Lower-modular elements of the lattice of semigroup varieties*

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Abstract

We call a semigroup variety *modular* [*upper-modular, lower-modular, neutral*] if it is a modular [respectively upper-modular, lower-modular, neutral] element of the lattice of all semigroup varieties. It is proved that if $\mathcal{V}$ is a lower-modular variety then either $\mathcal{V}$ coincides with the variety of all semigroups or $\mathcal{V}$ is periodic and the greatest nil-subvariety of $\mathcal{V}$ may be given by 0-reduced identities only. We completely determine all commutative lower-modular varieties. In particular, it turns out that a commutative variety is lower-modular if and only if it is neutral. A number of corollaries of these results are obtained.

Key words: semigroup, variety, periodic variety, nil-variety, 0-reduced variety, lattice of varieties, lower-modular element, modular element, upper-modular element, strongly modular element, neutral element.

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Introduction and summary

The class of all varieties of semigroups forms a lattice under the following naturally defined operations: for varieties $\mathcal{X}$ and $\mathcal{Y}$, their *join* $\mathcal{X} \vee \mathcal{Y}$ is the variety generated by the set-theoretical union of $\mathcal{X}$ and $\mathcal{Y}$ (as classes of semigroups), while their *meet* $\mathcal{X} \wedge \mathcal{Y}$ coincides with the set-theoretical intersection of $\mathcal{X}$ and $\mathcal{Y}$. Special elements of different types in lattices of varieties of semigroups and universal algebras have been examined in several articles (see, for instance, [4, 5, 10, 14, 16]). Here we continue these investigations. Recall the definitions of special elements of lattices considered in this article.

An element x of a lattice $\langle L; \vee, \wedge \rangle$ is called *modular* if

$$\forall\, y, z \in L\colon \quad y \leq z \longrightarrow (x \vee y) \wedge z = (x \wedge z) \vee y,$$

and *lower-modular* if

$$\forall\, y, z \in L\colon \quad x \leq y \longrightarrow (z \vee x) \wedge y = (z \wedge y) \vee x.$$

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Upper-modular elements are defined dually to lower-modular ones. Following [16], we call an element x of a lattice L *strongly modular* if x is simultaneously a modular, an upper-modular, and a lower-modular element of L. An element x of a lattice L is called *neutral* if

$$\forall y, z \in L\colon \quad (x \vee y) \wedge (y \vee z) \wedge (z \vee x) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x).$$

As is well known, x is neutral if and only if, for all $y, z \in L$, the sublattice of L generated by x, y, and z is distributive (see, for instance, [2, Theorem III.2.4]).

We denote by $\mathcal{SEM}$ the variety of all semigroups. A semigroup variety $\mathcal{V}$ is called *proper* if $\mathcal{V} \neq \mathcal{SEM}$. For convenience, we call a semigroup variety *modular* [*upper-modular, lower-modular, strongly modular, neutral*] if it is a modular [upper-modular, lower-modular, strongly modular, neutral] element of the lattice $\mathbb{SEM}$ of all semigroup varieties. A number of results about varieties of these five types have been obtained in [5,11–14,16]. In this article we present some new information about lower-modular varieties. In particular, we prove that every proper lower-modular variety is periodic (Theorem 1) and completely determine all lower-modular commutative varieties (Theorem 2).

To formulate the main results, we need some more definitions and notation. A semigroup S with 0 is said to be a *nilsemigroup* if, for every $s \in S$, there exists a positive integer n with $s^n = 0$. A semigroup variety $\mathcal{V}$ is called a *nil-variety* if each member of $\mathcal{V}$ is a nilsemigroup. A semigroup is called *periodic* if each of its cyclic subsemigroup is finite. As is well known, every semigroup variety is either *periodic* (that is, consists of periodic semigroups) or *overcommutative* (that is, contains the variety of all commutative semigroups). It is easy to see that an arbitrary periodic semigroup variety $\mathcal{V}$ contains a greatest nil-subvariety. We denote this subvariety by $\mathrm{Nil}(\mathcal{V})$. Clearly, if w is a semigroup word then a semigroup S satisfies the identity system $wu = uw = w$, where u runs over the set of all words, if and only if S contains a zero element 0 and all values of the word w in S equal 0. We adopt the usual convention of writing $w = 0$ as a short form of such a system and referring to the expression $w = 0$ as to a single identity. Such identities are called 0-*reduced*. A semigroup variety is called 0-*reduced* if it can be defined by 0-reduced identities only (note that in [16] such varieties are called *Rees* varieties). Every 0-reduced variety is clearly a nil-variety. It can be easily deduced from the proof of Theorem 3.1 of the article [16] that a lower-modular nil-variety is a 0-reduced variety. The following theorem generalizes this statement.

