LOWER-MODULAR ELEMENTS OF THE LATTICE OF SEMIGROUP VARIETIES. III

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Abstract. We completely determine all lower-modular elements of the lattice of all semigroup varieties. As a corollary, we show that a lower-modular element of this lattice is modular.

1. Introduction and summary

The collection **SEM** of all semigroup varieties forms a lattice with respect to the class-theoretical inclusion. Special elements of different types in this lattice have been studied in several articles. An overview of results obtained in these articles is given in the recent survey [6, Section 14]. Recall the definitions of special elements mentioned in this paper. An element x of a lattice $\langle L; \vee, \wedge\rangle$ is called *modular* if

$$\forall\, y, z \in L\colon \quad y \le z \longrightarrow (x \vee y) \wedge z = (x \wedge z) \vee y,$$

lower-modular if

$$\forall\, y, z \in L\colon \quad x \le y \longrightarrow x \vee (y \wedge z) = y \wedge (x \vee z),$$

distributive if

$$\forall y, z \in L\colon \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Upper-modular elements are defined dually to lower-modular ones. It is evident that a distributive element is lower-modular.

We call a semigroup variety *modular* [*lower-modular, distributive*] if it is a modular [lower-modular, distributive] element of the lattice **SEM**. Distributive varieties are completely determined by the authors in [9, Theorem 1.1]. Here we consider a wider class of lower-modular varieties. These varieties were mentioned for the first time in [10] (see Lemma 2.4 below) and examined systematically in [7,8]. Here we complete this examination. The main result of this article gives a complete classification of lower-modular varieties. To formulate this result, we need a few definitions and notation.

A pair of identities $wx = xw = w$ where the letter x does not occur in the word w is usually written as the symbolic identity $w = 0$. (This notation is justified because a semigroup with such identities has a zero element and all

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values of the word w in this semigroup are equal to zero.) Identities of the form $w = 0$ as well as varieties given by such identities are called 0-*reduced*. By $\mathcal{T}$, $\mathcal{SL}$, and $\mathcal{SEM}$ we denote the trivial variety, the variety of all semilattices, and the variety of all semigroups, respectively. The main result of the article is the following

Theorem 1.1. *A semigroup variety $\mathcal{V}$ is lower-modular if and only if either $\mathcal{V} = \mathcal{SEM}$ or $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$, while $\mathcal{N}$ is a 0-reduced variety.*

Theorem 1.1, together with [4, Proposition 1.1] (see also Lemmas 2.4 and 2.5 below), immediately implies

Corollary 1.2. *A lower-modular semigroup variety is modular.* □

Theorem 1.1 and Corollary 1.2 give answers to Question 14.2 from [6] and Questions 1 and 2 from [8]. Besides, Theorem 1.1 solves Problems 3 and 4 from [8]. It is verified in [9, Corollary 1.2] that every distributive variety is modular. Clearly, this claim is generalized by Corollary 1.2.

The article consists of three sections. Section 2 contains some auxiliary results, while Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminaries

2.1. Some properties of lower-modular, upper-modular and modular elements in abstract lattices and the lattice SEM.

We start with two easy lattice-theoretical observations. If L is a lattice and $a \in L$ then $[a)$ stands for the *principal coideal* generated by a, that is, the set $\{x \in L \mid x \geq a\}$.

Lemma 2.1. *If x is a lower-modular element of a lattice L and $a \in L$ then the element $x \vee a$ is a lower-modular element of the lattice $[a)$.*

Proof. Let $y, z \in [a)$ and $x \vee a \leq y$. Then

$$\begin{aligned}
(x \vee a) \vee (y \wedge z) &= a \vee \big(x \vee (y \wedge z)\big) \\
&= a \vee \big(y \wedge (x \vee z)\big) && \text{because } x \text{ is lower-modular} \\
& && \text{and } x \leq x \vee a \leq y \\
&= y \wedge (x \vee z) && \text{because } a \leq y \wedge (x \vee z) \\
&= y \wedge \big(x \vee (a \vee z)\big) && \text{because } a \leq z \\
&= y \wedge \big((x \vee a) \vee z\big).
\end{aligned}$$

Thus $(x \vee a) \vee (y \wedge z) = y \wedge \big((x \vee a) \vee z\big)$, and we are done. □

Lemma 2.2. *Let L be a lattice and φ a surjective homomorphism from L onto a lattice L'. If x is an upper-modular element of L then $\varphi(x)$ is an upper-modular element of L'.*

