LOWER-MODULAR ELEMENTS OF THE LATTICE OF SEMIGROUP VARIETIES. II

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Dedicated to a memory of my father

ABSTRACT. A semigroup variety is called *modular* [*upper-modular, lower-modular, neutral*] if it is a modular [respectively upper-modular, lower-modular, neutral] element of the lattice of all semigroup varieties. We classify all lower-modular varieties in the class of varieties of semigroups with a completely regular power, in the class of varieties of index ≤ 2, and in the class of varieties satisfying an identity of the form $x_1x_2\cdots x_n = x_{1\pi}x_{2\pi}\cdots x_{n\pi}$ where π is a permutation on the set $\{1, 2, \dots, n\}$ with $1\pi \neq 1$ and $n\pi \neq n$. It turns out that every lower-modular variety is modular in all these three classes. Moreover, for varieties of index ≤ 2, the properties of being lower-modular, modular and neutral are equivalent. We completely determine also all semigroup varieties that are both upper-modular and lower-modular. It turns out that all such varieties are neutral.

INTRODUCTION

The class of all semigroup varieties forms a lattice under the following naturally defined operations: for varieties $\mathcal{X}$ and $\mathcal{Y}$, their *join* $\mathcal{X} \vee \mathcal{Y}$ is the variety generated by the set-theoretical union of $\mathcal{X}$ and $\mathcal{Y}$ (as classes of semigroups), while their *meet* $\mathcal{X} \wedge \mathcal{Y}$ coincides with the set-theoretical intersection of $\mathcal{X}$ and $\mathcal{Y}$. Special elements of different types in lattices of varieties of semigroups or universal algebras have been examined in several articles (see, for instance, [4, 5, 9–14, 16]). Here we continue these investigations. Recall the definitions of special elements considered in this article.

An element x of a lattice $\langle L; \vee, \wedge\rangle$ is called *modular* if

$$\forall\, y, z \in L\colon \quad y \leq z \longrightarrow (x \vee y) \wedge z = (x \wedge z) \vee y,$$

and *lower-modular* if

$$\forall\, y, z \in L\colon \quad x \leq y \longrightarrow (z \vee x) \wedge y = (z \wedge y) \vee x.$$

Upper-modular elements are defined dually to lower-modular ones. An element x of a lattice L is called *neutral* if

$$\forall y, z \in L\colon \quad (x \vee y) \wedge (y \vee z) \wedge (z \vee x) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x).$$

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As is well known, x is neutral if and only if, for all $y, z \in L$, the sublattice of L generated by x, y and z is distributive (see [3, Theorem III.2.4], for instance).

For convenience, we call a semigroup variety *modular* [*lower-modular, upper-modular, neutral*] if it is a modular [lower-modular, upper-modular, neutral] element of the lattice $\mathbb{SEM}$ of all semigroup varieties. A number of results about varieties of these four types have been obtained in [5, 10–14, 16]. In particular, lower-modular varieties have been examined in [11]. This article is a direct continuation of [11].

In [11], we have found a necessary condition for a semigroup variety to be lower-modular (see [11, Theorem 1] or Proposition 1.3 below), completely determined all commutative lower-modular varieties (see [11, Theorem 2] or Corollary 4.2 below) and described lower-modular nil-varieties (see [11, Corollary 2.7]). Recall some definitions. A semigroup variety is called *completely regular* if it consists of *completely regular* semigroups (that is, unions of groups). A semigroup variety is called *combinatorial* if all its groups are singleton. A semigroup variety is said to be *proper* if it differs from the variety $\mathcal{SEM}$ of all semigroups. The following open problems and questions were formulated in [11].

Problem 1 ([11, Problem 3.1]). *Describe the completely regular semigroup varieties that are lower-modular elements of the lattice* $\mathbb{SEM}$.

Problem 2 ([11, Problem 3.3]). *Describe the semigroup varieties that are both upper-modular and lower-modular elements of the lattice* $\mathbb{SEM}$.

Question 1 ([11, Question 3.2]). *Let* $\mathcal{V}$ *be a proper semigroup variety and a lower-modular element of the lattice* $\mathbb{SEM}$. *Is the variety* $\mathcal{V}$ *combinatorial*?

Question 2 ([11, Question 3.7]). *Does there exist a semigroup variety that is a lower-modular but not modular element of the lattice* $\mathbb{SEM}$?

Note that an interest to Problem 2 is strengthened by the fact that semigroup varieties that are both modular and upper-modular have been completely determined in [14, Theorem 1], while varieties that are both modular and lower-modular have been described in [16, Theorem 3.1].

In this article we solve Problems 1 and 2 in the general case and answer Questions 1 and 2 in three special cases. The article contains three main results. To characterize these results, we need some new definitions. A semigroup variety $\mathcal{V}$ is called a *variety of semigroups with a completely regular power* (CRP-*variety*, for brevity) if there exists a positive integer n such that, for every member S of $\mathcal{V}$, the semigroup S^n is completely regular. It is evident that every completely regular variety is a CRP-variety. The first main result of the article (Theorem 2.4) gives a description of lower-modular CRP-varieties. A semigroup variety is said to be a *variety of finite index* [*of index* n] if all its nilsemigroups are nilpotent of index $\le n$ for some n [and n is a least number with such a property]. As is well known, the varieties of index 1 are precisely the completely regular varieties. The second main result of the article (Theorem 3.3) gives a description of lower-modular varieties of index ≤ 2. As we have already mentioned, commutative lower-modular varieties are completely determined in [11, Theorem 2]. A natural generalization of the commutative law is a *permutation* identity, that is, an identity of the form

$$x_1x_2\cdots x_n = x_{1\pi}x_{2\pi}\cdots x_{n\pi} \tag{1}$$

where π is a non-trivial permutation on the set $\{1, 2, \dots, n\}$. The third main result of the article (Theorem 4.1) gives a description of lower-modular varieties satisfying an identity of the form (1) with $1\pi \ne 1$ and $n\pi \ne n$.

Each of Theorems 2.4 and 3.3 immediately implies a solution of Problem 1 (see Corollary 3.4). Theorem 3.3, together with a result of [11], implies a solution of Problem 2 (see Corollary 3.5). In particular, it turns out that a semigroup variety is both upper-modular and lower-modular if and only if it is neutral. Besides that, Theorems 2.4, 3.3 and 4.1 imply an affirmative answer to Question 1 and a negative answer to Question 2 in the classes of varieties considered in these theorems[1]. For one of these classes, we have a result that is essentially stronger than the statement following from a negative answer to Question 2. Namely, Theorem 3.3 shows that, for varieties of index ≤ 2, the properties of being lower-modular, modular and neutral are equivalent.