Theorem 1. *If a semigroup variety $\mathcal{V}$ is a lower-modular element of the lattice $\mathbb{SEM}$ then either $\mathcal{V} = \mathcal{SEM}$ or $\mathcal{V}$ is a periodic variety and $\mathrm{Nil}(\mathcal{V})$ is a 0-reduced variety.*

By Theorem 1 a proper lower-modular variety is periodic. According to results of [5, 11], the same is true for modular varieties and for upper-modular ones (see Lemmas 2.2 and 2.3 below). Note also that Theorem 1 together with results of [5, 13, 14] imply two results obtained earlier in [16], namely the description of semigroup varieties that both modular and lower-modular, and the description of strongly modular semigroup varieties (see Corollaries 2.9 and 2.10 below).

If Σ is a system of identities then we denote by $\operatorname{var}\Sigma$ the variety of all semigroups satisfying Σ. Put

$$\mathcal{T} = \operatorname{var}\{x = y\},\ \mathcal{SL} = \operatorname{var}\{x^2 = x,\, xy = yx\},\ \text{and } \mathcal{ZM} = \operatorname{var}\{xy = 0\}.$$

Our second main result is the following

Theorem 2. *Let $\mathcal{V}$ be a commutative semigroup variety. The following are equivalent:*

(i) *$\mathcal{V}$ is a lower-modular element of the lattice $\mathbb{SEM}$;*

(ii) *$\mathcal{V}$ is a strongly modular element of the lattice $\mathbb{SEM}$;*

(iii) *$\mathcal{V}$ is a neutral element of the lattice $\mathbb{SEM}$;*

(iv) *$\mathcal{V}$ is one of the varieties $\mathcal{T}$, $\mathcal{SL}$, $\mathcal{ZM}$, $\mathcal{SL} \vee \mathcal{ZM}$.*

Note that, according to [16, Proposition 4.1], assertions (ii) and (iii) of this theorem are equivalent for arbitrary semigroup varieties, and varieties satisfying these conditions are exhausted by the varieties listed in assertion (iv) and the variety $\mathcal{SEM}$ (see Corollary 2.10 below). Note also that commutative modular and commutative upper-modular varieties are completely described in [12] and [11] respectively. In particular, it is verified in [12] that every commutative modular variety is upper-modular.

The article consists of 3 sections. Section 1 contains preliminary information about lattices and semigroup varieties. In Section 2 we prove Theorems 1 and 2 and obtain several corollaries of Theorem 1; for instance, we essentially sharpen Theorem 1 for varieties that are both upper-modular and lower-modular (Corollary 2.4), characterize lower-modular nil-varieties (Corollary 2.7), and verify that every lower-modular nil-variety is modular (Corollary 2.8). Section 3 contains several open questions.

1 Preliminaries

We start with the following lattice-theoretical

Lemma 1.1. *Let L be a lattice with 0, $x \in L$, and let a be an atom and a neutral element of L. Then x is a modular [lower-modular] element of L if and only if $x \vee a$ is a modular [lower-modular] element of L.*

Proof. The 'modular half' of this assertion is verified in [14, Proposition 1.3(i)]. It is proved in [16, Lemma 1.3] that if x_1 and x_2 are lower-modular elements of a lattice then so is $x_1 \vee x_2$. It remains to verify that x is a lower-modular element of L whenever so is $x \vee a$.

So, let $x \vee a$ be a lower-modular element of L. Since a is an atom of L, we have that, for any $z \in L$, $z \ngeq a$ if and only if $z \wedge a = 0$. Because a is a neutral element of L, we have that if $b, c \in L$ and $b \wedge a = c \wedge a = 0$ then $(b \vee c) \wedge a = (b \wedge a) \vee (c \wedge a) = 0$. In other words,

$$\forall\, b, c \in L\colon \quad b \ngeq a \,\&\, c \ngeq a \longrightarrow b \vee c \ngeq a. \tag{1.1}$$

Further, it is known that if e is a neutral element of a lattice L then

$$\forall\, f,g \in L\colon \quad f \wedge e = g \wedge e \ \&\ f \vee e = g \vee e \longrightarrow f = g$$

(see, for instance, [2, Theorem III.2.4]). Therefore

$$\forall\, b,c \in L\colon \quad b \ngeq a \ \&\ c \ngeq a \ \&\ b \vee a = c \vee a \longrightarrow b = c. \tag{1.2}$$

Let $y,z \in L$ with $x \leq y$. We may assume that $x \ngeq a$ because otherwise $x \vee a = x$. Note that $x \vee a \leq y \vee a$ because $x \leq y$. We have

$$\begin{aligned}
\big((z \vee x) \wedge y\big) \vee a &= \big((z \vee x) \vee a\big) \wedge (y \vee a) && \text{because } a \text{ is neutral}\\
&= \big(z \vee (x \vee a)\big) \wedge (y \vee a) &&\\
&= \big(z \wedge (y \vee a)\big) \vee (x \vee a) && \text{because } x \vee a \leq y \vee a \text{ and}\\
& && x \vee a \text{ is lower-modular}\\
&= \big((z \wedge y) \vee (z \wedge a)\big) \vee (x \vee a) && \text{because } a \text{ is neutral}\\
&= \big((z \wedge y) \vee x\big) \vee \big((z \wedge a) \vee a\big) &&\\
&= \big((z \wedge y) \vee x\big) \vee a && \text{by the absorbtion law.}
\end{aligned}$$

