THEORY OF CROSS-CONNECTIONS

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Preface

Two distinct approaches to the study of regular semigroups came into existence in the seventies. One, inspired by Munn's work on inverse semigroups in the sixties, uses the set of idempotents as the basic object relative to which the structure of the given semigroup is analyzed. Munn showed that if E is a semilattice, then the set $T(E)$ of all principal-ideal isomorphisms of E is a *fundamental inverse semigroup* and that any fundamental inverse semigroup whose semilattice is isomorphic to E, is isomorphic to a full-subsemigroup of $T(E)$ (see [11]). In [15] (see also [13]) the set of idempotents of a regular semigroup was abstractly characterized as a biordered set and an appropriate generalization of Munn's result was obtained for *fundamental regular semigroups.* In [15] we have also been able to extend the theory to arbitrary regular semigroups using the concept of *inductive groupoids.*

The second approach, initiated by Hall [4], uses the ideal-structure of a regular semigroup to analyse its structure. If $\Lambda(S)$ and $I(S)$ respectively denote the partially ordered sets of principal left and right ideals of a regular semigroup S, it was shown in [4] that the *fundamental representation* is a homomorphism of S into the semigroup of certain pairs of transformations on $\Lambda(S)$ and $I(S)$ respectively. This implies that the *fundamental image* of S is a semigroup constructible as a semigroup of pairs of transformations on these partially ordered sets. Grillet refined Hall's theory by characterizing abstractly those partially ordered sets that arise as the partially ordered sets of left [or right] ideals of regular semigroups. Given a regular semigroup S, he showed that the partially ordered sets $\Lambda(S)$ and $I(S)$ are *regular* and if I is any regular partially ordered set, then the set $S(I)$ of all normal mappings on I is a regular semigroup such that $\Lambda(S(I))$ is order isomorphic to I (see [3]). Moreover, given two regular partially ordered sets Λ and I, the relation that should exist between them so that they are respectively order-isomorphic to the partially ordered sets of left and right ideals of a regular semigroup was characterized in terms of a pair of mappings $\Gamma\colon I \to \Lambda^\circ$ and $\Delta\colon \Lambda \to I^\circ$ where Λ° [I°] denote the regular partially ordered set of all normal equivalence relations on Λ [I] satisfying certain axioms. Grillet [3] calls such a pair (Γ, Δ)

of mappings as a *cross-connection* between I and Λ. Any regular semigroup S induces, in a natural fashion, a cross-connection between $I(S)$ and $\Lambda(S)$. Grillet showed that if (Γ, Δ) is a cross-connection between regular partially ordered sets I and Λ, then the set $U(I, \Lambda; \Gamma, \Delta)$ of all pairs (f, g) of mappings in $S(\Lambda) \times S(I)^{op}$ that "respects" the given cross-connection is a subsemigroup of $S(\Lambda) \times S(I)^{op}$ and is the universal fundamental regular semigroup inducing the given cross-connection (see [3] for details). (Here $S(I)^{op}$ denote the left-right dual of the semigroup $S(I)$.)

Our aim here is to extend Grillet's theory to arbitrary regular semigroups. It is clear that two regular semigroups having isomorphic biordered sets determine isomorphic regular partially ordered sets of left and right ideals and isomorphic cross-connections. Therefore the data provided by two regular partially ordered sets and a cross-connection between them is insufficient to characterize arbitrary regular semigroups. Consequently, we replace regular partially ordered sets $\Lambda(S)$ and $I(S)$ of left and right ideals in Grillet's theory by categories $\mathcal{L}(S)$ and $\mathcal{R}(S)$ of left and right ideals with morphisms as appropriate translations (see §III.3) and replace the cross-connection (pair of mappings) in Grillet's theory by a local isomorphism of $\mathcal{R}(S)$ to the *normal dual* of $\mathcal{L}(S)$ (see §III.4 and IV.1). In this set-up, we show that Grillet's theory can be extended to arbitrary regular semigroups. In addition, the more general theory also provides new insights into the fundamental case.

The material presented here is a revised form of some of the results contained in [16] (sections 1,3,4 and 5). In this revision, we have expanded several sketchy expositions in [16] and have elaborated many proofs. Also, apart from correcting several misprints we have also corrected two errors in Lemma 4.4 and Proposition 5.5 of [16]. These are minor errors in the sense that they do not affect the validity of the main results. In any case they have now been corrected. Since we are making extensive use of categorical terminology and results, it is felt that a preliminary chapter devoted to a brief discussion of categories will be appropriate. Therefore, we have added a new chapter (Chapter I) containing the required definitions and results; in this we have, as far as possible, followed MacLane [10].

In addition to a few minor changes in notation and terminology, we have also made two important changes in this revision. One of them is the definition of *subobject relations* in categories. The author's work (with E. Krishnan) on Fredholm operators [9] clarified some of the problems associated with the definition of subobjects in topological categories. We have incorporated this definition in Chapter II (which is also on lines suggested by Prof. Francis Pastijn in a personal communication to the author). We also discuss in Chapter II, the concept of regularity in categories. Natural categories such as **Set** (category of sets), $\mathbf{Vct}_K$ (category of vector spaces over the field K), or **Frd** (category of Fredholm operators between locally convex spaces), are regular. It turns out that normal categories of [16] are precisely small regular categories. We have therefore changed the term normal categories here to mean categories which are normal and reductive in the sense of [16]. Results of Chapter II are also new.

Chapter III discusses normal categories (that is, normal and reductive categories of [16]). Given any regular semigroup S, the category $\mathcal{L}(S)$ of left ideal and the category $\mathcal{R}(S)$ of right ideals of S are normal. Also if $\mathcal{C}$ is any normal category, we show that the set $T\mathcal{C}$ of all normal cones with suitable multiplication is a regular semigroup such that $\mathcal{L}(T\mathcal{C})$ is isomorphic to $\mathcal{C}$ and that there is an embedding of $\mathcal{R}(T\mathcal{C})$ into the category $\mathcal{C}^*$ of all set-valued functors on $\mathcal{C}$. The image $N^*\mathcal{C}$ of this embedding is called the normal dual of $\mathcal{C}$ (see III.4). In Chapter IV, we study the concept of cross-connections between normal categories. A cross-connection between two normal categories $\mathcal{C}$ and $\mathcal{D}$ is a local isomorphism $\Gamma\colon \mathcal{D} \to N^*\mathcal{C}$ such that the image of Γ is total in $N^*\mathcal{C}$. We show that every cross-connection Γ determines a unique regular semigroup $\tilde{S}\Gamma$ and conversely, every regular semigroup is isomorphic to a semigroup of the form $\tilde{S}\Gamma$ for a suitable cross-connection Γ. In Chapter V, we consider morphism of cross-connections. Cross-connections together with their morphisms determine a category, which we denote by $\boldsymbol{Cr}$. We show that to each morphism $m\colon \Gamma \to \Gamma'$ there corresponds a unique homomorphism $\tilde{S}m\colon \tilde{S}\Gamma \to \tilde{S}\Gamma'$ such that the assignments

$$\Gamma \mapsto \tilde{S}\Gamma \qquad m \mapsto \tilde{S}m$$

gives a functor $\tilde{S}\colon \boldsymbol{Cr} \to \boldsymbol{RS}$ which is a category equivalence of the category of cross-connections with the category of regular semigroups. The material contained in Chapters III and IV is an expanded form of results in §3, §4 and §5 of [16].