Note that Theorem 2.4 plays the key role in the article. In fact, two other theorems are proved by a reduction to Theorem 2.4. It turns out that, within the classes of varieties considered in Theorems 3.3 and 4.1, every lower-modular variety is a CRP-variety.

CRP-varieties and varieties of index ≤ 2 are varieties of finite index. The success in determining the lower-modular varieties within these two classes of varieties inspires the following

Problem 3. *Describe the semigroup varieties of finite index that are lower-modular elements of the lattice* $\mathbb{SEM}$.

A semigroup variety is called *permutational* if it satisfies a permutation identity. The following problem seems to be natural in view of Theorem 4.1.

Problem 4. *Describe the permutational semigroup varieties that are lower-modular elements of the lattice* $\mathbb{SEM}$.

Note that a solution of Problem 3 would imply a solution of Problem 4 (see Corollary 1.4 below).

The article is structured as follows. It consists of 4 sections. Section 1 contains some preliminary information. In Sections 2, 3 and 4, we prove Theorems 2.4, 3.3 and 4.1, respectively.

1. Preliminaries

We denote by $\mathcal{T}$ the trivial semigroup variety, by $\mathcal{SL}$ the variety of all semilattices and by $\mathcal{ZM}$ the variety of all semigroups with zero multiplication. The following lemma is well known (see [1], for instance).

Lemma 1.1. *The varieties $\mathcal{SL}$ and $\mathcal{ZM}$ are atoms of the lattice* $\mathbb{SEM}$. $\square$

[1] Note that Question 2 is also answered in the negative for nil-varieties [11, Corollary 2.8].

We will denote by F the free semigroup over a countably infinite set. Clearly, if w is a semigroup word then a semigroup S satisfies the identity system $wu = uw = w$, where u runs over F, if and only if S contains a zero element 0 and all values of the word w in S equal 0. We adopt the usual convention of writing $w = 0$ as a short form of such a system and referring to the expression $w = 0$ as to a single identity. A semigroup variety is called a *nil-variety* if it satisfies an identity of the form $x^n = 0$ for some positive integer n. The following proposition readily follows from [5, Proposition 1.6] (a deduction of Proposition 1.2 from the mentioned result of [5] is given explicitly in [10, Proposition 2.1]).

Proposition 1.2. *If a proper semigroup variety $\mathcal{V}$ is modular then $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$, while $\mathcal{N}$ is a nil-variety.* $\square$

Note that Proposition 1.2 has been essentially sharpened in [10, Theorem 2.5].

Recall that a semigroup variety is called *periodic* if every member of it is periodic. It is well known and easy to see that an arbitrary periodic variety $\mathcal{V}$ contains a greatest nil-subvariety. We denote this subvariety by $\mathrm{Nil}(\mathcal{V})$. The identities of the form $w = 0$ we call 0*-reduced*. A semigroup variety is called 0*-reduced* if it can be defined by 0-reduced identities only. (In the literature such varieties were sometimes referred to as *Rees*, see [16], for instance.)

Proposition 1.3 ([11, Theorem 1])**.** *If a proper semigroup variety $\mathcal{V}$ is a lower-modular element of the lattice $\mathbb{SEM}$ then $\mathcal{V}$ is periodic and the variety $\mathrm{Nil}(\mathcal{V})$ is 0-reduced.* $\square$

A semigroup variety is called *nilpotent* [*of index m*] if it satisfies an identity of the form $x_1x_2\cdots x_m = 0$ for some positive integer m [and m is a least number with such a property]. For a permutation identity (1), the number n is called a *length* of the identity. It is evident that if a 0-reduced variety $\mathcal{N}$ satisfies a permutation identity of length n then $\mathcal{N}$ is nilpotent of index $\le n$. Therefore Proposition 1.3 implies the following

Corollary 1.4. *If a semigroup variety $\mathcal{V}$ is a lower-modular element of the lattice $\mathbb{SEM}$ and $\mathcal{V}$ satisfies a permutation identity of length n then $\mathcal{V}$ is a variety of index $\le n$.* $\square$

Proposition 1.5 ([16, Proposition 2.4])**.** *A semigroup variety is a neutral element of the lattice $\mathbb{SEM}$ if and only if it is one of the varieties $\mathcal{T}$, $\mathcal{SL}$, $\mathcal{ZM}$, $\mathcal{SL} \vee \mathcal{ZM}$ or $\mathcal{SEM}$.* $\square$

We need some additional notation. The symbol $\equiv$ stands for the equality relation on the semigroup F. If w is a word and x is a letter then we denote by $\ell(w)$ the length of w, by $\ell_x(w)$ the number of occurrences of x in w, by $c(w)$ the set of all letters occurring in w, and by $t(w)$ the last letter of w. We denote by $\mathcal{LZ}$ [respectively $\mathcal{RZ}$] the variety of all left [right] zero semigroups. As usual, we denote by $\mathrm{var}\,\Sigma$ the semigroup variety given by the identity system Σ. Put

$$\mathcal{C} = \mathrm{var}\{x^2 = x^3,\, xy = yx\},$$
$$\mathcal{P} = \mathrm{var}\{xy = x^2y,\, x^2y^2 = y^2x^2\}.$$

It is well known and may be easily verified that the variety $\mathcal{C}$ is generated by the semigroup $\{0, c, 1\}$ where $\{0, c\}$ is the 2-element semigroup with zero multiplication and 1 is the unit element. It is well known also that the variety $\mathcal{P}$ is generated by the semigroup $\{e, a, 0\}$ where $e^2 = e$, $ea = a$ and all other products are equal to 0. By $\overleftarrow{\mathcal{P}}$ we denote the semigroup variety dual to $\mathcal{P}$. The claims (i) and (ii) of the following lemma are well known and may be easily verified, the claim (iii) was proved in [2, Lemma 7].