We see that

$$\big((z \vee x) \wedge y\big) \vee a = \big((z \wedge y) \vee x\big) \vee a. \tag{1.3}$$

Suppose at first that $y \ngeq a$. This implies $(z \vee x) \wedge y \ngeq a$ and $z \wedge y \ngeq a$. Recall that $x \ngeq a$. By (1.1) we conclude that $(z \wedge y) \vee x \ngeq a$. Now we may apply (1.2) and (1.3) concluding that $(z \vee x) \wedge y = (z \wedge y) \vee x$, that is x is a lower-modular element. The case when $z \ngeq a$ may be considered in quite a similar way. Finally, if $y \geq a$ and $z \geq a$ then we may apply (1.3) and conclude that

$$(z \vee x) \wedge y = \big((z \vee x) \wedge y\big) \vee a = \big((z \wedge y) \vee x\big) \vee a = (z \wedge y) \vee x.$$

Thus in this case x is a lower-modular element as well. □

Note that the analogue of Lemma 1.1 for upper-modular elements also holds (see [14, Proposition 1.3(ii)]).

The following lemma contains two important properties of the varieties $\mathcal{SL}$ and $\mathcal{ZM}$. The first property is well known (see, for instance, [1]), while the second one is proved in [16, Proposition 2.4].

Lemma 1.2. *The varieties $\mathcal{SL}$ and $\mathcal{ZM}$ are atoms and neutral elements of the lattice* $\mathbb{SEM}$. □

Lemmas 1.1 and 1.2 imply the following

Corollary 1.3. *Let $\mathcal{V}$ be a semigroup variety and let $\mathcal{A}$ be one of the varieties $\mathcal{SL}$, $\mathcal{ZM}$ or $\mathcal{SL} \vee \mathcal{ZM}$. The variety $\mathcal{V}$ is a modular [lower-modular] element of the lattice $\mathbb{SEM}$ if and only if the variety $\mathcal{V} \vee \mathcal{A}$ is a modular [lower-modular] element of $\mathbb{SEM}$.* □

Note that the analogue of Corollary 1.3 for upper-modular varieties also holds (this immediately follows from Lemma 1.2 of this paper and Proposition 1.3(ii) of [14]).

It is well known that an arbitrary periodic semigroup variety $\mathcal{V}$ contains a greatest group subvariety. We denote this subvariety by $\mathrm{Gr}(\mathcal{V})$. Recall that a semigroup variety $\mathcal{V}$ is called *combinatorial* if every group in $\mathcal{V}$ is singleton. The following observation will be helpful.

Lemma 1.4. *Let $\mathcal{V}$ be an arbitrary semigroup variety and $\mathcal{K}$ a combinatorial semigroup variety. Then every group from the variety $\mathcal{V} \vee \mathcal{K}$ belongs to $\mathcal{V}$. If, besides that, the variety $\mathcal{V}$ is periodic then* $\mathrm{Gr}(\mathcal{V} \vee \mathcal{K}) = \mathrm{Gr}(\mathcal{V})$.

Proof. Let $u = v$ be an arbitrary identity that holds in the variety $\mathcal{V}$. Since the variety $\mathcal{K}$ is combinatorial, it satisfies an identity of the form $x^n = x^{n+1}$ for some n. Then the variety $\mathcal{V} \vee \mathcal{K}$ satisfies the identity $u^{n+1}v^n = u^n v^{n+1}$. Therefore the identity $u = v$ holds in every group from $\mathcal{V} \vee \mathcal{K}$. Thus each group from $\mathcal{V} \vee \mathcal{K}$ lies in $\mathcal{V}$. The first assertion of our lemma is proved. If the variety $\mathcal{V}$ is periodic then we have $\mathrm{Gr}(\mathcal{V} \vee \mathcal{K}) \subseteq \mathcal{V}$, and therefore $\mathrm{Gr}(\mathcal{V} \vee \mathcal{K}) \subseteq \mathrm{Gr}(\mathcal{V})$. The opposite inclusion is evident. □

2 Proofs

2.1 Theorem 1

Let $\mathcal{V}$ be a proper lower-modular semigroup variety. We have to verify that $\mathcal{V}$ is periodic and the variety $\mathrm{Nil}(\mathcal{V})$ is a 0-reduced variety.