The aim of this work is to present the results leading to the basic construction of cross-connection semigroups and the proof of the category equivalence of categories $\boldsymbol{Cr}$ and $\boldsymbol{RS}$. Hence we have omitted results contained in §2, §6 and §7 of [16] since they are not relevant to this theme. The author believes that these results are of independent interest, especially in applications of cross-connection theory to other areas (like operator theory, theory of representations of groups,etc.). However, to achieve such applications additional research may be needed.

The author is happy to acknowledge the help he received from several quarters during the revision. Francis Pastijn read the original version carefully and pointed out a number of errors and misprints. He pointed out the error in Lemma 4.4 of [16] which convinced the author about the need for this revision. Also my colleagues and students participated in a series of seminars (given by D. Rajendran and R. Chettiyar) in which the entire work was reviewed. Needless to say that these seminars helped the author very much in this revision. In addition, P. G. Romeo, who generalized the cross-connection theory to include concordant semigroups, also gave a series of seminars on the more general theory which was also been of great help to the author.

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Contents

CHAPTER I

Categories

The aim of this chapter is to list some preliminary definitions and results about categories; this will enable us to set up notations and conventions to be followed in the sequel. In the first section we review some definitions from category theory for the convenience of later use. The remainder of the chapter is devoted to describing certain results and constructions of category theory needed later. Most of these results are quite standard and can be found in any standard work on categories. In our formulation of these results, we have followed Mac Lane[10] as far as possible.

1 DEFINITIONS AND NOTATIONS

In the following we shall generally follow notations and terminology established in [16] (except for some occasional modifications). However, for completeness, we shall reproduce most of them here. For those notation and / or terminology not explicitly defined here, the reader should refer [1,5,9,10] or [16].

1.1 Categories and functors

Let $\mathcal{C}$ be a category. If a and b are vertices (objects) in $\mathcal{C}$, then, as usual, we shall write $\mathcal{C}(a, b)$ for the set of all morphisms (arrows) with domain a and codomain b; also, $\mathcal{C}(a)$ will denote $\mathcal{C}(a, a)$ (see [9,10]). Unless otherwise stated, 'set' will always mean small set—see [10], pp 21–24.

In this work we adopt the following conventions: if $\mathcal{C}$ is a category $\mathcal{C}$ will also denote the morphism class of $\mathcal{C}$. The vertex class (object class) of $\mathcal{C}$ will be denoted by $v\mathcal{C}$. As in [16], *composition of morphisms* will be written in the order in which they appear in commutative diagrams. Identifying vertices with the corresponding identity morphisms on vertices, $v\mathcal{C}$ becomes a subclass of $\mathcal{C}$. Recall that the category $\mathcal{D}$ is a *subcategory* of $\mathcal{C}$, written $\mathcal{D} \subseteq \mathcal{C}$, if

$vD \subseteq vC$, the morphism class of D is a subclass of the morphism class of C and the composition in D is the restriction of the composition in C.

Observe that with any class X, we can trivially associate a category C with $vC = X$ and for $a, b \in X$, $C(a,b)$ is empty if $a \neq b$ and $C(a) = \{1_a\}$ where 1_a denotes the identity morphism on a. Since no confusion is likely, we shall denote this *trivial category on* X by X itself. Given a category C, considering the class vC as a category in this way, vC becomes a subcategory of C.

In the following, a *functor* always mean a *covariant functor*. If $F: C \to D$ is a functor, F will also denote the morphism map of F and the vertex map will be denoted by vF (see [16]). Note that, if D is a subcategory of C then the inclusions of vertex set and morphism set of D in those of C induces a functor of D into C which will be called the *inclusion functor*. In particular the inclusion $vC \subseteq C$ can be regarded as a category inclusion. If $F: C \to D$ any functor, we have $vF = F|vC$ and $vF: vC \to vD$ is a functor. For any category C, there always exists a functor, denoted by 1_C, whose vertex map is the identity map on the vertex set of C and whose morphism map is the identity map on the morphism set of C. Let **Set** denote the category of sets whose objects are sets and morphisms are functions. Then for fixed $c \in vC$ and $f: c' \to c'' \in C$, let $C(c,f)$ denote the function from $C(c,c')$ to $C(c,c'')$ defined as follows:

(1) $$C(c,f)(g) = gf.$$

for all $g \in C(c,c')$. Then the assignments

(2) $$c' \mapsto C(c,c') \qquad f \mapsto C(c,f)$$

for all $c' \in vC$ and $f: c' \to c'' \in C$, defines a functor $C(c,-)$ from C to the category **Set**. $C(c,-)$ is called the *covariant hom-functor* determined by c.

In particular, these conventions imply that a small category C is a partial algebra with composition as the partial binary operation and a map $F: C \to D$ between two small categories is a functor if and only if it is a partial algebra homomorphism such that $vF = F|vC$ maps vC to vD. Moreover the assignments

$$C \mapsto vC \quad \text{and} \quad F \mapsto vF$$

is a functor v from **Cat** (the category of small categories) to the category **Set**.

Recall that for any category $\mathcal{C}$, $\mathcal{C}^{op}$ denotes the category with the same vertex set and in which all arrows are reversed; that is, $v\mathcal{C} = v\mathcal{C}^{op}$ and for all $a, b \in v\mathcal{C}$, $\mathcal{C}^{op}(a,b) = \mathcal{C}(b,a)$. A (covariant) functor $F: \mathcal{C}^{op} \to \mathcal{D}$ is called a *contravariant* functor of $\mathcal{C}$ to $\mathcal{D}$. Notice that a contravariant functor of small categories is a partial algebra anti-homomorphism which preserves identities. Also, as in the covariant case, for fixed $c \in v\mathcal{C}$ and $f: c' \to c'' \in \mathcal{C}$, let $\mathcal{C}(f,c)$ denote the function from $\mathcal{C}(c'',c)$ to $\mathcal{C}(c',c)$ defined as follows:

$$(1^*) \qquad \mathcal{C}(f,c)(g) = fg.$$

for all $g \in \mathcal{C}(c'',c)$. As before, the assignments

$$(2^*) \qquad c' \mapsto \mathcal{C}(c',c) \qquad f \mapsto \mathcal{C}(f,c)$$

for all $c' \in v\mathcal{C}$ and $f: c' \to c'' \in \mathcal{C}$ defines a contravariant functor $\mathcal{C}(-,c): \mathcal{C} \to$ **Set** which is called the contravariant hom-functor.

