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PSEUDO-SEMILATTICES AND BIORDERED SETS, I

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In this paper we characterize the pseudo-semilattices that are biordered sets, and we show that these pseudo-semilattices form a variety.

B. M. Schein [6] defines a pseudo-semilattice to be a structure (E,ω^1,ω^r) consisting of a set E together with two quasi-orders ω^1 and ω^r on E satisfying the condition that for all $e,f \in E$ there exists a unique $h \in E$ such that

$$(1) \qquad \omega^1(e) \cap \omega^r(f) = \omega(h) .$$

Here $\omega = \omega^r \cap \omega^1$ and $\rho(e) = \{ e' \in E : e' \; \rho \; e\}$ for $\rho = \omega^1, \omega^r$ or ω. If we set

$$(2) \qquad f \wedge e = h$$

then $\wedge$ becomes a binary operation on E. The binary algebras arising in this way are also called pseudo-semilattices. It is shown in [6] that an associative pseudo-semilattice is a normal band, and that a commutative pseudo-semilattice is a semilattice. This result shows that there exists a non-trivial class of pseudo-semilattices that are biordered sets [4]. We have obtained a characterization of all biordered sets that are pseudo-semilattices ([4] Theorem 7.6). In this paper we give the converse characterization of all those pseudo-semilattices that are biordered sets. In [6] Schein shows that pseudo-semilattices form a variety of binary algebras defined by a set of identities. In Section 2 of this paper, we show that these pseudo-semilattices that are biordered sets form a subvariety of this variety. We also obtain the set of identities that determine this variety.

1. PSEUDO-SEMILATTICES THAT ARE BIORDERED SETS

In this paper we use the notations an the terminology established in [1] and [4]. In particular, we refer the reader to [4] for definitions of biordered sets and related concepts and results. For concepts related to bands, the reader is refered to [2].

If (E,ω^l,ω^r) is a pseudo-semilattice (p-semilattice), then

(3) $\quad e\,\omega_1^r\, f \Leftrightarrow f \wedge e = e \quad , \quad e\,\omega_1^l\, f \Leftrightarrow e \wedge f = e$

defines quasiorders on E such that $(E,\omega_1^l,\omega_1^r)$ is also a p-semilattice. Moreover, the binary operation $\wedge$ defined by (2) relative to $(E,\omega_1^l,\omega_1^r)$ and (E,ω^l,ω^r) coincide [6]. Thus we may assume without loss of generality that the quasiorders ω^l and ω^r of a p-semilattice (E,) satisfy (3). A p-semilattice satisfying this condition is said to be regular and we shall assume, in what follows, that all p-semilattices considered are regular.

It may be noted that the binary operation $\wedge$ defined in [6] is dual to the one defined by (2). In [4] as well as in an earlier draft of this paper we had used Schein's definition of $\wedge$. However, the definition adopted here (cf. (2)) is more convenient in comparing p-semilattices and biordered sets. We also note that the authors in [3] also use this definition.

As already noted, our objective is to characterize these p-semilattices that are biordered sets. In this connection we shall say that a p-semilattice E is a biordered set if the restriction of the binary operation $\wedge$ of E to the relation

(4) $\quad D_E = (\omega^l \cup \omega^r) \cup (\omega^l \cup \omega^r)^{-1}$

is the basic product of a biordered set. Likewise we shall say that a biordered set E is a p-semilattice if the basic product of

E can be extended to a binary operation $\wedge$ on E in such a way that $(E,\wedge)$ is a p-semilattice. It is clear, in view of (1), that if there exists such an extension, then it must be unique and that $\wedge$ must satisfy the condition

(5) $\quad S(e,f) = f \wedge e$

for all $e,f \in E$.

We begin by recalling the characterization of biordered sets that are p-semilattices.

Recall that a biordered set E is right regular if $\omega^l \subseteq \omega^r$ (see [4], Theorem 7.5) . This is equivalent with the fact that E is the biordered set of a right regular band. It follows that E is a semilattice if $\omega^l = \omega^r$.