Lemma 1.6. *The identity $u = v$ holds in the variety*:

(i) $\mathcal{RZ}$ *if and only if* $t(u) \equiv t(v)$;

(ii) $\mathcal{C}$ *if and only if* $c(u) = c(v)$ *and, for every letter* $x \in c(u)$, *either* $\ell_x(u), \ell_x(v) > 1$ *or* $\ell_x(u) = \ell_x(v) = 1$;

(iii) $\mathcal{P}$ *if and only if* $c(u) = c(v)$ *and either* $\ell_{t(u)}(u), \ell_{t(v)}(v) > 1$ *or* $\ell_{t(u)}(u) = \ell_{t(v)}(v) = 1$ *and* $t(u) \equiv t(v)$. $\square$

Recall that a semigroup variety is called a *variety with central idempotents* if it satisfies the quasiidentity $e^2 = e \longrightarrow ex = xe$. It is verified in [15, Lemma 2] that if a semigroup variety $\mathcal{V}$ contains none of the varieties $\mathcal{P}$, $\overleftarrow{\mathcal{P}}$, $\mathcal{LZ}$ and $\mathcal{RZ}$ then $\mathcal{V}$ is a variety with central idempotents. Further, it follows from the proof of [15, Proposition 1] that a periodic semigroup variety with central idempotents is the join of a variety generated by some monoid and a nil-variety. Thus we have the following

Lemma 1.7. *If a periodic semigroup variety $\mathcal{V}$ contains none of the varieties $\mathcal{P}$, $\overleftarrow{\mathcal{P}}$, $\mathcal{LZ}$ and $\mathcal{RZ}$ then $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where the variety $\mathcal{M}$ is generated by some monoid and $\mathcal{N} = \mathrm{Nil}(\mathcal{V})$.* $\square$

The following claim was announced by A. P. Birjukov in 1981 (see [7, Section 8], for instance). But its proof is published first in [12, Proposition 2.11], as far as we know.

Lemma 1.8. *Let n be a positive integer. A semigroup variety is a variety of index $\leq n$ if and only if it satisfies an identity of the form*

$$x_1 \cdots x_n = x_1 \cdots x_{i-1}(x_i \cdots x_j)^{r+1} x_{j+1} \cdots x_n$$

for some positive integer r and some i, j with $1 \leq i \leq j \leq n$. $\square$

If $\mathcal{V}$ is a CRP-variety then a minimal number n such that, for every $S \in \mathcal{V}$, the semigroup S^n is completely regular will be denoted by $\mathrm{crp}(\mathcal{V})$. For a semigroup variety $\mathcal{V}$, we write $\mathrm{ind}(\mathcal{V}) = n$ if $\mathcal{V}$ is a variety of (finite) index n, and $\mathrm{ind}(\mathcal{V}) = \infty$ if $\mathcal{V}$ is not a variety of finite index.

Lemma 1.9. *Let n be a positive integer. For a semigroup variety $\mathcal{V}$, the following are equivalent*:

a) $\mathcal{V}$ *is a variety of semigroups with a completely regular power and* $\mathrm{crp}(\mathcal{V}) \leq n$;

b) $\mathcal{V}$ *satisfies an identity of the form*

$$x_1 x_2 \cdots x_n = (x_1 x_2 \cdots x_n)^{r+1} \tag{2}$$

for some positive integer r;

c) $\mathrm{ind}(\mathcal{V}) \le n$ *and* $\mathcal{V}$ *contains none of the varieties* $\mathcal{P}$ *and* $\overleftarrow{\mathcal{P}}$.

Proof. As is well known, a semigroup variety is completely regular if and only if it satisfies an identity of the form $x = x^{r+1}$ for some positive integer r. This immediately implies the equivalence of the claims a) and b).

b) $\Longrightarrow$ c). If $\mathcal{V}$ satisfies an identity of the form (2) then $\mathcal{V}$ contains none of the varieties $\mathcal{P}$ and $\overleftarrow{\mathcal{P}}$ by Lemma 1.6(iii) and its dual, and $\mathrm{ind}(\mathcal{V}) \le n$ by Lemma 1.8.

The implication c) $\Longrightarrow$ b) was proved in [8, Theorem 2]. $\square$

2. Varieties of semigroups with a completely regular power

To prove the main result of this section (Theorem 2.4), we need the technique developed by Sapir in [6]. We introduce the basic notation from that paper. Let $\mathcal{G}$ be a periodic group variety and $\{v_i = 1 \mid i \in I\}$ a basis of identities of $\mathcal{G}$ (as a variety of groups) where v_i are semigroup words. Let $r = \exp(\mathcal{G})$ where $\exp(\mathcal{G})$ stands for the exponent of the variety $\mathcal{G}$. For a letter x, put $x^0 = x^{r(r+1)}$. Let

$$S(\mathcal{G}) = \mathrm{var}\{xyz = xy^{r+1}z,\ x^0y^0 = y^0x^0,\ x^2 = x^{r+2},\ xv_i^2y = xv_iy \mid i \in I\}.$$

As it is shown in [6], the variety $S(\mathcal{G})$ does not depend on the particular choice of the basis $\{v_i = 1 \mid i \in I\}$. Furthermore, let $F(\mathcal{G})$ be the free group of countably infinite rank in $\mathcal{G}$. A subset X of $F(\mathcal{G})$ is called *verbal* if it is closed under all endomorphisms of $F(\mathcal{G})$. Clearly, a verbal subset X of $F(\mathcal{G})$ is a set of all values in $F(\mathcal{G})$ of some set W of words; in this case we write $X = \mathcal{G}(W)$. If X is a verbal subset in $F(\mathcal{G})$ and $X = \mathcal{G}(W)$ then we put

$$S(\mathcal{G}, X) = S(\mathcal{G}) \wedge \mathrm{var}\bigl\{xwx = (xwx)^{r+1} \mid w \in W\bigr\}.$$

If $X = \{1\}$ where 1 is the unit element of $F(\mathcal{G})$ then we will write $S(\mathcal{G}, 1)$ rather than $S\bigl(\mathcal{G}, \{1\}\bigr)$. It is convenient to consider the empty set as a verbal subset in $F(\mathcal{G})$ and put $S(\mathcal{G}, \varnothing) = S(\mathcal{G})$.