Proof. Let $x' = \varphi(x)$ and let y', z' be elements of L' with $y' \leq x'$. Then there are $y, z \in L$ such that $y' = \varphi(y)$ and $z' = \varphi(z)$. We may assume that $y \leq x$. Indeed, if this is not the case then we may consider the element $x \wedge y$ rather than y because $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y) = x' \wedge y' = y'$. Since the element x

is upper-modular in L, we have $(z \wedge x) \vee y = (z \vee y) \wedge x$. This implies that $(z' \wedge x') \vee y' = (z' \vee y') \wedge x'$ that completes the proof. $\square$

Now we provide some known partial results about lower-modular varieties. It is well-known that if a semigroup variety $\mathcal{V}$ is *periodic* (that is, consists of periodic semigroups) then it contains the greatest nil-subvariety. We denote this subvariety by $\mathrm{Nil}(\mathcal{V})$. A semigroup variety $\mathcal{V}$ is called *proper* if $\mathcal{V} \neq \mathcal{SEM}$.

Lemma 2.3 ([7, Theorem 1]). *If a proper semigroup variety $\mathcal{V}$ is lower-modular then $\mathcal{V}$ is periodic and the variety $\mathrm{Nil}(\mathcal{V})$ is 0-reduced.* $\square$

Lemma 2.4 ([10, Corollary 3]). *A 0-reduced semigroup variety is modular and lower-modular.* $\square$

Note that the 'modular half' of Lemma 2.4 was rediscovered (in some other terms) in [4, Proposition 1.1].

Lemma 2.5. *A semigroup variety $\mathcal{V}$ is [lower-]modular if and only if the variety $\mathcal{V} \vee \mathcal{SL}$ is such.* $\square$

This fact was proved in [11, Corollary 1.5(i)] for modular varieties and in [7, Corollary 1.3] for lower-modular ones.

2.2. **Decomposition of some varieties into the join of subvarieties.** We denote by $\mathcal{LZ}$ [respectively $\mathcal{RZ}$] the variety of all left [right] zero semigroups. If Σ is a system of semigroup identities then $\operatorname{var}\Sigma$ stands for the semigroup variety given by Σ. Put

$$\mathcal{P} = \operatorname{var}\{xy = x^2y, x^2y^2 = y^2x^2\},$$
$$\overleftarrow{\mathcal{P}} = \operatorname{var}\{xy = xy^2, x^2y^2 = y^2x^2\}.$$

Lemma 2 of the article [12] and the proof of Proposition 1 of the same article imply the following

Lemma 2.6. *If a periodic semigroup variety $\mathcal{V}$ contains none of the varieties $\mathcal{LZ}$, $\mathcal{RZ}$, $\mathcal{P}$, and $\overleftarrow{\mathcal{P}}$ then $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where the variety $\mathcal{M}$ is generated by a monoid and $\mathcal{N} = \mathrm{Nil}(\mathcal{V})$.* $\square$

For any natural m, we put $\mathcal{C}_m = \operatorname{var}\{x^m = x^{m+1}, xy = yx\}$. In particular, $\mathcal{C}_1 = \mathcal{SL}$. For notational convenience, we define also $\mathcal{C}_0 = \mathcal{T}$.

Lemma 2.7 ([3]). *If a semigroup variety $\mathcal{M}$ is generated by a commutative monoid then $\mathcal{M} = \mathcal{G} \vee \mathcal{C}_m$ for some Abelian periodic group variety $\mathcal{G}$ and some $m \geq 0$.* $\square$

2.3. **Identities of certain semigroup varieties.** In the course of proving our results it will be convenient to have at our disposal a description of the identities of several concrete semigroup varieties. We denote by F the free semigroup over a countably infinite alphabet. The equality relation on F is denoted by $\equiv$. If u is a word and x is a letter then $c(u)$ stands for the set of all letters occurring in u, $\ell(u)$ is the length of u, $\ell_x(u)$ denotes the number of occurrences of x in u, while $t(u)$ is the last letter of u. The statements (i) and (ii) of the following lemma are well-known and can be easily verified. The statement (iii) was proved in [2, Lemma 7].