THEOREM 1. ([4], Theorem 7.6). *Let E be a biordered set. Then the following statements are equivalent.*

(a) *E is a p-semilattice* ,

(b) *for all* $e,f \in E$, $S(e,f)$ *contains exactly one element*,

(c) *for all* $e \in E$, *the biordered subset* $\omega^r(e)$ *is right regular and* $\omega^l(e)$ *is left regular*,

(d) *for all* $e \in E$, *the biordered subset* $\omega(e)$ *is a semilattice*.

THEOREM 2. *Let E be a p-semilattice. Then E is a biordered set if and only if E satisfies the following conditions and their duals. For all* $f,g \in \omega^r(e)$,

(PA1) $\quad (g \wedge e) \wedge f = g \wedge f$

(PA2) $\quad (f \wedge e) \wedge (g \wedge e) = f \wedge (g \wedge e) = (f \wedge g) \wedge e$.

Proof. First assume that E is a biordered set. Then by Theorem 1 (c), $\omega^r(e)$ is right regular for all $e \in E$. Hence the basic product of $\omega^r(e)$ can be extended in such a way that $\omega^r(e)$ becomes a right regular band. Now for $f,g \in \omega^r(e)$, if fg denotes the product in this band, $fg \in S(g,f)$. But by (5), $S(g,f) = \{f \wedge g\}$ and so $fg = f \wedge g$. Thus $(\omega^r(e),\wedge)$ is a right regular band. Identities (PA1) and (PA2) now follow from the associativity of $\wedge$ on $\omega^r(e)$.

Let us conversely assume that the p-semilattice E satisfies (PA1) and (PA2). If the basic product in E is defined in the restriction of $\wedge$ to D_E, it is clear that E satisfies axiom (B1) of

Definition 1.1 of [4] . Axioms (B21), (B22), (B31) and (B32) are immediate from (PA1) and (PA2). Let $e, f \in E$, and consider $g \in \omega^l(e) \cap \omega^r(e) = \omega(f \wedge e)$. From (PA2) and its dual we have

$$(e \wedge (f \wedge e)) \wedge (e \wedge g) = e \wedge ((f \wedge e) \wedge g) = e \wedge g$$

and

$$(g \wedge f) \wedge ((f \wedge e) \wedge f) = (g \wedge (f \wedge e)) \wedge f = g \wedge f \ .$$

Thus $f \wedge e \in S(e,f)$ and axiom (R) holds. Let x be any element of $S(e,f)$. Then $x \in \omega(f \wedge e)$, and so

$$\begin{aligned} x &= ((f \wedge e) \wedge x) \wedge (f \wedge e) \\ &= (((f \wedge e) \wedge x) \wedge f) \wedge (f \wedge e) && \text{by (PA1)} \\ &= (((f \wedge e) \wedge f) \wedge (x \wedge f)) \wedge (f \wedge e) && \text{by (PA2)} \\ &= ((f \wedge e) \wedge f) \wedge (f \wedge e) && \text{by the definition of } S(e,f) \\ &= f \wedge e && \text{by (PA1)} . \end{aligned}$$

We conclude that $S(e,f) = \{ f \wedge e \}$. For $f, g \in \omega^r(e)$, we have

$$S(f \wedge e, g \wedge e) = \{ (g \wedge e) \wedge (f \wedge e) \}$$

and

$$S(f,g) \wedge e = \{ (g \wedge f) \wedge e \} .$$

By (PA2) we have $(g \wedge e) \wedge (f \wedge e) = (g \wedge f) \wedge e$ and hence axiom (B4) holds. This completes the proof of Theorem 2.

The p-semilattices satisfying conditions (PA1) and (PA2) will be called partially associative p-semilattices (or PAp-semilattices). It may be noted that in [3] the authors use the term weak pseudo-semilattices to indicate p-semilattices as defined here and they call p-semilattices satisfying (PA1) and (PA2) pseudo-semilattices. Theorems 1 and 2 suggest that PAp-semilattices can be constructed from semilattices. For such constructions we refer the reader to [3] and [5].