As usual, if $\mathcal{X}$ is a variety then $L(\mathcal{X})$ stands for the subvariety lattice of $\mathcal{X}$. A distinguished role in the sequel play the following

Lemma 2.1 (the part a) of the main theorem in [6])**.** *Let $\mathcal{G}$ be a variety of periodic groups. The interval $\bigl[S(\mathcal{T}, 1),\ S(\mathcal{G})\bigr]$ of the lattice $L\bigl(S(\mathcal{G})\bigr)$ consists of all varieties of the form $S(\mathcal{H}, X)$ where $\mathcal{H} \subseteq \mathcal{G}$ and X is a (possibly empty) verbal subset of $F(\mathcal{G})$. Here, for varieties $S(\mathcal{H}, X)$ and $S(\mathcal{H}', X')$ from the interval $\bigl[S(\mathcal{T}, 1),\ S(\mathcal{G})\bigr]$, the inclusion $S(\mathcal{H}', X') \subseteq S(\mathcal{H}, X)$ holds if and only if $\mathcal{H}' \subseteq \mathcal{H}$ and there exists a set of words W such that $X = \mathcal{H}(W)$ and $\mathcal{H}'(W) \subseteq X'$.* $\square$

It is well known and easy to see that an arbitrary periodic variety $\mathcal{V}$ contains a greatest group subvariety. We denote this subvariety by $\mathrm{Gr}(\mathcal{V})$.

Lemma 2.2. *If a variety of semigroups with a completely regular power $\mathcal{V}$ is a lower-modular element of the lattice $\mathbb{SEM}$ then $\mathcal{V}$ is a combinatorial variety.*

Proof. Put $\mathcal{G} = \mathrm{Gr}(\mathcal{V})$, $\mathcal{Y} = \mathcal{V} \vee S(\mathcal{G}, 1)$ and $\mathcal{Z} = S(\mathcal{T})$. Lemma 2.1 implies that the lattice $L\bigl(S(\mathcal{G})\bigr)$ contains a sublattice shown in Fig. 1.

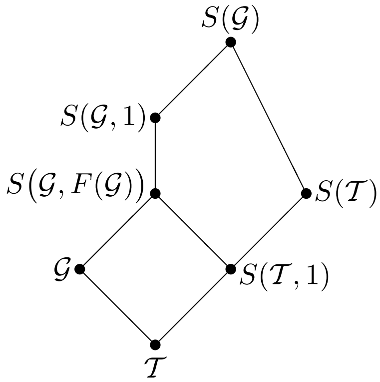

FIGURE 1. A sublattice of $L\bigl(S(\mathcal{G})\bigr)$

Using the equality $S(\mathcal{G}) = S(\mathcal{T}) \vee \mathcal{G}$ (see Fig. 1) and the inclusion $\mathcal{G} \subseteq \mathcal{V}$, we have

$$\mathcal{Y} = S(\mathcal{G},1) \vee \mathcal{V} \subseteq S(\mathcal{G}) \vee \mathcal{V} = S(\mathcal{T}) \vee \mathcal{G} \vee \mathcal{V} = S(\mathcal{T}) \vee \mathcal{V} = \mathcal{Z} \vee \mathcal{V}.$$

Therefore $(\mathcal{Z} \vee \mathcal{V}) \wedge \mathcal{Y} = \mathcal{Y}$. Since the variety $\mathcal{V}$ is lower-modular and $\mathcal{V} \subseteq \mathcal{Y}$, we have $(\mathcal{Z} \wedge \mathcal{Y}) \vee \mathcal{V} = (\mathcal{Z} \vee \mathcal{V}) \wedge \mathcal{Y}$, whence

$$(\mathcal{Z} \wedge \mathcal{Y}) \vee \mathcal{V} = \mathcal{Y}. \tag{3}$$

Furthermore, $S(\mathcal{T},1) = S(\mathcal{T}) \wedge S(\mathcal{G},1)$ (see Fig. 1), and therefore

$$S(\mathcal{T},1) = S(\mathcal{T}) \wedge S(\mathcal{G},1) \subseteq S(\mathcal{T}) \wedge \bigl(\mathcal{V} \vee S(\mathcal{G},1)\bigr) = \mathcal{Z} \wedge \mathcal{Y} \subseteq \mathcal{Z} = S(\mathcal{T}),$$

that is, $S(\mathcal{T},1) \subseteq \mathcal{Z} \wedge \mathcal{Y} \subseteq S(\mathcal{T})$. It is evident that the group $F(\mathcal{T})$ contains only two verbal subsets, namely $\varnothing$ and $\{1\}$. Therefore Lemma 2.1 implies that the interval $\bigl[S(\mathcal{T},1),\, S(\mathcal{T})\bigr]$ of the lattice $L\bigl(S(\mathcal{T})\bigr)$ consists of the varieties $S(\mathcal{T},1)$ and $S(\mathcal{T})$ only. Thus either $\mathcal{Z} \wedge \mathcal{Y} = S(\mathcal{T},1)$ or $\mathcal{Z} \wedge \mathcal{Y} = S(\mathcal{T})$. Let us consider these two cases separately.

Case 1: $\mathcal{Z} \wedge \mathcal{Y} = S(\mathcal{T},1)$. Note that $S(\mathcal{T},1) \vee \mathcal{G} = S\bigl(\mathcal{G},F(\mathcal{G})\bigr)$ (see Fig. 1). Using the equality (3) and the inclusion $\mathcal{G} \subseteq \mathcal{V}$, we have

$$\begin{aligned} S\bigl(\mathcal{G},F(\mathcal{G})\bigr) \vee \mathcal{V} &= S(\mathcal{T},1) \vee \mathcal{G} \vee \mathcal{V} = S(\mathcal{T},1) \vee \mathcal{V} \\ &= (\mathcal{Z} \wedge \mathcal{Y}) \vee \mathcal{V} = \mathcal{Y} = S(\mathcal{G},1) \vee \mathcal{V}, \end{aligned}$$

that is,

$$S\bigl(\mathcal{G},F(\mathcal{G})\bigr) \vee \mathcal{V} = S(\mathcal{G},1) \vee \mathcal{V}. \tag{4}$$

By Lemma 1.9, the variety $\mathcal{V}$ satisfies the identity (2) for some positive integers n and r. Let r be a minimal number with such a property. Substituting 1 for $x_2, \dots, x_n$ in (2), we obtain that every group in $\mathcal{V}$ satisfies the identity $x = x^{r+1}$. Therefore $\exp(\mathcal{G})$ divides r. Moreover, the choice of r implies that $\exp(\mathcal{G}) = r$. Let w be an arbitrary word. The definition of the variety $S\bigl(\mathcal{G},F(\mathcal{G})\bigr)$ shows that it satisfies the identity

$$xwx = (xwx)^{r+1}. \tag{5}$$

Let k be a positive integer with

$$k \geq \frac{n-2}{\ell(w)} \quad \text{and} \quad k = mr+1 \ \text{ for some } m.$$

