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PSEUDO-SEMILATTICES AND BIORDERED SETS - II
PSEUDO-INVERSE SEMIGROUPS

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In [8] we have characterized those pseudo-semilattices that are biordered sets. Since every biordered set is the biordered set of some regular semigroup (cf. [10]), the class of pseudo-semilattices that are biordered sets determines a class of regular semigroups. The semigroups belonging to this class will be called pseudo-inverse semigroups. The principal result of this paper will be a structure theorem for pseudo-inverse semigroups, which is a generalization of Schein's structure theorem for inverse semigroups. Our generalization consists in replacing groupoids in Schein's theory by pseudo-groupoids (that is, disjoint unions of Rees groupoids), and semilattices by pseudo-semilattices.

1. PSEUDO-INVERSE SEMIGROUPS

In this paper we shall use standard notations and terminologies of semigroup theory as presented in [1] and [2]. We shall further assume that the reader is familiar with the results and notations of [6] and [8]. In addition we shall use the following convention. (cf. [3]) : if $\mathcal{C}$ is a category, the statement "$A \in \mathcal{C}$" means that A is an object of $\mathcal{C}$ and the statement "ϕ in $\mathcal{C}$" means that ϕ is a morphism in $\mathcal{C}$. We shall also sometimes use the symbol representing a category as an abbreviation for the name of the set of objects of that category.

If E is a regular pseudo-semilattice (see [8] and [12] for a definition), then by Theorem 2 of [8], E is a biordered set if and only if it satisfies the conditions (PA1), (PA2) and their duals. In this paper we shall assume that all pseudo-semilattices considered satisfy these conditions. We shall denote by PSL the category of all such pseudo-semilattices. It is easy to see that PSL is a full subcategory of the category RB of all regular biordered sets.

Since every $E \in$ PSL is a biordered set, by Corollary 4.15 of [6], E is isomorphic to the biordered set of idempotents of some regular semigroup. Thus there is a class of regular semigroups whose biordered sets are pseudo-semilattices (in PSL). Semigroups belonging to this class are called pseudo-inverse semigroups and we denote the full subcategory of RS whose objects are pseudo-inverse semigroups by PIS. It may be noted that some authors use the term locally inverse semigroups for semigroups belonging to PIS (cf. [13]). Several well-known classes of semigroups belong to PIS. For example, primitive regular semigroups (in particular, completeley 0-simple semigroups), inverse semigroups, generalized inverse semigroups [14], regular locally testable semigroups [15] etc. , are all semigroups belonging to PIS.

We first give some equivalent characterizations of semigroups in PIS. The equivalence of statements (c) and (d) with (b) below is due to B. M. Schein (private communication).

THEOREM 1.1. For $S \in$ RS, the following statements are equivalent.
(a) $S \in$ PIS,
(b) for every $e \in E(S)$, eSe is an inverse semigroup,
(c) for all $e,s,t \in S$, $e, ese, ete \in E(S)$ implies $esete = etese$,
(d) S does not contain subsemigroups isomorphic to a left-zero [right-zero] semigroup of order 2 with an identity adjoined.

Proof. Immediate from Theorem 7.6 of [6].

It is easy to see that the class of pseudo-inverse semigroups is closed for taking regular subsemigroups, homomorphic images

and direct products (see [7], Theorem 3.2).

Let S be a regular semigroup. The natural partial order on S is the relation $\leqslant$ defined as follows [7]. For $x,y \in S$

$$x \leqslant y \Longleftrightarrow xS \subseteq yS \text{ and } x = fy \text{ for some } f \in E(R_x) . \qquad (1.2)$$

For later use we recall two results from [7]. The first one (Proposition 1.2) lists a few important properties of the natural partial order on regular semigroups. Even though the definition (1.2) is one-sided, one can show that the relation $\leqslant$ is self-dual.

PROPOSITION 1.2 ([7], Proposition 1.2). The following statements about two elements x and y of a regular semigroup S are equivalent.
(a) $x \leqslant y$,
(b) for every $f \in E(R_y)$, there exists $e \in E(R_x)$ such that $e \,\omega\, f$ and $x = ey$,
(c) for every $f' \in E(L_y)$, there exists $e' \in E(L_x)$ such that $e' \,\omega\, f'$ and $x = ye'$,
(d) $H_x \leqslant H_y$ and $xy'x = x$ for some [for all] $y' \in i(y)$.

(here $H_x \leqslant H_y$ means $R_x \leqslant R_y$ and $L_x \leqslant L_y$).

The most important property of the natural partial order on a semigroup in PIS is that it is compatible with the multiplication of the semigroup. In fact, this property characterizes semigroups in PIS. We have

THEOREM 1.3 ([7], Theorem 3.3). The following conditions on a regular semigroup S are equivalent.
(a) $S \in$ PIS,
(b) if $x \leqslant y$ then for every $(y_1,y_2) \in L_y \times R_y$, there exists a unique pair $(x_1,x_2) \in L_x \times R_x$ such that $x_i \leqslant y_i$, $i = 1,2$,
(c) $x,y,u,v \in S$, $x \leqslant u$, $y \leqslant v$ implies $xy \leqslant uv$,
(d) if $y \in S$, $y' \in i(y)$ and $x \leqslant y$ then there exists a unique $x' \in i(x)$ such that $x' \leqslant y'$.

Zalcstein [15] introduced the concept of locally testable semigroups as a simultaneous generalization of both normal bands and nilpotent semigroups. He has shown that a regular semigroup S is locally testable if and only if (i) S is periodic and (ii) for all $e \in E(S)$, eSe is a semilattice ([15], Theorem 4). This implies, by Theorem 1.1, that a regular locally testable semigroup is in particular a pseudo-inverse semigroup. In the following theorem we give a characterization of such a semigroup in terms of its natural partial order. The result further shows that condition (ii) above implies condition (i).

