Some facts about traces

Recall that the *trace* $\mathrm{tr}(A)$ of a square matrix A is the sum of the elements on the main diagonal of A: if $A = (\alpha_{ij})_{n\times n}$, then $\mathrm{tr}(A) := \sum_{i=1}^{n} \alpha_{ii}$.

Clearly, the trace of a self-conjugate matrix is a real number. In the lecture we claimed more: the trace of any **product** of two self-conjugate matrices is a real number. Observe that the product of two self-conjugate matrices need not be self-conjugate; more precisely, the product of two self-conjugate matrices A and B is self-conjugate if and only if A and B commute. So, the claim is not trivial; let us prove it.

Proposition. *If A and B are self-conjugate matrices, then* $\mathrm{tr}(AB) \in \mathbb{R}$.

Proof. We give two proofs. The first one is via a direct entry-wise calculation. Let $A = (\alpha_{ij})_{n\times n}$, $B = (\beta_{ij})_{n\times n}$. Then

$$\mathrm{tr}(AB) = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij}\beta_{ji} = \sum_{i,j=1}^{n} \alpha_{ij}\beta_{ji}.$$

On the other hand, we have

$$\begin{aligned}
\overline{\mathrm{tr}(AB)} &= \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{\alpha_{ij}}\,\overline{\beta_{ji}} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ji}\beta_{ij} && \text{since } A = A^* \text{ and } B = B^* \\
&= \sum_{i,j=1}^{n} \alpha_{ji}\beta_{ij}.
\end{aligned}$$

Clearly, the final expressions $\sum_{i,j=1}^{n} \alpha_{ij}\beta_{ji}$ and $\sum_{i,j=1}^{n} \alpha_{ji}\beta_{ij}$ represent the same sum so that $\mathrm{tr}(AB) = \overline{\mathrm{tr}(AB)} \in \mathbb{R}$.

The second proof does not explicitly appeal to the entries of A and B:

$$\begin{aligned}
\overline{\mathrm{tr}(AB)} &= \mathrm{tr}(\overline{A}\,\overline{B}) \\
&= \mathrm{tr}(A^T B^T) && \text{since } A = A^* = \overline{A}^T \text{ and } B = B^* = \overline{B}^T \\
&= \mathrm{tr}((BA)^T) && \text{since } A^T B^T = (BA)^T \\
&= \mathrm{tr}(BA) && \text{since } \mathrm{tr}(C) = \mathrm{tr}\, C^T \text{ for every matrix } C \\
&= \mathrm{tr}(AB) && \text{since } \mathrm{tr}(AB) = \mathrm{tr}(BA).
\end{aligned}$$

Thus, $\mathrm{tr}(AB) = \overline{\mathrm{tr}(AB)} \in \mathbb{R}$. □

In the second proof we made use of the well-known property $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. If fact, this property holds in a more general situation when A is a $n \times k$-matrix and B is a $k \times n$-matrix. This can be verified by a direct entry-wise calculation. We can use the property to easily deduce the formula

$$\mathrm{tr}(|x\rangle\langle x|A) = \langle x|A|x\rangle$$

which is proved in the textbook (p. 14) in a more complicated way and under additional assumptions. Indeed, taking the column $|x\rangle$ as the first factor and the row $\langle x|A$ as the second factor, we conclude that

$$\mathrm{tr}(|x\rangle\langle x|A) = \mathrm{tr}(\langle x|A|x\rangle),$$

but since $\langle x|A|x\rangle$ is a 1×1-matrix, that is, a number,

$$\mathrm{tr}(\langle x|A|x\rangle) = \langle x|A|x\rangle.$$