Lemma 2.8. *The identity $u = v$ holds in the variety:*

(i) $\mathcal{RZ}$ *if and only if* $t(u) \equiv t(v)$*;*

(ii) $\mathcal{C}_2$ *if and only if* $c(u) = c(v)$ *and, for every letter* $x \in c(u)$*, either* $\ell_x(u) > 1$ *and* $\ell_x(v) > 1$ *or* $\ell_x(u) = \ell_x(v) = 1$*;*

(iii) $\mathcal{P}$ *if and only if* $c(u) = c(v)$ *and either* $\ell_{t(u)}(u) > 1$ *and* $\ell_{t(v)}(v) > 1$ *or* $\ell_{t(u)}(u) = \ell_{t(v)}(v) = 1$ *and* $t(u) \equiv t(v)$*.* □

2.4. **Verbal subsets of free groups.** Similarly to the articles [7–9], we need here the technique developed by Sapir in [5]. We introduce the basic notation from that paper. Let $\mathcal{G}$ be a periodic group variety and $\{v_i = 1 \mid i \in I\}$ a basis of identities of $\mathcal{G}$ (as a variety of groups) where v_i are semigroup words. Let $r = \exp(\mathcal{G})$ where $\exp(\mathcal{G})$ stands for the exponent of the variety $\mathcal{G}$. For a letter x, put $x^0 = x^{r(r+1)}$. Let

$$S(\mathcal{G}) = \mathrm{var}\{xyz = xy^{r+1}z,\, x^0y^0 = y^0x^0,\, x^2 = x^{r+2},\, xv_i^2y = xv_iy \mid i \in I\}.$$

As it is shown in [5], the variety $S(\mathcal{G})$ does not depend on the particular choice of the basis $\{v_i = 1 \mid i \in I\}$ (see Remark 2.10 below). Furthermore, let $F(\mathcal{G})$ be the free group of countably infinite rank in $\mathcal{G}$. A subset X of $F(\mathcal{G})$ is called *verbal* if it is closed under all endomorphisms of $F(\mathcal{G})$. Clearly, a verbal subset X of $F(\mathcal{G})$ is a set of all values in $F(\mathcal{G})$ of some set W of non-empty words; in this case we write $X = \mathcal{G}(W)$. If X is a verbal subset in $F(\mathcal{G})$ and $X = \mathcal{G}(W)$ then we put

$$S(\mathcal{G}, X) = S(\mathcal{G}) \wedge \mathrm{var}\bigl\{xwx = (xwx)^{r+1} \mid w \in W\bigr\}.$$

If $X = \{1\}$ where 1 is the unit element of $F(\mathcal{G})$ then we will write $S(\mathcal{G}, 1)$ rather than $S\bigl(\mathcal{G}, \{1\}\bigr)$. It is convenient to consider the empty set as a verbal subset in $F(\mathcal{G})$ and put $S(\mathcal{G}, \varnothing) = S(\mathcal{G})$. If $\mathcal{H}$ is a subvariety of $\mathcal{G}$ and X is a verbal subset of $F(\mathcal{G})$ then we put

$$S(\mathcal{H}, X) = S(\mathcal{H}) \wedge S(\mathcal{G}, X). \tag{1}$$

To avoid a possible confusion, we note that the paper [5] does not contain an explicit definition of the variety $S(\mathcal{H}, X)$ where X is a verbal subset of $F(\mathcal{G})$ with $\mathcal{G} \ne \mathcal{X}$. But one can trace the argument of [5] to see that the equality (1) is what Sapir tacitly meant by this definition but failed to explicitly define.

As usual, if $\mathcal{X}$ is a variety then $L(\mathcal{X})$ stands for the subvariety lattice of $\mathcal{X}$. To prove Theorem 1.1, we need the following

Lemma 2.9 ([5]). *Let* $\mathcal{G}$ *be a variety of periodic groups. The interval* $\bigl[S(\mathcal{T}, 1),\, S(\mathcal{G})\bigr]$ *of the lattice* $L\bigl(S(\mathcal{G})\bigr)$ *consists of all varieties of the form* $S(\mathcal{H}, X)$ *where* $\mathcal{H} \subseteq \mathcal{G}$ *and* X *is a (possibly empty) verbal subset of* $F(\mathcal{G})$*. Here, for varieties* $S(\mathcal{H}, X)$ *and* $S(\mathcal{H}', X')$ *from the interval* $\bigl[S(\mathcal{T}, 1),\, S(\mathcal{G})\bigr]$*, the inclusion* $S(\mathcal{H}', X') \subseteq S(\mathcal{H}, X)$ *holds if and only if* $\mathcal{H}' \subseteq \mathcal{H}$ *and there exists a set of words* W *such that* $X = \mathcal{H}(W)$ *and* $\mathcal{H}'(W) \subseteq X'$*.* □