THEOREM 1.4. For a regular semigroup S, the following conditions are equivalent.

(1) S is locally testable,

(2) S is a combinatorial semigroup such that for any two $\mathcal{D}$-classes D and D' of S and $x \in D$ there exists at most one $y \in D'$ such that $y \leq x$,

(3) for every $e \in E(S)$, eSe is a semilattice.

Proof. (1) ⇒ (2). By Theorem 4 of [15] S is combinatorial. So assume that D and D' are two $\mathcal{D}$-classes of S and $x \in D$. If y_1 and y_2 belong to D' and $y_i \leq x$, then for $e \in E(R_x)$ by Proposition 1.2, there exists $f_i \in E(R_{y_i})$ such that $f_i \,\omega\, e$ and $y_i = f_i x$, $i = 1,2$.

Now $f_1, f_2 \in D'$ and so $L_{f_1} \cap R_{f_2} \neq \square$. If u is the element in $L_{f_1} \cap R_{f_2}$, then $eue = ef_2uf_1e = f_2uf_1 = u$ and so $u \in eSe$. Since eSe is a semilattice (by [15], Theorem 4), it follows that $u \in E(S)$. Then, since $f_2 \,\mathcal{R}\, u \,\mathcal{L}\, f_1$, we conclude that $f_2 = u = f_1$. Therefore $y_1 = y_2$.

(2) ⇒ (3). If S is a regular semigroup satisfying (2), then for all $e \in E(S)$, the regular subsemigroup eSe must also satisfy this condition. So, no $\mathcal{D}$-classes of eSe contain more than one idempotent. Since eSe is combinatorial, it follows that every $\mathcal{D}$-class of eSe consists of exactly one idempotent. Hence eSe is a semilattice.

(3) ⇒ (1). By Theorem 4 of [15], it is enough if we show that S is periodic whenever it satisfies (3). To this end assume that $a \in S$, $e \in E(L_a)$ and $f \in E(R_a)$. By Theorem 1.1, E(S) is a pseudo-semilattice. If $h = f \wedge e$, then $S(e,f) = \{h\}$ and by Theorem 1.2 of [6]

$$a^2 = (ah)(ha) \in R_{ah} \cap L_{ha} .$$

If a' is the inverse of a in $R_e \cap L_f$, then eh, $a'ha \in \omega(e) = eSe$ and $eh \,\mathcal{D}\, a'ha$. Hence $eh = a'ha$ so that

$$ah = a(eh) = (aa')ha = fha = ha .$$

Since $ah \,\mathcal{L}\, h \,\mathcal{R}\, ha$ holds we may now conclude that ah, h, ha and a^2 belong to the same $\mathcal{H}$-class. Since hSh is a semilattice, we have $ah = h = ha = a^2$. Hence $a^3 = a(a^2) = a(ha) = a^2$. This completes the proof.

COROLLARY 1.5. Let E be a biordered set and let $(\mathcal{L} \cup \mathcal{R})^t = \delta_o$ be the transitive closure of $\mathcal{L} \cup \mathcal{R}$. Then E is the biordered set of a regular locally testable semigroup if and only if for every $e \in E$, $\delta \cap \omega(e)$ contains at most one element for every $\delta \in E/\delta_o$.

Proof. First assume that S is a regular locally testable semigroup such that $E = E(S)$. Then $\delta_o \subseteq \mathcal{D}$ where $\mathcal{D}$ denotes Green's relation on S (see [6], p.103). Hence by Theorem 1.4 (2), δ_o satisfies the required conditions.

Now conversely, assume that E satisfies the given condition and let $S = B_1(E)$, where $B_1(E)$ is the fundamental idempotent generated regular semigroup generated by E. Let γ be any E-cycle in E, and $e \in \omega(e_\gamma)$. Then e and $e\tau(\gamma)$ are δ_o-related elements in $\omega(e_\gamma)$ and so $e = e\tau(\gamma)$. Hence $\tau(\gamma) = 1_{\omega(e_\gamma)}$ and so γ is τ-commutative. Hence S is combinatorial (by [6], Theorem 7.3). Moreover, δ_o is the restriction to E of Green's relation $\mathcal{D}$ of S, so that the given condition implies that S satisfies condition (2) of Theorem 1.4. Thus S is locally testable. This completes the proof.

2. STRUCTURE OF PSEUDO-INVERSE SEMIGROUPS

The partial algebra obtained by restricting the binary operation of a completely 0-simple semigroup S to $S \setminus \{0\}$ (the non-zero $\mathcal{D}$-class of S) is called a Rees groupoid and a disjoint union

$$P = \cup\{P_\alpha : \alpha \in I\}$$

of Rees groupoids P_α is called a pseudo-groupoid. Here and elsewhere below, the symbol $\cup$ is used to denote a disjoint union. A mapping $\phi : P \to Q$ of a pseudo-groupoid P into a pseudo-groupoid Q is a homomorphism of pseudo-groupoids if it is a homomorphism of the partial algebra P into the partial algebra Q; that is, ϕ satisfies the condition that for all $x,y \in P$ such that the product xy exists in P, the product $(x\phi)(y\phi)$ exists in Q and $(xy)\phi = (x\phi)(y\phi)$. It is easy to see that pseudo-groupoids together with homomorphisms defined above, form a category which we shall denote by PG.

