SIMON STEVIN, A Quarterly Journal of Pure and Applied Mathematics
Volume 56 (1982), Number 4 (October 1982)

# PSEUDO-SEMILATTICES AND BIORDERED SETS - III
## REGULAR LOCALLY TESTABLE SEMIGROUPS

K.S.S. Nambooripad

Communicated by F. Pastijn

In [ 9 ], Theorem 2.10, we have obtained a structure theorem for pseudo-inverse semigroups in terms of inductive pseudo-groupoids. An inductive pseudo-groupoid is a disjoint union of Rees groupoids endowed with a partial order satisfying axioms (I1)-(I4) of Definition 2.1 of [ 9 ]. In this paper we shall obtain a refinement of this result in the case of regular locally testable semigroups. Our structure theorem differs from the structure theorem for regular locally testable semigroups given in [ 10 ].

In Section 1, we obtain a characterization of pseudo-semilattices as a residuated subset of the product of two partially ordered sets. This result is different from the construction of pseudo-semilattices given by Meakin and Pastijn in [ 5 ]. We then use this result to obtain the proposed refinement. In the last section we describe the (isomorphism) class of all regular locally testable semigroups determined by a given biordered set.

Whenever possible, we shall use the notation and the terminology of [ 2 ] and [ 3 ]. Since this paper is a continuation of [ 9 ], we shall freely use notations and terminologies introduced there, without further comment.

## 1. PSEUDO-SEMILATTICES

Let I be a partially ordered set. A subset A of I is said to be residuated if the inclusion mapping $inc_A : A \subseteq I$ is residuated ([1], p. 11). If $inc_A^+$ denotes the residual of $inc_A$, then by Theorem 2.6 of [1], $inc_A^+ : I \to A$ is the unique surjective mapping such that $inc_A^+ \circ inc_A \leqslant 1_I$, $inc_A \circ inc_A^+ = 1_A$. Hence if $F_A = inc_A^+ \circ inc_A$, Then $F_A = F_A^2 \leqslant 1_I$ and so $F_A$ is a dual closure mapping of I into I (see [1]). Further im $F_A = A$. Conversely, if $F : I \to I$ is a dual closure mapping, then $A = \text{im } F$ is a residuated subset of I. For if $\phi : I \to A$ is the surjective mapping determined by F, then $\phi \circ inc_A = F \leqslant 1_I$ and $inc_A \circ \phi = \phi | A = F | A = 1_A$ (since F is an idempotent). Hence $inc_A$ is residuated with $inc_A^+ = \phi$. Further if B is any ordered set and $g : B \to I$ is an injective, residuated mapping, then by Theorem 2.6 of [1], $g^+ : I \to B$ is surjective and $g \circ g^+ = 1_B$. Hence if $F = g^+ \circ g$, then $F \leqslant 1_I$ and $F^2 = g^+ \circ (g \circ g^+) \circ g = g^+ \circ 1_B \circ g = g^+ \circ g = F$. Thus F is a dual closure mapping so that im $F = A = \text{im } g$ is a residuated subset of I. It is easy to see that the surjective mapping $\bar{g} : B \to A$ determined by g is an order isomorphism of B onto A. We summarize the foregoing discussion as follows.

<u>LEMMA</u> 1.1. <u>Let</u> I <u>be a partially ordered set and</u> $A \subseteq I$. <u>Then</u> A <u>is a residuated subset of</u> I <u>if and only if there exists a dual closure mapping</u> $F_A : I \to I$ <u>such that</u> $A = \text{im } F_A$. <u>When</u> $F_A$ <u>exists as above, it is unique</u>.

<u>Moreover, if</u> $g : B \to I$ <u>is any injective residuated mapping of an ordered set</u> B <u>into</u> I, <u>then</u> B <u>is order isomorphic to a residuated subset of</u> I.

The lemma shows that there exists a one-to-one correspondence between residuated subsets of I and dual closure mappings of I. Since dual closure mappings are idempotents, residuated subsets are retracts of I in the usual sense.

Let I and $\Lambda$ be partially ordered sets. If $P \subseteq I \times \Lambda$, we say that $e, f \in P$ are connected if there exist a positive integer n and elements $e_r \in P$, $r = 0, \ldots, 2n$, such that $e_0 = e$, $e_{2n} = f$ and for

$s = 1,\dots,n$, $e_{2s-1}p_I = e_{2s-2}p_I$ and $e_{2s}p_\Lambda = e_{2s-1}p_\Lambda$, where $p_I : I \times \Lambda \to I$ and $p_\Lambda : I \times \Lambda \to \Lambda$ are the projections.

If $A$ is any partially ordered set and $x \in A$, in the following we shall denote by $[-,x]_A$ the principal ideal of $A$ generated by $x$. Thus if $A \subseteq I$, then $[-,x]_A = [-,x]_I \cap A$.

THEOREM 1.2. *Let $I$ and $\Lambda$ be partially ordered sets and $E$ be a residuated subset of $I \times \Lambda$ such that for all $e \in E$, the mappings*

(a) $p_I : [-,e]_E \to [-,ep_I]_I$ *and* $p_\Lambda : [-,e]_E \to [-,ep_\Lambda]_\Lambda$

*are order isomorphisms. For $e,f \in E$, define*

$$e \wedge f = F_E(ep_I, fp_\Lambda) \tag{1.1}$$

*where $F_E$ denotes the dual closure mapping associated with $E$. (1.1) defines a binary operation on $E$ such that $(E,\wedge)$ is a partially associative pseudo-semilattice. $E$ is locally testable if and only if*

(b) *for all $e \in E$, no two distinct elements in $[-,e]_E$ are connected.*

*Conversely, every $E \in PSL$ is isomorphic to one constructed as above.*

Before proving the theorem we shall prove two lemmas in which we use the notations established above. In particular we assume that $E$ is a residuated subset of $I \times \Lambda$ satisfying condition (a).

LEMMA 1.3. *For $e,f \in E$,*

$$e \wedge f = f \iff fp_I \leqslant ep_I .$$

*If $e$ and $f$ satisfy this condition we have $(f \wedge e)p_I = fp_I$.*

*Proof*. If $e \wedge f = f$, then $F_E(ep_I, fp_\Lambda) = f$ and since $F_E \leqslant 1_{I \times \Lambda}$, we have $f \leqslant (ep_I, fp_\Lambda)$. Thus $fp_I \leqslant ep_I$.