Then the variety $S\bigl(\mathcal{G}, F(\mathcal{G})\bigr)$ satisfies the identity

$$xw^kx = (xw^kx)^{r+1}. \tag{6}$$

Since $\mathcal{V}$ satisfies the identity (2) and $\ell(xw^kx) \geq n$, the identity (6) holds in the variety $S\bigl(\mathcal{G}, F(\mathcal{G})\bigr) \vee \mathcal{V}$. The equality (4) implies that (6) holds in $S(\mathcal{G}, 1) \vee \mathcal{V}$, and therefore in $S(\mathcal{G}, 1)$. By the definition of the variety $S(\mathcal{G})$, it satisfies the identity $xyz = xy^{r+1}z$. This identity implies

$$\begin{aligned} xwx &= xw^{r+1}x \equiv x \cdot w \cdot w^r x = x \cdot w^{r+1} \cdot w^r x \equiv xw^{2r+1}x \equiv x \cdot w \cdot w^{2r} x \\ &= x \cdot w^{r+1} \cdot w^{2r} x \equiv xw^{3r+1}x = \cdots = xw^{mr+1}x \equiv xw^k x. \end{aligned}$$

We see that the identity

$$xwx = xw^kx \tag{7}$$

holds in the variety $S(\mathcal{G})$, and therefore in $S(\mathcal{G}, 1)$. Combining (6) and (7), we have that the identity (5) holds in $S(\mathcal{G}, 1)$. Thus the variety $S(\mathcal{G}, 1)$ satisfies the identity (5) for every word w. The variety $S\bigl(\mathcal{G}, F(\mathcal{G})\bigr)$ is given within $S(\mathcal{G})$ by the set of all identities of the form (5) where w runs over F. Since all these identities hold in $S(\mathcal{G}, 1)$, the inclusion $S(\mathcal{G}, 1) \subseteq S\bigl(\mathcal{G}, F(\mathcal{G})\bigr)$ is the case. The opposite inclusion is evident, whence $S(\mathcal{G}, 1) = S\bigl(\mathcal{G}, F(\mathcal{G})\bigr)$. By Lemma 2.1, this means that $F(\mathcal{G}) = \{1\}$. In other words, $\mathcal{G} = \mathcal{T}$, and we are done.

Case 2: $\mathcal{Z} \wedge \mathcal{Y} = S(\mathcal{T})$. As we have already noted above, $S(\mathcal{G}) = S(\mathcal{T}) \vee \mathcal{G}$ (see Fig. 1). Taking into account the equality (3) and the inclusion $\mathcal{G} \subseteq \mathcal{V}$, we have

$$S(\mathcal{G}, 1) \vee \mathcal{V} = \mathcal{Y} = (\mathcal{Z} \wedge \mathcal{Y}) \vee \mathcal{V} = S(\mathcal{T}) \vee \mathcal{V} = S(\mathcal{T}) \vee \mathcal{G} \vee \mathcal{V} = S(\mathcal{G}) \vee \mathcal{V}.$$

We see that

$$S(\mathcal{G}, 1) \vee \mathcal{V} = S(\mathcal{G}) \vee \mathcal{V}. \tag{8}$$

Let w be an arbitrary word such that the variety $\mathcal{G}$ satisfies (as a variety of groups) the identity $w = 1$, and k a positive integer with

$$k \geq \frac{n-2}{\ell(w)} \quad \text{and} \quad k = 2m \ \text{ for some } \ m.$$

The variety $\mathcal{G}$ satisfies the identity $w^k = 1$. The definition of the variety $S(\mathcal{G}, 1)$ shows that it satisfies the identity (6). Since $\mathcal{V}$ satisfies the identity (2) and $\ell(xw^kx) \geq n$, the identity (6) holds in the variety $S(\mathcal{G}, 1) \vee \mathcal{V}$. The equality (8) implies that (6) holds in $S(\mathcal{G}) \vee \mathcal{V}$, and therefore in $S(\mathcal{G})$. We always may include the identity $w = 1$ in the identity basis of $\mathcal{G}$. By the definition of the variety $S(\mathcal{G})$, it satisfies the identity $xwx = xw^2x$, and therefore the identities

$$\begin{aligned} xwx &= xw^2x = xw^4x \equiv x(w \cdot w^2 \cdot w)x = x(w \cdot w^4 \cdot w)x \equiv xw^6x \\ &\equiv x(w^2 \cdot w^2 \cdot w^2)x = x(w^2 \cdot w^4 \cdot w^2)x \equiv xw^8x = \cdots = xw^{2m}x \equiv xw^kx. \end{aligned}$$

We see that $S(\mathcal{G})$ satisfies the identity (7). Combining (6) and (7), we have that the identity (5) holds in $S(\mathcal{G})$. Thus if $\mathcal{G}$ satisfies the identity $w = 1$ then

$S(\mathcal{G})$ satisfies the identity (5). Therefore $S(\mathcal{G}) \subseteq S(\mathcal{G}, 1)$ but this inclusion contradicts Lemma 2.1.

Lemma is proved. $\square$

Lemma 2.3. *If a semigroup variety $\mathcal{V}$ is a lower-modular element of the lattice $\mathbb{SEM}$ and $\mathcal{V}$ satisfies an identity of the form*

$$x_1 \cdots x_n = x_1 \cdots x_{i-1}(x_i \cdots x_n)^{r+1}$$

for some positive integers n and r and some i with $1 \le i \le n$ then $\mathcal{V}$ does not contain the variety $\mathcal{RZ}$.

Proof. Arguing by contradiction, suppose that $\mathcal{RZ} \subseteq \mathcal{V}$. Put

$$\mathcal{V}^* = \mathrm{var}\bigl\{x_1 \cdots x_{n+1} = x_1 \cdots x_{i-1}(x_i \cdots x_n)^{r+1} x_{n+1}\bigr\}.$$

Clearly, $\mathcal{V} \subseteq \mathcal{V}^*$. Let us consider the variety $\mathcal{C} \wedge \mathcal{V}^*$. Put

$$\mathcal{C}_m = \mathrm{var}\{x^2 = x_1 x_2 \cdots x_m = 0,\, xy = yx\},$$
$$\mathcal{C}_\omega = \mathrm{var}\{x^2 = 0,\, xy = yx\}$$