Suppose that $\mathcal{V}$ is not periodic. Then $\mathcal{V}$ contains the variety $\mathcal{COM}$ of all commutative semigroups. It is proved by M. V. Volkov (see [11, Lemma 1.16]) that the join of all the minimal non-abelian periodic group varieties coincides with the variety $\mathcal{SEM}$. Therefore there exists a minimal non-abelian periodic group variety $\mathcal{G}$ with $\mathcal{G} \nsubseteq \mathcal{V}$. Put $\overline{\mathcal{V}} = \mathcal{V} \vee \mathcal{G}$. Clearly, $\mathcal{V} \subset \overline{\mathcal{V}}$. Recall that a semigroup S is said to be *nilpotent* if it satisfies the identity $x_1 x_2 \cdots x_n = 0$ for some n. A semigroup variety is called *nilpotent* if all its members are nilpotent. As is well known, every overcommutative semigroup variety is generated by all its nilpotent members. Hence there exists a nilpotent variety $\mathcal{N}$ such that $\mathcal{N} \subseteq \overline{\mathcal{V}}$ but $\mathcal{N} \nsubseteq \mathcal{V}$. Put $\mathcal{Y} = \mathcal{V} \vee \mathcal{N}$. Clearly, $\mathcal{V} \subset \mathcal{Y} \subseteq \overline{\mathcal{V}}$. Since $\mathcal{G} \nsubseteq \mathcal{V}$, Lemma 1.4 implies that $\mathcal{G} \nsubseteq \mathcal{V} \vee \mathcal{N} = \mathcal{Y}$. Therefore the variety $\mathcal{G} \wedge \mathcal{Y}$ is commutative, whence $\mathcal{G} \wedge \mathcal{Y} \subseteq \mathcal{COM} \subseteq \mathcal{V}$. Since $\mathcal{V}$ is lower-modular and $\mathcal{V} \subseteq \mathcal{Y}$, we have

$$\mathcal{V} = (\mathcal{G} \wedge \mathcal{Y}) \vee \mathcal{V} = (\mathcal{G} \vee \mathcal{V}) \wedge \mathcal{Y} = \overline{\mathcal{V}} \wedge \mathcal{Y} = \mathcal{Y},$$

a contradiction with $\mathcal{V} \subset \mathcal{Y}$. We have proved that the variety $\mathcal{V}$ is periodic.

Put $\mathcal{K} = \mathrm{Nil}(\mathcal{V})$. Suppose that $\mathcal{K}$ is not a 0-reduced variety. The proof of Theorem 3.1 in [16] implies that then there is a periodic group variety $\mathcal{H}$ such that $\mathrm{Nil}(\mathcal{H} \vee \mathcal{K}) \supset \mathcal{K}$. Set $\mathcal{K}' = \mathrm{Nil}(\mathcal{H} \vee \mathcal{K})$ and $\mathcal{V}' = \mathcal{V} \vee \mathcal{K}'$. If $\mathcal{X}$ is a periodic variety then clearly $\mathcal{H} \wedge \mathcal{X} = \mathcal{H} \wedge \mathrm{Gr}(\mathcal{X})$. We have

$$\begin{aligned}
\mathcal{K}' &\subseteq \mathrm{Nil}\big((\mathcal{H} \vee \mathcal{V}) \wedge \mathcal{V}'\big) && \text{because } \mathcal{K}' \subseteq (\mathcal{H} \vee \mathcal{V}) \wedge \mathcal{V}' \text{ and } \mathcal{K}' \text{ is a nil-variety} \\
&= \mathrm{Nil}\big((\mathcal{H} \wedge \mathcal{V}') \vee \mathcal{V}\big) && \text{because } \mathcal{V} \subseteq \mathcal{V}' \text{ and } \mathcal{V} \text{ is lower-modular} \\
&= \mathrm{Nil}\big(\big(\mathcal{H} \wedge \mathrm{Gr}(\mathcal{V}')\big) \vee \mathcal{V}\big) && \text{because } \mathcal{H} \wedge \mathcal{V}' = \mathcal{H} \wedge \mathrm{Gr}(\mathcal{V}') \\
&= \mathrm{Nil}\big(\big(\mathcal{H} \wedge \mathrm{Gr}(\mathcal{V})\big) \vee \mathcal{V}\big) && \text{because } \mathrm{Gr}(\mathcal{V}') = \mathrm{Gr}(\mathcal{V}) \text{ by Lemma 1.4} \\
&= \mathrm{Nil}\big((\mathcal{H} \wedge \mathcal{V}) \vee \mathcal{V}\big) && \text{because } \mathcal{H} \wedge \mathrm{Gr}(\mathcal{V}) = \mathcal{H} \wedge \mathcal{V} \\
&= \mathrm{Nil}(\mathcal{V}) = \mathcal{K},
\end{aligned}$$

a contradiction with $\mathcal{K} \subset \mathrm{Nil}(\mathcal{H} \vee \mathcal{K}) = \mathcal{K}'$. Thus $\mathcal{K}$ is a 0-reduced variety. Theorem 1 is proved. □

2.2 Corollaries

Theorem 1 and results of articles [5, 11, 13] imply several corollaries. First of all, we reproduce results of [5, 11, 13] that will be used.

Lemma 2.1 ([13, Corollary 3])**.** *An arbitrary* 0*-reduced variety is a modular and a lower-modular element of the lattice* $\mathbb{SEM}$*.* □

The 'modular half' of this lemma was rediscovered in [5, Proposition 1.1]. Note that this assertion immediately follows from [4, Proposition 2.2] (just this argument was used in [13]) but was not mentioned in [4] explicitly.