Conversely, if $fp_I \leqslant ep_I$, then $f = (fp_I, fp_\Lambda) \leqslant (ep_I, fp_\Lambda)$, and since $F_E$ is order preserving and idempotent, we have $f = F_E(fp_I, fp_\Lambda) \leqslant F_E(ep_I, fp_\Lambda) = e \wedge f$. Then $fp_\Lambda \leqslant (e \wedge f)p_\Lambda \leqslant fp_\Lambda$ and so $(e \wedge f)p_\Lambda = fp_\Lambda$. Since $f \in [-, e \wedge f]_E$, by (a) we get $e \wedge f = f$.

Since $fp_I \in [-,ep_I\,]_I$, by (a) there exists $f' \in [-,e\,]_E$ such that $f'p_I = fp_I$. Thus $f' \leqslant (fp_I,ep_\Lambda)$ and so $f' = F_E(f') \leqslant F_E(fp_I,ep_\Lambda) = f \wedge e$. Hence $f'p_I = fp_I \leqslant (f \wedge e)p_I \leqslant fp_I$, and we conclude $(f \wedge e)p_I = fp_I$.

<u>LEMMA</u> 1.4. <u>Let</u> $e,f,g \in E$ <u>and</u> $h = f \wedge g$. <u>If</u> $fp_I, gp_I \in [-,ep_I\,]_I$, <u>then</u> $hp_I \leqslant gp_I$.

<u>Proof</u>. Since $hp_\Lambda \leqslant gp_\Lambda$, by (a), there exists $h' \in [-,g\,]_E$ such that $h'p_\Lambda = hp_\Lambda$. Then $h'p_I \leqslant gp_I \leqslant ep_I$, and so by Lemma 1.3 $e \wedge h' = h'$. Similarly $hp_I \leqslant fp_I \leqslant ep_I$ and so $e \wedge h = h$. But since $hp_\Lambda = h'p_\Lambda$, we have

$$h = e \wedge h = F_E(ep_I,hp_\Lambda) = F_E(ep_I,h'p_\Lambda) = e \wedge h' = h' \quad .$$

Hence $hp_I = h'p_I \leqslant gp_I$.

<u>Proof of Theorem</u> 1.2. Define the relations $\omega^r$ and $\omega^l$ on E as follows : for $e,f \in E$,

$$e\ \omega^r\ f \iff ep_I \leqslant fp_I$$

and

$$e\ \omega^l\ f \iff ep_\Lambda \leqslant fp_\Lambda \quad .$$

Clearly, $\omega^r$ and $\omega^l$ are quasi-orders on E. If $e,f \in E$ are arbitrary, and $h = e \wedge f$, then $h = F_E(ep_I,fp_\Lambda) \leqslant (ep_I,fp_\Lambda)$, and so $hp_I \leqslant ep_I$ and $hp_\Lambda \leqslant fp_\Lambda$. Hence $h \in \omega^r(e) \cap \omega^l(f)$. If $g \in \omega^r(e) \cap \omega^l(f)$, then $g\ \omega^r\ e$, so that $gp_I \leqslant ep_I$. Similarly $gp_\Lambda \leqslant fp_\Lambda$. Hence $g = (gp_I,gp_\Lambda) \leqslant (ep_I,fp_\Lambda)$. Since $F_E$ is an idempotent order preserving mapping with $\operatorname{im} F_E = E$, we have

$$g = F_E(g) \leqslant F_E(ep_I,fp_\Lambda) = h \quad .$$

Thus $gp_I \leqslant hp_I$, $gp_\Lambda \leqslant hp_\Lambda$. Therefore $g \in \omega^r(h) \cap \omega^l(h) = \omega(h)$, where $\omega = \omega^r \cap \omega^l$. It follows that $(E,\omega^l,\omega^r)$ is a pseudo-semilattice in which the binary operation $\wedge$ is defined by (1.1).

By Lemma 1.3, $(E,\wedge)$ is a regular pseudo-semilattice. Hence to show that E is a biordered set, it suffices to prove the following : for all $e \in E$, $f,g \in \omega^r(e)$,

(1) $(f \wedge e) \wedge g = f \wedge g$ ,

(2) $(f \wedge g) \wedge e = f \wedge (g \wedge e) = (f \wedge e) \wedge (g \wedge e)$

(see [ 8 ], Theorem 2). By Lemma 1.3, we have $(f \wedge e)p_I = fp_I$ and so

$$(f \wedge e) \wedge g = F_E((f \wedge e)p_I, gp_\Lambda) = F_E(fp_I, gp_\Lambda) = f \wedge g \ ,$$

and

$$f \wedge (g \wedge e) = F_E(fp_I, (g \wedge e)p_\Lambda) = F_E((f \wedge e)p_I, (g \wedge e)p_\Lambda)$$
$$= (f \wedge e) \wedge (g \wedge e) \ .$$

To prove that $(f \wedge g) \wedge e = f \wedge (g \wedge e)$, let $h = f \wedge g$ and $h' = h \wedge e$, $g' = g \wedge e$ and $k = f \wedge g'$. Then $kp_I \leqslant fp_I$ and $kp_\Lambda \leqslant g'p_\Lambda \leqslant ep_\Lambda$. By Lemma 1.3 and Lemma 1.4 $kp_I \leqslant g'p_I = gp_I$. Hence there exists $k_1 \in [-,g]_E$ such that $kp_I = k_1p_I$ . Then $k_1p_I \leqslant fp_I$ and $k_1p_\Lambda \leqslant gp_\Lambda$. Therefore $k_1 = F_E(k_1) \leqslant F_E(fp_I, gp_\Lambda) = f \wedge g = h$. It follows that $kp_I \leqslant hp_I$ . Since $kp_\Lambda \leqslant ep_\Lambda$, it follows that $k \leqslant F_E(hp_I, ep_\Lambda) = h \wedge e = h'$.

Now by Lemma 1.3, $h'p_I = hp_I$, $g'p_I = gp_I$ . Also, $hp_I \leqslant gp_I$ by Lemma 1.4 . Hence there exists $\bar{h} \in [-,g']_E$ such that $\bar{h}p_I = hp_I = h'p_I$ . But $\bar{h}, h' \in [-,e]_E$ and so $\bar{h} = h'$. Hence $h' \ \omega \ g'$. So $h'p_\Lambda \leqslant g'p_\Lambda$, $h'p_I = hp_I \leqslant fp_I$ . Hence $h' \leqslant F_E(fp_I, g'p_\Lambda) = f \quad g' = k$ and this proves (2).

It is easy to see from the definition and Lemma 1.3 that $f,g \in [-,e]_E$ are connected if and only if there exists an E-chain C such that $e_C = f$ and $f_C = g$ . Thus $f,g \in [-,e]_E$ are connected if and only if $f,g \in \omega(e) \cap \delta_1$ for some $\delta_1 \in E/\delta_o$, and so the condition (b) of Theorem 1.2 is equivalent to the condition that for all $\delta_1 \in E/\delta_o$ , $\delta_1 \cap \omega(e)$ contains at most one element. Hence by Corollary 1.5 of [ 9 ] E is locally testable if and only if E satisfies (b).

