the operation of taking inverse element and completely regular semigroups with the operation of taking inverse element in a maximal subgroup. The class of all inverse semigroups and the class of all completely regular semigroups considered as algebras of type (2,1) form varieties that both include the variety of all groups as a subvariety. Varieties of inverse semigroups occupy a significant place in the monograph [68] where, in particular, issues related to their lattice are comprehensively treated. The monograph [73] pays much attention to varieties of completely regular semigroups, also from the viewpoint of varietal lattices. We notice that several pieces of information concerning varieties of inverse and completely regular semigroups were inserted in the texts of the surveys [94] and [93] that were mainly devoted to semigroup varieties in the plain semigroup setting. In the last years, one has begun to consider yet another type of unary semigroups, namely epigroups. Recall that an *epigroup* is a semigroup S such that some power of each element of S lies in some subgroup of S. An epigroup may be turned into

[1]We notice that the comprehensive introduction to [94] contains fairly detailed historical comments and gives a general picture of investigations on semigroup varieties up to the mid 1980s.

[2]It is appropriate to note here that yet another group of topics, namely algorithmic problems (not only for semigroup varieties but also for varieties of groups, associative or Lie algebras and, to a certain extent, for varieties of arbitrary universal algebras), has become the subject of the fundamental survey [35] written by students of the first author.

a unary semigroup in a natural way (see Subsection 2.1); unary completely regular semigroups turn out to be a particular type of epigroups. The idea to treat epigroups as unary semigroups was promoted in [90]. This approach allows one to pose various questions about epigroups also within the framework of the theory of varieties. In [90], the first author has presented some initial facts about epigroup varieties and formulated a number of questions concerning possible further developments in this direction; an essential part of this information has been reproduced in the survey [91], where also some advancements achieved during last few years have been reviewed. In this survey we touch upon a part of this information related to lattices of varieties. The survey includes also some important information about lattices of completely regular semigroup varieties that was left beyond the monograph [73] and a few facts about lattices of inverse semigroup varieties.

The table of contents clearly outlines the structure of the article. In general, one can say that the material under review deals mainly with the following three aspects: examining properties of the lattice of all semigroup varieties or of some of its important sublattices; characterizing varieties with given properties of their subvariety lattices; describing varieties that are, in a sense, special elements of the lattice of all semigroup varieties. When the goal of describing a lattice under consideration is set, in some cases it turns out to be possible to obtain a description in terms of explicit lattice-theoretical constructions and even to draw the corresponding diagram (such situations will appear in Sections 1 and 4); in some other cases a description is formulated in terms of a reduction to certain related lattices of some algebraic structures that appear to be "more transparent" and more convenient for further considerations (such situations will appear in Sections 5–7 and 12).

We notice that some principal results in the area under review (for instance, ones giving a complete classification of varieties with certain properties) have very long formulations. In such cases, in order to save space, we restrict ourself to presenting the essence of the corresponding result and refer the interested reader to the original source for details.

0.2. Terminology and notation. We suppose that the reader is acquainted with standard general-algebraic information used in the article as well as with textbook notions of the theory of semigroups, the theory of lattices and the theory of varieties of universal algebras. As the main reference sources, one can mention the monographs [13], [22], [55], and especially the handbooks [17] and [98]. Basically, we follow the terminology adopted in these[3]. We continue to follow the agreement adopted in [94] and [93] that an

[3]In this connection, one can note that in many publications (apparently, starting with the book [28]) and in particular in the previous surveys [94] and [93], completely regular semigroups, that is unions of groups, were called *Clifford* semigroups. At the same time, in a number of papers the term "Clifford semigroup" has begun to be used (less luckily, in our opinion) for a particular type of completely regular semigroups, namely semilattices of groups (see terminology commentaries on this occasion in Subsection 2.1 of the survey [91] or more detailed ones in Subsection I.2 of the monograph [92]). Here we have switched

adjective indicating a property shared by all semigroups of a given variety is applied to the variety itself; the expressions like "commutative variety", "periodic variety", "nilvariety", etc. are understood in this sense. In this connection, it is worth noticing that the terms "completely regular variety of semigroups" and "variety of completely regular semigroups" have different meanings: the former stands for a variety in the plain semigroup setting consisting of completely regular semigroups, the latter means that we speak about unary semigroups.

Let us recall definitions of some types of semigroup varieties and identities. A variety is called *proper* if it differs from the variety of all semigroups. An identity $u = v$ is called *balanced* if every letter occurs in the words u and v the same number of times. The length of each of the parts of a balanced identity is called the *length* of this identity. An identity of the form $x_1x_2\cdots x_n = x_{1\pi}x_{2\pi}\cdots x_{n\pi}$ where π is a non-trivial permutation on the set $\{1, 2, \ldots, n\}$ is called a *permutation* identity. A *permutational variety* is a variety satisfying some permutation identity. A pair of identities $wx = xw = w$ where the letter x does not occur in the word w is usually written as the symbolic identity $w = 0$. (This notation is justified because a semigroup with such identities has a zero element and all values of the word w in this semigroup are equal to zero.) An identity of the form $w = 0$ as well as a variety given by identities of such a form are called 0-*reduced*. A semigroup variety $\mathcal{V}$ is called a variety of *finite degree* if all nilsemigroups in $\mathcal{V}$ are nilpotent; $\mathcal{V}$ is called a variety of *degree* n if nilpotency degrees of nilsemigroups in $\mathcal{V}$ are bounded by the number n and n is the least number with this property[4]. A semigroup variety is called *finitely generated* if it is generated by a finite semigroup.

Let us recall now definitions of some types of lattices and their elements. One says that an element x of a partially ordered set $\langle S; \leq\rangle$ *covers* an element $y \in S$ if $y < x$ and there exists no element $z \in S$ with $y < z < x$. A lattice $\langle L; \vee, \wedge\rangle$ is called [*weakly*] *upper semimodular* if, for all $x, y \in L$, $x \vee y$ covers y whenever x covers $x \wedge y$ [respectively, x and y cover $x \wedge y$]. [*Weakly*] *lower semimodular* lattices are defined dually. A lattice is called *upper semidistributive* if it satisfies the quasiidentity $x\vee y = x\vee z \longrightarrow x\vee y = x \vee (y \wedge z)$. *Lower semidistributive* lattices are defined dually. An element x of a lattice L is called *neutral* if, for any $y, z \in L$ the elements x, y and z generate a distributive sublattice in L. An element x of a lattice L is

to the term "completely regular semigroup" taking into account that it has become more common.

[4]Note that in several papers, in particular, in the survey [93], varieties with such properties were called varieties of finite index and varieties of index n respectively. However this contradicts to the generally accepted use of the term "index" in a completely different sense in considerations of epigroups and in particular periodic semigroups (see Subsection 2.1). On the other hand, the term "degree of a variety" in the aforementioned sense is completely coordinated with the notion of the nilpotency degree of a semigroup. Altogether, these arguments explain our decision to choose the term "degree".

said to be *modular* if $(x \vee y) \wedge z = (x \wedge z) \vee y$ for all $y, z \in L$ with $y \leq z$, and *upper-modular* if $(z \wedge x) \vee y = (z \vee y) \wedge x$ for all $y, z \in L$ with $y \leq x$. *Lower-modular* elements are defined dually to upper-modular ones.

Let us recall also that a sublattice of a lattice is called an *ideal* [*coideal*] if this sublattice contains lower [upper] bounds of each of its elements. By $[a)$ we will denote the *principal coideal* of a given lattice L generated by an element $a \in L$; by the definition, $[a) = \{x \in L \mid x \geq a\}$.

The semigroup variety given by an identity system Σ is denoted by var Σ. Let us list several concrete semigroup varieties that will appear in the sequel many times: the variety of all semigroups $\mathcal{SEM}$, the variety of all commutative semigroups $\mathcal{COM} = \mathrm{var}\,\{xy = yx\}$, the variety of all abelian groups whose exponent divides n $\mathcal{A}_n = \mathrm{var}\,\{x^n y = y,\, xy = yx\}$, the variety of all semilattices $\mathcal{SL} = \mathrm{var}\,\{x^2 = x,\, xy = yx\}$, the variety of all left zero semigroups $\mathcal{LZ} = \mathrm{var}\,\{xy = x\}$, the variety of all right zero semigroups $\mathcal{RZ} = \mathrm{var}\,\{xy = y\}$, the variety of all null semigroups $\mathcal{ZM} = \mathrm{var}\,\{xy = 0\}$, the trivial variety $\mathcal{T}$.

We denote by $L(\mathcal{K})$ the lattice of all varieties contained in the class of algebraic systems $\mathcal{K}$. The lattice $L(\mathcal{SEM})$ will be denoted by **SEM** throughout.

CHAPTER I. The first layer of information

1. The lattice of all semigroup varieties

The lattice **SEM** possesses all textbook properties of the subvariety lattices of varieties of universal algebras: it is complete, atomic and coalgebraic, and its cocompact elements are precisely finitely based varieties. In 1955 Kalicki and Scott [34] described the atoms of the lattice **SEM**: these are precisely the varieties $\mathcal{A}_p$ for all prime p, $\mathcal{LZ}$, $\mathcal{RZ}$, $\mathcal{SL}$, $\mathcal{ZM}$. The varieties $\mathcal{SL}$ and $\mathcal{ZM}$ have yet another remarkable property: they are neutral elements of the lattice **SEM** (see Theorem 14.2). Results of [2] show that if semigroup varieties $\mathcal{X}$ and $\mathcal{Y}$ do not contain an atom $\mathcal{A}$ of the lattice **SEM**, then also $\mathcal{X} \vee \mathcal{Y} \not\supseteq \mathcal{A}$. This implies that the lattice **SEM** is 0-*distributive*, that is, satisfies the following implication: $x \wedge z = y \wedge z = 0 \longrightarrow (x \vee y) \wedge z = 0$. There are no coatoms in the lattice **SEM** (see Theorem 3.4), moreover, it is not difficult to verify that each of its nontrivial coideals is uncountable.

If $\mathcal{V}$ is a semigroup variety, then we denote by $\overleftarrow{\mathcal{V}}$ the variety consisting of semigroups dual (that is, antiisomorphic) to semigroups from $\mathcal{V}$. It is evident that the mapping δ of the lattice **SEM** into itself given by the rule $\delta(\mathcal{V}) = \overleftarrow{\mathcal{V}}$ for every variety $\mathcal{V}$ is an automorphism of the lattice **SEM**. The following question is still open.

Question 1.1. *Are there non-trivial automorphisms of the lattice* **SEM** *different from* δ?

We notice that there exist infinitely many non-trivial injective endomorphisms of the lattice **SEM**. Namely, let $\mathcal{V} = \mathrm{var}\,\{u_i = v_i \mid i \in I\}$, let m and n be positive integers and $x_1, \ldots, x_m, y_1, \ldots, y_n$ letters that do not occur in the words u_i and v_i. Put

$$\mathcal{V}^{m,n} = \mathrm{var}\,\{x_1 \cdots x_m u_i y_1 \cdots y_n = x_1 \cdots x_m v_i y_1 \cdots y_n \mid i \in I\}.$$

It is verified by Kopamu [41] that the variety $\mathcal{V}^{m,n}$ does not depend on the choice of an identity basis of the variety $\mathcal{V}$ and the mapping $\mathcal{V} \longmapsto \mathcal{V}^{m,n}$ is an injective endomorphism of the lattice **SEM**. We will meet these endomorphisms again in Subsection 10.1. We call them the *Kopamu endomorphisms.*

The structure of the lattice **SEM** as a whole is extremely complex. It is sufficient to mention that it contains an interval antiisomorphic to the partition lattice of a countably infinite set [10, 29]. In view of known properties of partition lattices, this implies that the subvariety lattice of an arbitrary variety of universal algebras of at most countably infinite type is embeddable in **SEM**, that **SEM** does not satisfy any non-trivial quasiidentity, and that **SEM** is uncountable. One of a few "positive" properties of this lattice is the covering property (see Theorem 3.1).