Lemma 2.2 ([5, Proposition 1.6][1])**.** *If a semigroup variety* $\mathcal{V}$ *is a modular element of the lattice* $\mathbb{SEM}$ *then either* $\mathcal{V} = \mathcal{SEM}$ *or* $\mathcal{V} \subseteq \mathcal{SL} \vee \mathcal{N}$ *for some nil-variety* $\mathcal{N}$*.* □

Lemma 2.3 ([11, Theorem 1])**.** *If a semigroup variety* $\mathcal{V}$ *is an upper-modular element of the lattice* $\mathbb{SEM}$ *then either* $\mathcal{V} = \mathcal{SEM}$ *or* $\mathcal{V}$ *is a periodic variety and the variety* $\mathrm{Nil}(\mathcal{V})$ *is commutative and satisfies the identity* $x^2y = xy^2$*.* □

A semigroup variety $\mathcal{V}$ is said to be a *variety of index* n if every nilsemigroup in $\mathcal{V}$ satisfies the identity $x_1x_2\cdots x_n = 0$ and n is the least number with such a property. Since a commutative 0-reduced variety is contained in the variety $\mathcal{ZM}$, Theorem 1 and Lemma 2.3 imply the following

Corollary 2.4. *If a semigroup variety* $\mathcal{V}$ *is both an upper-modular and a lower-modular element of the lattice* $\mathbb{SEM}$ *then either* $\mathcal{V} = \mathcal{SEM}$ *or* $\mathcal{V}$ *is a variety of index* ≤ 2*.* □

This corollary and Lemma 1.2 imply the following

Corollary 2.5. *Let* $\mathcal{V}$ *be a nil-variety. The following are equivalent:*

[1] One should note that the paper [5] has dealt with the lattice of equational theories of semigroups, that is, the dual of $\mathbb{SEM}$ rather than the lattice $\mathbb{SEM}$ itself. When reproducing results from [5], we adapt them to the terminology of the present article. Note that the definition of a modular element of a lattice is selfdual, whence modular elements of the lattice of equational theories precisely correspond to modular varieties.

(i) $\mathcal{V}$ *is both an upper-modular and a lower-modular element of the lattice* $\mathbb{SEM}$*;*

(ii) $\mathcal{V}$ *is a strongly modular element of the lattice* $\mathbb{SEM}$*;*

(iii) $\mathcal{V}$ *is a neutral element of the lattice* $\mathbb{SEM}$*;*

(iv) $\mathcal{V}$ *is one of the varieties* $\mathcal{T}$ *or* $\mathcal{ZM}$*.* □

Theorem 1 and Lemma 2.1 imply the following three corollaries.

Corollary 2.6. *If a periodic semigroup variety* $\mathcal{V}$ *is a lower-modular element of the lattice* $\mathbb{SEM}$ *then so is the variety* $\mathrm{Nil}(\mathcal{V})$. □

Corollary 2.7. *A nil-variety is a lower-modular element of the lattice* $\mathbb{SEM}$ *if and only if it is a* 0*-reduced variety.* □

Corollary 2.8. *If a nil-variety is a lower-modular element of the lattice* $\mathbb{SEM}$ *then it is a modular element of this lattice.* □

The following two corollaries were proved firstly in [16].

Corollary 2.9 ([16, Theorem 3.1])**.** *A semigroup variety* $\mathcal{V}$ *is both a modular and a lower-modular element of the lattice* $\mathbb{SEM}$ *if and only if either* $\mathcal{V} = \mathcal{SEM}$ *or* $\mathcal{V}$ *is a* 0*-reduced variety or* $\mathcal{V} = \mathcal{SL} \vee \mathcal{R}$ *for some* 0*-reduced variety* $\mathcal{R}$*.*

Proof. The 'if' part immediately follows from Lemma 2.1 and Corollary 1.3. Let us verify the 'only if' part. Let $\mathcal{V}$ be a modular and a lower-modular variety with $\mathcal{V} \neq \mathcal{SEM}$. Since $\mathcal{V}$ is modular, Lemma 2.2 implies that $\mathcal{V} \subseteq \mathcal{SL} \vee \mathcal{N}$ for some nil-variety $\mathcal{N}$. Applying Lemma 1.2, we have

$$\mathcal{V} = \mathcal{V} \wedge (\mathcal{SL} \vee \mathcal{N}) = (\mathcal{V} \wedge \mathcal{SL}) \vee (\mathcal{V} \wedge \mathcal{N}).$$

Put $\mathcal{R} = \mathcal{V} \wedge \mathcal{N}$. By Lemma 1.2 $\mathcal{V}$ coincides with one of the varieties $\mathcal{R}$ or $\mathcal{SL} \vee \mathcal{R}$. Taking into account Corollary 1.3, we see that the variety $\mathcal{R}$ is lower-modular in both the cases. Since $\mathcal{R}$ is a nil-variety, it remains to refer to Corollary 2.7. □

Corollary 2.10 ([16, Proposition 4.1])**.** *For a semigroup variety* $\mathcal{V}$ *the following are equivalent:*

(i) $\mathcal{V}$ *is a strongly modular element of the lattice* $\mathbb{SEM}$*;*