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. We shall say that a functor $F: \mathcal{C} \to \mathcal{D}$ is *v–injective* if vF is injective and F is *v–surjective* if vF is surjective. F is said to be *faithful* if the morphism map is injective on each hom-set of $\mathcal{C}$ and F is *injective* if it is faithful and v–injective. Note that this is equivalent to requiring that F is injective as a partial algebra homomorphism. We shall say that F is *full* if its morphism map is surjective on each hom-set of F. It is *surjective* if it is surjective as a partial algebra homomorphism (or, its morphism map is surjective). In this case, it is easy to see that F is v–surjective. F is *strictly full* if it is full and v–surjective. If F is strictly full then it is clearly surjective. We shall say that F is an *embedding* if it is *fully-faithful* and v–injective. An *isomorphism* of categories is an embedding in which vF is a bijection. If F is an isomorphism, the *inverse* F^{-1} exists and is also an isomorphism of categories.

1.2 Natural transformation

Let $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{C} \to \mathcal{D}$ be two functors (with the same domain and codomain). Recall that a natural transformation $\eta: F \to G$ is a map $a \mapsto \eta_a$ from the vertex class $v\mathcal{C}$ of $\mathcal{C}$ to the morphism class of $\mathcal{D}$ (which by the convention introduced above is denoted by $\mathcal{D}$ itself) such that for each $a \in v\mathcal{C}$,

$\eta_a \colon F(a) \to G(a)$ is a morphism in $\mathcal{D}$ (called the component of η at $a \in v\mathcal{C}$) and the following diagram commutes for all $f \colon a \to b$ in $\mathcal{C}$:

$$\begin{array}{ccc} F(a) & \xrightarrow{\eta_a} & G(a) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(b) & \xrightarrow[\eta_b]{} & G(b) \end{array}$$

If every component of η is an isomorphism, then η is called a natural isomorphism. Functors F and G from $\mathcal{C}$ to $\mathcal{D}$ are *naturally equivalent* (written $F \cong G$) if there is a natural isomorphism $\eta \colon F \to G$. Notice that for any functor $F \colon \mathcal{C} \to \mathcal{D}$ the map $a \mapsto 1_{F(a)}$ is a natural isomorphism which is denoted by 1_F.

1.3 Equivalence of categories

We say that two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there exist functors $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{C}$ such that $1_\mathcal{C} \cong FG$ and $GF \cong 1_\mathcal{D}$. In this case, F is said to be an equivalence of $\mathcal{C}$ to $\mathcal{D}$ (and similarly, G is an equivalence from $\mathcal{D}$ to $\mathcal{C}$). Also if $\eta \colon 1_\mathcal{C} \to FG$ and $\epsilon \colon GF \to 1_\mathcal{D}$ are the corresponding natural isomorphisms, we say that

$$(F, G, \eta, \epsilon) : \mathcal{C} \to \mathcal{D}$$

is an *adjoint equivalence* of $\mathcal{C}$ to $\mathcal{D}$ and G is called the *adjoint inverse* of F (and F is the adjoint inverse of G). Notice that an inverse is, in particular, an adjoint inverse.

2 FUNCTOR CATEGORIES

Recall that, given two small categories $\mathcal{C}$ and $\mathcal{D}$, we have a category $[\mathcal{C}, \mathcal{D}]$ whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are natural transformations (see [10]). If S and T are functors from $\mathcal{C}$ to $\mathcal{D}$, we shall also use the more usual notation $\mathrm{Nat}(S,T)$ to denote the set $[\mathcal{C}, \mathcal{D}](S,T)$ of all morphisms (natural transformations) in $[\mathcal{C}, \mathcal{D}]$ from S to T. Notice that composition in this category is defined as the component-wise product of natural transformations: if $\eta \in \mathrm{Nat}(S,T)$ and $\zeta \in \mathrm{Nat}(T,U)$, then $\eta\zeta \in \mathrm{Nat}(S,U)$ is the natural transformation defined by

$$(\eta\zeta)_c = \eta_c\zeta_c. \tag{3}$$

for all $c \in v\mathcal{C}$ (see [10]). Any subcategory of $[\mathcal{C}, \mathcal{D}]$ will be called a *functor category* (or a category of functors) from $\mathcal{C}$ to $\mathcal{D}$.

2.1 Bifunctors and bifunctor criterion

Let $\mathcal{C}$, $\mathcal{D}$ be categories. Recall that the *product category* $\mathcal{C} \times \mathcal{D}$ is the category with vertex class $v\mathcal{C} \times v\mathcal{D}$, morphism class $\mathcal{C} \times \mathcal{D}$ and in which composition of morphisms are defined componentwise; that is, if $(f,g)\colon (c,d) \to (c',d')$ and $(f',g')\colon (c',d') \to (c'',d'')$ are morphisms in $\mathcal{C} \times \mathcal{D}$, then the equation

$$(f,g)(f',g') = (ff',gg').$$

defines the composition in the category $\mathcal{C} \times \mathcal{D}$. A *bifunctor* or a functor in two variables is a (covariant) functor $B\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ (where $\mathcal{E}$ is another category). A bifunctor $B\colon \mathcal{C}^{op} \times \mathcal{D} \to \mathcal{E}$ is said to be contravariant in the first variable and covariant in the second. In an obvious manner, the definition above can be extended to functors in n variables which is contravariant in r variables.

The following principle, called the *bifunctor criterion*, is useful in checking whether a given assignments of functors and natural transformations constitute a bifunctor:

Bifunctor criterion 1 *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories. For all $c \in \boldsymbol{v}\mathcal{C}$ and $d \in \boldsymbol{v}\mathcal{D}$ let*

$$F_d\colon \mathcal{C} \to \mathcal{E} \quad \text{and} \quad G_c\colon \mathcal{D} \to \mathcal{E}$$

be functors such that $F_d(c) = G_c(d)$ for all $c \in \boldsymbol{v}\mathcal{C}$ and $d \in \boldsymbol{v}\mathcal{D}$. Then there exists a bifunctor $B\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ with $B(c,-) = G_c$ for all c and $B(-,d) = F_d$ for all d if and only if for every pair of morphisms $f\colon c \to c' \in \mathcal{C}$ and $g\colon d \to d' \in \mathcal{D}$ the following diagram commutes:

$$\begin{array}{ccc} F_d(c) & \xrightarrow{G_c(g)} & G_c(d') \\ {\scriptstyle F_d(f)}\downarrow & & \downarrow{\scriptstyle F'_d(f)} \\ F_d(c') & \xrightarrow[G_{c'}(g)]{} & G_{c'}(d') \end{array}$$

If this holds, then B is defined by the assignments:

(4a)
$$B(c,d) = F_d(c) = G_c(d)$$

for all $(c,d) \in \boldsymbol{v}\mathcal{C} \times \boldsymbol{v}\mathcal{D}$ and

(4b)
$$B(f,g) = F_d(f)G_{c'}(g) = G_c(g)F_{d'}(f)$$

for all $(f,g)\colon (c,d) \to (c',d') \in \mathcal{C} \times \mathcal{D}$. □

We refer the reader to [10], Proposition 1 on page 37 for further information about this principle. Given any category $\mathcal{C}$, it is easy to check that the contra- and co-variant hom-functors

$$\mathcal{C}(-,c)\colon \mathcal{C}^{op} \to \mathbf{Set} \qquad \mathcal{C}(c,-)\colon \mathcal{C} \to \mathbf{Set}$$

(cf. (2) and (2*)) satisfy the bifunctor criterion above and hence determines a unique bifunctor $\mathcal{C}(-,-)\colon \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$. $\mathcal{C}(-,-)$ is called the hom-functor. Notice that $\mathcal{C}(-,-)$ sends each $(c,d) \in \boldsymbol{v}\mathcal{C} \times \boldsymbol{v}\mathcal{C}$ to the set $\mathcal{C}(c,d)$ and $(f,g) \in \mathcal{C}(c',c) \times \mathcal{C}(d,d')$ to the function $\mathcal{C}(f,g)$ defined by

(5)
$$\mathcal{C}(f,g)\colon h \mapsto fhg.$$

Clearly the bifunctor $\mathcal{C}(-,-)$ is contravariant in the first variable and covariant in the second.