Main sublattices of the lattice **SEM** and their mutual location are shown in Fig. 1. First of all, **SEM** is divided into two large sublattices with essentially different properties: the coideal **OC** of all *overcommutative* varieties (that is, varieties containing $\mathcal{COM}$) and the ideal **Per** of all periodic varieties.

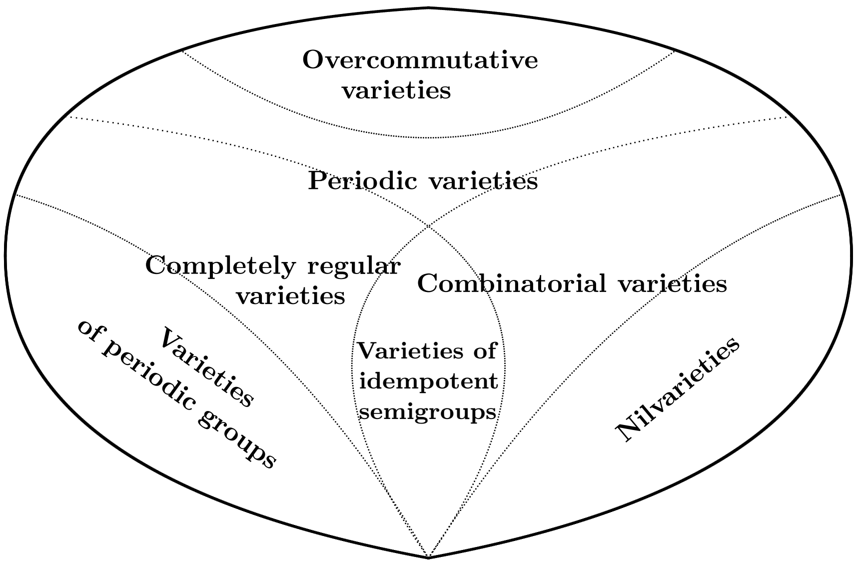

FIGURE 1. "The map" of the lattice of all semigroup varieties

The lattice **OC** admits a relatively easy and concise description in terms of congruences of unary algebras of some special type (see Subsection 5.1). Thus the most complex part of the lattice **SEM** is the lattice **Per**.

There are two large ideals with very different properties in the lattice **Per**: the ideal **CR** of all periodic completely regular varieties and the ideal **Comb** of all *combinatorial* varieties (that is varieties all of whose groups are singleton). Their intersection is the lattice **I** of all idempotent semigroups that was completely described independently by A. P. Birjukov [8], Fennemore [16] and Gerhard [18]. Another proof of this result was published by Gerhard and Petrich [20]. The lattice **I** is countably infinite and distributive. It is shown in Fig. 2.

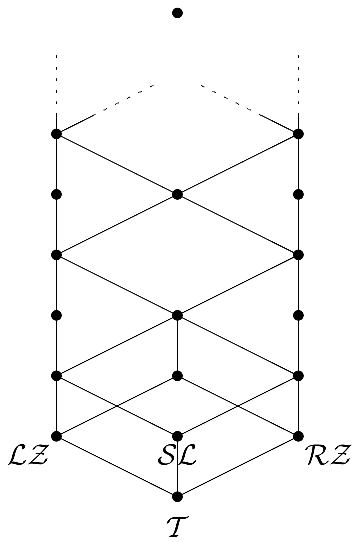

FIGURE 2. The lattice of varieties of idempotent semigroups

Speaking about the lattice **CR**, one should mention that it may be considered as a sublattice of the lattice of varieties of unary completely regular semigroups (see Section 6). The latter lattice was the subject of intensive studies in the 1980s and its structure is quite well investigated by now. In Section 6, we present key results concerning this lattice.

Essentially less information is known about the structure of the lattice **Comb**. The aforementioned results of [10, 29] deal in fact with this lattice, whence we can claim that the lattice **Comb** is as complex as the whole lattice **SEM**. We may say the same about the ideal **Nil** of the lattice **Comb** consisting of all nilvarieties since the results of [29] deal with this ideal. On the other hand, the lattice **Nil**, as well as the lattice **OC**, admits some characterization in terms of congruences of unary algebras of some special type (see Section 7).

All the aforementioned types of semigroup varieties admit a characterization in the language of atoms of the lattice **SEM** (see Table 1).

2. Varieties of epigroups

2.1. General remarks. For an element a of a given epigroup, we denote by e_a the unit element of the maximal subgroup G that contains some power

A semigroup variety is	if and only if it
an overcommutative variety	contains $\mathcal{A}_p$ for all prime p
a periodic variety	does not contain $\mathcal{A}_p$ for some prime p
a completely regular variety	does not contain $\mathcal{ZM}$
a group variety	does not contain $\mathcal{LZ}, \mathcal{RZ}, \mathcal{SL}, \mathcal{ZM}$
a combinatorial variety	does not contain $\mathcal{A}_p$ for all prime p
a variety of idempotent semigroups	does not contain $\mathcal{ZM}$ and $\mathcal{A}_p$ for all prime p
a nilvariety	does not contain $\mathcal{LZ}, \mathcal{RZ}, \mathcal{SL}$ and $\mathcal{A}_p$ for all prime p

Table 1. A characterization of some types of varieties

of a. It is known that $ae_a = e_a a$ and $ae_a \in G$. We denote by $\overline{a}$ the element inverse to ae_a in G. The element $\overline{a}$ is called the *pseudo-inverse* of a, and the mapping $a \longmapsto \overline{a}$ turns an epigroup to a unary semigroup. An epigroup has *index* n if the nth power of every its element lies in some of its subgroups and n is the least number with this property. We denote by $\mathcal{E}$ the class of all epigroups and by $\mathcal{E}_n$ the class of all epigroups of index $\leq n$. We see that $\mathcal{E}_1$ is just the class of all completely regular semigroups. Note that in the realm of completely regular semigroups it is usual to denote the pseudo-inverse element for an element a by a^{-1}.

Every periodic semigroup is an epigroup and every periodic semigroup variety may be considered as a variety of epigroups with the operation of pseudo-inversion in the signature. Indeed, each periodic variety satisfies an identity of the form $x^n = x^{n+d}$, and the operation of pseudo-invertion is definable in the semigroup signature in this case: it is easy to see that $\overline{x} = x^{nd-1}$. Therefore, the lattice **Per** is naturally embedded into $L(\mathcal{E})$; speaking slightly informally, we will assume that **Per** is a sublattice of the lattice $L(\mathcal{E})$. Thus results about periodic semigroup varieties, as well as results about varieties of completely regular semigroups may be interpreted as results about epigroup varieties too. So, when we examine epigroup varieties *per se*, it is natural to assume that we consider questions not within the classes of periodic or completely regular semigroups.

Results about epigroup varieties that are known so far mainly concern with equational and structural aspects (see corresponding results in [90] and [91]). As to considerations of the varietal lattices, here only the first steps are made and we hope that the main successes here will be obtained in the future. For each n, the class $\mathcal{E}_n$ is a variety; it is defined by the identities

$$(xy)z = x(yz),\ \ x\overline{x} = \overline{x}x,\ \ x\overline{x}^2 = \overline{x},\ \ x^{n+1}\overline{x} = x^n.$$

The class $\mathcal{E}$ is not a variety, so $L(\mathcal{E})$ is a lattice without the greatest element. The chain $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_n \subset \cdots$ in the lattice $L(\mathcal{E})$ may be considered (non-formally speaking) as a "spine" of this lattice because for every epigroup variety $\mathcal{V}$ there is a number n with $\mathcal{V} \subseteq \mathcal{E}_n$. It is easy to see that the lattice $L(\mathcal{E})$ has the same atoms as **SEM**, and for any n the principal coideal $[\mathcal{E}_n)$ of the lattice $L(\mathcal{E})$ has a unique atom: the join of $\mathcal{E}_n$ and the variety generated by the cyclic nilpotent semigroup with $n+1$ elements; this atom is contained in every variety of $[\mathcal{E}_n)$ different from $\mathcal{E}_n$. Two results (and one open problem) concerning the covering property for the lattice $L(\mathcal{E})$ and the lattices $L(\mathcal{E}_n)$ are presented in Subsection 3.1.

Considerations about further possible study of epigroup varieties have been presented in [90] (in Subsection 4° from Section 1); most of these considerations have been repeated in the survey [91] (Subsection 2.3), a part of them concerns lattices of varieties. We do not repeat here these considerations and refer the interested reader to the above sources. We formulate only two of the questions mentioned in [90] and [91]; the first of them is formulated there as a conjecture.

Question 2.1. *Are there coatoms in the lattice $L(\mathcal{E}_n)$ for arbitrary n?*

Question 2.2. a) *What are properties of the intervals $[\mathcal{E}_n, \mathcal{E}_{n+1}]$ in the lattice $L(\mathcal{E})$ (for instance, what are the ordinals of maximal chains and the cardinalities of maximal antichains in these intervals)?* b) *What are interactions between these intervals? In particular, are the lattices $[\mathcal{E}_m, \mathcal{E}_{m+1}]$ and $[\mathcal{E}_n, \mathcal{E}_{n+1}]$ non-isomorphic whenever $m \neq n$, and is the lattice $[\mathcal{E}_m, \mathcal{E}_{m+1}]$ embeddable into $[\mathcal{E}_n, \mathcal{E}_{n+1}]$ whenever $m \leq n$?*

With respect to Question 2.2a, we note that it may be easily verified that the interval $[\mathcal{E}_1, \mathcal{E}_2]$ contains a chain isomorphic to the chain of real numbers with the usual order.

2.2. Complete congruences on lattices $L(\mathcal{E}_n)$. One of standard methods of analyzing varietal lattices is a study of their complete congruences. Each such congruence partitions the corresponding lattice into intervals whose structure may be more transparent than the structure of the whole lattice, and the greatest and the least elements of these intervals often have interesting properties and may serve as "basis points" for further investigations. As a model example, we mention here the complete congruence on the lattice **Inv** of all varieties of inverse semigroups introduced by E. I. Kleiman [38]; the classes of this congruence consist of all inverse semigroup varieties with the fixed join with the variety of all groups. It is shown by Reilly [83] that each class of this congruence is modular while the lattice **Inv** itself is non-modular. This approach was successfully applied in the 1980s to the lattice of varieties of completely regular semigroups, and later to lattices of different classes of finite or regular semigroups related to varieties (pseudovarieties, e-varieties).

It has been shown in the interesting paper by Pastijn [64] that a natural generality for many constructions applied earlier in the finite and regular cases can be reached in the lattice $L(\mathcal{E}_n)$. In particular, the following relations τ and γ considered earlier for lattices of e-varieties in [7] and for lattices of varieties of completely regular semigroups in [71, 72, 76], turn out to be complete congruences on $L(\mathcal{E}_n)$: $\mathcal{V}\,\tau\,\mathcal{W}$ if and only if the varieties $\mathcal{V}$ and $\mathcal{W}$ contain the same fundamental epigroups (an epigroup is called *fundamental* if the restriction of any of its non-trivial congruences to the set of all idempotents is non-trivial); $\mathcal{V}\,\gamma\,\mathcal{W}$ if and only if the varieties $\mathcal{V}$ and $\mathcal{W}$ contain the same idempotent generated semigroups. The complete congruence $\iota = \tau \vee \gamma$ on $L(\mathcal{E}_n)$ also is considered in [64]. It is proved there that $\mathcal{V}\,\iota\,\mathcal{W}$ if and only if the varieties $\mathcal{V}$ and $\mathcal{W}$ contain the same fundamental epigroups and the same idempotent generated semigroups, and that ι coincides with the kernel of a complete homomorphism of the lattice $L(\mathcal{E}_n)$ onto the lattice of all varieties of so-called idempotent algebras of epigroups from $\mathcal{E}_n$.