Conversely, assume that $E \in PSL$ and let $I = E/\mathcal{R}$ and $\Lambda = E/\mathcal{L}$ . Define $\theta : E \to I \times \Lambda$ and $\phi : I \times \Lambda \to E$ as follows.

$e\theta = (R_e, L_e)$ for every $e \in E$ ,

and

$(R_e, L_f)\phi = e \wedge f$, for every $(R_e, L_f) \in I \times \Lambda$ .

It is easy to see that $\theta$ is an injective, order preserving mapping of E into $I \times \Lambda$. If $R_e = R_{e'}$, and $L_f = L_{f'}$, we have $e \wedge f = e' \wedge f'$ (see Proposition 2.5 of [ 6 ]). Thus $\phi$ is well-defined. It is also order preserving. Further for all $e \in E$, $(e\theta)\phi = (R_e, L_e)\phi = e \wedge e = e$, so that $\theta\phi = 1_E$. For $(R_e, L_f) \in I \times \Lambda$, $(R_e, L_f)\phi\theta = (e \wedge f)\theta = (R_{e \wedge f}, L_{e \wedge f})$. Since $R_{e \wedge f} \leqslant R_e$, $L_{e \wedge f} \leqslant L_f$ it follows that $\phi\theta \leqslant 1_{I \times \Lambda}$. Hence $\theta$ is residuated and $\phi$ is its residual. Hence by Lemma 1.1, $F = \phi\theta$ is a dual closure mapping of $I \times \Lambda$ and so $E' = \operatorname{im} F$ is a residuated subset of $I \times \Lambda$. If $e' = (R_e, L_e)$, with $e \in E$, is any element of $E'$, then $p_I : [-, e']_{E'} \to [-, R_e]_I$ is an order isomorphism. For, clearly $\theta : \omega(e) \to [-, e']_{E'}$ is an isomorphism and it follows from Theorem 1.3 (b) of [ 9 ] that the mapping $g \to R_g$ is an isomorphism of the semilattice $\omega(e)$ onto $[-, R_e]_I$. Similarly, $p_\Lambda : [-, e']_{E'} \to [-, L_e]_\Lambda$ is also an isomorphism. Thus $E'$ satisfies condition (a). If $e, f \in E$, $(e \wedge f)\theta = (R_{e \wedge f}, L_{e \wedge f})$. Now $(R_{e \wedge f}, L_{e \wedge f}) = F(R_e, L_f) = (R_e, L_e) \wedge (R_f, L_f) = e\theta \wedge f\theta$ by (1.1). Thus $\theta$ is an isomorphism of E onto $E'$. This completes the proof of Theorem 1.2 .

<u>REMARK</u>. If I, $\Lambda$ and E are as in the statement of Theorem 1.2, then it is easy to see that the mappings

$$R_e \to ep_I, \qquad L_e \to ep_\Lambda,$$

are order isomorphisms of $E/\mathcal{R}$ and $E/\mathcal{L}$ onto subsets of I and $\Lambda$ respectively. It is easy to see that they are not in general isomorphisms of $E/\mathcal{R}$ onto I and $E/\mathcal{L}$ onto $\Lambda$. Thus E does not uniquely determine the partially ordered sets I and $\Lambda$. It can be seen that these mappings are isomorphisms if and only if E is a subdirect product of I and $\Lambda$ in the sense that the mappings $p_I | E$ and $p_\Lambda | E$ are surjective.

## 2. STRUCTURE OF REGULAR LOCALLY TESTABLE SEMIGROUPS

A Rees groupoid G is said to be combinatorial if the subgroups of G are trivial. It is clear that a Rees groupoid G = $\mathcal{M}^0(H;I,\Lambda;P)\setminus\{0\}$ is combinatorial if and only if the group H contains exactly one element. This leads to a convenient representation of combinatorial Rees groupoids given in the following lemma whose proof is routine .

LEMMA 2.1. (a) *Let* I *and* $\Lambda$ *be sets and* R *a subdirect product of* I *and* $\Lambda$. *Let* $G(I,\Lambda;R)$ *be the partial algebra on the set* $I \times \Lambda$ *with partial binary operation defined as follows.*

$$(i,j)(i',j') = \begin{cases} (i,j') \text{ if } (i',j) \in R \\ \text{undefined otherwise} \end{cases} \quad . \tag{2.1}$$

*Then* $G(I,\Lambda;R)$ *is a combinatorial Rees groupoid. Conversely, every combinatorial Rees groupoid is isomorphic to one constructed in this way.*

(b) *A mapping* $\phi : G(I,\Lambda;R) \to G(I',\Lambda';R')$ *of combinatorial Rees groupoids is a homomorphism if and only if there exist mappings* $\theta : I \to I'$ *and* $\psi : \Lambda \to \Lambda'$ *such that* $\theta \times \psi | R$ *is a mapping of* R *into* R' *and* $\phi = \theta \times \psi$ .

(c) *If* $\phi_1 : G(I,\Lambda;R) \to G(I',\Lambda';R')$ *and* $\phi_2 : G(I',\Lambda';R') \to G(I'',\Lambda'';R'')$ *are homomorphisms of combinatorial Rees groupoids, and if* $\phi_i = \theta_i \times \psi_i$, $i = 1,2$, *then* $\phi_1\phi_2 = \theta_1\theta_2 \times \psi_1\psi_2$ .

Let $P : \mathcal{L} \to \underline{\text{Set}}$ be a set-valued functor. A functor $P' : \mathcal{L} \to \underline{\text{Set}}$ is a subfunctor of P if for all $a \in V(\mathcal{L})$, $P'(a)$ is a subset of $P(a)$, and the inclusion $inc_a : P'(a) \subseteq P(a)$ is natural in a. Here, as in [ 6 ], $V(\mathcal{L})$ denotes the vertex class of $\mathcal{L}$ ; the morphism class of $\mathcal{L}$ is denoted by $\mathcal{L}$ itself. We shall write $P' \subseteq P$ to mean that P' is a subfunctor of P. It may be noted that $P' \subseteq P$ if and only if for all $f : a \to b$ in $\mathcal{L}$, $P'(f) = P(f)|P'(a)$. If $P,Q : \mathcal{L} \to \underline{\text{Set}}$ are functors, the universal properties of the cartesian properties of sets implies the assignments

$$a \to P(a) \times Q(a) \ (a \in V(\mathcal{L})), \quad f \to P(f) \times Q(f) \ (f \text{ in } \mathcal{L})$$

yield a functor $P \times Q : \mathcal{L} \to \underline{\text{Set}}$ and that $P \times Q$ is the product of P and Q in the functor category [ $\mathcal{L}$,Set ] (see also [ 4 ]).