(in particular, $\mathcal{C}_1 = \mathcal{T}$ and $\mathcal{C}_2 = \mathcal{ZM}$). It is well known that the lattice $L(\mathcal{C})$ has the form shown in Fig. 2 (see [1], for instance). By Lemma 1.8, $\mathrm{ind}(\mathcal{V}^*) \le n+1$. Fig. 2 shows that if $\mathcal{X} \subseteq \mathcal{C}$ and $\mathrm{ind}(\mathcal{X}) \le m$ for some positive integer m then $\mathcal{X} \subseteq \mathcal{SL} \vee \mathcal{C}_m$. Therefore $\mathcal{C} \wedge \mathcal{V}^* \subseteq \mathcal{SL} \vee \mathcal{C}_{n+1}$. Since the variety $\mathcal{V}$ is lower-modular and $\mathcal{V} \subseteq \mathcal{V}^*$, we have

$$(\mathcal{C} \vee \mathcal{V}) \wedge \mathcal{V}^* = (\mathcal{C} \wedge \mathcal{V}^*) \vee \mathcal{V} \subseteq \mathcal{SL} \vee \mathcal{C}_{n+1} \vee \mathcal{V}.$$

The variety $\mathcal{SL} \vee \mathcal{C}_{n+1} \vee \mathcal{V}$ satisfies the identity

$$x_1 \cdots x_{n+1} = x_1 \cdots x_i(x_{i+1} \cdots x_{n+1})^{r+1},$$

whence this identity holds in the variety $(\mathcal{C} \vee \mathcal{V}) \wedge \mathcal{V}^*$. Then there exists a sequence of words $u_0, u_1, \ldots, u_m$ such that

$$u_0 \equiv x_1 \cdots x_{n+1}, \;\; u_m \equiv x_1 \cdots x_i(x_{i+1} \cdots x_{n+1})^{r+1}$$

and, for every $j = 0, 1, \ldots, m-1$, the identity $u_j = u_{j+1}$ holds in one of the varieties $\mathcal{C} \vee \mathcal{V}$ or $\mathcal{V}^*$. Both these varieties contain the variety $\mathcal{RZ}$. Now Lemma 1.6(i) implies that $t(u_j) \equiv x_{n+1}$ for all $j = 0, 1, \ldots, m$. Since $\ell_{x_{n+1}}(u_0) = 1$ and $\ell_{x_{n+1}}(u_m) > 1$, there exists an index k with $0 < k \le m$ such that $\ell_{x_{n+1}}(u_{k-1}) = 1$ and $\ell_{x_{n+1}}(u_k) > 1$. Lemma 1.6(ii) implies that the identity $u_{k-1} = u_k$ fails in the variety $\mathcal{C}$, and therefore in $\mathcal{C} \vee \mathcal{V}$. Further, $u_{k-1} = u_k$ is false in the variety $\mathcal{P}$ by Lemma 1.6(iii). Since $\mathcal{P} \subseteq \mathcal{V}^*$ (by the definition of $\mathcal{V}^*$), the variety $\mathcal{V}^*$ also does not satisfy $u_{k-1} = u_k$. We see that this identity fails in both the varieties $\mathcal{C} \vee \mathcal{V}$ and $\mathcal{V}^*$, contradicting the choice of the words $u_0, u_1, \ldots, u_m$. This completes the proof. $\square$

The main result of this section is the following

Theorem 2.4. *Let $\mathcal{V}$ be a variety of semigroups with a completely regular power. The following are equivalent*:

a) *$\mathcal{V}$ is a lower-modular element of the lattice* $\mathbb{SEM}$;
b) *$\mathcal{V}$ is a modular and lower-modular element of the lattice* $\mathbb{SEM}$;

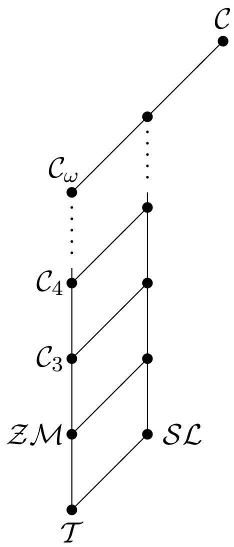

FIGURE 2. The lattice $L(\mathcal{C})$

c) $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ *where* $\mathcal{M}$ *is one of the varieties* $\mathcal{T}$ *or* $\mathcal{SL}$, *while* $\mathcal{N}$ *is a* 0*-reduced and nilpotent variety of index* $\le n$, *where* $n = \mathrm{crp}(\mathcal{V})$.

Proof. a) $\Longrightarrow$ c). Let $n = \mathrm{crp}(\mathcal{V})$. By Lemma 1.9, the variety $\mathcal{V}$ satisfies the identity (2) for some r and contains none of the varieties $\mathcal{P}$ and $\overleftarrow{\mathcal{P}}$. Furthermore, Lemma 2.3 and its dual imply that $\mathcal{V}$ contains none of the varieties $\mathcal{LZ}$ and $\mathcal{RZ}$. It is evident that every CRP-variety is periodic. Now Lemma 1.7 applies with the conclusion that $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where the variety $\mathcal{M}$ is generated by some monoid and $\mathcal{N} = \mathrm{Nil}(\mathcal{V})$. Lemma 1.8 and the fact that $\mathcal{V}$ satisfies the identity (2) imply that the variety $\mathcal{N}$ is nilpotent of index $\le n$. Besides, $\mathcal{N}$ is 0-reduced by Proposition 1.3. Substituting 1 for $x_2, \dots, x_n$ in the identity (2), we obtain that every monoid in $\mathcal{V}$ satisfies the identity $x = x^{r+1}$, whence it is completely regular. But $\mathcal{V}$ does not contain non-trivial groups (by Lemma 2.2) and none of the varieties $\mathcal{LZ}$ and $\mathcal{RZ}$. This implies that every completely regular subvariety of $\mathcal{V}$ is contained in $\mathcal{SL}$ (see [1], for instance). Therefore $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ (see Lemma 1.1).

c) $\Longrightarrow$ b). Every 0-reduced variety is both modular and lower-modular by [13, Corollary 3]; by [16, Corollary 2.5], the join of every 0-reduced variety and $\mathcal{SL}$ also is modular and lower-modular[2].

The implication b) $\Longrightarrow$ a) is evident. □

3. VARIETIES OF INDEX ≤ 2

To prove the main result of this section (Theorem 3.3), we need two auxiliary results. In fact, the first of them is well known. We provide its proof for the sake of completeness.

[2]Note that the 'modular half' of both these claims has been noted also in [5, Proposition 1.1] in some other terminology. Moreover, the assertion that a 0-reduced variety is modular readily follows from [4, Proposition 2.2]. However, this assertion was not formulated in [4] explicitly.