The general theory of complete congruences on lattices of varieties has been developed by Pastijn and Trotter [65] who have also constructed some further complete congruences on $L(\mathcal{E}_n)$.

3. The cover relation

Studying the cover relation in varietal lattices had attracted considerable attention on the early stage of development of the theory of varieties. It appears that this attention was due to anticipations that the structure of lattices of varieties may be revealed by moving "upward": from the trivial variety to its covers, that is atoms, from the atoms to their covers, etc. Although this hope with respect to "big" varietal lattices such as **SEM** has turned out to be somewhat naive, investigations of the cover relation in **SEM** and related varietal lattices have brought a number of interesting results.

3.1. The covering property. General properties of coalgebraic lattices imply that every proper subvariety in $\mathcal{SEM}$ defined by a finite number of identities has a cover. But there are subvarieties of $\mathcal{SEM}$ that can not be defined by a finite number of identities. The question of whether or not every proper subvariety of $\mathcal{SEM}$ has a cover was posed in the survey [15]; the affirmative answer follows from a more general result by A. N. Trahtman [99]. To formulate and discuss this result we need the following definition. We say that a lattice has the *covering property* if every its element different from the greatest element has a cover.

Theorem 3.1 (A. N. Trahtman [99]). *The subvariety lattice of an overcommutative semigroup variety has the covering property.*

A simple proof of Theorem 3.1 is given in [128], see also [136].

Recall that a lattice is called *strongly atomic* if every its non-singleton interval contains an atom. Theorem 3.1 implies

Corollary 3.1. *The lattice* **OC** *is strongly atomic.*

It was natural to ask whether or not the subvariety lattice of every semigroup variety has the covering property. A. N. Trahtman [100] has answered this question in the negative; Trahtman's counter-example is the variety var $\{xy^4x = xy^5x\}$. Later Pollák [78] has shown that the covering property fails also in the subvariety lattice of the Burnside variety $\mathcal{B}_{m,n} = \mathrm{var}\,\{x^m = x^{m+n}\}$ with any $m > 1$. For the sake of completeness, we note that the subvariety lattice of the variety $\mathcal{B}_{1,1}$, that is the variety of all idempotent semigroups has the covering property (see Fig. 2), while the question of whether or not the lattice $L(\mathcal{B}_{1,n})$ with $n > 1$ has this property is still open. M. V. Sapir [87] has shown that the property we discuss may fail even in the subvariety lattice of a finitely generated variety.

Quite a mixed picture arises when the covering property is studied for lattices of varieties of semigroups equipped by various additional operations. It turns out that this property fails in the lattice of monoid varieties [78] and in the lattice of varieties of inverse semigroups [39]. Covers in the lattice of epigroup varieties have been investigated in [136] where the following results have been obtained.

Theorem 3.2 (M. V. Volkov [136]). *The lattice $L(\mathcal{E})$ has the covering property, while the lattices $L(\mathcal{E}_n)$ with $n > 1$ do not have it.*

The following question was posed by the first author 30 years ago and still remains open.

Question 3.1 ([97], Problem 2.62b). *Does the lattice $L(\mathcal{E}_1)$ of all varieties of completely regular semigroups possess the covering property*?

Note that every periodic variety of completely regular semigroups has a cover in $L(\mathcal{E}_1)$. But it is easy to see that this does not imply an affirmative answer to the question of whether or not the covering property holds in the lattice $L(\mathcal{B}_{1,n})$.

3.2. The number of covers. Since every proper semigroup variety has a cover, the question about the number of covers arises naturally. The problem of description of semigroup varieties with a finite [countably infinite, uncountable] set of covers was noted in the survey [4]. For overcommutative varieties, it was solved long ago by the following result.

Theorem 3.3 (A. Ya. Aĭzenštat [1]). *A proper overcommutative semigroup variety has a finite* [*countably infinite*] *set of covers whenever it has a finite* [*infinite*] *identity basis.*

In the periodic case, the question about the number of covers turns out to be much more complex. Since every periodic variety contains only a finite number of atoms, one may conjecture that every periodic variety would have infinitely many covers. However it turns out (and this is one of interesting manifestations of non-modularity of the lattice **SEM**) that there exist periodic varieties with a finite number of covers. One may verify using the results of the article [129] that this property holds for an arbitrary non 0-reduced nilvariety that is not contained in the variety $\mathrm{var}\{x^2y = xyx = yx^2 = 0\}$. It was noted in the survey [4] that a finite number of covers occurs also for an arbitrary semigroup variety given by one identity $u = v$ such that the words u and v depend on the same letters and either lengths of these words are equal or v does not coincide with a word of the kind $a\xi(u)b$ where a and b are (may be empty) words and ξ is an endomorphism of the free semigroup. Some cases where the number of covers of a periodic variety is infinite are indicated in [9]. The first example of a variety with uncountably many covers in **SEM** has been found by A. N. Trahtman [99]. Recent results by P. A. Kozhevnikov [44, 45] imply that even the atoms $\mathcal{A}_p$ with sufficiently large prime p have uncountably many covers (see Theorem 10.2).

3.3. Other results. As known, if a variety $\mathcal{X}$ has an independent basis within some variety $\mathcal{V}$ that strongly contains $\mathcal{X}$ then $\mathcal{X}$ has a cover in the lattice $L(\mathcal{V})$. A. N. Trahtman has formulated the question of whether or not every independently based semigroup variety $\mathcal{X}$ has a cover in the subvariety lattice of any variety that strongly contains $\mathcal{X}$ ([97], Problem 2.55a). A negative answer to this question has been obtained by V. Yu. Popov [80].

Let $\langle S; \leq \rangle$ be a partially ordered set and $x, y \in S$. Generalizing the notion of covering of one element by another one, we say that the *distance between x and y is finite* if either $x = y$ or $x < y$ and there exist elements $z_0, z_1, \ldots, z_n \in S$ such that $z_0 = x$, $z_n = y$ and for each $i = 0, 1, \ldots, n-1$ z_{i+1} covers z_i or $y < x$ and the condition dual to the previous one holds. Otherwise we say that the distance between x and y is infinite. Further, we say that the *distance between x and y equals ω* if the distance between x and y is infinite and either $x < y$ and for every $z \in S$ with $x \leq z < y$ the distance between x and z is finite or the dual condition holds. The following is true.

Theorem 3.4 (M. V. Volkov [127])**.** *If $\mathcal{V}$ is a proper semigroup variety, then there exists a semigroup variety $\mathcal{W}$ such that $\mathcal{V} \subseteq \mathcal{W}$ and the distance between $\mathcal{V}$ and $\mathcal{W}$ equals ω.*

4. Subvariety lattices of certain varieties

There are many papers in which the subvariety lattice of some concrete semigroup variety is described. We do not aim to list all these papers and survey here only such that appear to deserve attention by some reason.

As we have already noted in Section 1, the lattice $\mathbf{I}$ of all varieties of idempotent semigroups has been described in [8, 16, 18] (see Fig. 2). Later the following description of the lattice of all varieties of semigroups with idempotent square, that is the subvariety lattice of the variety $\mathrm{var}\,\{xy = (xy)^2\}$, has been obtained.

Theorem 4.1 (Gerhard [19])**.** *The lattice $L(\mathrm{var}\,\{xy = (xy)^2\})$ is a subdirect product of the lattice $\mathbf{I}$ and the lattice shown in Fig.* 3.

In particular, the lattice $L(\mathrm{var}\,\{xy = (xy)^2\})$ is distributive.

The description of the lattice of all varieties of semigroups whose square is a rectangular semigroup, that is satisfies the identity $xyx = x$, was obtained earlier by I. I. Mel'nik [58]. This lattice is shown in Fig. 4, where $\mathcal{LZM}$ and

$\mathcal{RZM}$ denote the varieties var $\{xyz = xy\}$ and var $\{xyz = yz\}$ respectively. This result served as a starting-point for several papers whose authors considered weaker restrictions to a variety than those in [58]. So, Petrich in [67] has described the lattice of all varieties of semigroups whose square is an orthodox normal band of groups of exponent dividing a fixed number n. It has turned out that this lattice is the direct product of the lattice shown in Fig. 4, the lattice of all varieties of periodic groups of exponent dividing n, and the 2-element chain. The lattice of varieties of semigroups whose cube is a rectangular semigroup has been studied in [40, 60]. A complete description of this lattice has not been obtained but a wide sublattice has been determined in [40] and the subvariety lattice of the variety var $\{xyz = xywyz\}$ has been described in [60].

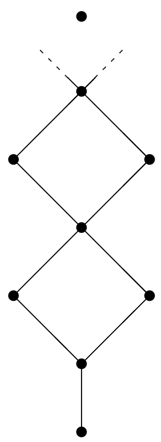

FIGURE 3

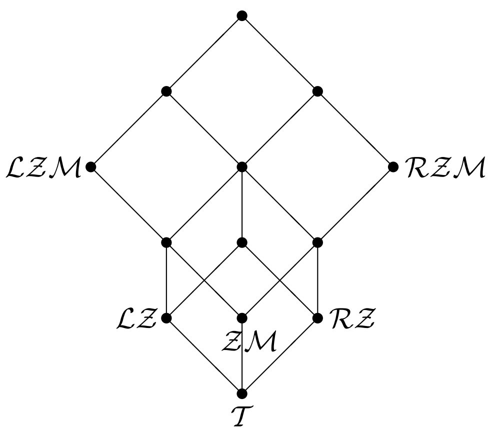

FIGURE 4. The lattice of varieties of semigroups whose square is a rectangular semigroup

The article by Petrich [69] contains a description of the subvariety lattice of the variety $\mathcal{D}_p^1$ generated by the semigroup D_p^1 where D_p is the Rees matrix semigroup over the cyclic group of prime order p with the sandwich matrix $\left(\begin{smallmatrix} e & e \\ e & a \end{smallmatrix}\right)$ where a is a non-identity element of the group while e is its identity element. It turns out that this lattice is distributive and consists of 32 elements; its diagram is shown in [69].

The variety $\mathcal{D}_p^1$ is interesting by the following reason. When one studies finitely based semigroup varieties, it is important to find examples of *limit* (that is, minimal non finitely based) varieties. Only a few explicit examples of such varieties are known so far (see [79, 87, 126] and [94, 139]), and there are no completely regular varieties among them. But completely regular limit varieties exist (this follows from the existence of completely regular non finitely based varieties and Zorn's Lemma). The question of whether or not the variety $\mathcal{D}_p^1$ is finitely based was formulated in the survey [94] (Question 8.1) and is still open. Results of [69] imply that all proper subvarieties of $\mathcal{D}_p^1$ are finitely based. Thus if the answer to the question will be negative then the variety $\mathcal{D}_p^1$ will become the first explicit example of a completely regular limit semigroup variety.

The first attempt to find a boundary between modularity and non-modularity, as well as between distributivity and non-distributivity, in lattices of nilvarieties was made by I. I. Mel'nik [59]. In particular, the lattice of varieties of 5-nilpotent commutative varieties was completely determined there. The diagram of this lattice is shown in [59], it contains 32 elements.

CHAPTER II. The main sublattices of the lattice of semigroup varieties

As we have already noted in Section 1, the lattice **SEM** is partitioned into the sublattices **OC** and **Per**. The lattice **OC** is considered in Section 5. It is quite difficult to study the lattice **Per** as a whole because some parts of this lattice have extremely different properties. First of all, this concerns the lattices **CR** and **Nil**. The first of them is modular, while the second one does not satisfy any non-trivial lattice identity. Interactions between these two lattices also are quite complex. So, even in the case when the variety $\mathcal{V}$ has the form $\mathcal{K} \vee \mathcal{N}$ where $\mathcal{K} \in$ **CR** and $\mathcal{N} \in$ **Nil**, the structure of the lattice $L(\mathcal{V})$, as a rule, is not determined by the structure of the lattices $L(\mathcal{K})$ and $L(\mathcal{N})$. We note that the rare situations when $L(\mathcal{K} \vee \mathcal{N}) \cong L(\mathcal{K}) \times L(\mathcal{N})$ have been completely determined (see Proposition 13.1).