We observe that $P = \cup P_\alpha \in$ PG in which each P_α is a Brandt-groupoid is a groupoid in the usual sense, that is, a small category in which all morphisms are isomorphisms. Also, if S is a regular semigroup and if D is a $\mathcal{D}$-class of S, then by Theorem 3.4 of [1], the trace $D(*)$ of D is a Rees groupoid and so $S(*) \in$ PG, where

$$S(*) = \cup\{D(*) : D \in S/\mathcal{D}\} \quad .$$

Also any homomorphism $\phi : S \to T$ of regular semigroups naturally induces a homomorphism $\phi : S(*) \to T(*)$. However, it must be noted that, not all homomorphisms of $S(*)$ into $T(*)$ extend to a homomorphism of S into T.

We may define Green's relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ in pseudo-groupoids in the following way. Let $\mathcal{K}$ denote any of Green's relations $\mathcal{L}$, $\mathcal{R}$ or $\mathcal{H}$. Let $P = \cup\{P_\alpha : \alpha \in I\} \in$ PG, and for $\alpha \in I$, let $\mathcal{K}_\alpha$ denote the restriction of Green's relation $\mathcal{K}$ on the completely 0-simple semigroup P_α^0 to the $\mathcal{D}$-class P_α of non-zero elements of P_α^0. We then set

$$\mathcal{K}(P) = \mathcal{K} = \cup\{\mathcal{K}_\alpha : \alpha \in I\}.$$

It is clear that $\mathcal{K}$ is always an equivalence relation on P. Also, if $\phi : P \to Q = \cup\{Q_\beta : \beta \in I\}$ is in PG, then it is clear that it preserves the relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$, so that ϕ induces a homomorphism ϕ of the Rees groupoid P_α into some Rees groupoid Q_β.

Recall from [7] that an equivalence relation ρ on a partially ordered set $(X,\leqslant)$ is reflecting if $(\leqslant\circ\rho) \subseteq (\rho\circ\leqslant)$, that is, for all $x,y,z \in X$ such that $x \leqslant y \rho z$, there exists $z' \in X$, such that $x \rho z' \leqslant z$. A reflecting equivalence relation ρ is strictly compatible (with $\leqslant$) if $\leqslant \cap \rho = \iota_X$, that is, if for all $x,y \in X$, $x \leqslant y$ and $x \rho y$, we have $x = y$.

<u>DEFINITION</u> 2.1. Let $P \in$ PG and $\leqslant$ be a partial order on P. We say that $(P,\leqslant)$ is an inductive pseudo-groupoid if the following axioms hold.

(I1) If $x,y,u,v \in P$, $x \leqslant u$, $y \leqslant v$ and if xy and uv exist in P, then $xy \leqslant uv$.

(I2) The relations $\mathcal{L}$ and $\mathcal{R}$ are strictly compatible.

(I3) The set of idempotents $E(P)$ of P is an (order) ideal of $(P,\leqslant)$.

(I4) $(E(P),\omega^l,\omega^r) \in$ PSL where ω^l [ω^r] is the smallest quasi-order on $E(P)$ containing the restrictions of $\leqslant$ and $\mathcal{L}$ [$\mathcal{R}$] to $E(P)$.

<u>DEFINITION</u> 2.2. Let $P,Q \in$ PG be inductive with respect to partial orders $\leqslant$ and $\prec$ on P and Q respectively. A homomorphism $\phi : P \to Q$ is said to be inductive if it is order preserving and $E(\phi) = \phi|E(P)$ is a pseudo-semilattice homomorphism of $E(P)$ into $E(Q)$.

It is clear that if $\phi : P \to Q$, $\psi : Q \to R$ are homomorphisms, then $\phi\psi : P \to R$ is also a homomorphism. Since 1_P is obviously a homomorphism, we have a category IPG in which objects are inductive pseudo-groupoids and morphisms are homomorphisms defined above. Our objective here is to show that the category IPG is naturally equivalent to the category PIS of all pseudo-inverse semigroups.

We begin by constructing an embedding π : PIS $\to$ IPG (that is, a fully faithful functor of PIS into IPG).

<u>THEOREM</u> 2.3. (i) <u>Let $S \in$ PIS. Then</u> $\pi(S) = (S(*),\trianglelefteq)$, <u>where</u> $\leqslant$ <u>denotes the natural partial order on S, is an inductive pseudo-groupoid.</u>

(ii) <u>Let $S,T \in$ PIS. Then $\phi : S \to T$ is a homomorphism of semigroups if and only if ϕ is a homomorphism of $\pi(S)$ into $\pi(T)$.</u>

(iii) For $S \in$ PIS, let $\pi(S) = (S(\star), \trianglelefteq)$ (as in (i)) and for $\phi : S \to T$ in PIS, let $\pi(\phi)$ be the homomorphism of $\pi(S)$ into $\pi(T)$ induced by ϕ (as in (ii)). Then π : PIS $\to$ IPG is an embedding.

Proof. (i) Since $S \in$ PIS, it follows from Proposition 1.2 (b) and (c) and Proposition 1.3 (b) and (c) that $\pi(S)$ satisfies axioms (I1),(I2) and (I3). To prove (I4), we first observe that $E(\pi(S)) = E(S)$, $\mathscr{R}(\pi(S)) = \mathscr{R}(S)$ and $\mathscr{L}(\pi(S)) = \mathscr{L}(S)$. Also $E(S) \in$ PSL and the restriction of $\mathscr{R}(S)$ to $E(S)$ is $\omega^r \cap (\omega^r)^{-1} = \mathcal{R}$, where ω^r is the right quasi-order of $E(S)$. Similarly the restriction of $\leqslant$ to $E(S)$ is $\omega = \omega^r \cap \omega^l$ (by Proposition 1.1 of [7]). Hence by axiom (B21) of [6] it follows that $\omega^r = \mathcal{R} \circ \omega$ and ω^r is the smallest quasi-order on $E(S)$ containing $\mathcal{R}$ and ω. Similarly the left quasi-order ω^l is the smallest quasi-order on $E(S)$ containing $\omega^l \cap (\omega^l)^{-1} = \mathcal{L}$ and ω. Thus $\pi(S)$ satisfies axiom (I4).