Remark 2.10. Lemma 2.9 shows that the construction of the variety $S(\mathcal{G}, X)$ is in fact independent of the actual choice of the 'generator' W of the verbal subset X; it is only X that really matters, as different choices of W will result in the same variety. In particular, by the definition of the variety $S(\mathcal{G}, X)$, it satisfies the identity $xwx = (xwx)^{r+1}$ whenever $w \in W$. In view of Lemma 2.9, this identity holds in $S(\mathcal{G}, X)$ not only for $w \in W$ but for any word w representing an element of X.

2.5. **Overcommutative varieties.** We denote by $\mathcal{COM}$ the variety of all commutative semigroups. A semigroup variety $\mathcal{V}$ is called *overcommutative* if $\mathcal{V} \supseteq \mathcal{COM}$. The lattice of all overcommutative varieties is denoted by $\mathbf{OC}$. The structure of this lattice was clarified by Volkov in [13]. It turns out that the lattice $\mathbf{OC}$ admits a concise and transparent description in terms of congruence lattices of unary algebras of some special type, called G-sets. This description plays an essential role in the proof of Theorem 1.1. To reproduce the result from [13], we need some new definitions and notation.

Let A be a non-empty set. We denote by $\mathbf{S}_A$ the group of all permutations on A. If $A = \{1, 2, \ldots, m\}$ then we will write $\mathbf{S}_m$ rather than $\mathbf{S}_{\{1,2,\ldots,m\}}$. A *$G$-set* is a unary algebra on a set A where the unary operations form a group of permutations on A (that is, the subgroup of $\mathbf{S}_A$). The congruence lattice of a G-set A is denoted by $\mathrm{Con}(A)$.

Let m and n be positive integers with $2 \le m \le n$. A sequence $\lambda = (\ell_1, \ell_2, \ldots, \ell_m)$ of positive integers such that

$$\ell_1 \ge \ell_2 \ge \cdots \ge \ell_m \text{ and } \sum_{i=1}^{m} \ell_i = n$$

is said to be a *partition of the number n into m parts.* For a word u, we put $\mathrm{part}(u) = \bigl(\ell_{x_1}(u), \ell_{x_2}(u), \ldots, \ell_{x_m}(u)\bigr)$ where $m = \max\bigl\{i \mid x_i \in c(u)\bigr\}$. Let us fix positive integers m and n with $2 \le m \le n$ and a partition $\lambda = (\ell_1, \ell_2, \ldots, \ell_m)$ of the number n into m parts. Put

$$W_\lambda = \bigl\{u \in F \mid \ell(u) = n,\ c(u) = \{x_1, x_2, \ldots, x_m\} \text{ and } \mathrm{part}(u) = \lambda\bigr\},$$
$$\mathbf{S}_\lambda = \{\sigma \in \mathbf{S}_m \mid \ell_i = \ell_{i\sigma} \text{ for all } i = 1, 2, \ldots, m\}.$$

Clearly, $\mathbf{S}_\lambda$ is a subgroup in $\mathbf{S}_m$.

If $u \equiv x_{i_1} x_{i_2} \cdots x_{i_n}$ where $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$ are (not necessarily different) letters and $\pi \in \mathbf{S}_{c(u)}$ then we denote by $u\pi$ the word $x_{i_1\pi} x_{i_2\pi} \cdots x_{i_n\pi}$. It is clear that if $u \in W_\lambda$ and $\pi \in \mathbf{S}_\lambda$ then $u\pi \in W_\lambda$. For every $\pi \in \mathbf{S}_\lambda$, we define the unary operation π^* on W_λ by letting $\pi^*(u) \equiv u\pi$ for any word $u \in W_\lambda$. Obviously, the set W_λ with the collection of unary operations $\{\pi^* \mid \pi \in \mathbf{S}_\lambda\}$ is an $\mathbf{S}_\lambda$-set. The description of the lattice $\mathbf{OC}$ mentioned above is given by the following

Proposition 2.11 ([13])**.** *The lattice* $\mathbf{OC}$ *is anti-isomorphic to a subdirect product of congruence lattices* $\mathrm{Con}(W_\lambda)$ *where λ runs over the set of all partitions.* □

3. Proof of Theorem 1.1

Sufficiency immediately follows from Lemmas 2.4 and 2.5 and the evident fact that the variety $\mathcal{SEM}$ is lower-modular.