2.2 An isomorphism of functor categories

It is well-known that, if $\mathcal{C}$, $\mathcal{D}$ are small categories and $\mathcal{E}$ is any category, we have the following category isomorphisms:

(6) $$[\mathcal{C},\ [\mathcal{D},\ \mathcal{E}]] \cong [\mathcal{C} \times \mathcal{D},\ \mathcal{E}] \cong [\mathcal{D},\ [\mathcal{C},\ \mathcal{E}]]$$

(see [10]). In fact the first isomorphism is defined by the assignments:

(*) $$F \mapsto F(-,-); \qquad \text{and} \qquad \eta \mapsto \eta_{-,-}.$$

Here $F(-,-)$ is defined, for any functor $F \in \boldsymbol{v}\,[\mathcal{C},\ [\mathcal{D},\ \mathcal{E}]]$, as follows. For each $c \in \boldsymbol{v}\mathcal{C}$, let $G_c = F(c)$. By hypothesis $G_c\colon \mathcal{D} \to \mathcal{E}$ is a functor. Also for each $d \in \boldsymbol{v}\mathcal{D}$, let F_d be defined by the assignments

$$c \mapsto F(c)(d) \quad \text{and} \quad f \mapsto F(f)(d).$$

It is easy to see that $F_d = F(-)(d)\colon \mathcal{C} \to \mathcal{E}$ is a functor. If $f\colon c \to c' \in \mathcal{C}$, then $F(f)\colon F(c) \to F(c')$ is a natural transformation and hence the following diagram commutes for each $g\colon d \to d' \in \mathcal{D}$:

(**) $$\begin{array}{ccc} F(c,d) & \xrightarrow{F(f)_d} & F(c',d) \\ {\scriptstyle F(c)(g)}\big\downarrow & & \big\downarrow{\scriptstyle F(c')(g)} \\ F(c,d') & \xrightarrow[F(f)_{d'}]{} & F(c',d') \end{array}$$

It follows from bifunctor criterion (see 2.1) that the functors F_d and G_c determines a unique bifunctor $F(-,-)\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ defined as follows:

(7a) $$F(c,d) = F(c)(d) \qquad \text{and}$$

for each $(c,d) \in \boldsymbol{v}\mathcal{C} \times \boldsymbol{v}\mathcal{D}$, and for $(f,g)\colon (c,d) \to (c',d')$, let

(7b) $$F(f,g) = (F(f)_d)(F(c')(g)) = (F(c)(g))(F(f)_{d'}).$$

Then we clearly have $F(c,-) = F(c)$ and $F(-,d) = F(-)(d)$ for all c and d.

Similarly if η is a natural transformation in $[\mathcal{C}, [\mathcal{D}, \mathcal{E}]](F, G)$, and if we define

$$\eta_{c,d} = (\eta_c)_d \tag{7c}$$

then it is easily seen that $\eta_{-,-}: F(-,-) \to G(-,-)$ is a natural transformation of bifunctors.

Conversely, let $F(-,-) \in \boldsymbol{v}[\mathcal{C} \times \mathcal{D}, \mathcal{E}]$ and $\eta_{-,-} \in [\mathcal{C} \times \mathcal{D}, \mathcal{E}]$. For each $c \in \boldsymbol{v}\mathcal{C}$, $F(c,-): \mathcal{D} \to \mathcal{E}$ is a functor and for each $f: c \to c' \in \mathcal{C}$, by the bifunctor criterion, $F(f,-): F(c,-) \to F(c',-)$ is a natural transformation. Define $\tilde{F}$ and $\tilde{\eta}$ as follows:

$$\begin{gathered} \tilde{F}(c) = F(c,-); \quad \tilde{F}(f) = F(f,-); \\ \tilde{\eta}_c = \eta_{c,-} \end{gathered} \tag{8}$$

It can be shown that $\tilde{F}: \mathcal{C} \to [\mathcal{D}, \mathcal{E}]$ is the unique functor such that the bifunctor $\tilde{F}(-,-)$ determined by $\tilde{F}$ as above (using Equations (7a) and (7b)) coincides with $F(-,-)$. Also it is easy to see that $\tilde{\eta}: \tilde{F} \to \tilde{G}$ is the unique natural transformation such that the natural transformation of bifunctors determined by $\tilde{\eta}$ (as in Equation (7c)) is the same as $\eta_{-,-}$. It follows that the assignments given by $(*)$ is a category isomorphism. Since categories $\mathcal{C} \times \mathcal{D}$ and $\mathcal{D} \times \mathcal{C}$ are isomorphic, the second isomorphism of Equation (6) can be obtained in the obvious way.

Remark 1 Notice that even if $\mathcal{C}$ and $\mathcal{D}$ are not small $[\mathcal{C}, \mathcal{D}]$ can still be interpreted as a category though the hom-sets of this category is no longer small; also Equation (6) remains valid where the isomorphisms are isomorphisms of "large" categories (that is, categories whose hom-sets belongs to a higher universe so that they are not small sets—see [10], pp 21–24). In any case, given any bifunctor $F: \mathcal{C} \times \mathcal{D} \to \mathcal{E}$, Equation (8) gives a *representation* $\tilde{F}$ sending each object in $\mathcal{C}$ to a functor from $\mathcal{D}$ to $\mathcal{E}$ and morphisms to natural transformations between such functors and this assignment is functorial in the sense that it preserve identities and composition. When $\mathcal{C}$ and $\mathcal{D}$ are not small, $\tilde{F}$ will be a functor from a category with small hom-sets to a category whose hom-sets may not be small sets.