(ii) $\mathcal{V}$ *is a neutral element of the lattice* $\mathbb{SEM}$*;*

(iii) $\mathcal{V}$ *is one of the varieties* $\mathcal{T}$*,* $\mathcal{SL}$*,* $\mathcal{ZM}$*,* $\mathcal{SL} \vee \mathcal{ZM}$ *or* $\mathcal{SEM}$*.*

Proof. The implication (iii) $\Longrightarrow$ (ii) follows from Lemma 1.2 and the well-known fact that the join of neutral elements is neutral itself (see, for instance, [2, Theorem III.2.9]). The implication (ii) $\Longrightarrow$ (i) is evident. It remains to verify the implication (i) $\Longrightarrow$ (iii). Let $\mathcal{V}$ be a strongly modular variety with $\mathcal{V} \neq \mathcal{SEM}$. By Corollary 2.9 either $\mathcal{V} = \mathcal{R}$ or $\mathcal{V} = \mathcal{SL} \vee \mathcal{R}$ for some 0-reduced variety $\mathcal{R}$. On the other hand, Corollary 2.4 implies that $\mathcal{V}$ is a variety of index ≤ 2. Therefore $\mathcal{R}$ is a nil-variety of index ≤ 2, whence $\mathcal{R} \subseteq \mathcal{ZM}$. By Lemma 1.2, $\mathcal{R}$ is one of the varieties $\mathcal{T}$ or $\mathcal{ZM}$. Corollary is proved. □

One more corollary of Theorem 1 is related to the notion of a definable subset of a lattice. A subset D of a lattice L is said to be *definable* in L if there exists a first order formula $\mathsf{F}(x)$ with one free variable such that evaluating the variable at an element $a \in L$ yields a true sentence if and only if $a \in D$; in this situation we say also that the formula $\mathsf{F}(x)$ *defines* the set D. A deep study of definable subsets of the lattice $\mathbb{SEM}$ has been carried out by Ježek and McKenzie in [5]. In particular, it was shown in [5, Theorem 1.11] that the class of all 0-reduced varieties is definable in the lattice $\mathbb{SEM}$. This fact was established via a sequence of lemmas involving rather complicated and somewhat artificial formulas, and in [5] there is no any explicitly written first order formula defining the class of all 0-reduced varieties. A simple and transparent formula of such a kind was constructed in [16, Proposition 4.5]. Now we can write some simpler formula defining the class of all 0-reduced varieties. First of all, one can reproduce the formula constructed in [16]. For this, we define some auxiliary formulas. Let $\mathsf{Mod}(x)$ [respectively $\mathsf{LMod}(x)$, $\mathsf{A}(x)$] be the formula defining the set of all modular elements [lower-modular elements, atoms] of a lattice, and $\mathsf{P}(x)$ the formula that says that x is not a greatest element of a lattice. We will not write these four formulas explicitly because they are evident. Set

$$\mathsf{M}(x,y,z) \;\rightleftharpoons\; y \le z \longrightarrow (x \vee y) \wedge z = (x \wedge z) \vee y;$$
$$\mathsf{Nil}(x) \;\rightleftharpoons\; \forall y \quad \mathsf{A}(y) \;\&\; y \le x \;\longrightarrow\; \bigl(\forall a,b,c \quad a,b,c \ngeq y \;\longrightarrow\; \mathsf{M}(a,b,c)\bigr).$$

Lemma 2.11. *The formula* $\mathsf{Nil}(x)$ *defines the class of all nil-varieties in the lattice* $\mathbb{SEM}$.

Proof. Let $\mathcal{V}$ be a non-trivial nil-variety. If $\mathcal{A}$ is a minimal non-trivial semigroup variety with $\mathcal{A} \subseteq \mathcal{V}$ then $\mathcal{A} = \mathcal{ZM}$. Furthermore, if $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \not\supseteq \mathcal{ZM}$ then $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are *completely regular* varieties (that is, varieties consisting of completely regular semigroups — unions of groups). Since the lattice of all completely regular varieties is modular [7], the sentence $\mathsf{Nil}(\mathcal{V})$ is true. Clearly, the sentence $\mathsf{Nil}(\mathcal{T})$ is true as well. Finally, let $\mathcal{W}$ be a semigroup variety and $\mathcal{W}$ is not a nil-variety. Then there exists a minimal non-trivial semigroup variety $\mathcal{A}$ with $\mathcal{A} \subseteq \mathcal{V}$ and $\mathcal{A} \ne \mathcal{ZM}$. The lattice of all nil-varieties is not modular. (This fact may be easily deduced already from the pioneer article by Schwabauer [9]. In the explicit form, it was mentioned firstly, probably, by Mel'nik in [6].) Hence there exist three (for instace, nil-)varieties $\mathcal{K}, \mathcal{M}, \mathcal{N}$ such that $\mathcal{K}, \mathcal{M}, \mathcal{N} \not\supseteq \mathcal{A}$ and the sentence $\mathsf{M}(\mathcal{K}, \mathcal{M}, \mathcal{N})$ is false. Therefore the sentence $\mathsf{Nil}(\mathcal{W})$ is false too. $\square$