Necessity. Let $\mathcal{V}$ be a proper lower-modular semigroup variety. Lemma 2.1 implies that the variety $\mathcal{W} = \mathcal{V} \vee \mathcal{COM}$ is a lower-modular element of the lattice $\mathbf{OC}$. The variety $\mathcal{W}$ is proper because the variety $\mathcal{SEM}$ is not decomposable into the join of any two proper varieties [1].

Recall that an identity $u = v$ is called *balanced* if each letter occurs in u and v the same number of times. It is well-known that if an overcommutative variety satisfies some identity then this identity is balanced.

Being proper, the variety $\mathcal{W}$ satisfies a non-trivial balanced identity $u = v$. Let $|c(u)| = m$. We may assume that $c(u) = \{x_1, x_2, \dots, x_m\}$ and $\ell_{x_1}(u) \ge \ell_{x_2}(u) \ge \cdots \ge \ell_{x_m}(u)$ (otherwise we may rename letters). Put $\ell_i = \ell_{x_i}(u)$ for all $i = 1, 2, \dots, m$. Then $\mathrm{part}(u) = \mathrm{part}(v) = (\ell_1, \ell_2, \dots, \ell_m)$. We may assume that $\ell_1 > \ell_2 > \cdots > \ell_m > 1$ (if it is not the case, we may multiply $u = v$ by an appropriate word from the right). Let x and y be arbitrary letters with $x, y \notin c(u)$ and $\lambda = \mathrm{part}(xyu) = (\ell_1, \ell_2, \dots, \ell_m, 1, 1)$ (we identify here x and y with x_{m+1} and x_{m+2} respectively). We denote by ν the fully invariant congruence on F corresponding to the variety $\mathcal{W}$ and by α the restriction of ν to W_λ. Then

$$xyu \,\alpha\, xyv,\ yxu \,\alpha\, yxv,\ xuy \,\alpha\, xvy,\ yux \,\alpha\, yvx.$$

Proposition 2.11 implies that there is a surjective homomorphism from the lattice dual to $\mathbf{OC}$ onto $\mathrm{Con}(W_\lambda)$. Now Lemma 2.2 applies with the conclusion that α is an upper-modular element of the lattice $\mathrm{Con}(W_\lambda)$. Since $\ell_1 > \ell_2 > \cdots > \ell_m > 1$, the group $\mathbf{S}_\lambda$ consists of two elements. Let β be the equivalence relation on W_λ with only two non-singleton classes $\{xyu, xyv\}$ and $\{yxu, yxv\}$, and let γ be the equivalence relation on W_λ with only four non-singleton classes $\{xyu, xuy\}$, $\{xyv, xvy\}$, $\{yxu, yux\}$, and $\{yxv, yvx\}$. It is evident that β and γ are congruences on W_λ and $\beta \subseteq \alpha$. Therefore $(\gamma \vee \beta) \wedge \alpha = (\gamma \wedge \alpha) \vee \beta$.

Note that $xuy \,\gamma\, xyu \,\beta\, xyv \,\gamma\, xvy$. Therefore $(xuy, xvy) \in \gamma \vee \beta$, whence

$$(xuy, xvy) \in (\gamma \vee \beta) \wedge \alpha = (\gamma \wedge \alpha) \vee \beta.$$

This means that there is a word $w \in W_\lambda$ such that $xuy \not\equiv w$ and either $(xuy, w) \in \gamma \wedge \alpha$ or $xuy \,\beta\, w$. But the latter contradicts the choice of the congruence β. Hence $(xuy, w) \in \gamma \wedge \alpha$. In particular, $xuy \,\gamma\, w$. The definition of γ implies that $w \equiv xyu$. Thus $(xuy, xyu) \in \gamma \wedge \alpha$. In particular, we have verified that $xuy \,\alpha\, xyu$.

Let γ' be the equivalence relation on W_λ with only four non-singleton classes $\{xyu, yux\}$, $\{xyv, yvx\}$, $\{yxu, xuy\}$, and $\{yxv, xvy\}$. Clearly, γ' is a congruence on W_λ. Now we may repeat almost literally arguments from the previous paragraph with using γ' instead of γ. As a result, we obtain that $xyu \,\alpha\, yux$. Thus $xuy \,\alpha\, xyu \,\alpha\, yux$. Since $\alpha \subseteq \nu$, we have $xuy \,\nu\, yux$. This means that the identity

$$xuy = yux \tag{2}$$

holds in the variety $\mathcal{W}$. Therefore this identity holds in the variety $\mathcal{V}$ as well.