Nevertheless, some information about the whole lattice **Per** can be obtained (see Subsection 2.2). The structure of the lattices **CR** and **Nil** is considered in Sections 6 and 7 respectively. Sections 8 and 9 deal with two further wide sublattices of the lattice **SEM**. The first of them consists of commutative varieties, while the second one consists of varieties that are contained in periodic varieties generated by 0-simple semigroups; the latter varieties are called *Rees–Sushkevich varieties.*

5. Overcommutative varieties

5.1. The structure of the lattice OC. Here and in Section 7, we need the notion of a G-set. Recall that a unary algebra with the carrier A and the set of operations G is called a *G-set* if G is equipped with a structure of a group and, for any $g, h \in G$ and $a \in A$ the equalities $g\bigl(h(a)\bigr) = (gh)(a)$ and $e(a) = a$ hold where e is the identity element of the group G. Some preliminary information about G-sets and in particular about their congruences may be found, for instance, in the monograph [57].

It is proved in [135] that the lattice **OC** is decomposable into a subdirect product of certain intervals and each of these intervals is antiisomorphic to the congruence lattice of a certain G-set. To give the exact formulation of this result, we need some notation.

We will consider semigroup words over a countably infinite alphabet $\{x_1, x_2, \dots, x_n, \dots\}$. For any n put $X_n = \{x_1, x_2, \dots, x_n\}$. If u is a word, then $\ell(u)$ denotes its length, $\ell_i(u)$ is a number of occurrences of the letter x_i in

u and $c(u)$ stands for the set of all letters occurring in u. By $\mathbb{S}_n$ we denote the symmetric group on the set $\{1,2,\ldots,n\}$. If $c(u)=X_n$ and $\pi\in\mathbb{S}_n$, then we denote by $u\pi$ the word obtained from u by changing of x_i to $x_{i\pi}$ for all $i=1,2,\ldots,n$. Let m and n be integers with $2\le m\le n$. A *partition of a number n into m parts* is a sequence of positive integers $(\lambda_1,\lambda_2,\ldots,\lambda_m)$ such that $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_m$ and $\sum\limits_{i=1}^{m}\lambda_i=n$. We denote by $\Lambda_{n,m}$ the set of all partitions of the number n into m parts and by Λ the union of sets $\Lambda_{n,m}$ for all natural numbers m and n with $2\le m\le n$. Let u be a word such that $c(u)=X_m$ and $\ell_i(u)\ge\ell_{i+1}(u)$ for all $i=1,2,\ldots,m-1$. The partition $\bigl(\ell_1(u),\ell_2(u),\ldots,\ell_m(u)\bigr)$ of the number $\ell(u)$ into m parts is denoted by $\mathrm{part}(u)$.

Now let us fix integers m and n with $2\le m\le n$ and a partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_m)\in\Lambda_{n,m}$. We denote by W_λ the set of all words u such that $\ell(u)=n$, $c(u)=X_m$, $\ell_i(u)\ge\ell_{i+1}(u)$ for all $i=1,2,\ldots,m-1$ and $\mathrm{part}(u)=\lambda$, and by $\mathbb{S}_\lambda$ the set of all permutations $\sigma\in\mathbb{S}_m$ such that $\lambda_i=\lambda_{i\sigma}$ for all $i=1,2,\ldots,m$. It is clear that $\mathbb{S}_\lambda$ is a subgroup in $\mathbb{S}_m$. For a permutation $\sigma\in\mathbb{S}_\lambda$, let us define the unary operation σ^* on the set W_λ by the rule: $\sigma^*(u)=u\sigma$ for any $u\in W_\lambda$. It is clear that the set W_λ with the collection of operations $\{\sigma^*\mid\sigma\in\mathbb{S}_\lambda\}$ is an $\mathbb{S}_\lambda$-set. We denote by $\mathcal{S}_n$ the variety defined by all balanced identities of length $\ge n$ and by $\mathcal{S}_{n,m}$ the subvariety of the variety $\mathcal{S}_{n+1}$ defined within $\mathcal{S}_{n+1}$ by all balanced identities of length n depending on $\le m$ letters. Put also $\mathcal{S}_{n,1}=\mathcal{S}_{n+1}$. By $\mathcal{S}_\lambda$ we denote the subvariety of the variety $\mathcal{S}_{n,m-1}$ defined within $\mathcal{S}_{n,m-1}$ by all balanced identities of the form $u=v$ with $u\in W_\lambda$. The interval $[\mathcal{S}_\lambda,\,\mathcal{S}_{n,m-1}]$ of the lattice $\mathbf{OC}$ is denoted by I_λ.

Theorem 5.1 (M. V. Volkov [135]). *The lattice* $\mathbf{OC}$ *is a subdirect product of intervals of the kind I_λ where λ runs over Λ, while an interval I_λ is antiisomorphic to the congruence lattice of the $\mathbb{S}_\lambda$-set W_λ.*

This theorem shows that, for further studies of the lattice $\mathbf{OC}$, it is useful to understand the structure of congruence lattices of G-sets. As we have already noted at the beginning of the section, some information about this is contained in [57] (see Lemma 4.20 there). In more detail these lattices have been investigated in [104]. This paper contains also a characterization of G-sets with a number of (lattice or multiplicative) properties of their congruences, while special elements of several types in congruence lattices of G-sets have been considered in [106, 107].

Theorem 5.1 readily implies

Corollary 5.1 ([135]). *The lattice* $\mathbf{OC}$ *is residually small.*

Also, it is shown in [135] that, for an arbitrary overcommutative variety $\mathcal{V}$, the interval $[\mathcal{COM},\,\mathcal{V}]$ of the lattice $L(\mathcal{V})$ is similar to the lattice $\mathbf{OC}$, that is, it is decomposable into a subdirect product of certain intervals and each of these intervals is antiisomorphic to the congruence lattice of a certain G-set.

5.2. Identities and related conditions. The description of intervals of the form $[\mathcal{COM},\,\mathcal{V}]$ mentioned at the end of the previous subsection, permits to classify overcommutative varieties $\mathcal{V}$ such that these intervals are modular or distributive (see [105]). In fact, an essentially stronger result has been obtained in course of this classification. To formulate this result, let us denote by M_k the lattice consisting of zero, identity and k atoms, and by $M_{k,n}$ the lattice shown in Fig. 5 (here $k, n \geq 3$). We say that two lattices are *quasiequationally equivalent* if they satisfy the same quasiidentities.

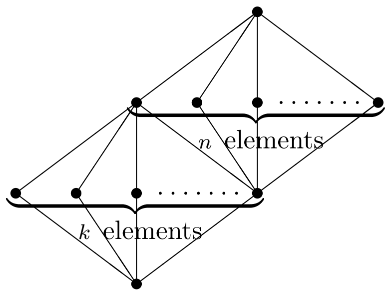

FIGURE 5. The lattice $M_{k,n}$

Proposition 5.1. *Let $\mathcal{V}$ be an overcommutative semigroup variety different from $\mathcal{COM}$. If the interval $[\mathcal{COM},\,\mathcal{V}]$ is modular then it is quasiequationally equivalent to one of the following lattices*: 1) *the two-element chain*; 2) M_3; 3) M_4; 4) $M_{4,3}$.

Since the lattice $M_{4,3}$ is arguesian, this proposition means, in particular, that the arguesian law is equivalent to the modular one in lattices of overcommutative varieties. For each of the four cases mentioned in Proposition 5.1, corresponding overcommutative varieties have been completely determined in [105].

For intervals of the kind $[\mathcal{COM},\,\mathcal{V}]$, modularity, upper semimodularity and weak upper semimodularity are equivalent, while lower semimodularity and weak lower semimodularity are equivalent to each other but not equivalent to modularity. These results and a classification of varieties $\mathcal{V}$ such that the interval $[\mathcal{COM},\,\mathcal{V}]$ is lower semimodular also have been obtained in [105].

The lattice M_3 is subdirectly indecomposable, whence there is the largest quasivariety of lattices that does not contain M_3. We denote this quasivariety by $\overline{\mathsf{M}}_3$. The following is true.

Theorem 5.2 (B. M. Vernikov [108])**.** *For an overcommutative semigroup variety $\mathcal{V}$, the following are equivalent*: (a) $[\mathcal{COM},\,\mathcal{V}] \in \overline{\mathsf{M}}_3$; (b) *the interval $[\mathcal{COM},\,\mathcal{V}]$ is upper semidistributive*; (c) *the interval $[\mathcal{COM},\,\mathcal{V}]$ is lower semidistributive*; (d) *the interval $[\mathcal{COM},\,\mathcal{V}]$ is distributive.*

Together with the mentioned results of [105], this theorem gives a complete description of overcommutative varieties $\mathcal{V}$ such that the interval $[\mathcal{COM},\,\mathcal{V}]$

is (upper or lower) semidistributive or belongs to an arbitrary quasivariety of lattices that does not contain the lattice M_3.

5.3. Special elements. An element x of a lattice L is called *distributive* if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $y, z \in L$. A *codistributive* element is defined dually. Every neutral element is distributive and codistributive but the converse implications are false in abstract lattices. Nevertheless, the following is true.

Theorem 5.3 (B. M. Vernikov [107])**.** *For an overcommutative semigroup variety $\mathcal{V}$, the following are equivalent*: (a) *$\mathcal{V}$ is a distributive element of the lattice* **OC**; (b) *$\mathcal{V}$ is a codistributive element of* **OC**; (c) *$\mathcal{V}$ is a neutral element of* **OC**; (d) *$\mathcal{V}$ is one of the varieties $\mathcal{SEM}$, $\mathcal{S}_n$, $\mathcal{S}_{n,m}$ or $\mathcal{S}_\lambda$, where m and n are arbitrary integers with $2 \le m \le n$, while $\lambda \in \Lambda$.*

The following problem seems to be interesting.

Problem 5.1. *Classify varieties that are* a) *modular elements*, b) *upper-modular elements*, c) *lower-modular elements of the lattice* **OC**.

6. Completely regular varieties

Here we denote the lattice $L(\mathcal{E}_1)$ of all varieties of (unary) completely regular semigroups by **UCR**, thus resembling the notation **CR** for the lattice of all completely regular varieties of (plain) semigroups. According to the observation in the second paragraph of Subsection 2.1, the lattice **CR**, being a sublattice in **Per**, may be considered as a sublattice of the lattice **UCR**. Therefore all properties of the latter lattice inherited by sublattices persist in the lattice **CR**. Practically all information about the lattice **CR** that is known so far arises as "projection" on **SEM** of results about the lattice **UCR**. By this reason, throughout this section we will speak about the latter lattice.

The lattice **UCR** is partitioned into the union of the coideal $[\mathcal{SL})$ and the ideal **UCS** that consists of all varieties of completely simple semigroups. Numerous results about the lattice **UCR** and certain its sublattices (in particular, about the lattice **UCS**) have been systematized in the monograph [73]. Here we present some fundamental achievements that were left beyond [73]. First of all, these are the results about the structure of the lattice **UCR** obtained in a cycle of papers by Polák [74–76].

The variety $\mathcal{SL}$ is a neutral element of the lattice **UCR**, whence this lattice is a subdirect product of the coideal $[\mathcal{SL})$ and the 2-element chain. It is the coideal $[\mathcal{SL})$ that has been studied in [74–76]. To formulate the corresponding results, we need some notation. We denote by U the free unary semigroup over a countably infinite alphabet with the unary operation ${}^{-1}$. Elements of U are called *unary words*. As for the plain free semigroup, we denote by $c(u)$ the set of all letters occurring in the unary word $u \in U$. Following Clifford [12], for a unary word $u \in U$ with $|c(u)| > 1$ we denote by $0(u)$ [respectively $1(u)$] the unary word obtained from the longest initial [terminal] segment of the word u containing $|c(u)| - 1$ letters by omitting all opening brackets such that the segment does not contain the corresponding closing ones [respectively all expressions of the form $)^{-1}$ such that the segment not contain the corresponding opening brackets]. For example, if $u = x((yx)^{-1}z)^{-1}x$, then $0(u) = x(yx)^{-1}$ and $1(u) = xzx$.