According to [16, Proposition 4.5] (see also Corollary 2.9 above) the class of all 0-reduced varieties is definable in the lattice $\mathbb{SEM}$ by the formula

$$\mathsf{Mod}(x) \;\&\; \mathsf{LMod}(x) \;\&\; \mathsf{P}(x) \;\&\; \mathsf{Nil}(x)$$

(this formula was written in [16] in other notation). Note that we may eliminate the conjunct $\mathsf{P}(x)$ from this formula because, for a semigroup variety, the property 'to be proper' is guaranteed already by the conjunct $\mathsf{Nil}(x)$. Moreover, Corollary 2.7 and Lemma 2.11 imply the following

Corollary 2.12. *The class of all* 0*-reduced varieties is definable in the lattice* $\mathbb{SEM}$ *by the formula* $\mathsf{LMod}(x)$ & $\mathsf{Nil}(x)$. $\square$

2.3 Theorem 2

The implication (iv) $\Longrightarrow$ (iii) of Theorem 2 follows from Corollary 2.10. The implications (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) are evident. It remains to verify the implication (i) $\Longrightarrow$ (iv).

We need some additional notation. If S is a semigroup then we denote by S^1 the semigroup S with a new unit element adjoined. If a variety $\mathcal{V}$ contains semigroups of the form N^1, where N is a nilsemigroup, then $\mathrm{Nil}^1(\mathcal{V})$ denotes the variety generated by all semigroups of such a form; otherwise $\mathrm{Nil}^1(\mathcal{V}) = \mathcal{T}$.

Now let $\mathcal{V}$ be a commutative lower-modular semigroup variety. Theorem 1 implies that $\mathcal{V}$ is periodic. Results of [3] and the proof of Proposition 1 in [15] imply that $\mathcal{V} = \mathcal{G} \vee \mathcal{M} \vee \mathcal{N}$ where $\mathcal{G} = \mathrm{Gr}(\mathcal{V})$, $\mathcal{M} = \mathrm{Nil}^1(\mathcal{V})$, and $\mathcal{N} = \mathrm{Nil}(\mathcal{V})$. Every commutative 0-reduced variety is contained in the variety $\mathcal{ZM}$. Theorem 1 and the fact that the variety $\mathcal{N}$ is commutative (because $\mathcal{V}$ is) imply that $\mathcal{N} \subseteq \mathcal{ZM}$. Further, it is well known that if N is a singleton semigroup [a 2-element semigroup with zero multiplication] then the semigroup N^1 generates the variety $\mathcal{SL}$ [respectively the variety $\mathcal{C} = \mathrm{var}\{x^2 = x^3,\, xy = yx\}$]. It is evident that $\mathrm{Nil}\,(\mathcal{C}) \nsubseteq \mathcal{ZM}$. The inclusion $\mathcal{N} \subseteq \mathcal{ZM}$ implies now that $\mathcal{M} \nsupseteq \mathcal{C}$, whence $\mathcal{M} \subseteq \mathcal{SL}$. Lemma 1.2 implies now that $\mathcal{N}$ is one of the varieties $\mathcal{T}$ or $\mathcal{ZM}$, while $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$. It remains to verify that $\mathcal{G} = \mathcal{T}$. We denote by $\mathcal{A}_k$ (where $k > 1$) the variety of all abelian groups of exponent dividing k. If $\mathcal{G} \neq \mathcal{T}$ then $\mathcal{G} = \mathcal{A}_k$ for some $k > 1$. In view of Corollary 1.3, it suffices to prove that the variety $\mathcal{A}_k$ for an arbitrary positive integer $k > 1$ is not lower-modular. Here we need some notation introduced in [8]. Let $\mathcal{G}$ be a periodic group variety and $\{v_i = 1 \mid i \in I\}$ a basis of identities of $\mathcal{G}$ where v_i are semigroup words. Let us denote by r the exponent of the variety $\mathcal{G}$. For a letter x, put $x^0 = x^{r(r+1)}$. Furthermore, let $F(\mathcal{G})$ be the free group of countably infinite rank in $\mathcal{G}$. A subset X of $F(\mathcal{G})$ is called *verbal* if it is closed under all endomorphisms of $F(\mathcal{G})$. Clearly, a verbal subset X of $F(\mathcal{G})$ is a set of all values in $F(\mathcal{G})$ of some set V of words; in this case we write $X = \mathcal{G}(V)$. Put

$$S(\mathcal{G}) = \mathrm{var}\{xyz = xy^{r+1}z,\, x^0y^0 = y^0x^0,\, x^2 = x^{r+2},\, xv_i^2y = xv_iy \mid i \in I\}$$

and

$$S(\mathcal{G}, X) = S(\mathcal{G}) \wedge \mathrm{var}\{xvy = (xvy)^{r+1} \mid v \in V\},$$

where X is a verbal subset in $F(\mathcal{G})$ and $X = \mathcal{G}(V)$. It is verified in [8] that the subvariety lattice of the variety $S(\mathcal{A}_k)$ contains a sublattice shown in Fig. 1 (see [8, Fig. 1][2]). We see that

$$\big(S(\mathcal{T}) \wedge S(\mathcal{A}_k, 1)\big) \vee \mathcal{A}_k = S\big(\mathcal{A}_k, F(\mathcal{A}_k)\big),$$

while

$$\big(S(\mathcal{T}) \vee \mathcal{A}_k\big) \wedge S(\mathcal{A}_k, 1) = S(\mathcal{A}_k, 1).$$