2.3 Yoneda lemma

For any category $\mathcal{C}$, we use the notation $\mathcal{C}^*$ to denote the functor category $[\mathcal{C}, \mathbf{Set}]$. If $\mathcal{C}$ and $\mathcal{D}$ are any two categories, by Equation (6), we have the following isomorphisms:

$$(9) \qquad [\mathcal{C}, \mathcal{D}^*] \cong (\mathcal{C} \times \mathcal{D})^* \cong [\mathcal{D}, \mathcal{C}^*].$$

(see the remarks above). By Equation (9) and Remark 1, there are functors (representations) $H_\mathcal{C}$ and $H^\mathcal{C}$, such that

$$(10) \qquad H_\mathcal{C}(-,-) = \mathcal{C}(-,-) = H^\mathcal{C}(-,-)$$

(see Equations (7a), (7b), and (8)). It follows that

$$H_\mathcal{C} : \mathcal{C}^{op} \to \mathcal{C}^*$$

is a unique contravariant representation of $\mathcal{C}$ into $\mathcal{C}^*$ (see Equation (9)). Similarly

$$H^\mathcal{C} : \mathcal{C} \to (\mathcal{C}^{op})^*$$

is a unique covariant representation.

Let $F \in \mathcal{C}^*$ and $u \in F(c)$ with $c \in \boldsymbol{v}\mathcal{C}$. It is easy to see that for each $c' \in \boldsymbol{v}\mathcal{C}$ and $f \in \mathcal{C}(c, c')$,

$$(11) \qquad \zeta^u_{c'}(f) = F(f)(u)$$

defines a map $\zeta^u_{c'} : \mathcal{C}(c, c') \to F(c')$ such that the assignment $c' \mapsto \zeta^u_{c'}$ is a natural transformation ζ^u of $\mathcal{C}(c, -)$ to F. Every element of $\mathrm{Nat}(\mathcal{C}(c, -), F)$ is of this form. This leads to the following well-known result, due to N. Yoneda [21], which we shall need in the sequel (see also [10], pp 59–62).

Yoneda Lemma 2 *Let $\mathcal{C}$ be a category, $c \in \boldsymbol{v}\mathcal{C}$ and $F \in \boldsymbol{v}\mathcal{C}^*$. Then the map*

$$Y_{c,F} : u \mapsto \zeta^u$$

is a bijection of $F(c)$ onto $\mathrm{Nat}(\mathcal{C}(c, -), F)$ *which is* natural *in c and F.* □

The last statement that $Y_{c,F}$ is natural in c and F may be explained as follows. Let $\boldsymbol{E}_\mathcal{C}$ be defined on objects and morphisms of the category $\mathcal{C} \times \mathcal{C}^*$ as follows:

$$(12) \qquad \boldsymbol{E}_\mathcal{C}(c, F) = F(c), \quad \boldsymbol{E}_\mathcal{C}(f, \eta) = F(f)\eta_{c'} = \eta_c G(f)$$

where $f \in \mathcal{C}(c, c')$ and $\eta \in \mathrm{Nat}(F, G)$. The equality $F(f)\eta_{c'} = \eta_c G(f)$ follows from the fact that η is a natural transformation. It is easy to see that $\boldsymbol{E}_{\mathcal{C}}$ is a set-valued bifunctor on $\mathcal{C} \times \mathcal{C}^*$ and is called the *evaluation functor*. Similarly, $\mathbf{N}_{\mathcal{C}}$ defined on objects and morphisms of $\mathcal{C} \times \mathcal{C}^*$ to **Set** by

$$\mathbf{N}_{\mathcal{C}}(c, F) = \mathrm{Nat}(H_{\mathcal{C}}(c), F), \quad \mathbf{N}_{\mathcal{C}}(f, \eta) = \mathcal{C}^*(H_{\mathcal{C}}(f), \eta) \tag{13}$$

is a bifunctor. Here $H_{\mathcal{C}}$ denotes the functor from $\mathcal{C}^{op}$ to $\mathcal{C}^*$ satisfying Equation (10) and $\mathcal{C}^*(H_{\mathcal{C}}(f), \eta)$ is the function defined by Equation (5). Yoneda lemma is equivalent to the following:

Corollary 3 *The assignment*

$$Y\colon (c, F) \mapsto Y_{c,F}$$

is a natural isomorphism $Y\colon \boldsymbol{E}_{\mathcal{C}} \to \mathbf{N}_{\mathcal{C}}$. □

Another consequence of Yoneda lemma is that it gives some useful representations, called *Yoneda representations*. In fact, the functors $H_{\mathcal{C}}$ and $H^{\mathcal{C}}$ are embedding of categories; $H_{\mathcal{C}}$ is called the *contravariant Yoneda representation (embedding)* and $H^{\mathcal{C}}$ is called the *covariant Yoneda representation (embedding)*.

3 UNIVERSAL ARROWS, REPRESENTABLE FUNCTORS AND LIMITS

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Recall that a *universal arrow* from $d \in \boldsymbol{v}\mathcal{D}$ to the functor F is a pair (c, g) where $c \in \boldsymbol{v}\mathcal{C}$ and $g \in \mathcal{D}(d, F(c))$ such that given any pair (c', g') with $c' \in \boldsymbol{v}\mathcal{C}$ and $g \in \mathcal{D}(d, F(c'))$, there is a unique $f \in \mathcal{C}(c, c')$ such that $g' = g \circ F(f)$ (cf. [10], p 55). In this case, we say that the morphism g is *universal* from d to F. A universal arrow from F to d is defined dually.

The remainder of this section deals with some applications of this concept which we shall find useful later.

3.1 Universal elements

Let $F \in \mathcal{C}^*$ and let (c, g) be a universal arrow from a one point set $*$ to F. Then the map $g\colon * \to F(c)$ is uniquely determined by the element $x = g(*)$. In this case the pair (c, x) (or, the element x alone, if the object c is clear from the context) is called a *universal element* for F. Note that $x \in F(c)$ is a universal element for F if and only if for every $c' \in \mathfrak{v}\mathcal{C}$ and $y \in F(c')$, there is a unique $f\colon c \to c'$ such that $F(f)(x) = y$. It is easy to see that the natural transformation ζ^x defined by Equation (11) is a natural isomorphism if and only if the element $x \in F(c)$ is a universal element for F. By Yoneda lemma every natural isomorphism of F with a covariant hom-functor $\mathcal{C}(c, -)$ is obtained in this manner.

3.2 Representable functors

A functor $F \in \mathcal{C}^*$ is said to be *representable* if F is naturally isomorphic to some $\mathcal{C}(c, -)$; in this case the object $c \in \mathfrak{v}\mathcal{C}$ is called a *representing object* for F. Remarks above imply that c is a representing object for F if and only if $F(c)$ contains a universal element for F. In particular, F is representable if and only if F has a universal element.

