Every non-negative operator is self-conjugate

An operator ρ on a Hilbert space H is said to be *non-negative* if $\langle u, \rho u\rangle$ is a non-negative real number for every $u \in H$. In the lecture we claimed that a non-negative operator is always self-conjugate. Let us prove this claim.

We need a little lemma.

Lemma. *If $\langle u, \rho u\rangle = 0$ for every $u \in H$, then $\rho = 0$.*

Proof. Take $u = x + y$. Then we have

$$\begin{aligned} 0 &= \langle x+y, \rho(x+y)\rangle \\ &= \langle x, \rho x\rangle + \langle x, \rho y\rangle + \langle y, \rho x\rangle + \langle y, \rho y\rangle \\ &= \langle x, \rho y\rangle + \langle y, \rho x\rangle \qquad \text{since } \langle x, \rho x\rangle = \langle y, \rho y\rangle = 0. \end{aligned}$$

Now take $u = x - iy$. Then we have

$$\begin{aligned} 0 &= \langle x-iy, \rho(x-iy)\rangle \\ &= \langle x, \rho x\rangle + i\langle x, \rho y\rangle - i\langle y, \rho x\rangle - \langle y, \rho y\rangle \\ &= i(\langle x, \rho y\rangle - \langle y, \rho x\rangle) \qquad \text{since } \langle x, \rho x\rangle = \langle y, \rho y\rangle = 0. \end{aligned}$$

Hence

$$\begin{aligned} \langle x, \rho y\rangle + \langle y, \rho x\rangle &= 0, \\ \langle x, \rho y\rangle - \langle y, \rho x\rangle &= 0. \end{aligned}$$

Summing up these two equalities, we get $\langle x, \rho y\rangle = 0$. Since x can be chosen as any vector in H, we conclude that $\rho y = 0$, and since y can be chosen as any vector in H, we conclude that $\rho = 0$. □

Observe that the lemma fails for operators on Euclidian spaces: say, if ρ is the $\frac{\pi}{2}$-rotation of the plane $\mathbb{R}^2$, we have $\langle u, \rho u\rangle = 0$ for every $u \in \mathbb{R}^2$.

Now we are ready to prove the claim; in fact, we prove a stronger result:

Proposition. *If an operator ρ on a Hilbert space H is such that $\langle u, \rho u\rangle$ is a real number for every $u \in H$, then ρ is self-conjugate.*

Proof. Since $\langle u, \rho u\rangle$ is a real number, we have

$$\langle u, \rho u\rangle = \overline{\langle u, \rho u\rangle} = \langle \rho u, u\rangle = \langle u, \rho^* u\rangle.$$

Hence $\langle u, (\rho - \rho^*)u\rangle = 0$ for every $u \in H$. Now the above lemma yields $\rho - \rho^* = 0$, that is, $\rho = \rho^*$. □