Lemma 2.8 and its dual imply that the identity (2) fails in the varieties $\mathcal{LZ}$, $\mathcal{RZ}$, $\mathcal{P}$, and $\overleftarrow{\mathcal{P}}$. Thus $\mathcal{V}$ does not contain these varieties. By Lemma 2.3 the variety $\mathcal{V}$ is periodic. Now Lemma 2.6 applies with the conclusion that $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where the variety $\mathcal{M}$ is generated by a monoid and $\mathcal{N} = \mathrm{Nil}(\mathcal{V})$. Lemma 2.3 implies that the variety $\mathcal{N}$ is 0-reduced. It remains to verify that $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$.

Since $\mathcal{M}$ is generated by a monoid, the set of its identities is closed for deleting letters; therefore, by deleting of all letters from $c(u)$ in (2), we obtain that $\mathcal{M}$ is commutative. Now we can apply Lemma 2.7 and conclude that $\mathcal{M} = \mathcal{G} \vee \mathcal{C}_m$ for some Abelian periodic group variety $\mathcal{G}$ and some $m \geq 0$. Suppose that $m \geq 2$. Then $\mathcal{V} \supseteq \mathcal{C}_2$. It is easy to deduce from Lemma 2.8 that $\mathcal{C}_2 \vee \mathcal{RZ} \supseteq \mathcal{P}$. Hence $\mathcal{V} \vee \mathcal{RZ} \supseteq \mathcal{C}_2 \vee \mathcal{RZ} \supseteq \mathcal{P}$. Put $\mathcal{U} = \mathcal{V} \vee \mathcal{P}$. Note that $\mathcal{V}, \mathcal{P} \not\supseteq \mathcal{RZ}$. As is well-known, the variety $\mathcal{RZ}$ is an atom of the lattice **SEM**. Therefore $\mathcal{V} \wedge \mathcal{RZ} = \mathcal{P} \wedge \mathcal{RZ} = \mathcal{T}$. It is well-known also that the lattice **SEM** is 0-*distributive*, that is, satisfies the condition

$$\forall x, y, z\colon \quad x \wedge z = y \wedge z = 0 \longrightarrow (x \vee y) \wedge z = 0$$

(see [6, Section 1], for instance). Therefore $\mathcal{U} \wedge \mathcal{RZ} = (\mathcal{V} \vee \mathcal{P}) \wedge \mathcal{RZ} = \mathcal{T}$. Combining these observations, we have

$$\begin{aligned}
\mathcal{V} &= \mathcal{V} \vee \mathcal{T} && \\
&= \mathcal{V} \vee (\mathcal{U} \wedge \mathcal{RZ}) && \text{because } \mathcal{U} \wedge \mathcal{RZ} = \mathcal{T} \\
&= \mathcal{U} \wedge (\mathcal{V} \vee \mathcal{RZ}) && \text{because } \mathcal{V} \text{ is lower-modular and } \mathcal{V} \subseteq \mathcal{U} \\
&\supseteq \mathcal{P} && \text{because } \mathcal{U} = \mathcal{V} \vee \mathcal{P} \supseteq \mathcal{P} \text{ and } \mathcal{V} \vee \mathcal{RZ} \supseteq \mathcal{P}.
\end{aligned}$$

Thus $\mathcal{V} \supseteq \mathcal{P}$. A contradiction shows that $m \leq 1$, whence $\mathcal{M} = \mathcal{G} \vee \mathcal{K}$ where $\mathcal{K}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$. It remains to check that $\mathcal{G} = \mathcal{T}$.

The remaining part of the proof is based on Lemma 2.9. Note that in what follows we combine (with slight modifications) arguments from the proofs of [8, Lemma 2.2] and [9, Proposition 3.1].

Suppose that $\mathcal{G} \neq \mathcal{T}$. Put $\mathcal{Y} = \mathcal{V} \vee S(\mathcal{G}, 1)$ and $\mathcal{Z} = S(\mathcal{T})$. Further considerations are illustrated by Fig. 1.