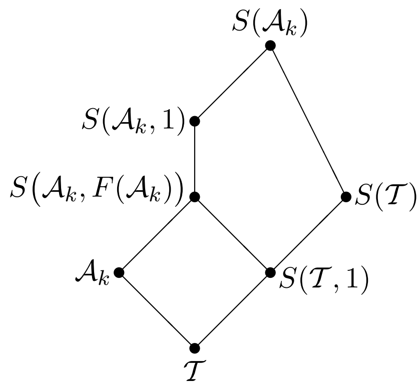

Figure 1: A sublattice of the subvariety lattice of $S(\mathcal{A}_k)$

Since $\mathcal{A}_k \subseteq S(\mathcal{A}_k, 1)$ and $S\big(\mathcal{A}_k, F(\mathcal{A}_k)\big) \neq S(\mathcal{A}_k, 1)$, we are done. □

3 Open questions

By Theorem 1 any proper lower-modular variety is periodic. The class of all periodic varieties contains two wide subclasses that play a distinguished role in the theory of semigroup varieties and have very different properties in many aspects. We mean the class of all completely regular varieties and the class of all nil-varieties. Lower-modular nil-varieties are characterized by Corollary 2.7. This inspires the following

Problem 3.1. *Describe completely regular semigroup varieties that are lower-modular elements of the lattice* $\mathbb{SEM}$.

Note that by Lemma 2.2 there are two completely regular modular varieties only, namely the varieties $\mathcal{T}$ and $\mathcal{SL}$. On the other hand, according to [11, Theorem 2] every commutative completely regular variety is upper-modular.

As we have proved in Subsection 2.3, all non-trivial abelian periodic group varieties are not lower-modular. This permits to conjecture that a proper lower-modular variety does not contain non-singleton groups.

Question 3.2. *Let* $\mathcal{V}$ *be a proper semigroup variety and a lower-modular element of the lattice* $\mathbb{SEM}$*. Is the variety* $\mathcal{V}$ *combinatorial*?

Note that a proper modular variety is combinatorial by Lemma 2.2, while for upper-modular varieties this is false (see [11, Theorem 2]).

Semigroup varieties that are both modular and lower-modular were completely described in [16, Theorem 3.1] (see Corollary 2.9), while varieties that are both modular and upper-modular were completely determined in [14, Theorem 1]. This inspires the following

Problem 3.3. *Describe semigroup varieties that are both upper-modular and lower-modular elements of the lattice* $\mathbb{SEM}$.

[2] Formally speaking, this figure from [8] deals with the case when k is a prime number only. But all proofs in [8] remain true for arbitrary k.

Corollary 2.4 may be considered as a partial step in a solution of this problem. Theorem 2 and Corollary 2.5 solve Problem 3.3 within the classes of commutative varieties and nil-varieties respectively.

It seems to be probable that the classes of all modular, upper-modular, and lower-modular semigroup varieties are pairwise incomparable by inclusion. This supposition is almost completely confirmed by the following three examples.

Example 3.4. If $\mathcal{ZM} \subset \mathcal{V} \subseteq \mathrm{var}\{x^2y = 0,\, xy = yx\}$ then the variety $\mathcal{V}$ is modular and upper-modular by [14, Theorem 1] but not lower-modular (this follows from Theorem 1 of [14] and Corollary 2.4).

Example 3.5. If $\mathcal{V}$ is a 0-reduced variety and $\mathcal{V} \nsubseteq \mathcal{ZM}$ then $\mathcal{V}$ is modular and lower-modular by Corollary 2.9 but not upper-modular (this follows from Corollaries 2.4 and 2.9).

Example 3.6. The varieties $\mathrm{var}\{xy = x\}$ and $\mathrm{var}\{xy = y\}$ are upper-modular because they are minimal non-trivial semigroup varieties (see, for instance, [1]) but not modular by Lemma 2.2.

It remains to give an example of a lower-modular but not a modular variety. But we do not know whether or not such a variety exists. The list of semigroup varieties that are known to be lower-modular is very short: it includes the varieties mentioned in Corollary 2.9 only, and by this corollary all these varieties are modular as well.

Question 3.7. *Does there exist a semigroup variety that is a lower-modular element of the lattice* $\mathbb{SEM}$ *but not a modular element of this lattice*?

This question is answered in negative within the classes of commutative varieties and nil-varieties (see Theorem 2 and Corollary 2.8 respectively). Note that a negative answer to Question 3.7 in the general case would immediately imply a solution of Problem 3.3 (see Corollary 2.10) and an affirmative answer to Question 3.2 (see Lemma 2.2).

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