(ii) If $\phi : S \to T$ is a homomorphism in PIS, it is clearly a homomorphism of $S(\star)$ into $T(\star)$ and $E(\phi) : E(\pi(S)) = E(S) \to E(T) = E(\pi(T))$ is a homomorphism in PSL (cf. Theorem 1.1 of [6]). By Theorem 1.8 of [7] , ϕ preserves the partial order $\leqslant$ and so $\phi : \pi(S) \to \pi(T)$ is a homomorphism in IPG.

Conversely assume that $\phi : \pi(S) \to \pi(T)$ is a homomorphism. If $x \in S$, $e \in E(R_x)$ and $f\ \omega^r\ e$, then $fx \leqslant x$ and $fx \in R_f$. Hence $(fx)\phi \leqslant x$ and $(fx)\phi\ \mathscr{R}\ f\phi$ in T. But $E(\phi)$ is a regular bimorphism and so $f\phi\ \omega^r\ e\phi$. Hence $f\phi x\phi\ \mathscr{R}\ f\phi$ and $f\phi x\phi \leqslant x\phi$. By Proposition 1.3 (b) we conclude that $(fx)\phi = (f\phi)(x\phi)$. Similarly if $e' \in E(L_x)$ and $f'\ \omega^l\ e'$, then $(xf')\phi = (x\phi)(f'\phi)$. Now if $x,y \in S$ and $h = f \wedge e$ where $e \in E(L_x)$ and $f \in E(R_y)$, then by Theorem 1.2 of [6] $xy = (xh)(hy)$ and since $h\phi = f\phi \wedge e\phi$, we have $(x\phi)(y\phi) = ((x\phi)(h\phi))((h\phi)(y\phi))$. Since $h\ \omega^l\ e$ and $h\ \omega^r\ f$, by the result proved above, $((xh)(hy))\phi = (xh)\phi(hy)\phi = ((x\phi)(h\phi))((h\phi)(y\phi))$. Hence $(xy)\phi = (x\phi)(y\phi)$ and so $\phi : S \to T$ is a homomorphism.

(iii) π defined in the statement of the theorem is clearly a functor. The fact that it is fully faithful follows from (ii).

It will follow from the principal structure theorem of this section that π defined above is an equivalence of the two categories PIS and IPG.

Let $P \in$ IPG and $E(P) = E$. Then by axiom (I4), E is a biordered set and as in [6] we shall denote by $\mathcal{R}$, $\mathcal{L}$ and ω, the relations $\omega^r \cap (\omega^r)^{-1}$, $\omega^l \cap (\omega^r)^{-1}$ and $\omega^r \cap \omega^l$ respectively. In view of axiom (B21) for biordered sets, we have

$$\omega^r = \mathcal{R} \circ \omega , \quad \omega^l = \mathcal{L} \circ \omega .$$

Also on any biordered set, the relations $\mathcal{R}$ and $\mathcal{L}$ are strictly compatible with respect to ω.

LEMMA 2.4. The relations $\mathcal{L}$, $\mathcal{R}$ and ω are restrictions of $\mathscr{L}$, $\mathscr{R}$ and $\leqslant$ to $E = E(P)$ respectively.

Proof. We shall first show that $\omega^r = \mathcal{R}' \circ \omega'$ and $\omega^l = \mathcal{L}' \circ \omega'$ where $\mathcal{R}'$, $\mathcal{L}'$ and ω' are restrictions of $\mathscr{R}$, $\mathscr{L}$ and $\leqslant$ to E. Since ω^r is the quasi-order generated by $\mathcal{R}'$ and ω' we have $\mathcal{R}' \circ \omega' \subseteq \omega^r$. From axioms (I2) and (I3) it follows that $\mathcal{R}'$ is strictly compatible. Hence $\omega' \circ \mathcal{R}' \subseteq \mathcal{R}' \circ \omega'$ and so

$$\begin{aligned}(\mathcal{R}' \circ \omega) \circ (\mathcal{R}' \circ \omega) &= \mathcal{R}' \circ (\omega' \circ \mathcal{R}') \circ \omega' \\ &\subseteq \mathcal{R}' \circ (\mathcal{R}' \circ \omega) \circ \omega' \\ &= (\mathcal{R}' \circ \mathcal{R}') \circ (\omega' \circ \omega') \\ &\subseteq \mathcal{R}' \circ \omega' .\end{aligned}$$

Since $\mathcal{R}' \circ \omega'$ is clearly reflexive, this proves that it is a quasi-order containing $\mathcal{R}'$ and ω'. Thus $\omega^r = \mathcal{R}' \circ \omega'$. Similarly, $\omega^l = \mathcal{L}' \circ \omega'$.