To an arbitrary fully invariant congruence $\sim$ on U, we assign the relation $\eqsim$ on U defined recurrently by the following rule: $u \eqsim v$ if and only if $u \sim v$, $c(u) = c(v)$, and besides that $0(u) \eqsim 0(v)$ and $1(u) \eqsim 1(v)$ whenever $|c(u)| > 1$. It is verified in [74] that the relation $\eqsim$ is also a fully invariant congruence. If $\mathcal{V}$ is a variety of unary semigroups corresponding to the fully invariant congruence $\sim$, then we denote by $\overline{\mathcal{V}}$ the variety corresponding to $\eqsim$. One can define the relation ρ on the lattice **UCR** by the following rule: $\mathcal{V}\,\rho\,\mathcal{W}$ if and only if $\overline{\mathcal{V}} = \overline{\mathcal{W}}$. It is proved in [74] that this relation is a complete congruence on **UCR**.

Let L be a lattice with zero. We denote by $V \oplus L$ the lattice that is the ordinal sum of the three-element ordered set shown in Fig. 6 and the lattice L. Further, let us denote by Λ the ordered set shown in Fig. 7. The main result by Polák ([75], Theorem 3.6) states that the coideal $[\mathcal{SL})$ of the lattice **UCR** is embeddable into the lattice of all isotone mappings from Λ into the lattice $V \oplus \mathbf{UCR}/\rho$; the image of the coideal $[\mathcal{SL})$ under this embedding has been explicitly described in [75]. Thus studying the lattice **UCR** is reduced to studying the lattice $\mathbf{UCR}/\rho$. In particular, the construction by Polák implies that $[\mathcal{SL})$ is a subdirect product of countably many copies of the lattice $V \oplus \mathbf{UCR}/\rho$.

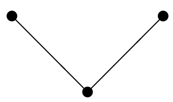

FIGURE 6

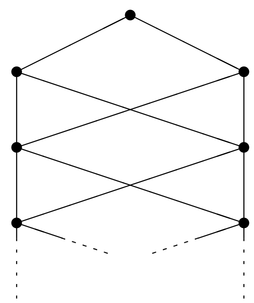

FIGURE 7

Recall that a completely regular semigroup is called *orthodox* if all its idempotents form a subsemigroup. The construction from [75] turns out to be especially transparent if restricted to the lattice **UOCR** of all orthodox varieties of completely regular semigroups. In this case the lattice $\mathbf{UOCR}/\rho$ turns out to be isomorphic to the lattice of all varieties of groups, and Polák [75] has obtained a presentation of **UOCR** as a precisely described sublattice of the direct product of countably many copies of the lattice of varieties of groups. We note that an analogous presentation for the lattice of all orthodox varieties in the context of periodic varieties was found earlier by V. V. Rasin [82].

The description of the lattice **I** of all varieties of idempotent semigroups mentioned in Section 1 is in fact a limit partial case of results of [75]. The lattice $\mathbf{I}/\rho$ is singleton and this gives a presentation of **I** as a certain lattice of isotone mappings from Λ into the four-element lattice $V \oplus 1$.

Another fundamental achievement in the study of the lattice **UCR** is

Theorem 6.1 (Pastijn [62, 63], Petrich and Reilly [70])**.** *The lattice* **UCR** *is modular, and moreover arguesian.*

In [62] this fact has been obtained as an application of the results by Polák [74–76] characterized above. The proofs in [63, 70] are based on an investigation of interactions between identities in varietal lattices and multiplicative properties of fully invariant congruences on free semigroups – we touch upon this subject in Subsection 11.4.

7. Nilvarieties

It is shown in [121, 122] that (as in the overcommutative case) subvariety lattices of nilvarieties may be characterized in terms of congruences of G-sets. The corresponding result has more complex formulation than the one for overcommutative varieties. We consider here a partial (but key) case in which lattices of nilvarieties have a relatively simple structure. The situation in the general case will be characterized at the end of the section.

We say that a semigroup variety is *homogeneous* if an identity $u = v$ where u and v are words of different length implies in this variety the identity $u = 0$. A variety is called *hereditarily homogeneous* if every its subvariety is homogeneous. Every hereditarily homogeneous variety is a nilvariety. They are subvariety lattices of hereditarily homogeneous varieties that will be characterized below.

We need some addition notation; the notation introduced in Subsection 5.1 will be used without special references. We denote by F the free semigroup over an alphabet $\{x_1, x_2, \dots, x_n, \dots\}$. Let $\mathcal{V}$ be a nilvariety of semigroups and m and n positive integers with $m \le n$. We denote by $F_{n,m}(\mathcal{V})$ the set of words $u \in F$ such that $\ell(u) = n$, $c(u) = X_m$ and $\mathcal{V}$ does not satisfy the identity $u = 0$. Let ν be the fully invariant congruence on the semigroup F corresponding to the variety $\mathcal{V}$. The restriction of ν to $F_{n,m}(\mathcal{V})$ is denoted by $\nu_{n,m}$. Clearly, $\nu_{n,m}$ is an equivalence relation. For each $\nu_{n,m}$-class, let us take an arbitrary element in this class. The set of all these words will be denoted by $W_{n,m}(\mathcal{V})$. Put $W^0_{n,m}(\mathcal{V}) = W_{n,m}(\mathcal{V}) \cup \{0\}$ where 0 is the zero of the $\mathcal{V}$-free semigroup of countable rank. If $u \in W_{n,m}(\mathcal{V})$ and $\sigma \in \mathbb{S}_m$ then $u\sigma \in F_{n,m}(\mathcal{V})$, whence there exists a unique word $(u\sigma)^* \in W_{n,m}(\mathcal{V})$ such that $\mathcal{V}$ satisfies the identity $u\sigma = (u\sigma)^*$. Now we define the unary operation σ^* on the set $W^0_{n,m}(\mathcal{V})$ by the rule: $\sigma^*(0) = 0$ and $\sigma^*(u) = (u\sigma)^*$ for every $u \in W_{n,m}(\mathcal{V})$. It is verified in [121] that the set $W^0_{n,m}(\mathcal{V})$ with the collection of operations $\{\sigma^* \mid \sigma \in \mathbb{S}_m\}$ is an $\mathbb{S}_m$-set, whenever the variety $\mathcal{V}$ is hereditarily homogeneous. Further, let $\mathcal{V}_n$ denote the subvariety of the variety $\mathcal{V}$ defined within $\mathcal{V}$ by the identity $x_1 x_2 \cdots x_n = 0$ and let $\mathcal{V}_{n,m}$ denote the subvariety of the variety $\mathcal{V}_{n+1}$ defined within $\mathcal{V}_{n+1}$ by all identities of the form $u = 0$ where $\ell(u) = n$ and $|c(u)| = m$. Put also $\mathcal{V}_{n,0} = \mathcal{V}_{n+1}$. Let $I_{n,m}(\mathcal{V})$ stand for the interval $[\mathcal{V}_{n,m}, \mathcal{V}_{n,m-1}]$ of the lattice $L(\mathcal{V})$.

Theorem 7.1 (B. M. Vernikov and M. V. Volkov [122])**.** *If $\mathcal{V}$ is a hereditarily homogeneous semigroup variety then the lattice $L(\mathcal{V})$ is a subdirect product of all intervals of the kind $I_{n,m}(\mathcal{V})$ where m and n are positive integers with $m \le n$, and an interval $I_{n,m}(\mathcal{V})$ is antiisomorphic to the congruence lattice of the $\mathbb{S}_m$-set $W^0_{n,m}(\mathcal{V})$.*

Theorem 7.1 reduces studying subvariety lattices of hereditarily homogeneous varieties to considering congruence lattices of G-sets. As in the overcommutative case, results of [104] help to apply this theorem.

If a nilvariety $\mathcal{V}$ is not hereditarily homogeneous then, as it is shown in [122], the lattice $L(\mathcal{V})$ is embedded into the dual of a subdirect product of congruence lattices of certain G-sets (this embedding is explicitly constructed in [122]). These G-sets differ from G-sets of the kind $W^0_{n,m}(\mathcal{V})$ but are similar to them: their elements also are some words and the zero and the role of G is also played the group $\mathbb{S}_m$.

8. Commutative varieties

Let us denote by **Com** the lattice of all commutative semigroup varieties. We notice that all elements of this lattice except the greatest one are periodic varieties and these elements form a sublattice of the lattice **Per**. A certain characterization of the lattice **Com** has been suggested by Kisielewicz [36][5]. It is impossible to reproduce this characterization in all detail and we restrict ourself to its brief description. Every commutative variety is encoded in [36] (see also [27]) by a quadruple (J, m, r, π) where J is a coideal in the quasi-ordered set of all finite sequences of non-negative integers, m is a non-negative integer, r is a positive integer, and π is an equivalence relation on the set of all finite sequences of non-negative integers that do not occur in J. Here J and π should satisfy several restrictions (some of these restrictions follow from other ones and therefore may be omitted as it was verified later in [23]). The variety corresponding to the quadruple (J, m, r, π) is denoted by $\mathcal{C}(J, m, r, \pi)$. It has been characterized in [36] when the inclusion $\mathcal{C}(J_1, m_1, r_1, \pi_1) \subseteq \mathcal{C}(J_2, m_2, r_2, \pi_2)$ holds and which quadruples of the mentioned kind encode the join or the meet of the varieties $\mathcal{C}(J_1, m_1, r_1, \pi_1)$ and $\mathcal{C}(J_2, m_2, r_2, \pi_2)$.

[5]This paper, as well as papers [23–27, 37], deals with the dual lattice of equational theories of commutative semigroups rather than the lattice **Com** itself. Presenting results of these papers, we will "translate" them from the language of equational theories into the varietal language.

Already in the first stage of investigations of the lattice of semigroup varieties, Schwabauer in [89] found a wide distributive sublattice of the lattice **Com**; it consists of all subvarieties of the variety $\mathcal{COM}$ that are defined within $\mathcal{COM}$ by identities of the form $u = uv$. It was shown by Nelson [61] that this sublattice is the largest modular sublattice in **Com**. Results of the article [36] show that the sublattice found by Schwabauer form some "skeleton" of the lattice **Com**. We will meet varieties from this sublattice again in Section 15. Following [36], we call them *Schwabauer* varieties.

Techniques developed in [36] serve as a base for [23–26, 37]. In [23], join- and meet-undecomposable elements of the lattice **Com** have been studied. In particular, it has been shown that the only element of this lattice that is both join- and meet-undecomposable is the variety $\mathcal{COM}$. An investigation of order properties of the lattice **Com** related to the notion of a well-quasi-ordered set and certain its modifications was started in [5] and continued in [24]. In [26] the cover relation in the lattice **Com** has been studied and it has been shown how the quadruples (J_1, m_1, r_1, π_1) and (J_2, m_2, r_2, π_2) are related in the case when one of the varieties $\mathcal{C}(J_1, m_1, r_1, \pi_1)$ and $\mathcal{C}(J_2, m_2, r_2, \pi_2)$ covers the other one. The contents of papers [25,37] will be discussed in Section 15.