Let $\mathbb{D}$ be a partially ordered set. In the following we shall regard $\mathbb{D}$ as a small category in the following way : $V(\mathbb{D})$ is the same as the set $\mathbb{D}$ and morphisms of the category $\mathbb{D}$ are pairs $(a,b)$ with $b \leqslant a$. We denote the morphism with domain a and codomain b by $b \leqslant a$. Note that this is dual to the convention adopted in [ 4 ].

THEOREM 2.2. Let $\mathbb{D}$ be a partially ordered set and $P,Q : \mathbb{D} \to \mathbf{Set}$ be functors such that for all $a,b \in \mathbb{D}$ with $a \neq b$, $P(a) \cap P(b) = \square = Q(a) \cap Q(b)$. Assume that $\Delta$ is a subfunctor of $P \times Q$ such that

(1) $\Delta(a)$ is a subdirect product of $P(a)$ and $Q(a)$ for all $a \in \mathbb{D}$, and

(2) for $i \in P(a)$ and $j \in Q(b)$ $(a,b \in \mathbb{D})$ there exists $d_{ij} \in \mathbb{D}$ such that

(i) $d_{ij} \leqslant a$, $d_{ij} \leqslant b$ and $(iP(a,d_{ij}), jQ(b,d_{ij})) \in \Delta(d_{ij})$,

(ii) if $d \leqslant a$, $d \leqslant b$ and $(iP(a,d), jQ(b,d)) \in \Delta(d)$, then $d \leqslant d_{ij}$.

Let $S(P,Q;\Delta) = \cup \{P(a) \times Q(a) : a \in \mathbb{D}\}$. Define a product in $S(P,Q;\Delta)$ by

$$(i,j)(i',j') = (iP(a,d_{i'j}), j'Q(b,d_{i'j})) \tag{2.2}$$

where $(i,j) \in P(a) \times Q(a)$ and $(i',j') \in P(b) \times Q(b)$. Then, with this product, $S(P,Q;\Delta)$ is a regular locally testable semigroup.

Conversely, every regular locally testable semigroup is isomorphic to one constructed in this way.

Proof. In view of condition (1), for every a, $G(a) = G(P(a),Q(a);\Delta(a))$ is a combinatorial Rees groupoid and since $\Delta$ is a subfunctor of $P \times Q$, for all $a' \leqslant a$, $G(a,a') = (P \times Q)(a,a') = P(a,a') \times Q(a,a')$ is a homomorphism of $G(a)$ into $G(a')$. Thus G is a functor of $\mathbb{D}$ to the category of combinatorial Rees groupoids and $\bar{G} = \cup \{G(a) : a \in \mathbb{D}\}$ is a pseudo-groupoid.

Now define the relation $\leqslant$ on $\bar{G}$ as follows : for $(i,j) \in G(a)$ and $(i',j') \in G(a')$,

$$(i',j') \leqslant (i,j) \iff a' \leqslant a \text{ and } (i',j') = ((i,j))G(a,a') . \tag{2.3}$$

It is easy to see that (2.3) defines a partial order on $\bar{G}$ such that for all $a,a' \in \mathbb{D}$ and $(i,j) \in G(a)$, $G(a')$ contains at most one element $(i',j')$ such that $(i',j') \leqslant (i,j)$. Such an element exists in $G(a')$ if and only if $a' \leqslant a$.

We proceed to show that $\overline{G}$ is inductive with respect to this partial order. If $x \in G(a)$ and $y \in G(a')$ with $y \leqslant x$, then for all $z \in L_x$, $zG(a,a') \mathcal{L}\ y$ and $zG(a,a') \leqslant z$. Since no two elements belonging to the same component of $\overline{G}$ (Rees subgroupoid of $\overline{G}$) are comparable, it follows in particular that $\mathcal{L}(G)$ and $\mathcal{R}(G)$ are strictly compatible. Hence $\overline{G}$ satisfies axiom (I2) of Definition 2.1 of [ 9 ]. Axiom (I3) follows immediately from the fact that $G(a,a')$ is a homomorphism of $G(a)$ into $G(a')$ $(a' \leqslant a)$. If $x,y \in G(a)$, $x',y' \in G(a')$, $x' \leqslant x$, $y' \leqslant y$, and $xy$ exists, then $(xy)G(a,a') = (xG(a,a'))(yG(a,a')) = x'y'$. Thus $x'y' \leqslant xy$, and axiom (I1) holds.

Let $\overline{P} = \cup \{P(a) : a \in \mathbb{D}\}$ and $\overline{Q} = \cup \{Q(a) : a \in \mathbb{D}\}$. Then $\overline{P}$ and $\overline{Q}$ are partially ordered sets with respect to relations defined as follows : for $i \in P(a)$, $j \in P(b)$,

$$i \leqslant j \iff a \leqslant b \text{ and } i = jP(a,b) \quad . \qquad (2.4)$$

The relation on $\overline{Q}$ is defined similarly. It may be noted that $\overline{G} \subseteq \overline{P} \times \overline{Q}$ and the partial order on $\overline{G}$ defined by (2.3) coincides with the restriction of the product order on $\overline{P} \times \overline{Q}$ to $\overline{G}$. Furthermore, if $\overline{p} : \overline{P} \times \overline{Q} \to \overline{P}$ and $\overline{q} : \overline{P} \times \overline{Q} \to \overline{Q}$ are projections, then for all $x \in \overline{G}$, $\overline{p} : [-,x]_{\overline{G}} \to [-,x\overline{p}]_{\overline{P}}$ and $\overline{q} : [-,x]_{\overline{G}} \to [-,x\overline{q}]_{\overline{Q}}$ are order isomorphisms.