2. THE VARIETY OF PAp-SEMILATTICES

If (d) is any identity, then its (left-right) dual will be denoted by $(d)^*$.

The following characterization of p-semilattices as a variety is due to Schein [6].

THEOREM 3 ([6], Theorem 2). *An algebra $(E,\wedge)$ is a p-semilattice if and only if it satisfies the following identities and their duals.*

(a) $\quad x \wedge x = x$,

(b) $\quad (x \wedge y) \wedge (x \quad z) = (x \wedge y) \wedge z$,

(c) $\quad x \wedge ((x \wedge y) \wedge z) = (x \wedge y) \wedge z$.

Let P denote the set of identities in the theorem above. Since $(a)^* = (a)$, P contains five distinct identities. It is shown in [6] that they are independent. We proceed to show that PAp-semilattices can also be defined by a set of five distinct independent identities.

THEOREM 4. *Let $E = (E,\wedge)$ be an algebra. Then E is a PAp-semilattice if and only if it satisfies the following identities and their duals.*

(a) $\quad x \wedge x = x$,

(b) $\quad (x \wedge y) \wedge (x \wedge z) = (x \quad y) \wedge z$,

(c) $\quad ((x \wedge y) \wedge (x \wedge z)) \wedge (x \wedge u) = (x \wedge y) \wedge ((x \wedge z) \wedge (x \wedge u))$.

These identities are independent.

Proof. First assume that E is an algebra satisfying the given identities and their duals. Applying (b) and (d), we obtain

(d') $\quad (x \wedge y) \wedge ((x \wedge z) \wedge u) = ((x \wedge y) \wedge z) \wedge (x \wedge u))$.

Taking x = y in (d') we obtain, using (a) and (b), that

$$x \wedge ((x \wedge z) \wedge u) = (x \wedge z) \wedge (x \wedge u) = (x \wedge z) \wedge u \ .$$

Thus E satisfies (c), and $(c)^*$ is proved dually. Hence E is a p-semilattice. If $f,g,h \in \omega^r(e)$, we get from (d) that

$$(f \wedge g) \wedge h = f \wedge (g \wedge h)$$

and so $\omega^r(e)$ is associative. In particular, E satisfies (PA1) and (PA2). The dual statements $(PA1)^*$ and $(PA2)^*$ are proved similarly and hence E is a PAp-semilattice.

If E is a PAp-semilattice, then it follows from Theorem 3 that E satisfies (a) and (b). The identity (d) is equivalent to the statement that for all $e \in E$, $\omega^r(e)$ is associative. Thus, by Theorem 1, E satisfies (d). The fact that E satisfies $(d)^*$ follows dually.

To see that these identities are independent, we first note that any zero semigroup S such that $|S| > 1$, satisfies all identities except (a). A band B satisfies (a), (d) and $(d)^*$. If B is also left regular but not left normal (see [2] for definitions), then B satisfies (b) but not $(b)^*$. Similarly a right regular band which is not right normal satisfies all identities except (b). Independence of (d) or $(d)^*$ from the other identities follows from the following example of a non-associative p-semilattice [6].

<u>EXAMPLE</u>. Let $E = \{a,b,c,d\}$. Define ω^l and ω^r on E by

$$\omega^r(a) = \omega^l(a) = \{a\},$$
$$\omega^r(b) = \omega^l(b) = \{a,b\},$$
$$\omega^r(c) = \omega^l(c) = \{a,b,c\},$$

and $\omega^r(d) = \{a,b,d\}$ and $\omega^l(d) = \{a,d\}$.

Then ω^r and ω^l are quasi-orders and (E,ω^l,ω^r) is a p-semilattice. Thus E satisfies (a), (b) and $(b)^*$. Since $\omega^l(a)$, $\omega^l(b)$, $\omega^l(c)$ and $\omega^l(d)$ form semilattices for $\wedge$, it follows that E satisfies $(d)^*$. But $(b \wedge d) \wedge b = a$ and $b \wedge (d \wedge b) = b$. Thus $\omega^r(d)$ is not associative and hence E does not satisfy (d).