3.3 Limits

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $d \in \mathfrak{v}\mathcal{D}$. In the following we denote by Δd the *constant functor* from $\mathcal{C}$ to $\mathcal{D}$ with value d; that is, the functor which sends every object of $\mathcal{C}$ to d and every morphism to 1_d. By a *cone* we mean a natural transformation σ belonging to either $\mathrm{Nat}[F, \Delta d]$ or $\mathrm{Nat}[\Delta d, F]$ where $F\colon \mathcal{C} \to \mathcal{D}$ is a functor. If $\sigma \in \mathrm{Nat}[F, \Delta d]$ then it is called a *cone from the base F to the vertex d*. Clearly $\sigma\colon c \mapsto \sigma_c$ is a function from $\mathfrak{v}\mathcal{C}$ to $\mathcal{D}$ such that for any $f\colon c \to c' \in \mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} F(c) & \xrightarrow{F(f)} & F(c') \\ {\scriptstyle \sigma_C}\downarrow & & \downarrow{\scriptstyle \sigma_{c'}} \\ d & = & d \end{array}$$

We shall write $\sigma\colon F \overset{\cdot}{\to} d$ to mean that σ is a cone from the base F to d. In particular, if F is the inclusion functor of $\mathcal{C}$ in $\mathcal{D}$, we shall say that σ is a cone from the base $\mathcal{C}$ to d; in this case we write $\sigma\colon \mathcal{C} \to d$.

Dually if $\sigma \in \mathrm{Nat}[\Delta d, F]$ then it is called a *cone to the base F from the vertex d* (see [10], pp 62–71). In this case, we write $\eta\colon F \stackrel{\cdot}{\leftarrow} d$ to indicate this natural transformation.

A cone $\sigma\colon F \stackrel{\cdot}{\to} d$ is a *universal cone* if for each cone $\tau\colon F \stackrel{\cdot}{\to} d'$ there is a unique $g\colon d \to d'$ such that the following diagram commutes for every $c \in \mathfrak{v}\mathcal{C}$:

$$\begin{array}{ccc} F(c) & \xrightarrow{\sigma_c} & d \\ \| & & \downarrow{\scriptstyle g} \\ F(c) & \xrightarrow[\tau_c]{} & d' \end{array}$$

Now it is easy to see that the construction of the constant functor Δd above can be extended to a functor $\Delta\colon \mathcal{D} \to [\mathcal{C}, \mathcal{D}]$. A cone $\sigma\colon F \stackrel{\cdot}{\to} d$ is universal if and only if the natural transformation $\sigma\colon F \to \Delta d$ is a universal arrow from F to the functor Δ in the sense defined earlier in this section. The *direct limit (or inductive limit or colimit)* of F is a pair (d, σ) where $d \in \mathfrak{v}\mathcal{D}$ and $\sigma\colon F \stackrel{\cdot}{\to} d$ is a universal cone (see [10] pp 67–68). In this case we write

$$d = \varinjlim F$$

and σ is called the *limiting cone*. Dually the *limit (or inverse limit or projective limit)* of F is a pair $(\varprojlim F, \tau)$ where $\varprojlim F \in \mathfrak{v}\mathcal{D}$ and $\tau\colon F \stackrel{\cdot}{\leftarrow} \varprojlim F$ is universal to F from $\varprojlim F$.

It is well-known that if $\mathcal{C}$ is a small category then for any functor $F\colon \mathcal{C} \to$ **Set**, both $\varinjlim F$ and $\varprojlim F$ exists, since the category **Set** is complete and cocomplete (see [10], pp 105–108). In fact, let X denote the disjoint union of sets $\{\, F(c) : c \in \mathfrak{v}\mathcal{C} \,\}$ and let ρ denote the smallest equivalence relation containing the relation

$$\{\, (x, y) \in X \times X : F(f)(x) = y \quad \text{for some} \quad f \in \mathcal{C} \,\}.$$

Also, let $\rho^\natural\colon X \to X/\rho$ denote the quotient map. Then it can be checked that

$$\varinjlim F = X/\rho$$

and the map $c \mapsto \rho^\natural | F(c)$ gives the limiting cone. The inverse limit of F can be constructed in a similar fashion (see [10], Theorem 1, p 106).

4 MONOMORPHISMS AND EPIMORPHISMS

Recall that a morphism f in a category $\mathcal{C}$ is a *monomorphism* if for $g, h \in \mathcal{C}$, $gf = hf$ implies $g = h$; that is, f is a monomorphism if it is right cancelable . A morphism $f \in \mathcal{C}(c, c')$ is called a *split monomorphism* if there exists a morphism $g \in \mathcal{C}(c', c)$ with $fg = 1_c$ in which case g is called a *right inverse* of f. It is easy to see that a split monomorphism is a monomorphism; but not all monomorphisms are split. Dually, $f \in \mathcal{C}(c, c')$ is an *epimorphisn* if f is left cancelable and f is a *split epimorphism* if there is $g \in \mathcal{C}(c', c)$ such that $gf = 1_{c'}$. As before a split epimorphism is an epimorphism and f is a split epimorphism if and only if its left inverse is a split monomorphism. A morphism f is a *balanced morphism* if it is both a monomorphism and an epimorphism. It is useful to observe that, for $f, g \in \mathcal{C}$, if fg is a monomorphism then f is a monomorphism and the dual observation holds for epimorphisms. Further, a balanced morphism which is a split monomorphism or a split epimorphism is an isomorphism.

On the class mono$\,\mathcal{C}$ of all monomorphisms in $\mathcal{C}$ define the relation

$$(14) \qquad f \preceq g \quad \Longleftrightarrow \quad f = hg \quad \text{for some} \quad h \in \mathcal{C}.$$

Clearly if $f \preceq g$ then f and g have the same codomain and the morphism h such that $f = hg$ is also a monomorphism. Also the relation $\preceq$ is a *quasi-order* (that is, a reflexive and transitive relation) and so $\sim\, = \,\preceq \cap \preceq^{-1}$ is an equivalence relation on mono$\,\mathcal{C}$. Two monomorphisms f and g are said to be *equivalent* if $f \sim g$. Thus

$$(15) \qquad f \sim g \quad \Longleftrightarrow \quad f = hg \text{ and } g = kf \text{ for some } h, k \in \mathcal{C}.$$

Clearly, if $f \sim g$, the morphism h above with $f = hg$ is an isomorphism and $k = h^{-1}$ (see [10], p 122). Definitions dual to the above give a quasi-order and an equivalence relation on the class epi$\,\mathcal{C}$ of all epimorphisms in $\mathcal{C}$; since there is no possibility of confusion we shall use the same notations $\preceq$ and $\sim$ to denote these relations on epi$\,\mathcal{C}$ as well. Two epimorphisms related by $\sim$ are said to be equivalent.

CHAPTER II

Subobjects and Regularity

In this chapter, we introduce the important preliminary notion of the *subobject relation* in a category (in § 1). Most of the familiar categories such as **Set**, **Grp** and **Top** have a natural choice of subobjects. Moreover morphisms in these categories also satisfy a factorization property which leads to the definition of the image of a morphism as a universal subobject of its codomain. In a category endowed with subobject relation we discuss various factorization properties and the notion of images (§ 2). We also discuss a sufficient condition for a category having factorization property to have images. In § 3 we study the notion of regularity in a category with factorization property. We show that a morphism is regular if and only if it has normal factorization and its image is a retract of its codomain. Further, every regular category is balanced.