Now $\mathcal{R}'$ is an equivalence relation on E and so $\mathcal{R}' = (\mathcal{R}')^{-1} \subseteq (\omega^r)^{-1}$. Thus $\mathcal{R}' \subseteq \omega^r \cap (\omega^r)^{-1} = \mathcal{R}$. Similarly $\mathcal{L}' \subseteq \mathcal{L}$ and $\omega' \subseteq \omega$. Now if $(e,f) \in \mathcal{L}$, then $(e,f) \in \mathcal{L}' \circ \omega' = \mathcal{L} \circ \omega$ and so there exists $e' \in E$ with $e\ \mathcal{L}'\ e'\ \omega' f$. Then $f\ \mathcal{L}\ e\ \mathcal{L}\ e'$ and $e'\ \omega\ f$. Since $\mathcal{L}$ is strictly compatible with respect to ω, we conclude that $e' = f$. Hence $e\ \mathcal{L}'\ f$, thus $\mathcal{L} = \mathcal{L}'$. Similarly $\mathcal{R} = \mathcal{R}'$. To prove that $\omega = \omega'$ consider $(e,f) \in \omega$. Then $(e,f) \in \omega^l = \mathcal{L}' \circ \omega'$ and so there exists e' such that $e\ \mathcal{L}'\ e'\ \omega' f$. Since $\mathcal{L}' = \mathcal{L}$ it follows that $e\ \mathcal{L}\ e'$ and $e,e' \in \omega(f)$. Since $E \in$ PSL, $\omega(f)$ is a semilattice, and we conclude that $e = e'$.

LEMMA 2.5. Let $P \in$ IPG. Then we have the following (and their duals).

(a) Let $e,f \in E(P)$ and $e\ \omega^r f$. Then for all $y \in R_f$ there exists a unique $x \in R_e$ such that $x \leqslant y$.

(b) Let $x, y \in P$, $x \leqslant y$ and $y' \in i(y)$. Then there exists a unique $x' \in i(x)$ such that $x' \leqslant y'$.

Proof. (a) Since $e \,\omega^r f$, $ef \,\omega\, f$ and so $e \,\mathcal{R}\, ef \leqslant f$ by Lemma 2.4. Hence $ef \leqslant f \,\mathcal{R}\, y$ and so by axiom (I2) there exists an element x such that $e \,\mathcal{R}\, ef \,\mathcal{R}\, x \leqslant y$. To prove that x is unique, assume that $x_1, x_2 \in R_e = R_{ef}$ and $x_i \leqslant y$, $i = 1,2$. Choose $y' \in i(y) \cap L_f$. Then as before, there exists $x' \in L_{ef}$ such that $x' \leqslant y'$. Then the products $x'x_i$ exist in P and so by axiom (I1), $g_i = x'x_i \leqslant y'y$. By axiom (I3) we get $g_i \in E(P)$ for $i = 1,2$. Then $g_1 \,\mathcal{R}\, g_2$ and $g_1, g_2 \in \omega(y'y)$. Since $\omega(y'y)$ is a semilattice, we have $g_1 = g_2$. Thus $x_1 \,\mathcal{L}\, g_1 = g_2 \,\mathcal{L}\, x_2$ and the products $x_i x'$ exist in P. Since $x_i x' \leqslant yy' = f$, we conclude by axiom (I3) and (I4) that $x_1 x' = x_2 x' = ef$. Therefore

$$x_1 = efx_1 = x_2 x' x_1 = x_2 g_1 = x_2 g_2 = x_2 .$$

(b) Let x, y and y' be as in the statement of the theorem and write $f = yy'$ and $f' = y'y$. Then $x \leqslant y \,\mathcal{R}\, f$ and so by axioms (I2) and (I3) there exists $e \in E(P)$ such that $x \,\mathcal{R}\, e \,\omega\, f$. Since $y' \,\mathcal{L}\, f$, by (a), there exists a unique $x' \in L_e$ such that $x' \leqslant y'$. Then the product $x'x$ exists in P and so by axioms (I1) and (I3), $e' = x'x \leqslant y'y = f'$ and $e' \in E(P)$. Also, $x = xe' = xx'x$, $x' = e'x' = x'xx'$. Hence $x' \in i(x)$.

To prove the uniqueness, assume that $x'' \in i(x)$ and $x'' \leqslant y'$. Then $e'' = x''x \,\mathcal{L}\, x'x = e'$ and $e'', e' \in \omega(f')$. Since $\omega(f')$ is a semilattice, we have $e'' = e'$. Hence $x' \,\mathcal{R}\, x''$. Dually we have $x' \,\mathcal{L}\, x''$ and hence x' and x'' are $\mathcal{H}$-equivalent inverses of x. Therefore $x' = x''$.

Suppose that $e, f \in E(P)$, $e \,\omega^r f$ and $y \in R_f$. Then by the foregoing lemma, there exists a unique $x \in R_e$ such that $x \leqslant y$. We shall denote this element by $e \star y$. Dually, if $e \,\omega^l f$ and $y \in L_f$, then the unique element x in L_e such that $x \leqslant y$ will be denoted by $y \star e$. In particular, if $e \,\omega\, f$, and $y \in R_f$, then there exists at least one inverse y' of y in L_f and then $y' \star e \in i(e \star y)$. For, $e \star y \in R_e$ and $y' \star e \in L_e$, and so $(y' \star e)(e \star y)$ exists in P. Further $(y' \star e)(e \star y) \leqslant y'y$ and so $g =$

$(y' \star e)(e \star y) \in E(P)$. Then the product $(e \star y)(y' \star e)$ also exists and $(e \star y)(y' \star e) = e$. Hence $(e \star y) = e(e \star y) = (e \star y)(y' \star e)(e \star y)$ and $y' \star e = (y' \star e)e = (y' \star e)(e \star y)(y' \star e)$. From the uniqueness part of Lemma 2.5 it also follows that $y' \star e = g \star y'$ and $e \star y = y \star g$ so that $(y \star g)(g \star y') = e$. For convenience of later reference we state these results as :

LEMMA 2.6. Let $e, f \in E(P)$ with $e \,\omega\, f$ and let $y \in R_f$ and $y' \in i(y) \cap L_f$. Then $e \star y$ and $y' \star e$ are unique elements in R_e and L_e respectively such that $e \star y \leqslant y$, $y' \star e \leqslant y'$ and $y' \star e \in i(e \star y)$. Further, if $g = (y' \star e)(e \star y)$, then $g \leqslant y'y$, $y' \star e = g \star y'$, $e \star y = y \star g$ and $(y \star g)(g \star y') = e$.