9. Rees–Sushkevich varieties

The definition of Rees–Sushkevich varieties is given in the introduction to this chapter. Let $\mathcal{RS}_n$ denote the variety generated by all completely 0-simple semigroups satisfying the identity $x^2 = x^{n+2}$; clearly, $\mathcal{RS}_n \subseteq \mathcal{E}_2$. An arbitrary Rees–Sushkevich variety is contained in $\mathcal{RS}_n$ for some n. Let us denote the lattice $L(\mathcal{RS}_n)$ by $\mathbf{RS}_n$. During last few years, some progress in studying lattices of the kind $\mathbf{RS}_n$ has been achieved, see [46–49, 51–54, 84–86, 141]. A general approach to studying these lattices is based on a consideration of their complete congruences in flavor of the methodology discussed in Subsection 2.2. One of such congruences (whose consideration turns out to be most effective) is defined by the following: $\mathcal{V}\,\theta\,\mathcal{W}$ if and only if the varieties $\mathcal{V}$ and $\mathcal{W}$ contain the same completely simple semigroups and the same completely 0-simple semigroups with zero divisors. In [54], the congruence θ is considered on the lattice $\mathbf{RS}_1$ of all combinatorial Rees–Sushkevich varieties. In this work, it has been proved that θ partitions $\mathbf{RS}_1$ into 9 intervals and the extreme elements of these intervals have been described; these extreme elements form a 31-element distributive sublattice that serves as a "skeleton" of the lattice $\mathbf{RS}_1$. Some of θ-classes have been completely described, see [48]. It turns out that the least θ-class is the most complex one; its structure has been revealed in [49]. Using these results, Lee has proved that the lattice $\mathbf{RS}_1$ is countable [52].

CHAPTER III. Varieties with some types of subvariety lattices

A considerable number of articles have been devoted to semigroup varieties with different types of subvariety lattices. Two types of restrictions to the subvariety lattices have attracted most attention: finiteness conditions (that is, conditions that hold in every finite lattice) and identities (and related conditions). Considering conditions of the second type naturally leads to interest for varieties whose subvariety lattices contain "big" sublattices which clearly satisfy no non-trivial identity. Conditions of the three types are considered in the three first sections of this chapter. In the last section, we consider three more types of restrictions to the subvariety lattices: symmetry conditions (that is, conditions related to the notion of lattice dualism), complement conditions and some conditions related to them, decomposability into direct product.

10. Finiteness conditions

10.1. Small varieties. To start with finiteness conditions, it is natural to consider the strongest of them, namely the property of being a finite lattice. Varieties with finite subvariety lattice are called *small*. The problem of describing small varieties was posed in the survey [15] and attracted attention of many authors. Nevertheless this problem seems still to be very far from a complete solution.

A simple but important necessary condition for a subvariety lattice to be finite is contained in the following

Proposition 10.1 (A. Ya. Aĭzenštat [3]). *Every small semigroup variety is a variety of finite degree.*

The statement converse to Proposition 10.1 is false in general. The simplest counter-example is the variety of all idempotent semigroups whose degree equals 1 while the subvariety lattice is infinite, see Fig. 2. Nevertheless, the following is true.

Theorem 10.1 (S. A. Malyshev [56]). *A permutational semigroup variety is small if and only if it is a variety of finite degree.*

M. V. Sapir and E. V. Sukhanov have developed in [88] the structure theory of varieties of finite degree and, basing on this theory, they have shown that a classification of small varieties reduces to solutions of the following three subproblems: 1) to describe small completely regular varieties; 2) to find out under which conditions the Kopamu endomorphisms (see Section 1) preserve the property of being a small variety; 3) to find out under which conditions a variety whose semigroups are ideal extensions of semigroups from a small variety by semigroups from a nilvariety remains small.

We note that the subproblem 1) coincides with the corresponding problem for varieties of completely regular semigroups formulated by the first author of the survey in [97] (Problem 2.59c). We note also that this subproblem contains as a particular case the problem of classification of small varieties of periodic groups. In view of the following result, the latter problem seems to be extremely difficult.

Theorem 10.2 (P. A. Kozhevnikov [44,45]). *There exist uncountably many periodic group varieties with the three-element subvariety lattice.*

The same result was announced earlier by S. V. Ivanov but his announce has not been confirmed by a complete proof later. We note also that examples constructed in [44,45] to prove Theorem 10.2 have shown that a small variety is not necessarily locally finite.

In view of Theorem 10.2, the maximum that we may hope for in subproblem 1) is to classify small completely regular varieties modulo groups. However, essential difficulties appear also here. For example, recently Kaďourek [33] has found a completely simple semigroup variety whose subvariety lattice has the cardinality of the continuum and which contains only 5 group subvarieties. This means that a reduction of subproblem 1) to the case of group varieties is possible only under some additional restrictions. One of some restrictions appears in the following theorem.

Theorem 10.3 (V. V. Rasin [82]). *An orthodox semigroup variety $\mathcal{V}$ is small if (and, evidently, only if) its largest group subvariety is small and $\mathcal{V}$ does not contain* $\mathrm{var}\,\{x^2 = x\}$.

Subproblem 2) also seems to be quite non-trivial because it is known that the Kopamu endomorphisms in general do not preserve the property of being a small variety even for completely regular varieties. The corresponding example can be easily extracted from a construction given by M. V. Sapir in [87]. We note that the same construction provides examples that demonstrate extreme "fragility" of the class of small varieties: the join of two small varieties and a cover of a small variety may be not small varieties.

As to subproblem 3), there is no essential progress here so far; in particular, the following question still remains open.

Question 10.1. *Is a semigroup variety small whenever its semigroups are ideal extensions of semigroups from some small completely regular variety $\mathcal{V}$ by semigroups from some nilpotent variety $\mathcal{N}$?*

It may be verified that the answer is affirmative whenever either $\mathcal{V}$ consists of groups or $\mathcal{V}$ is an orthodox variety while $\mathcal{N} = \mathcal{ZM}$.

10.2. Ascending and descending chain conditions. For brevity, we say that a variety $\mathcal{V}$ *satisfies an ascending* [*descending*] *chain condition* if this condition is satisfied by the lattice $L(\mathcal{V})$.

First of all, we note that the two conditions are independent even for finitely generated varieties. An example of a finitely generated variety with ascending chain condition but without descending chain condition was given in the survey [15]: this is the variety $\mathrm{var}\,\{x^2 = x^3,\ xy = yx\}$ (it is generated by the three-element semigroup C^1 where C is the two-element null semigroup). An (essentially more difficult) example of a finitely generated variety with descending chain condition but without ascending chain condition was constructed by M. V. Sapir [87]. The latter example demonstrates also that the class of varieties with descending chain condition is not closed under taking of joins and covers. The corresponding questions for the ascending chain condition still remain open.

Question 10.2. a) *Does the join of two varieties with ascending chain condition satisfy this condition*? b) *Does a cover of a variety with ascending chain condition satisfy this condition*?

It is possible to show that both parts of Question 10.2 are answered in the affirmative for several important classes of semigroup varieties, for example, for completely regular varieties, nilvarieties, permutational varieties. We note also that the affirmative answer to the first part of Question 10.2 would imply the affirmative answer to its second part.

The following question has been formulated in the survey [4] and remains open for 30 years.

Question 10.3. *Does there exist a semigroup variety whose subvariety lattice is infinite but satisfies both the ascending chain condition and the descending chain condition*?

It is easy to see that a variety $\mathcal{V}$ satisfies the descending chain condition if and only if every subvariety of $\mathcal{V}$ is given within $\mathcal{V}$ by a finite number of identities. In particular, the descending chain condition is satisfied by every *hereditarily finitely based* variety, that is, a variety all of whose subvarieties are finitely based, and by every limit variety.

Hereditarily finitely based and limit varieties are intensely studied in the course of investigations on the finite basis problem; we refer the interested reader for detailed information about corresponding results to Chapter III of the survey [94] and to the recent surveys [137, 139].

10.3. The finiteness of width. A lattice L is said to have *finite width* [*width* n] if all antichains in this lattice are finite [and contain $\leq n$ elements and n is the least number with such a property]. We say that $\mathcal{V}$ is a variety of *finite width* if the width of the lattice $L(\mathcal{V})$ is finite. Results by M. V. Sapir and E. V. Sukhanov [88] readily imply that any semigroup variety of finite width is either periodic and permutational or a variety of finite degree. M. V. Sapir has observed that every periodic permutational variety has finite width (unpublished). Thus the problem of classifying of varieties of finite width reduces to the case of varieties of finite degree. We note that, among varieties with the latter property, there are varieties of infinite width and, moreover, varieties whose subvariety lattice contains uncountable antichains. So, by Theorem 10.2 there exist uncountable antichains in the lattice of varieties of groups of a sufficiently large prime exponent, while results by Kaďourek [32] imply that there exist such antichains in the lattice of combinatorial varieties of degree 2.

11. Identities and related conditions

11.1. Modularity. The problem of describing semigroup varieties with modular subvariety lattice was posed in the survey [15]. It was solved by the third author of this article at the beginning of the 1990s. A full and correct formulation of this result first appeared in the dissertation [134] and was published in [109]. It is impossible to reproduce this formulation here because it is quite lengthy (in particular, it includes the list of 146 maximal nilvarieties with modular subvariety lattice). So, we formulate explicitly only the following necessary condition for modularity of subvariety lattice.

Theorem 11.1 (M. V. Volkov [129])**.** *If a semigroup variety $\mathcal{V}$ has modular subvariety lattice, then one of the following holds*: (1) $\mathcal{V}$ *is a variety of degree* ≤ 2; (2) $\mathcal{V} \subseteq \mathcal{A}_n \vee \mathcal{C} \vee \mathcal{N}$ *for some* n, *where* $\mathcal{C} = \mathrm{var}\,\{x^2 = x^3,\, xy = yx\}$, *while the variety* $\mathcal{N}$ *satisfies the identities* $x^2y = xyx = yx^2 = 0$ *and a permutation identity of length* 4; (3) $\mathcal{V} \subseteq \mathcal{SL} \vee \mathcal{N}$, *where* $\mathcal{N}$ *is a nilvariety satisfying a permutation identity of length* 4.

Thus the further analysis of varieties with modular subvariety lattice reduces to consideration of three cases mentioned in Theorem 11.1. As is known [21], a semigroup variety has degree ≤ 2 if and only if it satisfies one of the identities $xy = (xy)^{n+1}$, $xy = x^{n+1}y$ and $xy = xy^{n+1}$ for some positive integer n. If a variety $\mathcal{V}$ satisfies the first of these identities, then the square of any semigroup in $\mathcal{V}$ is a completely regular semigroup. Varieties with such a property are called *varieties of semigroups with completely regular square*. Every variety of semigroups with completely regular square has modular subvariety lattice (M. V. Volkov and T. A. Ershova [142]). Varieties with modular subvariety lattice satisfying one of the identities $xy = x^{n+1}y$ and $xy = xy^{n+1}$ have been described in [132], while varieties with the same property satisfying condition (2) of Theorem 11.1 have been characterized in [131]. Case (3) reduces to studying nilvarieties with modular subvariety lattice. The origin version of the proof a complete description of such varieties is given in the dissertation [134] only. An essentially simpler version of the proof was given in the cycle of paper [109, 124, 138] as a part of the proof of certain stronger results that will be mentioned in Subsections 11.3 and 11.5.

Theorem 11.1 shows that varieties in question are periodic. It seems to be possible to generalize this result to the case of epigroup varieties. The corresponding problem already have been noted by the first author in [90] as well as in [91] (Problem 3.21).

Problem 11.1. *Describe varieties of epigroups with modular subvariety lattice.*

It is known (see [91], Corollary 3.19) that every epigroup variety with modular subvariety lattice consists of semilattices of archimedian semigroups.

11.2. Distributivity. The following problem was posed by the first author more than 30 years ago and it is not solved in the general case so far.

Problem 11.2 ([97], Problem 2.60a)**.** *Describe varieties of semigroups with distributive subvariety lattice.*

The most progress in this direction has been obtained in [129, 131–133]. In [129], it has been verified that a statement analogous to Theorem 11.1 holds true for varieties with distributive subvariety lattice, in [131], varieties with distributive subvariety lattice satisfying the condition (2) of Theorem 11.1 have been described, while in [132] an analogous result has been obtained for varieties satisfying one of the identities $xy = x^{n+1}y$ and $xy = xy^{n+1}$ (in this case a description is given "modulo groups"). Finally, nilvarieties with distributive subvariety lattice have been completely determined in [133]; a simpler and shorter proof of this result is contained in [123]. Thus to complete a description of varieties with distributive subvariety lattice, it remains to classify varieties of semigroups with completely regular square having the discussed property. This problem includes as a particular case the following problem that was posed by the first author simultaneously with Problem 11.2.