1 THE SUBOBJECT RELATION

According to the usual definition, *subobjects* in a category, are certain equivalence classes of monomorphisms (see [10], p 122). While this is quite adequate in **Set** and algebraic categories such as **Grp** (the category of groups), $\mathbf{Vct}_K$ the category of vector spaces over the field K) and $\mathbf{Mod}_R$ (the category of modules over the ring R), the natural subobject relation in topological categories such as **Top** (the category of topological spaces) or **Tvs** (the category of topological vector spaces over the field K where K is either $\boldsymbol{R}$ or $\boldsymbol{C}$) indicate embeddings rather than monomorphisms. We shall therefore give a new definition of subobject relation in categories to take this distinction into account (see [9]).

1.1 Preorders

Recall that a *preorder* P is a category such that for any p, $p' \in P$, the hom-set $P(p,p')$ contains utmost one morphism. In this case, the relation $\subseteq$ on the class $\boldsymbol{v}P$ defined by

$$(1) \qquad p \subseteq p' \quad \Longleftrightarrow \quad P(p,p') \neq \emptyset$$

is a quasiorder. When P is a preorder, $\boldsymbol{v}P$ will stand for the quasiordered class $(\boldsymbol{v}P, \subseteq)$. Conversely given a quasiorder $\leq$ on the class X, the subset

$$P = \{ (x,y) \in X \times X : x \leq y \}$$

of $X \times X$ (which is the graph of $\leq$) is a preorder such that the quasiordered class $\boldsymbol{v}P$ defined by Equation (1) is order-isomorphic with $(X, \leq)$ (see [10], p 11). Also it is clear that if $F \colon P \to P'$ is a functor, then $\boldsymbol{v}P \colon \boldsymbol{v}P \to \boldsymbol{v}P'$ is an order-preserving map and any order-preserving map from $\boldsymbol{v}P$ to $\boldsymbol{v}P'$ determines a unique functor from P to P'. It is easy to see that the category of all small preorders is naturally equivalent to the category of all quasiordered sets.

Notice that in a preorder P, two objects p and p' are isomorphic if and only if $P(p,p') \neq \emptyset \neq P(p',p)$. Therefore the relation defined by Equation (1) is a *partial order* if and only if P does not contain non-trivial isomorphisms (that is, identity morphisms are the only isomorphisms in P). If this holds, we shall say that P is a *strict preorder.* Clearly, a preorder induced by a partially ordered class in the sense above is a strict preorder and conversely. In particular, a small preorder is strict if and only if it is induced by a partially ordered set. Moreover the category of small strict preorders is equivalent to the category of partially ordered sets (with morphisms as order preserving maps).

1.2 Subobjects

The following is a modified form of Definitions (2.1) and (2.2) of [9].

Definition 1 *Let C be a category. A* choice of subobjects *in C is a subcategory $P \subseteq C$ satisfying the following:*

(a) *P is a strict preorder with $\boldsymbol{v}P = \boldsymbol{v}\mathcal{C}$.*
(b) *Every $f \in P$ is a monomorphism in $\mathcal{C}$.*
(c) *If $f, g \in P$ and if $f = hg$ for some $h \in \mathcal{C}$ then $h \in P$.*

When P satisfies these conditions, the pair $(\mathcal{C}, P)$ is called a category with subobjects.

In the following, to simplify the notation, we shall say that $\mathcal{C}$ is a category with subobjects to mean that it is a pair consisting of a category and subpreorder satisfying the axioms above. We shall denote this preorder by $\boldsymbol{\sigma}^*\mathcal{C}$. A morphism in $\boldsymbol{\sigma}^*\mathcal{C}$ will be called an *inclusion* in $\mathcal{C}$; if $C, D \in \boldsymbol{v}\mathcal{C}$ and $C \subseteq D$, we denote the unique morphism in $\boldsymbol{\sigma}^*\mathcal{C}$ from C to D by $\jmath_C^D$ and write $\jmath_C^D \colon C \subseteq D$; in this case, C is referred to as a *subobject* of D. (The preorder $\boldsymbol{\sigma}^*\mathcal{C}$ itself will be called the *preorder of inclusions* in $\mathcal{C}$. Since $\boldsymbol{\sigma}^*\mathcal{C}$ is a strict preorder, the relation defined by Equation (1) is a partial order on $\boldsymbol{v}\mathcal{C} = \boldsymbol{v}\boldsymbol{\sigma}^*\mathcal{C}$ and is called the *inclusion relation* in $\mathcal{C}$. Note that, in view of axiom (c) above, no two distinct inclusions can be equivalent. Any monomorphism equivalent to an inclusion will be called an *embedding* in $\mathcal{C}$. Moreover, if $f \colon C \to D$ is a morphism in $\mathcal{C}$ and if $C' \subseteq C$, we set

$$\jmath_{C'}^{C} f = f|C'; \tag{2}$$

and $f|C'$ is called the *restriction of f to C'*.

If $\mathcal{C}$ and $\mathcal{D}$ are two categories with subobjects, a functor $F \colon \mathcal{C} \to \mathcal{D}$ is said to be *inclusion preserving* $F|\boldsymbol{\sigma}^*\mathcal{C}$ is a functor from $\boldsymbol{\sigma}^*\mathcal{C}$ to $\boldsymbol{\sigma}^*\mathcal{D}$. In this case, we write $\boldsymbol{\sigma}^*F$ for $F|\boldsymbol{\sigma}^*\mathcal{C}$. It is easy to see that any inclusion preserving functor preserves embeddings. We shall say that two categories $\mathcal{C}$ and $\mathcal{D}$ with subobjects are *isomorphic* if there is a category isomorphism $F \colon \mathcal{C} \to \mathcal{D}$ such that $\boldsymbol{\sigma}^*F \colon \boldsymbol{\sigma}^*\mathcal{C} \to \boldsymbol{\sigma}^*\mathcal{D}$ is also an isomorphism. It is clear that $\boldsymbol{\sigma}^*$ may be regarded as a functor of the category of categories with subobjects to the category of preorders. Notice that these categories may not have small hom-sets. However, the category **Scat** of all small categories with subobjects, preserving functors as morphisms is a category with small hom-sets and $\boldsymbol{\sigma}^* \colon \mathbf{Scat} \to \mathbf{Preord}$ is a functor to the category **Preord** of small preorders.

Remark 1 A category $\mathcal{C}$ is said to be *concrete* if there is a *forgetful* functor $U: \mathcal{C} \to \mathbf{Set}$ which is faithful. All "set-based" categories such as **Set**, **Grp**, $\mathbf{Vct}_K$, **Top** and **Tvs** are all naturally concrete. In such categories, the set of monomorphisms given by

$$(\bullet) \qquad \{ f: C \to D \text{ in mono}\mathcal{C} : U(f): U(C) \subseteq U(D) \}$$

is a choice of subobjects for $\mathcal{C}$. Thus for the category **Grp**, the set of all group homomorphisms whose underlying maps are inclusions is a choice of subobjects for **Grp** so that subobjects correspond to subgroups in the usual sense. In this case all monomorphisms (which are injective homomorphisms) are embeddings in the sense defined above. Similar remarks hold for other "algebraic categories" such as $\mathbf{Vct}_K$, **Ab**, $\mathbf{Mod}_R$, etc.