Lemma 3.1. $\operatorname{ind}(\mathcal{P}\vee\overleftarrow{\mathcal{P}})=3$.

Proof. The variety $\mathcal{P}\vee\overleftarrow{\mathcal{P}}$ satisfies the identity $xyz=xy^2z$, whence Lemma 1.8 implies that $\operatorname{ind}(\mathcal{P}\vee\overleftarrow{\mathcal{P}})\le 3$. On the other hand, Lemma 1.6(iii) and its dual imply that the variety $\mathcal{P}\vee\overleftarrow{\mathcal{P}}$ does not satisfy any non-trivial identity of the form $xy=w$. By Lemma 1.8, $\operatorname{ind}(\mathcal{P}\vee\overleftarrow{\mathcal{P}})\ge 3$. $\square$

Lemma 3.2. *If a semigroup variety $\mathcal{V}$ is a lower-modular element of the lattice* $\mathbb{SEM}$ *and $\mathcal{V}$ contains at least one of the varieties $\mathcal{P}$ or $\overleftarrow{\mathcal{P}}$ then* $\operatorname{ind}(\mathcal{V})\ge 3$.

Proof. We may assume without any loss of generality that $\mathcal{V}\supseteq\overleftarrow{\mathcal{P}}$. If $\mathcal{V}\supseteq\mathcal{P}$ too then $\operatorname{ind}(\mathcal{V})\ge\operatorname{ind}(\mathcal{P}\vee\overleftarrow{\mathcal{P}})$, whence $\operatorname{ind}(\mathcal{V})\ge 3$ by Lemma 3.1. Now let $\mathcal{V}\not\supseteq\mathcal{P}$. Put $\mathcal{N}=\operatorname{var}\{x^2=0\}$ and $\mathcal{V}^*=\mathcal{V}\vee\mathcal{N}$. There exists an identity $u=v$ which holds in $\mathcal{V}$ but fails in $\mathcal{P}$. This identity holds in the variety $\overleftarrow{\mathcal{P}}$ because $\mathcal{V}\supseteq\overleftarrow{\mathcal{P}}$. By dual to Lemma 1.6(iii), $c(u)=c(v)$. Lemma 1.6(iii) implies that either $\ell_{t(u)}(u)>1$ but $\ell_{t(v)}(v)=1$ or $\ell_{t(u)}(u)=1$ but $\ell_{t(v)}(v)>1$ or $\ell_{t(u)}(u)=\ell_{t(v)}(v)=1$ but $t(u)\not\equiv t(v)$. Let x be a letter with $x\notin c(u)$. Put $u'\equiv x^2u$ and $v'\equiv x^2v$. The variety $\mathcal{V}^*$ satisfies the identity $u'=v'$. It is clear that either $\ell_{t(u')}(u')>1$ but $\ell_{t(v')}(v')=1$ or $\ell_{t(u')}(u')=1$ but $\ell_{t(v')}(v')>1$ or $\ell_{t(u')}(u')=\ell_{t(v')}(v')=1$ but $t(u')\not\equiv t(v')$. Lemma 1.6(iii) implies that the identity $u'=v'$ fails in the variety $\mathcal{P}$. Therefore $\mathcal{V}^*\not\supseteq\mathcal{P}$, whence $\mathcal{P}\wedge\mathcal{V}^*\subset\mathcal{P}$. It is well known that the lattice $L(\mathcal{P})$ has the form shown in Fig. 3. We see that $\mathcal{P}\wedge\mathcal{V}^*\subseteq\mathcal{SL}\vee\mathcal{ZM}\subseteq\overleftarrow{\mathcal{P}}\subseteq\mathcal{V}$, whence $(\mathcal{P}\wedge\mathcal{V}^*)\vee\mathcal{V}=\mathcal{V}$. Since the variety $\mathcal{V}$ is lower-modular and $\mathcal{V}\subseteq\mathcal{V}^*$, we have $\mathcal{V}=(\mathcal{P}\wedge\mathcal{V}^*)\vee\mathcal{V}=(\mathcal{P}\vee\mathcal{V})\wedge\mathcal{V}^*$. It is evident that $\operatorname{ind}(\mathcal{V}^*)\ge\operatorname{ind}(\mathcal{N})=\infty$, that is, $\operatorname{ind}(\mathcal{V}^*)=\infty$. In view of the inclusion $\mathcal{V}\supseteq\overleftarrow{\mathcal{P}}$ and Lemma 3.1, we have

$$\begin{aligned}\operatorname{ind}(\mathcal{V})&=\operatorname{ind}\big((\mathcal{P}\vee\mathcal{V})\wedge\mathcal{V}^*\big)=\min\big\{\operatorname{ind}(\mathcal{P}\vee\mathcal{V}),\operatorname{ind}(\mathcal{V}^*)\big\}\\&=\min\big\{\operatorname{ind}(\mathcal{P}\vee\mathcal{V}),\infty\big\}=\operatorname{ind}(\mathcal{P}\vee\mathcal{V})\ge\operatorname{ind}(\mathcal{P}\vee\overleftarrow{\mathcal{P}})=3.\end{aligned}$$

The lemma is proved. $\square$

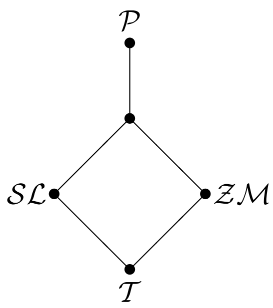

FIGURE 3. The lattice $L(\mathcal{P})$

The main result of this section is the following

Theorem 3.3. *For a semigroup variety $\mathcal{V}$ of index ≤ 2, the following are equivalent*:

a) $\mathcal{V}$ *is a lower-modular element of the lattice* $\mathbb{SEM}$;
b) $\mathcal{V}$ *is a modular element of the lattice* $\mathbb{SEM}$;
c) $\mathcal{V}$ *is a neutral element of the lattice* $\mathbb{SEM}$;
d) $\mathcal{V}$ *is one of the varieties* $\mathcal{T}$, $\mathcal{SL}$, $\mathcal{ZM}$ *or* $\mathcal{SL} \vee \mathcal{ZM}$.