Problem 11.3 ([97], Problem 2.60b)**.** *Describe varieties of completely regular semigroups with distributive subvariety lattice.*

It is natural to speak about a description "modulo groups" here. The most progress in this direction so far is due to V. V. Rasin [82]: an orthodox variety has distributive subvariety lattice if and only if the subvariety lattice of its largest group subvariety is distributive. The third author and T. A. Ershova have proved that an analogous result holds true for every variety of semigroups with completely regular square such that in any semigroup of the variety the set of all idempotents forms a subsemigroup (unpublished). This strengthens not only the mentioned result by V. V. Rasin but also the result by Gerhard [19] about distributivity of the subvariety lattice of the variety of all semigroups with idempotent square (see Theorem 4.1).

As to a description of periodic group varieties with distributive subvariety lattice, we note that this problem had attracted considerable attention in the past but there have been no significant progress here since the middle of the 1970s. Moreover, the aforementioned fact that there exists an uncountable set of group varieties with three-element subvariety lattice (see Theorem 10.2) causes certain pessimism here. Apparently, some progress in classifying of periodic group varieties with distributive subvariety lattice can be achieved only when some essentially new ideas would appear.

At the conclusion of this subsection, we mention the article [130] where commutative semigroup varieties with distributive subvariety lattice are completely classified.

11.3. Arguesity and semimodularity. According to Theorem 6.1, the lattice **CR** is arguesian. This assertion was strengthened in [142] where the arguesity of the subvariety lattice of an arbitrary variety of semigroups with completely regular square was verified. Both these facts have been generalized by the following statement.

Theorem 11.2 (B. M. Vernikov and M. V. Volkov [109, 124, 138])**.** *For a semigroup variety $\mathcal{V}$, the following are equivalent*: (a) *the lattice $L(\mathcal{V})$ is arguesian*; (b) *the lattice $L(\mathcal{V})$ is modular*; (c) *the lattice $L(\mathcal{V})$ is upper semimodular*; (d) *the lattice $L(\mathcal{V})$ is weakly upper semimodular.*

In view of results mentioned in Subsection 11.1, this theorem leads to a complete description of semigroup varieties with arguesian or [weakly] upper semimodular subvariety lattice.

Semigroup varieties whose subvariety lattice is [weakly] lower semimodular have also been completely determined in [109, 124, 138]. It turns out that, in lattices of semigroup varieties, lower semimodularity and weak lower semimodularity are equivalent to each other but not equivalent to modularity. The gap between the corresponding classes of varieties is very small: as is shown in [109, 124, 138], there are uncountably many minimal semigroup varieties with non-modular subvariety lattice but only one of them has lower semimodular subvariety lattice.

We note that there are many analogues between results of this subsection and results of Subsection 5.2. But this analogy is not complete. So, it is verified in [124] that these exists a non-trivial lattice identity that holds in all weakly lower semimodular lattices of subvarieties of semigroup varieties, but results of the work [105] allows one to readily find an example of an overcommutative variety $\mathcal{V}$ such that the interval $[\mathcal{COM}, \mathcal{V}]$ is lower semimodular but does not satisfy any non-trivial lattice quasiidentity.

11.4. Interactions with multiplicative properties of fully invariant congruences. Multiplicative properties of congruences are properties formulated in terms of the product of congruences as binary relations. When studying varieties, it is reasonable to focus on fully invariant congruences on free algebras because these congruences correspond to varieties. The simplest and "mostly popular" multiplicative property is permutability. We call a semigroup variety $\mathcal{V}$ [*almost*] *fi-permutable* if any two fully invariant congruences [contained in the least semilattice congruence] on every $\mathcal{V}$-free semigroup permute.

Since the variety $\mathcal{SL}$ is an atom and a neutral element in **SEM**, well-known properties of equivalence lattices imply that every almost fi-permutable semigroup variety has a modular and, moreover, arguesian subvariety lattice. This fact serves as a base for one of approaches to studying semigroup varieties with modular subvariety lattice. A striking confirmation of a fruitfulness of this approach is the articles [63, 70]. It has been proved there that every variety of completely regular semigroups is almost fi-permutable, whence the lattice of all such varieties is arguesian.

A complete description of fi-permutable and almost fi-permutable varieties is obtained in [120] and [123] respectively. These results show that there are some very unexpected relationships between the conditions we discuss and identities in varietal lattices. A typical example of such relationships is provided by the following

Proposition 11.1. *If a semigroup variety $\mathcal{V}$ is* [*almost*] *fi-permutable and is not completely simple* [*completely regular*] *then the lattice $L(\mathcal{V})$ is distributive.*

The part of this result concerning fi-permutable varieties has been obtained in [120], while the part concerning almost fi-permutable varieties has been proved in [123]. Results of such a kind dealing with other multiplicative restrictions to fully invariant congruences have been obtained in [110–112].

11.5. Quasiidentities. Combinatorial semigroup varieties whose subvariety lattice belongs to a fixed quasivariety of modular lattices have been described in [113]. The following fact has been proved there (the lattices that appear in the formulation of this fact are introduced in Subsection 5.2).

Proposition 11.2 (B. M. Vernikov [113])**.** *Let $\mathcal{V}$ be a non-trivial combinatorial semigroup variety. If the lattice $L(\mathcal{V})$ is modular, then it is quasiequationally equivalent to one of the following lattices*: 1) *the two-element chain*; 2) M_3; 3) M_4; 4) $M_{3,3}$; 5) $M_4 \times M_{3,3}$; 6) $M_{4,3}$.

In case 1) the lattice $L(\mathcal{V})$ is distributive; a description of combinatorial varieties with distributive subvariety lattice readily follows from results of [124, 133]. A classification of combinatorial varieties corresponding to each of the cases 2)–5) of Proposition 11.2 is given in [113]. Finally, combinatorial varieties whose subvariety lattice is quasiequationally equivalent to the lattice $M_{4,3}$ have been determined in [109, 124, 138].

The modularity of the lattice **CR** immediately implies that (upper or lower) semidistributivity is equivalent to distributivity for subvariety lattices of completely regular varieties. But no a priori consideration allows one to state that this equivalence would take place for lattices of nilvarieties too. Nevertheless, the following analogue of Theorem 5.2 holds.

Theorem 11.3 (B. M. Vernikov [108])**.** *For a nilvariety $\mathcal{V}$, the following are equivalent*: (a) $L(\mathcal{V}) \in \overline{\mathsf{M}}_3$; (b) *the lattice $L(\mathcal{V})$ is upper semidistributive*; (c) *the lattice $L(\mathcal{V})$ is lower semidistributive*; (d) *the lattice $L(\mathcal{V})$ is distributive.*

In view of the results of [133] mentioned in Subsection 11.2, this theorem gives a complete description of nilvarieties of semigroups whose subvariety lattice is (upper or lower) semidistributive or belongs to any other quasivariety of lattices that does not contain the lattice M_3. We note that for arbitrary (and moreover for combinatorial) semigroup varieties none of the conditions (a)–(c) of Theorem 11.3 is equivalent to the condition (d).

11.6. "Narrowness" conditions. A limit strengthening of the distributive law is the property of being a chain. A *chain* variety is a variety whose subvariety lattice is a chain. Non-group chain semigroup varieties have been classified in [95], while results of the article [6] readily imply a description of locally finite chain group varieties. The problem of a classification of chain group varieties beyond the locally finite case seems to be extremely difficult in view of Theorem 10.2. Chains are lattices of width 1. Non-nilpotent and non-group semigroup varieties whose subvariety lattice is of width 2 have been determined in [96], while nilpotent varieties whose subvariety lattice has just one pair of non-comparable elements (the "extreme" particular case of lattices of width 2) have been classified in [102].

12. Lattice universality

A semigroup variety $\mathcal{V}$ is called *lattice universal* if the lattice $L(\mathcal{V})$ contains an interval dual to the partition lattice on a countably infinite set. We note that in this case the lattice $L(\mathcal{W})$ for an arbitrary variety of algebras $\mathcal{W}$ of at most countable similarity type is embeddable into $L(\mathcal{V})$, and this fact explains our terminology. The first example of a lattice universal semigroup variety was given by Burris and Nelson [10]: this is the variety $\mathrm{var}\{x^2 = x^3\}$. Later Ježek [29] showed that the smaller variety $\mathrm{var}\{x^2 = 0\}$ also is lattice universal. The third author and M. V. Sapir have described lattice universal varieties in rather a large class of varieties. The formulation of this (unpublished) result involves the Zimin words [144] defined by the following recurrent way: $Z_1 = x_1$, $Z_{n+1} = Z_n x_{n+1} Z_n$.

Theorem 12.1 (M. V. Volkov and M. V. Sapir)**.** *Suppose that a semigroup variety $\mathcal{V}$ is defined by identities depending on at most n variables and all periodic groups in $\mathcal{V}$ are locally finite. Then $\mathcal{V}$ is lattice universal if and only if it does not satisfy any non-trivial identity of the form $Z_{n+1} = w$.*

One can consider a weaker version of lattice universality: the variety $\mathcal{V}$ is said to be *finitely universal* if the lattice $L(\mathcal{V})$ contains the dual copy of the partition lattice of an arbitrary finite set (and therefore, according to a known result by Pudlak and Tůma [81], it contains an isomorphic copy of an arbitrary finite lattice). It is known [11] that the variety $\mathcal{COM}$ is finitely universal; moreover, I. O. Korjakov [43] has proved that the dual of the partition lattice of any finite set is embeddable even in the lattice of all commutative nilpotent semigroup varieties. It is easy to see that the variety $\mathcal{COM}$ is a minimal finitely universal variety. Another known example of a minimal finitely universal variety is the variety $\mathcal{H} = \mathrm{var}\{x^2 = xyx = 0\}$, see [121]. Results of [88] imply that if a semigroup variety contains neither $\mathcal{COM}$ nor $\mathcal{H}$ then it is either periodic and permutational or a variety of finite degree. This fact and results about varieties of finite width discussed in Subsection 10.3 imply that if a finitely universal variety does not contain $\mathcal{COM}$ and $\mathcal{H}$ then it must be a variety of finite degree. But the following question is open so far.

Question 12.1. *Does there exist a finitely universal semigroup variety of finite degree?*

Since the variety $\mathcal{H}$ plays an important role in problems we consider, the question about a structure of the lattice $L(\mathcal{H})$ is quite interesting. Every proper subvariety of the variety $\mathcal{H}$ is given within $\mathcal{H}$ by some family of permutation identities (according to a result by Pollák [77], we may assume that this family is finite) and/or an identity of nilpotency of some degree. This description of the set of all subvarieties of the variety $\mathcal{H}$ (without indicating of a lattice order on this set) is given in the paper [49], for instance. The more interesting fact is that, in spite of very complex local structure of the lattice $L(\mathcal{H})$, this lattice admits quite a simple global description in terms of subgroup lattices of symmetric groups. In order to formulate this description, we denote by L_n the subgroup lattice of the group $\mathbb{S}_n$ with the new greatest element adjoined. It can be easily deduced from Theorem 7.1 that the lattice $L(\mathcal{H})$ is dual to a subdirect product of the lattices L_n where n runs over all positive integers; results of [77] allows one to explicitly describe the image of $L(\mathcal{H})$ under its dual embedding into $\prod\limits_{n\in\mathbb{N}} L_n$.