LEMMA 2.7. Let $x, y \in P$ such that xy exists in P and let $z' \in i(xy)$. Then there exist $x' \in i(x)$ and $y' \in i(y)$ such that $z' = y'x'$. Further if $e \,\omega\, (xy)z'$, then

$$e \star xy = (e \star x)(g \star y)$$

and

$$z' \star e = (y' \star g)(x' \star e)$$

where $g = (x' \star e)(e \star x)$.

Proof. Let $f = (xy)z''$ and $f' = z'(xy)$. Then $f \,\mathcal{R}\, xy \,\mathcal{R}\, x$ and $f' \,\mathcal{L}\, xy \,\mathcal{L}\, y$. Let h be the idempotent in $L_x \cap R_y$. If x' is the inverse of x in $R_h \cap L_f$ and y' is the inverse of y in $L_h \cap R_{f'}$, then $h \in L_{y'} \cap R_{x'}$ and so the product $y'x'$ exists in P. Also $y'x' \in R_{y'} \cap L_{x'} = R_{f'} \cap L_f$ and so $z' \,\mathcal{H}\, y'x'$. Clearly $y'x' \in i(xy)$ and so $z' = y'x'$.

Now let $e \,\omega\, f$ and $g = (x' \star e)(e \star x)$. Then by Lemma 2.6, $g \,\omega\, x'x = h$ and $e \star x = x \star g$. Hence the product $(e \star x)(g \star y)$ exists in P and so by axiom (I1) $(e \star x)(g \star y) \leqslant xy$. Also $(e \star x)(g \star y) \,\mathcal{R}\, e \star x \,\mathcal{R}\, e$ and so by Lemma 2.5, $e \star xy = (e \star x)(g \star y)$. The equality $z' \star e = y'x' \star e = (y' \star g)(x' \star e)$ follows in a similar fashion.

We proceed to show that we can associate with every IPG an inductive groupoid $\mathcal{G}(P)$ in such a way that the correspondence

$P \to \mathcal{G}(P)$ is functorial. Let

$$\mathcal{G}(P) = \{(x,x') : x \in P \text{ and } x' \in i(x)\} .$$

As in the proof of Theorem 3.8 of [6] it can be shown that it is the morphism set of a groupoid whose vertex set is $E(P) = E$ if we define composition in $\mathcal{G}(P)$ as follows :

$$(x,x')(y,y') = \begin{cases} (xy,y'x') \text{ if } x'x = yy' \\ \text{undefined otherwise} \end{cases} . \tag{2.1}$$

Next, define the relation $\leqslant$ on $\mathcal{G}(P)$ as follows :

$$(x,x') \leqslant (y,y') \Longleftrightarrow x \leqslant y \text{ and } x' \leqslant y' . \tag{2.2}$$

Clearly $\leqslant$ is a partial order on $\mathcal{G}(P)$. It is also obvious that $\mathcal{G}(P)$ satisfies axiom (OG2 of Definition 3.1 of [6]. Lemma 2.6 shows that it satisfies axiom (OG3) when we set

$$(x,x') \mid e = (e \star x, x' \star e) \tag{2.3}$$

for any $e \;\omega\; xx'$. Note that in view of the definition of the product in $\mathcal{G}(P)$ we have $e_{(x,x')} = xx'$ and $f_{(x,x')} = x'x$. Here, as in [6] we identify $V(\mathcal{G}(P))$ with the set of identities of $\mathcal{G}(P)$. To show that $\mathcal{G}(P)$ satisfies axiom (OG1), suppose that $(x,x'),(y,y'),(u,u'),(v,v')$ belong to $\mathcal{G}(P)$, $(x,x') \leqslant (u,u')$, $(y,y') \leqslant (v,v')$ and the products $(x,x')(y,y')$ and $(u,u')(v,v')$ exist in $\mathcal{G}(P)$. Then the products xy, $y'x'$, uv, $v'u'$ exist in P, and by (2.2), $x \leqslant u$, $x' \leqslant u'$, $y \leqslant v$, $y' \leqslant v'$. Hence by axiom (I1), we have $xy \leqslant uv$ and $y'x' \leqslant v'u'$. Thus by (2.2),

$$(x,x')(y,y') = (xy,y'x') \leqslant (uv,v'u') = (u,u')(v,v') .$$

This proves that $\mathcal{G}(P)$ is an ordered groupoid. To prove that $\mathcal{G}(P)$ is inductive, we define an evaluation of $\mathcal{C}(E)$ into $\mathcal{G}(P)$ as follows . Since $V(\mathcal{C}(E)) = E$, we set $V(\epsilon_P) = 1_E$. For $C = C(e_o,\ldots,e_n) \in \mathcal{C}(E)$, we define

$$\epsilon_P(C) = (e_o \ldots e_n, e_n \ldots e_o) . \tag{2.4}$$

Observe that since $C = C(e_o,\ldots,e_n)$ is an E-chain in E, the products $e_o \ldots e_n$ and $e_n \ldots e_o$ exist in P and $e_n \ldots e_o \in i(e_o \ldots e_n)$. Hence $\epsilon_P(C)$ is a well-defined element of $\mathcal{G}(P)$. It is easy to see that $\epsilon_P : \mathcal{C}(E) \to \mathcal{G}(P)$ is a functor.