We now derive some alternative sets of identities for the variety of PAp-semilattices. From [6] we see that every p-semilattice satisfies the following identity and its dual.

(b') $\quad x \wedge (x \wedge y) = x \wedge y$.

In (d'), taking $x = z$ we obtain

$$((x \wedge y) \wedge x) \wedge (x \wedge u) = (x \wedge y) \wedge (x \wedge u)$$
$$= (x \wedge y) \wedge u \qquad \text{by (b)} \ .$$

Similarly, taking $x = u$ in (d'), we obtain

$$(x \wedge y) \wedge ((x \wedge z) \wedge x) = ((x \wedge y) \wedge z) \wedge x \ .$$

Thus PAp-semilattices satisfy the following identities and their duals.

(e1) $\quad ((x \wedge y) \wedge x) \wedge (x \wedge z) = (x \wedge y) \wedge z$,

(e2) $\quad (x \wedge y) \wedge ((x \wedge z) \wedge x) = ((x \wedge y) \wedge z) \wedge x$.

We put

$PA_1 = \{(a), (b), (b)^*, (d), (d)^*\}$,

$PA_2 = \{(a), (b), (b)^*, (d'), (d')^*\}$,

$PA_3 = \{(a), (b'), (b')^*, (d'), (d')^*\}$,

$PA_4 = \{(a), (b), (b)^*, (c), (c)^*, (e1), (e1)^*, (e2), (e2)^*\}$.

Also we denote the variety of algebras defined by PA_i by K_i .

THEOREM 5. $K_1 = K_2 = K_3 = K_4$, *and each one of them denotes the variety of PAp-semilattices.*

Proof. By Theorem 4, K_1 is the variety of PAp-semilattices. We have seen that (d') follows from (b) and (d). Similarly it can be seen that (d) follows from (b) and (d'). Thus $K_1 = K_2$. We have already observed that PAp-semilattices satisfy all identities in PA_3 and PA_4 . On the other hand we have

$$(x \wedge y) \wedge z = (x \wedge y) \wedge ((x \wedge y) \wedge z) \qquad \text{by (b')}$$
$$= ((x \wedge y) \wedge y) \wedge (x \wedge z) \qquad \text{by (d')}$$
$$= (x \wedge y) \wedge (x \wedge z) \qquad \text{by } (b')^* .$$

Hence (b) follows from the identities in PA_3 . Dually $(b)^*$ also follows from these identities and hence $K_3 = K_2$. Finally suppose that E is an algebra satisfying the identities in PA_4 . Then E is a p-semilattice by Theorem 3. Further, for $f,g \in \omega^r(e)$, and if we put $x = e$, $y = f$ and $z = g$ in (e1), then we obtain

$$(f \wedge e) \wedge g = f \wedge g \quad ;$$

from $(b)^*$ and (e2) it follows that

$$(f \wedge e) \wedge (g \wedge e) = f \wedge (g \wedge e) = (f \wedge g) \wedge e \quad .$$

Thus E satisfies (PA1) and (PA2) and so $E \in K_1$. This completes the proof.

We have shown that the identities in the set PA_1 are mutually independent. It is not known whether this is true for the other sets as well. It may be noted that the set PA_4 consists of identities involving three variables only.

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REFERENCES

1. Clifford, A.H., and G.B. Preston, The Algebraic Theory of Semigroups, Vol.1, Math. Surveys No. 7, Amer. Math. Soc., Providence (1961).

2. Kimura, N., Structure of idempotent semigroups I, Pac. J. Math. 8 (1953), 257-275.

3. Meakin J. and F. Pastijn, Structure of pseudo-semilattices, Algebra universalis, to appear.

4. Nambooripad, K. S. S., Structure of regular semigroups I, Mem. Amer. Math. Soc. 224 (1979).

5. Pastijn, F., Rectangular bands of inverse semigroups, submitted.

6. Schein, B.M., Pseudo-semilattices and pseudo-lattices, Izv. Vyss. Ucebn. Zav. Mat. 2 (117) (1972), 81-94 (in Russian).

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