There exist finite semigroups generating finitely universal varieties. For instance, the 5-element Brandt semigroup has this property (it generates the variety containing $\mathcal{H}$). The question arises what is the minimal number n with the property that there is an n-element semigroup generating a finitely universal variety. An answer to this question has been found by Lee [50] who has proved that $n = 4$ and has listed all 4-element semigroups generating a finitely universal variety. It turns out that there are only four such semigroups up to isomorphism and antiisomorphism. Subvarieties of varieties generated by each of these four semigroups have been classified in [48, 49, 143].

13. Other restrictions

13.1. Symmetry conditions. A semigroup variety $\mathcal{V}$ is called *selfdual* if its subvariety lattice is dual to itself, and is called *admitting dualism* if there is a semigroup variety $\mathcal{V}^*$ such that the lattices $L(\mathcal{V})$ and $L(\mathcal{V}^*)$ are dual to each other. A variety is called *hereditarily selfdual* [*hereditarily admitting dualism*] if all its subvarieties are selfdual [admit dualism]. In [103], hereditarily selfdual varieties have been described, it has been verified that a semigroup variety admitting dualism is periodic, and essential information about hereditarily admitting dualism varieties has been obtained. In particular, it has been verified there that the subvariety lattice of any such variety is lower semimodular. This statement and Theorem 11.2 easily imply the following essentially stronger fact: the subvariety lattice of any hereditarily admitting dualism semigroup variety is distributive.

13.2. Complementability and related conditions. Recall that a lattice L with 0 and 1 is called a *lattice with upper semicomplements* if, for any $x \in L \setminus \{0\}$, there is $y \in L \setminus \{1\}$ with $x \vee y = 1$.

Theorem 13.1. *For a semigroup variety $\mathcal{V}$, the following are equivalent*: (a) $L(\mathcal{V})$ *is a lattice with upper semicomplements*; (b) $L(\mathcal{V})$ *is a complemented lattice*; (c) $L(\mathcal{V})$ *is a finite Boolean algebra*; (d) $\mathcal{V}$ *is the join of a finite number of atoms of the lattice* **SEM**.

The equivalence of the assertions (b)–(d) was proved in [118], while the assertions (a) and (b) are equivalent for arbitrary varieties of universal algebras [14]. Since the description of atoms of the lattice **SEM** is known (see Section 1), Theorem 13.1 gives a complete description of semigroup varieties whose subvariety lattice is upper semicomplemented, complemented or satisfies all standard stronger versions of complementability (such as the relative complementness, the uniqueness of complements etc.).

Results of [14, 118] imply an analogue of Theorem 13.1 for varieties of inverse or completely regular semigroups.

The lower semicomplementness condition dual to upper semicomplementness one is not interesting from the viewpoint of varietal lattices since simplest lattice-theoretical considerations show that a complete atomic lattice is lower semicomplemented if and only if its largest element is the join of all its atoms.

13.3. Decomposability into the direct product. Semigroup varieties whose subvariety lattice is decomposable into a direct product have been studied in [101]. A necessary condition has been found, varieties with the mentioned property in several classes of semigroups varieties have been determined and the following result has been proved there.

Proposition 13.1 (B. M. Vernikov [101])**.** *If $\mathcal{K}$ is a completely regular semigroup variety and $\mathcal{N}$ is a nilvariety, then $L(\mathcal{K} \vee \mathcal{N}) \cong L(\mathcal{K}) \times L(\mathcal{N})$ if and only if either $\mathcal{K} \subseteq \mathcal{SL}$ or $\mathcal{N} \subseteq \mathcal{ZM}$ or the variety $\mathcal{K}$ is commutative and $\mathcal{N} \subseteq \mathrm{var}\{x^2y = xyx = yx^2 = 0\}$.*

Unfortunately, this result is not known enough, and this occasionally leads to incorrect statements in some papers. For example, it has been claimed in [42] that, for any n and k, the lattice of all varieties consisting of ideal extensions of rectangular bands of groups from $\mathcal{A}_k$ by n-nilpotent semigroups is the direct product of the lattice $L(\mathcal{A}_k)$ and the lattice of all varieties consisting of ideal extensions of rectangular semigroups by n-nilpotent semigroups. But Proposition 13.1 shows that this is incorrect whenever $k > 1$ and $n > 3$.

CHAPTER IV. SPECIAL ELEMENTS OF THE LATTICE OF SEMIGROUP VARIETIES

14. Modular and related elements

Informally speaking, results from Subsection 11.1 indicate the zones of "global modularity" in the lattice **SEM**. In order to investigate the phenomenon of modularity in **SEM**, the next natural step is to consider varieties that guarantee, so to speak, local modularity in their neighborhood. Here we have in mind studying modular elements of the lattice **SEM** and other types of its elements whose definitions are based on the modular law in some way.

Along with modular, upper-modular and lower-modular elements (whose definitions are given in Subsection 0.2), it seems to be natural to consider elements that have all these three properties simultaneously. We call such elements *strongly modular*. A semigroup variety is called *modular* [*upper-modular, lower-modular, strongly modular, neutral*] if it is a modular [upper-modular, lower-modular, strongly modular, neutral] element of the lattice **SEM**.

An essential step in investigation of modular varieties is the following

Theorem 14.1. *If a variety $\mathcal{V}$ is modular, then either $\mathcal{V} = \mathcal{SEM}$ or $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M} \subseteq \mathcal{SL}$ and $\mathcal{N}$ is a nilvariety.*

This statement (in a slightly weaker version and in another terminology) is contained in the paper [31] by Ježek and McKenzie[6]. The more precise formulation given above is taken from [114].

Theorem 14.1 completely reduces studying modular varieties to the nil-case. In this case there are one necessary and one sufficient condition, and a gap between these conditions seems to be not very large. We call an identity $u = v$ *substitutive* if the words u and v depend on the same letters and v may be obtained from u by renaming of letters. The following proposition is true.

Proposition 14.1. 1) *Every modular nilvariety may be defined by* 0*-reduced and substitutive identities only.* 2) *Every* 0*-reduced semigroup variety is modular.*

The first assertion of this proposition has been proved in [114], the second one has been observed in [119] and (in other terminology) in [31]. We note that proofs of both statements of Proposition 14.1 use some ideas from Ježek's paper [30].

To achieve further progress in studying modular varieties, one has to consider nilvarieties satisfying substitutive identities. The simplest particular case of substitutive identities is permutation identities. Thus the following problem arises naturally.

Problem 14.1. *Describe permutational modular semigroup varieties.*

Some first steps in solving of this problem have been made in [114]. In particular, commutative modular varieties have been described there.

Let us turn to upper-modular and lower-modular varieties. We note that necessary conditions for these two types of varieties have been given in [117]

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[6] This paper has dealt with the lattice of equational theories of semigroups rather than the lattice **SEM**. We note that the modular elements of the former lattice precisely correspond to the modular semigroup varieties because the definition of a modular element of a lattice is selfdual.

and [115] respectively. In particular, it turns out that a proper variety with any of these properties is periodic. A description of upper-modular and lower-modular varieties in the nil-case has been obtained in [125] and [115] respectively, while in the commutative case the same has been achieved in [117] and [115] respectively. The paper [116] contains a classification of lower-modular varieties in which some power of every semigroup is completely regular and of lower-modular varieties of degree ≤ 2; in the latter case the properties of being a lower-modular element and of being a neutral element turn out to be equivalent. We note that the proofs in [116] have made an essential use of techniques developed by M. V. Sapir in [87].

To advance further in studying upper-modular and lower-modular varieties, it is necessary, first of all, to answer the following questions.

Question 14.1. *Does there exist a non-upper-modular variety of semigroups with completely regular square*?

Question 14.2. *Does there exist* a) *a proper non-combinatorial lower-modular semigroup variety and* b) *a lower-modular but non-modular semigroup variety*?

In the connection with Question 14.2b, it is worth noting that all five other potentially possible implications between the properties to be modular, upper-modular and lower-modular varieties are false: there exist modular but not upper-modular, modular but not lower-modular, upper-modular but not modular, upper-modular but not lower-modular, and lower-modular but not upper-modular varieties.

Let us turn now to strongly modular and neutral varieties. We note that any neutral element of an arbitrary lattice is strongly modular, but the converse statement is false in general. Nevertheless, the following is true.

Theorem 14.2. *For a semigroup variety* $\mathcal{V}$, *the following are equivalent*: (a) $\mathcal{V}$ *is both upper-modular and lower-modular*; (b) $\mathcal{V}$ *is strongly modular*; (c) $\mathcal{V}$ *is neutral*; (d) $\mathcal{V}$ *coincides with one of the varieties* $\mathcal{T}$, $\mathcal{SL}$, $\mathcal{ZM}$, $\mathcal{SL} \vee \mathcal{ZM}$ *or* $\mathcal{SEM}$.

The equivalence of assertions (b)–(d) has been proved in [140], while the equivalence of assertions (a) and (d) has been verified in [116].

It remains to note that varieties that are both modular and upper-modular have been determined in [125], while varieties that are both modular and lower-modular have been classified in [140].

15. Definable sets of varieties

A subset A of a lattice $\langle L; \vee, \wedge \rangle$ is called *definable* in L if there exists a first order formula $\Phi(x)$ with one free variable x in the language of lattice operations $\vee$ and $\wedge$ which defines A in L. This means that, for an element $a \in L$, the sentence $\Phi(a)$ is true if and only if $a \in A$. If A consists of a single element, we speak about definability of this element. It is evident that, for a semigroup variety $\mathcal{V}$, if the sentence $\Phi(\mathcal{V})$ is true then the sentence $\Phi(\overleftarrow{\mathcal{V}})$ also is. A variety $\mathcal{V}$ is called *semidefinable* if the set $\{\mathcal{V}, \overleftarrow{\mathcal{V}}\}$ is definable in the lattice **SEM**.

A number of deep results about varieties and sets of varieties definable in **SEM** have been obtained in the paper [31] by Ježek and McKenzie. It has been conjectured there that every finitely based semigroup variety is semidefinable. It is easy to see that the affirmative answer to Question 1.1 would imply that this conjecture fails. The main results of [31] are summarized in the following theorem. In particular, this theorem confirms the above conjecture for locally finite varieties.

Theorem 15.1 (Ježek and McKenzie [31])**.** 1) *The sets of all finitely based, all locally finite, all finitely generated and all* 0*-reduced semigroup varieties are definable in the lattice* **SEM**. 2) *Every finitely based locally finite and every finitely generated semigroup variety is semidefinable.*

For each set of varieties from assertion 1) of this theorem, the paper [31] contains no explicit first order formula defining this set. Quite a simple formula defining the set of all 0-reduced varieties has been written down in [140]. An even simpler formula is exhibited in [115].

If $\mathcal{V}$ is a commutative variety then $\mathcal{V}$ is finitely based [66] and $\mathcal{V} = \overleftarrow{\mathcal{V}}$. If, besides that, $\mathcal{V} \neq \mathcal{COM}$ then $\mathcal{V}$ is locally finite. It is easy to verify that the variety $\mathcal{COM}$ is definable in the lattice **SEM**. Therefore assertion 2) of Theorem 15.1 implies the following

Corollary 15.1. *Every commutative semigroup variety is definable in the lattice* **SEM**.

On the other hand, there exist commutative varieties that are not definable in the lattice **Com**. It follows from the existence of non-trivial automorphisms of this lattice. First examples of such automorphisms have been given in [37]. In the recent paper [25], a description of the automorphism group of the lattice **Com** is obtained; it turns out that this group is uncountably infinite and satisfies the identity $x^2 = 1$ (whence it is abelian).

Definability of several varieties and sets of varieties of commutative semigroups in the lattice **Com** has been proved in [37]. In particular, the following is true.

Theorem 15.2 (Kisielewicz [37])**.** *The set of all varieties generated by abelian groups, every variety generated by an abelian group, the set of all Schwabauer varieties and every Schwabauer variety are definable in the lattice* **Com**.

The study of commutative varieties definable in the lattice **Com** has been completed in [25] where a characterization of all such varieties has been found.

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