To prove that ϵ_P is order preserving, consider $C(e_o,e_1) \in \mathcal{C}(E)$ and $e \;\omega\; e_o$. Then by (2.3) and (2.4), we have

$$\epsilon_P(C(e_o,e_1)) \mid e = (e \star e_o e_1, e_1 e_o \star e) .$$

Now if $e_o \mathcal{L} e_1$, then (by Lemma 2.7 or by a direct reasoning),

$$e \star e_o e_1 = g_o g_1 \quad \text{where } g_o = e \text{ and } g_1 = e_1 e e_1 ,$$

where the product on the right is a basic product in E. Similarly, we have

$$e_1 e_o \star e = g_1 g_o .$$

Hence

$$\epsilon_P(C(e_o,e_1)) \mid e = (g_o g_1, g_1 g_o) .$$

Since $e \star C(e_o,e_1) = C(g_o,g_1)$ it follows that

$$\epsilon_P(C(e_o,e_1)) \mid e = \epsilon_P(e \star C(e_o,e_1)) .$$

The case $e_o \mathcal{R} e_1$ can be treated in a dual way. Hence, in either case we have $\epsilon_P(C(e_o,e_1)) \mid e = \epsilon_P(e \star C(e_o,e_1))$. Using induction, we can easily show that

$$\epsilon_P(C) \mid e = \epsilon_P(e \star C)$$

for all $C \in \mathcal{C}(E)$ and $e \;\omega\; e_C$. Hence ϵ_P is order preserving.

Since $E \doteq E(P)$ is a pseudo-semilattice, any V-isomorphism of $\mathcal{C}(E)$ into $\mathcal{G}(P)$ satisfies axioms (IG1) and (IG2) of Definition 3.1 of [6]. We thus have the following.

<u>PROPOSITION</u> 2.8. <u>Let</u> $P \in$ IPG <u>and</u>

$$\mathcal{G}(P) = \{(x,x') : x \in P,\; x' \in i(x)\} .$$

<u>Then</u> $\mathcal{G}(P)$ <u>is the morphism set of an inductive groupoid (which we also denote by</u> $\mathcal{G}(P)$) <u>whose vertex set is</u> $E(P)$ <u>and in which composition, partial order and evaluation are defined by</u> (2.1), (2.2) <u>and</u> (2.4) <u>respectively</u>.

<u>PROPOSITION</u> 2.9. <u>Let</u> $P \in$ IPG <u>and</u> $S = S(\mathcal{G}(P))$. <u>Then</u> S <u>is a pseudo-inverse semigroup and the mapping</u> $\gamma_P : P \to \pi(S)$ <u>defined by</u>

$$x\gamma_P = \overline{(x,x')} \tag{2.5}$$

<u>where</u> $x' \in i(x)$ <u>and</u> $\overline{(x,x')}$ <u>denotes the</u> p-<u>class of</u> (x,x') <u>in</u> $\mathcal{G}(P)$, <u>is an isomorphism</u>.

<u>Proof</u>. Since $E(S)$ is isomorphic to $V(\mathcal{G}(P)) = E(P)$ and since $E(P) \in$ PSL, it follows that S is a pseudo-inverse semigroup. We can show exactly as in the proof of Lemma 4.11 of [6] that γ_P

is a bijection of P into $\pi(S)$. Also the mapping $\gamma_P | E(P)$: $e \to \overline{\epsilon_P(e,e)}$ is an isomorphism of E(P) onto E(S) (see [6], Lemma 4.10).

To show that γ_P is a homomorphism, consider x,y such that the product xy exists in P. Let h be the idempotent in $L_x \cap R_y$. If x' is an inverse of x in R_h, and y' an inverse of y in L_h, then the product (x,x')(y,y') exists in $\mathcal{G}(P)$ and y'x' ∈ i(xy). Also

$$\overline{(xy,y'x')} = \overline{(x,x')(y,y')} = \overline{(x,x')}\,\overline{(y,y')} .$$

Further,

$$\overline{(x',x)}\,\overline{(x,x')} = \overline{(x'x,x'x)} = \overline{(yy',yy')} = \overline{(y,y')}\,\overline{(y',y)} .$$

Hence the product $\overline{(x,x')}\,\overline{(y,y')}$ exists in $\pi(S)$. Thus γ_P is a homomorphism. Conversely assume that u,v ∈ $\pi(S)$ and uv exists in $\pi(S)$. Then there exist u' ∈ i(u) and v' ∈ i(v) such that u'u = vv' ; this implies that the product (u,u')(v,v') exists in $\mathcal{G}(S)$. Now, by Lemma 4.10 of [6], $\nu : \mathcal{G}(P) \to \mathcal{G}(S)$ is an isomorphism so that there exist (x,x'),(y,y') ∈ $\mathcal{G}(P)$ such that $\nu(x,x') = (u,u')$, $\nu(y,y') = (v,v')$. Hence $u = \overline{(x,x')} = x\gamma_P$ and $v = \overline{(y,y')} = y\gamma_P$. Since the product (x,x')(y,y') exists in $\mathcal{G}(P)$, x'x = yy', and so the product xy exists in P. Hence γ_P is an isomorphism of the pseudo-groupoid P onto the pseudo-groupoid $\pi(S)$.