Now define $F : \overline{P} \times \overline{Q} \to E = E(\overline{G}) = \{\Delta(a) : a \in \mathbb{D}\}$ as follows. If $i \in P(a) \subseteq \overline{P}$ and $j \in Q(b) \subseteq \overline{Q}$, then by condition (2) (i), there exists $d_{ij} \in \mathbb{D}$ such that $d_{ij} \leqslant a$, $d_{ij} \leqslant b$ and $(iP(a,d_{ij}),jQ(b,d_{ij})) \in \Delta(d_{ij})$. We define

$$F(i,j) = (iP(a,d_{ij}),jQ(b,d_{ij})) \quad .$$

Then $F$ is well-defined, and by condition (2) (ii), $F$ is order-preserving. By (2.3) and its dual we have

$$F(i,j) = (iP(a,d_{ij}),jQ(b,d_{ij})) \leqslant (i,j)$$

and so $F \leqslant 1_{\overline{P} \times \overline{Q}}$. Also if $(i,j) \in E$, then $a = b = d_{ij}$ and so $F(i,j) = (i,j)$. Hence $F$ is idempotent with $\operatorname{im} F = E$. Thus $F$ is a dual closure mapping onto $E$. Since $[-,e]_E = [-,e]_{\overline{G}} \subseteq E$ for all $e \in E$, by the remark above, $\overline{p} : [-,e]_E \to [-,e\overline{p}]_{\overline{P}}$ and $\overline{q} : [-,e]_E \to [-,e\overline{q}]_{\overline{Q}}$ are order isomorphisms. Hence by Theorem 1.2, E is a pseudo-semilattice where the operation is defined as follows. Let $e \in \Delta(a)$, $f \in \Delta(b)$, where $a,b \in D$. Then

$$e \wedge f = F(e\bar{p}, f\bar{q}) = (e\bar{p}\ P(a,d_{e\bar{p},f\bar{q}}),\ f\bar{q}\ Q(b,d_{e\bar{p},f\bar{q}})) \quad (2:5)$$

It is also easy to check that the quasi-order $\omega^r$ [$\omega^l$] of E is the relation generated by the restrictions of $\leqslant$ and $\mathcal{R}(\bar{G})$ [$\mathcal{L}(\bar{G})$] to E. Hence $\bar{G}$ satisfies axiom (I4) of [ 9 ], Definition 2.1, also.

Since $\bar{G} \in$ IPG, $\mathcal{P}(\bar{G})$ is a pseudo-inverse semigroup by Theorem 2.10 of [ 9 ]. By Theorems 2.3 and 2.10 of [ 9 ], the partial order on $\bar{G}$ defined by (2.3) coincides with the natural partial order on $\mathcal{P}(\bar{G})$ and so, $\mathcal{P}(\bar{G})$ satisfies condition (2) of Theorem 1.4 of [ 9 ]. Hence $\mathcal{P}(\bar{G})$ is locally testable.

It remains to prove that $\mathcal{P}(\bar{G})$ is the same as $S(P,Q;\Delta)$. Since the set $S(P,Q;\Delta)$ is the same as $\bar{G}$, it is enough if we show that the product in $\mathcal{P}(\bar{G})$ defined by (2.6) of [ 9 ] coincides with the product defined by (2.2). If $(i,j) \in G(a) \subseteq \mathcal{P}(\bar{G})$ and $(k,m) \in G(b) \subseteq \mathcal{P}(\bar{G})$, then by (2.6) of [ 9 ],

$$(i,j)(k,m) = ((i,j) \star h)(h \star (k,m))$$

where $h = f \wedge e$, $e \in E(L_{(i,j)})$ and $f \in E(R_{(k,m)})$. Since $e\ \mathcal{L}\ (i,j)$, we have $e\bar{q} = j$ and similarly $f\bar{p} = k$. Thus

$$h = F(f\bar{p}, e\bar{q}) = F(k,j) = (kP(b,d), jQ(a,d)),$$

where $d = d_{kj}$. The only element $(i',j') \in G(d)$ such that $(i',j') \leqslant (i,j)$ is $(i,j)G(a,d) = (iP(a,d), jQ(a,d))\ \mathcal{L}\ h$. Hence $(i,j) \star h = (iP(a,d), jQ(a,d))$. Similarly $h \star (k,m) = (kP(b,d), mQ(b,d))$ and so, by (2.1),

$$((i,j) \star h)(h \star (k,m)) = (iP(a,d), mQ(b,d)) .$$

Since the right hand side is the same as the product $(i,j)(k,m)$ defined by (2.2), it follows that the two products coincide. This completes the proof of the direct part.

Conversely, assume that S is a regular locally testable semigroup. We put $\mathbb{D} = S/\mathcal{D}$. We define the relation $\leqslant$ on $\mathbb{D}$ as follows.:

$$a' \leqslant a \iff \exists\, x \in a,\ x' \in a' \text{ such that } x' \leqslant x \quad (2.6)$$

where the latter relation $\leqslant$ is the natural partial order on S. It is clear that the relation defined by (2.6) is reflexive and transitive. By Theorem 1.4 (2) of [ 9 ] no two distinct $\mathcal{D}$-related elements of S are comparable in the natural partial order and so $\leqslant$ is antisymmetric.

We next construct functors P,Q and $\Delta$. For each $a \in \mathbb{D}$, let $P(a)$ denote te set of all $\mathcal{R}$-classes of S contained in the $\mathcal{D}$-class a. If $a' \leqslant a$, then by Lemma 2.2 of [7] and Theorem 1.4(2) of [9], for every $x \in a$ there exists a unique $x' \in a'$ such that $x' \leqslant x$ and so for every $R \in P(a)$ there exist a unique $R' \in P(a')$ such that $R' \leqslant R$. Hence $P(a,a') : R \to R'$ is a mapping of $P(a)$ into $P(a')$. Clearly $P(a,a) = 1_{P(a)}$ and $P(a,a')P(a',a'') = P(a,a'')$ whenever $a'' \leqslant a' \leqslant a$. Thus the assignments

$$a \to P(a),\ a' \leqslant a \to P(a,a')$$

is a set valued functor on $\mathbb{D}$. Similarly, if $a \in \mathbb{D}$, let $Q(a)$ denote the set of $\mathcal{L}$-classes of S contained in a, and if $a' \leqslant a$ in $\mathbb{D}$, let $Q(a,a')$ denote the map which sends $L \in Q(a)$ to $L' \in Q(a')$, where $L' \leqslant L$. Then

$$a \to Q(a),\ a' \leqslant a \to Q(a,a')$$

yields a set valued functor on $\mathbb{D}$. Next, let

$$\Delta(a) = \{(R_e, L_e) : e \in E(a)\}\ .$$

Then $\Delta(a) \subseteq P(a) \times Q(a)$. Obviously $\Delta(a)$ is a subdirect product of $P(a)$ and $Q(a)$. Also if $a' \leqslant a$ and $e \in E(a)$,

$$(R_eP(a,a'), L_eQ(a,a')) = (R_g, L_g)\ ,$$

where g is the unique idempotent in $a'$ such that $g\ \omega\ e$. Thus, $P(a,a') \times Q(a,a')|\Delta(a)$ is a map of $\Delta(a)$ into $\Delta(a')$. Hence if we set

$$\Delta(a,a') = P(a,a') \times Q(a,a')|\Delta(a)\ ,$$

then $\Delta : \mathbb{D} \to \underline{\text{Set}}$ becomes a subfunctor of $P \times Q$.