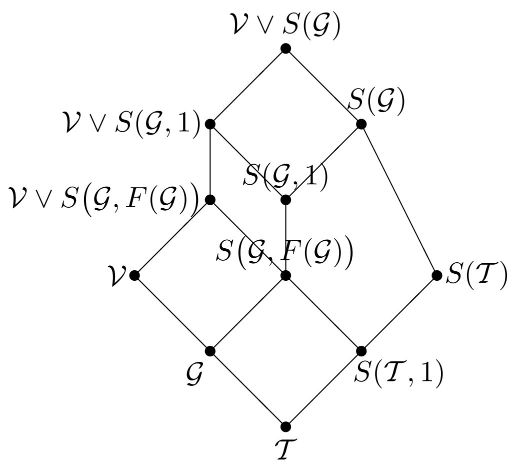

FIGURE 1. A fragment of the lattice $L(\mathcal{V} \vee S(\mathcal{G}))$

Lemma 2.9 implies that $S(\mathcal{G}) = S(\mathcal{T}) \vee \mathcal{G}$ (see Fig. 1). Using this equality and the inclusion $\mathcal{G} \subseteq \mathcal{V}$, we have

$$\mathcal{Y} = S(\mathcal{G}, 1) \vee \mathcal{V} \subseteq S(\mathcal{G}) \vee \mathcal{V} = S(\mathcal{T}) \vee \mathcal{G} \vee \mathcal{V} = S(\mathcal{T}) \vee \mathcal{V} = \mathcal{Z} \vee \mathcal{V}.$$

Therefore $(\mathcal{Z} \vee \mathcal{V}) \wedge \mathcal{Y} = \mathcal{Y}$. Since the variety $\mathcal{V}$ is lower-modular and $\mathcal{V} \subseteq \mathcal{Y}$, we have $(\mathcal{Z} \wedge \mathcal{Y}) \vee \mathcal{V} = (\mathcal{Z} \vee \mathcal{V}) \wedge \mathcal{Y}$, whence

$$(\mathcal{Z} \wedge \mathcal{Y}) \vee \mathcal{V} = \mathcal{Y}. \tag{3}$$

Furthermore, Lemma 2.9 implies that $S(\mathcal{T}, 1) = S(\mathcal{T}) \wedge S(\mathcal{G}, 1)$ (see Fig. 1). Therefore

$$S(\mathcal{T}, 1) = S(\mathcal{T}) \wedge S(\mathcal{G}, 1) \subseteq S(\mathcal{T}) \wedge \big(\mathcal{V} \vee S(\mathcal{G}, 1)\big) = \mathcal{Z} \wedge \mathcal{Y} \subseteq \mathcal{Z} = S(\mathcal{T}),$$

that is, $S(\mathcal{T}, 1) \subseteq \mathcal{Z} \wedge \mathcal{Y} \subseteq S(\mathcal{T})$. It is evident that the group $F(\mathcal{T})$ contains only two verbal subsets, namely $\varnothing$ and $\{1\}$. Therefore Lemma 2.9 implies that the interval $\big[S(\mathcal{T}, 1),\, S(\mathcal{T})\big]$ of the lattice $L\big(S(\mathcal{T})\big)$ consists of the varieties $S(\mathcal{T}, 1)$ and $S(\mathcal{T})$ only. Thus either $\mathcal{Z} \wedge \mathcal{Y} = S(\mathcal{T}, 1)$ or $\mathcal{Z} \wedge \mathcal{Y} = S(\mathcal{T})$. Let us consider these two cases separately. Let $\exp(\mathcal{G}) = r$.

Case 1: $\mathcal{Z} \wedge \mathcal{Y} = S(\mathcal{T}, 1)$. Lemma 2.9 implies that $S(\mathcal{T}, 1) \vee \mathcal{G} = S\big(\mathcal{G}, F(\mathcal{G})\big)$ (see Fig. 1). Using the equality (3) and the inclusion $\mathcal{G} \subseteq \mathcal{V}$, we have

$$\begin{aligned} S\big(\mathcal{G}, F(\mathcal{G})\big) \vee \mathcal{V} &= S(\mathcal{T}, 1) \vee \mathcal{G} \vee \mathcal{V} = S(\mathcal{T}, 1) \vee \mathcal{V} \\ &= (\mathcal{Z} \wedge \mathcal{Y}) \vee \mathcal{V} = \mathcal{Y} = S(\mathcal{G}, 1) \vee \mathcal{V}, \end{aligned}$$

that is,

$$S\big(\mathcal{G}, F(\mathcal{G})\big) \vee \mathcal{V} = S(\mathcal{G}, 1) \vee \mathcal{V}. \tag{4}$$