It remains to show that γ_P is an order isomorphism. Accordingly, assume that x,y ∈ P and x ⩽ y. Then by Lemma 2.6, for f ∈ E(R_y), there exists e ∈ E(R_x) such that x = e ⋆ y. Also if y' ∈ i(y) ∩ L_f then y' ⋆ e is an inverse of e ⋆ y in L_e and y' ⋆ e ⩽ y'. Hence (y,y')| e = (e ⋆ y,y' ⋆ e). Now as in the proof of Lemma 4.10 of [6] we see that (y,y')| e = e(y,y') where $\overline{e} = \overline{\epsilon_P(e,e)} = e\gamma_P$. It follows from the definition of the natural partial order that $\overline{e}\,\overline{(y,y')} \leqslant \overline{(y,y')}$. Hence $x\gamma_P = (e \star y)\gamma_P = \overline{(e \star y,y' \star e)} = \overline{(y,y')| e}$ $= \overline{e}\,\overline{(y,y')} \leqslant y\gamma_P$. Conversely, assume that u,v ∈ $\pi(S)$ and u ⩽ v. Then by Proposition 1.2, for $\overline{f} \in E(R_v)$ there exist $\overline{e} \in E(R_u)$ such that $\overline{e}\ \omega\ \overline{f}$ and $u = \overline{e}v$. Let e,f,y ∈ P such that $e\gamma_P = \overline{e}$, $f\gamma_P = \overline{f}$ and $y\gamma_P = v$. Since γ_P is an isomorphism, it follows that y ∈ R_f . Then e ⋆ y ∈ R_e and e ⋆ y ⩽ y. So $(e \star y)\gamma_P \in R_{\overline{e}}$ and $(e \star y)\gamma_P \leqslant y\gamma_P = v$. It follows from Theorem 1.3 (b) that $(e \star y)\gamma_P = u$. This completes the proof of Proposition 2.9 .

We now prove the principal structure theorem of this section.

THEOREM 2.10. Let P ∈ IPG. For x,y ∈ P define

$$xy = (x \star h)(h \star y) \tag{2.6}$$

where h = f ∧ e, e ∈ E(L_x) and f ∈ E(R_y). Then (2.6) defines a binary operation in P with respect to which P becomes a pseudo-inverse semigroup $\mathcal{P}(P)$ such that $\pi(\mathcal{P}(P)) = P$.

Proof. If x,y ∈ P and if h = f ∧ e where e ∈ E(L_x) and f ∈ E(R_y), then $\overline{e} = e\gamma_P \in E(L_{x\gamma_P})$, $\overline{f} \in E(R_{y\gamma_P})$ and $\overline{h} = \overline{f} \wedge \overline{e}$ by Proposition 2.9. Also $(x \star h)\gamma_P = x\gamma_P \star \overline{h} = x\gamma_P\overline{h}$ and $(h \star y)\gamma_P = \overline{h}y\gamma_P$. Hence (by Theorem 1.2 of [6] and Proposition 2.9)

$$\begin{aligned}((x \star h)(h \star y))\gamma_P &= (x \star h)\gamma_P(h \star y)\gamma_P \\ &= ((x\gamma_P)\overline{h})(\overline{h}(y\gamma_P)) \\ &= (x\gamma_P)(y\gamma_P) .\end{aligned}$$

Thus by (2.6),

$$xy = (x\gamma_P y\gamma_P)\gamma_P^{-1}$$

and so (2.6) defines a single-valued binary operation on P and γ_P is an isomorphism of the resulting semigroup $\mathcal{P}(P)$ onto S = S($\mathcal{G}(P)$) . Hence $\mathcal{P}(P)$ is a pseudo-inverse semigroup. Therefore, by Proposition 2.9, P coincides with $\pi(\mathcal{P}(P))$.

REMARK. It follows from Theorem 2.3 that $\mathcal{P}$ defined in Theorem 2.10 above extends to a functor $\mathcal{P}$: IPG → PIS which is the inverse of π : PIS → IPG. If ϕ : P → P' is a homomorphism in IPG, then by Theorem 2.3 (ii) ϕ is also a homomorphism of $\mathcal{P}(P) \to \mathcal{P}(P')$ since $\pi(\mathcal{P}(P)) = P$ and $\pi(\mathcal{P}(P')) = P'$. If we denote by $\mathcal{P}(\phi)$ the homomorphism of $\mathcal{P}(P)$ to $\mathcal{P}(P')$ determined by ϕ as above, it is clear that $\mathcal{P}$ becomes a functor. Since $\mathcal{P}(\phi) = \phi$ and $\pi(\mathcal{P}(P)) = P$ it follows that $\mathcal{P}$ is the inverse of π. Thus, in particular, π is an equivalence of the categories PIS and IPG.

REMARK. We have observed that Theorem 2.10, which describes the structure of a pseudo-inverse semigroup in terms of the trace S(⋆) and the natural partial order on S, is a generalization of Schein's structure theorem for inverse semigroups [11]. A similar description of the structure of an arbitrary regular semigroup in terms of its trace and the natural partial order is not possible. However, a generalization of the above structure theorem is possible in the following form. Observe that if $e,f \in E(S)$ with $e \,\omega^r f$ [$e \,\omega^l f$], by Lemma 2.5, $\sigma^r(e,f) : x \to e * x$ [$\sigma^l(f,e) : x \to x * e$] is a mapping of R_f into R_e [L_f into L_e]. The families of mappings

$$\sigma^r = \{\sigma^r(f,e) : e \,\omega^r f\}, \quad \sigma^l = \{\sigma^l(f,e) : e \,\omega^l f\}$$

are called structure mappings of S. We may rephrase Theorem 2.10 using structure mappings instead of the natural partial order. In this form the theorem admits a generalization (see [4] and [5]). Structure theorems for pseudo-inverse semigroups different from the one given here can be found in [9] and [10].

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