We have already seen that $\Delta$ satisfies condition (1) of Theorem 2.2. To prove (2), let $R \in P(a)$ and $L \in Q(b)$. Choose $e \in E(R)$ and $f \in E(L)$ and $h = e \wedge f$. Then $R_h \leqslant R$, $L_h \leqslant L$ and $(R_h, L_h) \in \Delta(d_h)$, where $d_h$ is the $\mathcal{D}$-class of S containing h. Therefore $d_{RL} = d_h$ satisfies condition (2) (i). If $d \leqslant a$, $d \leqslant b$ and if $(R',L') \in \Delta(d)$ where $R' \leqslant R$, $L' \leqslant L$, then for some $g \in E(d)$, $R' = R_g$, $L' = L_g$ so that $g \in \omega^r(e) \cap \omega^l(f)$. Hence $g\ \omega\ h$ and so $d \leqslant d_{RL}$. Thus $d_{RL}$ satisfies condition (2) (ii) also.

Since S is combinatorial, it is clear that the mapping $\psi : x \to (R_x, L_x)$ is a bijection of S onto $S(P,Q;\Delta)$. It is easy to see that $\psi$ is order preserving. Further if $x,y \in a$ and if $xy$ exists in the trace of S, then $L_x \cap R_y$ contains an idempotent, say e. Hence $(R_y,L_x) = (R_e,L_e) \in \Delta(a)$ and so

$$(x\psi)(y\psi) = (R_x,L_x)(R_y,L_y) = (R_x,L_y) = (R_{xy},L_{xy}) = (xy)\psi .$$

Thus $\psi$ is a trace homomorphism. Also if $e \in E(a)$, $f \in E(b)$ and $h = e \wedge f$, then $R_h \leqslant R_e$, $L_h \leqslant L_f$ and so $R_eP(a,d_h) = R_h$, $L_fQ(e,d_h) = L_h$. Hence by (2.5)

$$e\psi \wedge f\psi = (R_e,L_e) \wedge (R_f,L_f) = (R_h,L_h) = h\psi .$$

Hence $\psi|E(S)$ is a homomorphism of the pseudo-semilattice E(S) onto the pseudo-semilattice $E(S(P,Q;\Delta))$. Since $\psi$ is a bijection, it follows from Theorem 2.3 of [ 9 ] that $\psi$ is an isomorphism of S onto $S(P,Q;\Delta)$.

Since the structure of regular locally testable semigroups has been described in terms of three functors $P,Q,\Delta \in [\mathbb{D},\underline{\text{Set}}]$ (where $\mathbb{D}$ is a partially ordered set), it is of interest to know how these structural data transform under homomorphisms. We have the following.

THEOREM 2.3. Let $S = S(P,Q;\Delta)$, $S' = S(P',Q';\Delta')$ be regular locally testable semigroups. Suppose that $\theta : \mathbb{D} = S/\mathcal{D} \to \mathbb{D}' = S'/\mathcal{D}$ is an order preserving map and $\sigma : P \to \theta P'$, $\tau : Q \to \theta Q'$ are natural transformations such that for all $i \in P(a)$ and $j \in Q(b)$ $(a,b \in \mathbb{D})$,

$$(\star) \qquad \theta(d_{ij}) = d_{i\sigma(a),j\tau(b)} .$$

Define $\phi = \phi(\theta;\sigma,\tau) : S \to S'$ by

$$(i,j)\phi = (i\sigma(a),j\tau(a)),((i,j) \in P(a) \times Q(a)) . \qquad (2.7)$$

Then $\phi$ is a homomorphism of S into S'.

Conversely if $\phi : S \to S'$ is any homomorphism, then there exist an order preserving map $\theta : \mathbb{D} \to \mathbb{D}'$ and natural transformations $\sigma : P \to \theta P'$, $\tau : Q \to \theta Q'$ such that $\theta$, $\sigma$ and $\tau$ satisfy condition $(\star)$ above, and $\phi = \phi(\theta;\sigma,\tau)$.

Proof. The direct part is proved by a straightforward verification. Suppose that $\theta$, $\sigma$ and $\tau$ satisfy $(\star)$. If $(i,j) \in P(a) \times Q(a) \subseteq S$

and $(k,m) \in P(b) \times Q(b) \subseteq S$, then

$$((i,j)(k,m))\phi = (iP(a,d_{kj}),mQ(b,d_{kj}))\phi \qquad \text{(by (2.2))}$$

$$= (iP(a,d_{kj})\sigma(d_{kj}),mQ(b,d_{kj})\tau(d_{kj})) \quad \text{(by (2.7))}$$

$$= (i\sigma(a)P'(\theta(a),\theta(d_{kj})),m\tau(b)Q(\theta(b),\theta(d_{kj})))$$

since $\sigma$ and $\tau$ are natural transformations

$$= (i'P'(\theta(a),d_{k'j'}),m'Q'(\theta(b),d_{k'm'}))$$

by the condition (★), where we have written $(i',j') = (i,j)\phi$ and $(k',m') = (k,m)\phi$

$$= (i',j')(k',m') \qquad \text{(by (2.2))} .$$

Hence $\phi$ is a homomorphism.

To prove the converse part, let $\phi : S \to S'$ be a homomorphism. Since $\phi$ preserves Green's relations it follows that $\phi$ induces a mapping $\theta : \mathbb{D} \to \mathbb{D}'$ defined by $\theta(D_x) = D_{x\phi}$ for all $x \in S$. If $D_x \leqslant D_y$, then there exists an $x' \in D_x$ with $x' \leqslant y$, and so $x'\phi \leqslant y\phi$. Since $x'\phi \in D_{x\phi}$ it follows that $\theta(D_x) \leqslant \theta(D_y)$. Now for $a \in \mathbb{D}$ define $\sigma(a)$ and $\tau(a)$ by the following. For all $(i,j) \in P(a) \times Q(a)$,

$$(i,j)\phi = (i\sigma(a),j\tau(a)) .$$

If $(i,j)\phi = (i',j')$ and $(i,m)\phi = (i'',m')$, where $(i,j),(i,m) \in P(a) \times Q(a)$, then $(i,j) \mathcal{R} (i,m)$ and so $(i',j') = (i,j)\phi \mathcal{R} (i,m)\phi = (i'',m')$. Hence $i' = i''$. This shows that $\sigma(a)$ is single-valued. Similarly, $\tau(a)$ is single-valued. If $a' \leqslant a$ and if $i \in P(a)$ and $i' = iP(a,a')$, then $i' \leqslant i$. If $j \in Q(a)$ such that $(i,j) = e \in \Delta(a)$, then $f = (i',j') = (iP(a,a'),jQ(a,a'))\ \omega\ e$. Hence $f\phi\ \omega\ e\phi$. Thus