Being a nilvariety, $\mathcal{N}$ satisfies an identity $u = 0$ for some word u. Suppose that the variety $\mathcal{G}$ (considered as a variety of groups) satisfies the identity $u = 1$. Let x be a letter with $x \notin c(u)$. Since the variety $\mathcal{G}$ is non-trivial, it does not satisfy the identity $ux = 1$. It is evident that $ux = 0$ in $\mathcal{N}$. Thus there is a word w such that the variety $\mathcal{N}$ satisfies the identity $w = 0$ but the variety $\mathcal{G}$ does not satisfy the identity $w = 1$. Let x be a letter with $x \notin c(w)$. Remark 2.10 implies that $S\big(\mathcal{G}, F(\mathcal{G})\big)$ satisfies the identity

$$xwx = (xwx)^{r+1}. \tag{5}$$

This identity holds in the variety $\mathcal{V}$ as well because $\mathcal{V} \subseteq \mathcal{G} \vee \mathcal{SL} \vee \mathcal{N}$. Therefore the variety $\mathcal{V} \vee S\big(\mathcal{G}, F(\mathcal{G})\big)$ satisfies the identity (5). But (5) fails in the variety $S(\mathcal{G}, 1)$ by the definition of this variety, whence (5) fails in the variety $\mathcal{V} \vee S(\mathcal{G}, 1)$. We have a contradiction with the equality (4).

Case 2: $\mathcal{Z} \wedge \mathcal{Y} = S(\mathcal{T})$. As we have already noted above, $S(\mathcal{G}) = S(\mathcal{T}) \vee \mathcal{G}$ (see Fig. 1). Taking into account the equality (3) and the inclusion $\mathcal{G} \subseteq \mathcal{V}$, we have

$$S(\mathcal{G}, 1) \vee \mathcal{V} = \mathcal{Y} = (\mathcal{Z} \wedge \mathcal{Y}) \vee \mathcal{V} = S(\mathcal{T}) \vee \mathcal{V} = S(\mathcal{T}) \vee \mathcal{G} \vee \mathcal{V} = S(\mathcal{G}) \vee \mathcal{V}.$$

We see that

$$S(\mathcal{G}, 1) \vee \mathcal{V} = S(\mathcal{G}) \vee \mathcal{V}. \tag{6}$$

Let w be an arbitrary word such that the variety $\mathcal{G}$ satisfies (as a variety of groups) the identity $w = 1$. Being a nil-variety, $\mathcal{N}$ satisfies the identity $x^n = 0$ for some n. The variety $\mathcal{G}$ satisfies the identity $w^{2n} = 1$. Remark 2.10 implies that the variety $S(\mathcal{G}, 1)$ satisfies the identity

$$xw^{2n}x = (xw^{2n}x)^{r+1}. \tag{7}$$

This identity holds in the varieties $\mathcal{M}$ and $\mathcal{N}$ as well, whence it holds in $\mathcal{V}$, and therefore in $S(\mathcal{G}, 1) \vee \mathcal{V}$. The equality (6) implies that (7) holds in $S(\mathcal{G}) \vee \mathcal{V}$, and therefore in $S(\mathcal{G})$. We always may include the identity $w = 1$ in the identity basis of $\mathcal{G}$. By the definition of the variety $S(\mathcal{G})$, it satisfies the identity $xwx = xw^2x$, and therefore the identities

$$\begin{aligned} xwx = xw^2x = xw^4x \equiv x(w \cdot w^2 \cdot w)x = x(w \cdot w^4 \cdot w)x \equiv xw^6x \\ \equiv x(w^2 \cdot w^2 \cdot w^2)x = x(w^2 \cdot w^4 \cdot w^2)x \equiv xw^8x = \cdots = xw^{2n}x. \end{aligned}$$

Combining the identities $xwx = xw^{2n}x$ and (7), we have that the identities

$$xwx = xw^{2n}x = (xw^{2n}x)^{r+1} = (xwx)^{r+1}$$

hold in $S(\mathcal{G})$. Thus $S(\mathcal{G})$ satisfies the identity (5) whenever $\mathcal{G}$ satisfies $w = 1$. Therefore $S(\mathcal{G}) \subseteq S(\mathcal{G}, 1)$ but this inclusion contradicts Lemma 2.9.

We have verified that $\mathcal{G} = \mathcal{T}$ and completed the proof of Theorem 1.1. □

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