Proof. a) $\Longrightarrow$ d). By Lemma 3.2, $\mathcal{V}$ contains none of the varieties $\mathcal{P}$ and $\overleftarrow{\mathcal{P}}$. Now Lemma 1.9 applies with the conclusion that $\mathcal{V}$ is a CRP-variety. By Theorem 2.4, $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$, while $\mathcal{N}$ is a nil-variety. Since $\mathcal{V}$ is a variety of index ≤ 2, we have $\mathcal{N} \subseteq \mathcal{ZM}$. It remains to refer to Lemma 1.1.

b) $\Longrightarrow$ d). By Proposition 1.2, $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$, while $\mathcal{N}$ is a nil-variety. Now we may complete the proof of this implication by repeating the arguments from the proof of the implication a) $\Longrightarrow$ d).

The implication d) $\Longrightarrow$ c) is guaranteed by Proposition 1.5, while the implications c) $\Longrightarrow$ a) and c) $\Longrightarrow$ b) are evident. □

Theorem 3.3 immediately implies the following statement that gives a solution of Problem 1.

Corollary 3.4. *For a completely regular semigroup variety* $\mathcal{V}$, *the following are equivalent*:

a) $\mathcal{V}$ *is a lower-modular element of the lattice* $\mathbb{SEM}$;
b) $\mathcal{V}$ *is a modular element of the lattice* $\mathbb{SEM}$;
c) $\mathcal{V}$ *is a neutral element of the lattice* $\mathbb{SEM}$;
d) $\mathcal{V}$ *is one of the varieties* $\mathcal{T}$ *or* $\mathcal{SL}$. □

Note that this corollary follows also from Proposition 1.2 and Theorem 2.4. The following statement gives a solution of Problem 2.

Corollary 3.5. *For a semigroup variety* $\mathcal{V}$, *the following are equivalent*:

a) $\mathcal{V}$ *is both an upper-modular and a lower-modular element of the lattice* $\mathbb{SEM}$;
b) $\mathcal{V}$ *is a neutral element of the lattice* $\mathbb{SEM}$;
c) $\mathcal{V}$ *is one of the varieties* $\mathcal{T}$, $\mathcal{SL}$, $\mathcal{ZM}$, $\mathcal{SL} \vee \mathcal{ZM}$ *or* $\mathcal{SEM}$.

Proof. The conditions b) and c) are equivalent by Proposition 1.5. The implication a) $\Longrightarrow$ c) follows from Theorem 3.3 and the following fact: if a proper semigroup variety is both upper-modular and lower-modular then it is a variety of index ≤ 2 [11, Corollary 2.4]. Finally, the implication b) $\Longrightarrow$ a) is evident. □

It was proved in [16, Proposition 2.4] that a semigroup variety is simultaneously modular, upper-modular and lower-modular if and only if it is neutral. Corollary 3.5 strengthens this result.

4. Permutational varieties

The main result of this section is the following

Theorem 4.1. *Let* $\mathcal{V}$ *be a semigroup variety satisfying an identity of the form* (1) *with* $1\pi \ne 1$ *and* $n\pi \ne n$. *The following are equivalent*:

a) *$\mathcal{V}$ is a lower-modular element of the lattice* $\mathbb{SEM}$;
b) *$\mathcal{V}$ is a modular and lower-modular element of the lattice* $\mathbb{SEM}$;
c) $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ *where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$, while $\mathcal{N}$ is a* 0*-reduced and nilpotent variety of index* $\le n$.

Proof. a) $\Longrightarrow$ c). By Corollary 1.4, $\mathcal{V}$ is a variety of index $\le n$. Lemma 1.6(iii) and its dual imply that $\mathcal{V}$ contains none of the varieties $\mathcal{P}$ and $\overleftarrow{\mathcal{P}}$. Now Lemma 1.9 applies with the conclusion that $\mathcal{V}$ is a CRP-variety and $\operatorname{crp}(\mathcal{V}) \le n$. It remains to refer to Theorem 2.4.

The implication c) $\Longrightarrow$ b) may be verified by repeating the arguments from the proof of the analogous implication in Theorem 2.4.

The implication b) $\Longrightarrow$ a) is evident. □

Theorem 4.1 readily implies the following result proved earlier in [11, Theorem 2].

Corollary 4.2. *For a commutative semigroup variety $\mathcal{V}$, the following are equivalent*:

a) *$\mathcal{V}$ is a lower-modular element of the lattice* $\mathbb{SEM}$;
b) *$\mathcal{V}$ is a neutral element of the lattice* $\mathbb{SEM}$;
c) *$\mathcal{V}$ is one of the varieties $\mathcal{T}$, $\mathcal{SL}$, $\mathcal{ZM}$ or $\mathcal{SL} \vee \mathcal{ZM}$.*

Proof. The implication a) $\Longrightarrow$ c) immediately follows from Theorem 4.1, the evident fact that a commutative 0-reduced variety is contained in $\mathcal{ZM}$, and Lemma 1.1. The implication c) $\Longrightarrow$ b) is guaranteed by Proposition 1.5. Finally, the implication b) $\Longrightarrow$ a) is evident. □

One more corollary of Theorem 4.1 is the following

Corollary 4.3. *Let a semigroup variety $\mathcal{V}$ be a lower-modular element of the lattice* $\mathbb{SEM}$. *If $\mathcal{V}$ satisfies an identity of the form* (1) *with $1\pi \ne 1$ and $n\pi \ne n$ then $\mathcal{V}$ satisfies all permutation identities of length n.* □

Note that a claim stronger than Corollary 4.3 holds for upper-modular varieties. Namely, if an upper-modular variety satisfies an identity of the form (1) with $1\pi \ne 1$ and $n\pi \ne n$ then it is commutative (and all such varieties are completely determined in [12, Theorem 1.2], in fact). This follows from Lemma 1.7 and the following three claims: a) a variety satisfying an identity of the form (1) with $1\pi \ne 1$ and $n\pi \ne n$ contains none of the varieties $\mathcal{P}$, $\overleftarrow{\mathcal{P}}$, $\mathcal{LZ}$ and $\mathcal{RZ}$ (by the claims (i) and (iii) of Lemma 1.6 and their duals); b) if $\mathcal{V}$ is a proper upper-modular semigroup variety then $\mathcal{V}$ is periodic and the variety $\operatorname{Nil}(\mathcal{V})$ is commutative [12, Theorem 1.1]; c) every monoid satisfying a permutation identity is commutative. Also, a weak analogue of Corollary 4.3 holds for modular varieties. Namely, if a modular variety satisfies an identity of the form (1) where $n \ge 5$ and the permutation π is odd then it satisfies all permutation identities of length n [10, Theorem 4.5(ii)].

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