$$f\phi = (i'\sigma(a'),j'\tau(a'))$$

$$= (i\sigma(a))P'(\theta(a),\theta(a')),j\tau(a)Q'(\theta(a),\theta(a'))) .$$

Hence we get $iP(a,a')\sigma(a') = i\sigma(a)P'(\theta(a),\theta(a'))$. This proves that the mapping $a \to \sigma(a)$ is a natural transformation of $P$ to $\theta P'$. Similarly the mapping $a \to \tau(a)$ is a natural transformation of $Q$ to $\theta Q'$. If $i \in P(a)$, $j \in Q(b)$, then we can find $m$ and $k$ such that $e = (i,m) \in \Delta(a)$ and $f = (k,j) \in \Delta(b)$. Since $(e \wedge f)\phi = e\phi \wedge f\phi$, where $e\phi = (i\sigma(a),m\tau(a))$ and $f\phi = (k\sigma(b),j\tau(b))$, we have $\theta(d_{ij}) = d_{i\sigma(a),j\tau(b)}$. This proves that $\theta$, $\sigma$ and $\tau$ satisfy the condition (★). It follows from the definition of $\sigma$ and $\tau$ that

$\phi = \phi(\theta;\sigma,\tau)$. This completes the proof.

It follows from Theorem 2.2 that functors P, Q and $\Delta$ may be used to classify regular locally testable semigroups. For example, if $S = S(P,Q;\Delta)$ is an inverse semigroup then for every $a \in \mathbb{D} = S/\mathcal{D}$, $G(a) = G(P(a),Q(a);\Delta(a))$ is a Brandt groupoid and so $\Delta(a)$ is a bijection of $P(a)$ onto $Q(a)$. Also since $\Delta \subseteq P \times Q$, it follows easily that the map $a \to \Delta(a)$ is a natural isomorphism of P to Q. The converse is clear. It follows therefore that, identifying P and Q by means of the natural isomorphism $a \to \Delta(a)$, we can describe locally testable inverse semigroups as follows.

COROLLARY 2.4. Let $\mathbb{D}$ be a partially ordered set and $F : \mathbb{D} \to \mathbf{Set}$ be a functor such that for all $a,b \in \mathbb{D}$ with $a \neq b$, $F(a) \cap F(b) = \Box$ and for all $i \in F(a)$, $j \in F(b)$, there exists a $d_{ij} \in \mathbb{D}$ such that

$$\{d \in \mathbb{D} : d \leqslant a, d \leqslant b,\ iF(a,d) = jF(b,d)\} = [\,-,d_{ij}\,]_{\mathbb{D}}\ .$$

Then $S(F) = \cup\{F(a) \times F(a) : a \in \mathbb{D}\}$ is a locally testable inverse semigroup when we define the product in $S(F)$ by

$$(i,j)(k,m) = (iF(a,d_{kj}),mF(b,d_{kj}))$$

where $(i,j) \in F(a) \times F(a)$ and $(k,m) \in F(b) \times F(b)$.

Conversely, every locally testable inverse semigroup is isomorphic to one constructed in this way.

Moreover, taking $\sigma = \tau$ in Theorem 2.3, it can be seen that an homomorphism of the locally testable inverse semigroup $S(F)$ into the locally testable inverse semigroup $S(F')$ can be described in terms of an order preserving map $\theta : \mathbb{D} \to \mathbb{D}'$ and a natural transformation $\sigma : F \to \theta F'$ satisfying conditions $(\star)$.

Again if we take $\mathbb{D}$ as a semilattice and $P,Q : \mathbb{D} \to \mathbf{Set}$ are two functors, and if $\Delta = P \times Q$, then it is easy to see that P,Q and $\Delta$ satisfy the conditions (1) and (2) of Theorem 2.2. The resulting semigroup is a normal band. Thus we obtain as a corollary of the foregoing a structure theorem for normal bands as a strong semilattice of rectangular bands (see [ 3 ]).

## 3. REGULAR LOCALLY TESTABLE SEMIGROUPS ADMITTING A GIVEN BIORDERED SET

We observe that in $S(P,Q;\Delta)$ various components of the structure data are not mutually independent. For example, for a partially ordered set $\mathbb{D}$, the fact that functors $P,Q$ and $\Delta$ can be found satisfying conditions (1) and (2) of Theorem 2.2 imposes significant structural restrictions on $\mathbb{D}$. In particular, this implies that every principal ideal of $\mathbb{D}$ is a semilattice.

However, Theorem 2.2 can be used to obtain a complete description of the class of all regular locally testable semigroups $S$ admitting a given biordered set $E$ as the biordered set of $S$. Consider the relation

$$\delta(S) = \mathcal{D}(S) \cap (E(S) \times E(S)) \quad . \tag{3.1}$$

It is clear from Theorem 1.4 of [ 9 ] that $\delta(S)$ satisfies the following conditions :

(LTE 1) $\quad \delta_o \subset \delta(S)$ ,

(LTE 2) $\quad$ if $e \;\omega\; f$ $(e,f \in E)$ then for all $f' \in \delta(S)(f)$, there exists a unique $e' \in \delta(S)(e)$ such that $e' \;\omega\; f'$.

Any equivalence relation on $E$ satisfying the two conditions above will be called a locally testable equivalence relation on $E$. The set of all such relations on $E$ will be denoted by $\mathrm{LTEq}(E)$.

THEOREM 3.1. *Let $E$ be a locally testable biordered set. Then any equivalence relation $\delta$ on $E$ is the restriction to $E$ of the Green's relation $\mathcal{D}$ on a regular locally testable semigroup $S$ with $E(S) = E$ if and only if $\delta \in \mathrm{LTEq}(E)$.*

*Proof*. If a locally testable semigroup $S$ which is regular exists with $E(S) = E$ and $\delta(S) = \delta$ then it is clear from the discussion above that $\delta \in \mathrm{LTEq}(E)$.

Conversely let $\delta \in \mathrm{LTEq}(E)$. Then it is easy to see that the relation $\leqslant$ defined on $\mathbb{D} = E/\delta$ by

$$\delta(e) \leqslant \delta(f) \iff \exists\, e' \in \delta(e) \text{ such that } e' \;\omega\; f$$

is a partial order on $\mathbb{D}$. For each $a \in \mathbb{D}$, let $P(a) = \{R_e : e \in a\}$.

If $a' \leqslant a$, then by axiom (LTE 2) for each $e \in a$ there exists a unique $e' \in a'$ such that $e' \, \omega \, e$. So, for each $i \in P(a)$, there exists a $i' \in P(a')$ such that $i' \leqslant i$. Hence there exists a unique function $P(a,a') : P(a) \to P(a')$ which sends $i \in P(a)$ to $i' \in P(a')$ such that $i' \leqslant i$. It is easy to see that

$$a \to P(a), \; a' \leqslant a \to P(a,a')$$

is a functor of $\mathbb{D}$ to $\underline{\text{Set}}$. Similarly,

$$a \to Q(a) = \{L_e : e \in a\} \text{ and } a' \leqslant a \to Q(a,a')$$

where $Q(a,a')$ is the unique map sending $j \in Q(a)$ to $j' \in Q(a')$ such that $j' \leqslant j$, is a functor of $\mathbb{D}$ to $\underline{\text{Set}}$. Also

$$a \to \Delta(a) = \{(R_e, L_e) : e \in a\}$$

and

$$a' \leqslant a \to \Delta(a,a') = P(a,a') \times Q(a,a') | \Delta(a)$$

is a functor of $\mathbb{D}$ to $\underline{\text{Set}}$. Moreover, $\Delta \subseteq P \times Q$ and for each $a \in \mathbb{D}$, $\Delta(a)$ is a subdirect product of $P(a)$ and $Q(a)$. Thus $\Delta$ satisfies (1) of Theorem 2.2. To prove condition (2), let $i = R_e \in P(a)$ and $j = L_f \in Q(b)$. If $h = e \wedge f$, then $d_{ij} = \delta(h) \leqslant a$, $d_{ij} \leqslant b$. Further since $R_h \leqslant R_e$, $L_h \leqslant L_f$, we have $(iP(a,d_{ij}), jQ(a,d_{ij})) = (R_h, L_h) \in \Delta(d_{ij})$. If $d \leqslant a$, $d \leqslant b$ and $(iP(a,d), jQ(b,d)) \in \Delta(d)$, then $iP(a,d) = R_g$, $jQ(b,d) = L_g$ for some $g \in d$. It follows from $R_g \leqslant R_e$, $L_g \leqslant L_f$ that $g \in \omega^r(e) \cap \omega^l(f) = \omega(h)$. Hence $d = \delta(g) \leqslant \delta(h) = d_{ij}$. Hence condition (2) of Theorem 2.2 holds. Therefore $S = S(P,Q;\Delta)$ is a locally testable semigroup which is regular and such that $E(S) = \cup\{\Delta(a) : a \in \mathbb{D}\}$. As in the proof of Theorem 1.2, it can be seen that the map $e \to (R_e, L_e)$ is an isomorphism of $E$ onto $E(S)$. Also $e \, \delta \, f$ in $E$ if and only if $(R_e,L_e),(R_f,L_f) \in \Delta(\delta(e))$ ; that is $(R_e,L_e) \, \delta(S) \, (R_f,L_f)$. Hence identifying $E$ with $E(S)$ by the isomorphism $e \to (R_e,L_e)$ we obtain a regular locally testable semigroup $S$ such that $\delta(S) = \delta$. This completes the proof.

Theorem 3.1 shows that there exists a one-to-one correspondence between the set LTEq(E) and the set of isomorphism classes of regular locally testable semigroups $S$ with $E(S) = E$. Therefore any information about the structure of LTEq(E) will provide an insight into the structure of the class of all regular locally testable semigroups admitting E as its biordered set.

Clearly LTEq(E) is a partially ordered set under inclusion and contains a least element namely, $\delta_o = (\mathcal{R} \cup \mathcal{L})^t$.

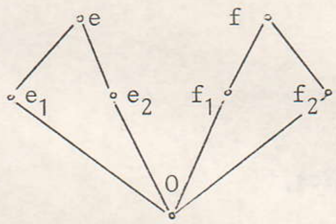
THEOREM 3.2. Let E be a locally testable pseudo-semilattice.

(i) Every non-empty subset of LTEq(E) which has an upper bound, has a join.

(ii) Every non-empty subset of LTEq(E) has a meet .

Proof. The meet of a non-empty subset of LTEq(E) coincides with the set-theoretical meet. The join of a non-empty subset X of LTEq(E) which has an upper bound, is the meet of the upper bounds of X, that is, the join of X in the lattice of all equivalence relations on E.

In general, LTEq(E) is not a lattice even when E is a semilattice. For, consider the semilattice E given by the following diagram :




It can be seen that the relations

$$\delta_1 = \{\{e,f\}, \{e_1,f_1\}, \{e_2,f_2\}, \{0\}\}$$

and

$$\delta_2 = \{\{e,f\}, \{e_1,f_2\}, \{e_2,f_1\}, \{0\}\}$$

both belong to LTEq(E). But $\delta_1 \vee \delta_2 \notin$ LTEq(E).

Finally, we observe that Theorem 3.1 provides an indirect characterization of the partially ordered sets $\mathbb{D}$ (of Theorem 2.2). For, they are precisely those partially ordered sets that arise as quotients $E/\delta$ where E is a locally testable pseudo-semilattice and $\delta \in$ LTEq(E). A direct characterization of $\mathbb{D}$ would be interesting.

## REFERENCES


1. Blyth, T.S. and M.F. Janowitz, Residuation Theory, Pergamon Press (1972).
2. Clifford, A.H. and G.B. Preston, The Algebraic Theory of Semigroups, Vol. I (1961), Vol. II (1967), Math. Surveys No.7, Amer. Math. Soc., Providence.
3. Howie, J.M., An Introduction to Semigroup Theory, Academic Press London (1976).
4. MacLane, S., Categories for the Working Mathematician, Springer Verlag (1971).
5. Meakin, J. and F. Pastijn, The structure of pseudo-semilattices, Algebra Universalis 13 (1981), 355-372.
6. Nambooripad, K.S.S., Structure of Regular Semigroups I, Mem. Amer. Math. Soc. 224 (1979).
7. Nambooripad, K.S.S., The natural partial order on a regular semigroup, Proc .Edinburgh Math. Soc. 23 (1980), 249-260.
8. Nambooripad, K.S.S., Pseudo-semilattices and biordered sets I, Simon Stevin 55 (1981), 103-110.
9. Nambooripad, K.S.S., Pseudo-semilattices and biordered sets II, Pseudo-inverse semigroups, Simon Stevin 56 (1982), 143-159.
10. Pastijn, F., Regular locally testable semigroups as semigroups of quasi-ideals, Acta Math. Acad. Sci. Hungaricae 36 (1980), 161-166 .
11. Zalcstein, Y., Locally testable semigroups, Semigroup Forum 5 (1973), 81-84.



Department of Mathematics,
University of Kerala,
Kariavattom 695 581
Trivandrum
India.



(received January 1981)