However, in the category **Top**, the choice of subobjects given by $(\bullet)$ consists of *all* continuous inclusions, so that subobjects of topological spaces (given by this choice) will include spaces other than the usual subspaces with the relative topology. In this case, the natural choice will be the collection of all inclusions which are homeomorphisms onto the range (or the subspace inclusions). With this choice of subobjects, it can be seen that not all monomorphisms in **Top** are embeddings in the sense above. Similar remarks hold for other "topological categories" such as **Tvs** or **Lcs**.

In the following, categories such as **Set**, **Grp**, **Ab**, $\mathbf{Vct}_K$ or **Top** will be assumed to have the natural choice of subobjects discussed above. Many familiar functors such as the construction of free objects in some of these categories are inclusion preserving. On the other hand, the construction of fundamental groups, homology groups, etc., are clearly not inclusion preserving.

Remark 2 Given a category $\mathcal{C}$, it is always possible to find some choice of subobjects which will make $\mathcal{C}$, a category with subobjects; for the class of all identity morphisms is a trivial choice of subobjects. This also implies that for a given abstract category, it may be possible to find more than one choice of subobjects.

2 FACTORIZATIONS

Let $\mathcal{C}$ be a category with subobjects. A morphism f in $\mathcal{C}$ has *factorization* if f can be expressed as $f = pm$ where p is an epimorphism and m is an embedding. Factorization of a morphism need not be unique. Let $f = pm$ be a factorization of f. Since m is an embedding, there exist an isomorphism u and an inclusion j such that $m = uj$. Then $q = pu$ is an epimorphism and $f = qj$. Thus every morphism f with factorization has at least one factorization of the form $f = qj$ where q is an epimorphism and j is an inclusion. We shall refer to such factorizations as *canonical* factorizations. A category $\mathcal{C}$ is said to have *factorization property* if every morphism of $\mathcal{C}$ has factorization.

2.1 Images

We now define the notion of the image of a morphism in a category with factorization property.

Definition 2 *The morphism f is said to have* image *if it has a canonical factorization $f = xj$ with the property that whenever $f = yj'$ is any other canonical factorization then there exists an inclusion j'' such that $y = xj''$. A category $\mathcal{C}$ is said to have* images *if every morphism in $\mathcal{C}$ has image.*

A canonical factorization $f = xj$ satisfying the property above is unique. For, if $f = yj'$ is any other canonical factorization, then the condition above implies that $A = \operatorname{cod} x \subseteq \operatorname{cod} y = B$. Hence if $f = yj'$ also satisfies this condition we have $A = B$ and so $j'' \in \mathcal{C}(A)$ where $y = xj''$. Since inclusions form a preorder and since $1_A \in \mathcal{C}(A)$, it follows that $j'' = 1_A$. Thus $x = y$ and since x is an epimorphism $j = j'$. Thus we have:

Lemma 1 *Let f be a morphism in $\mathcal{C}$ with factorization. Then f has image if and only if there exists a unique canonical factorization $f = xj$ such that for any canonical factorization $f = yj'$ there exists an inclusion j'' with $y = xj''$.*

□

When the morphism f has image, we will denote the unique canonical factorization of f satisfying the condition of the lemma above by $f = f^\circ j_f$ where f° denotes the unique *epimorphic component* and j_f is the *inclusion.* of f. Also the unique object $\operatorname{dom} j_f$ is called the *image* of f; we denote it by $\operatorname{im} f$.

We observe that all well-known "algebraic" categories such as **Set**, **Grp**, **Ab**, $\mathbf{Vct}_K$ or $\mathbf{Mod}_R$ and "topological" categories such as **Top**, **Tvs**, **Lcs**, etc. have images in the sense above. Also, trivially, a preorder has images.

It should be noted that the canonical factorization of a morphism with image need not be unique. For if Y is a dense subspace of X (in **Top**), then j^X_Y is an epimorphism in **Top** and so $j^X_Y 1_X$ is also a canonical factorization of j^X_Y. But in this case $1_Y j^X_Y$ is also a canonical factorization. Clearly, the factorization $1_Y j^X_Y$ satisfies the condition of the lemma above so that $\operatorname{im} j^X_Y = Y$ which agrees with the usual definition of images in **Top**.

Remark 3 Note that our definition of subobjects is different from the usual categorical definition (see for example, [10]). This is mainly because of the presence of a choice of subobjects; using this, we are able to replace monomorphisms (or subobjects in the sense of [10]) in the usual definition by objects in the category. However, our definition is more restrictive. For, it may be possible that a morphism has image as a monomorphism but this may fail to be an embedding.

The following observation will be of use later.

Lemma 2 *In a category C with subobjects, we have the following:*

(a) *A split inclusion $j^B_A \colon A \subseteq B$ is an epimorphism if and only if $A = B$ and $j = 1_B$.*
(b) *Let $\varrho \colon B \to A$ be a right inverse of j^B_A. Then ϱ is a monomorphism if and only if $\varrho = 1_B$.*

Proof (*a*): If j^B_A is an epimorphism, then it is a balanced morphism which is split as a monomorphism and so j^B_A is an isomorphism. Since inclusions form a strict preorder, j^B_A must be an identity. This proves (a).
(*b*): The hypothesis implies that ϱ is a balanced morphism which is split as an epimorphism and so ϱ is an isomorphism. Let v be the inverse of ϱ. Then $v\varrho = 1_A = j^B_A\varrho$. Since ϱ is a monomorphism, we have $v = j^B_A$ and by (a), $A = B$ and $j^B_A = 1_B = v$. Hence $\varrho = 1_B$. □

We end this section with the following sufficient condition for a category with factorization property to have images.

Proposition 3 *Let C be a category with factorization property such that every inclusion in C splits. Then every morphism in C has a unique canonical factorization. In particular, C has images.*

Proof Let $f = xj = yj'$ be two canonical factorizations of a morphism $f \in C$. Since j and j' are split, there exist $u, v \in C$ such that $ju = 1_A$ and $j'v = 1_B$ where $A = \operatorname{dom} j$ and $B = \operatorname{dom} j'$. Then

$$yj'uj = xjuj = xj = yj'$$

and since y is an epimorphism, we have $(j'u)j = j'$. Similarly, $(jv)j' = j$ and so, $j \sim j'$. Since j and j' are inclusions, we conclude that $j = j'$. Then $xj = yj$ and since j is a monomorphism we have $x = y$. It follows immediately from Definition 2 that every morphism in C has image. □

If $f\colon C \to D$ in C and $C' \subseteq C$, we shall write

$$f(C') = \operatorname{im}(\jmath^{C}_{C'} f) \tag{3}$$

if the image $\operatorname{im}(\jmath^{C}_{C'} f)$ exists. If C has images, then it follows from Definition 2 that $f(C') \subseteq \operatorname{im} f \subseteq D$. When $f(C')$ is defined, we shall refer to it as the image of C' under f.

Categories satisfying the conditions of the Proposition above have the following property:

Corollary 4 *Let C be a category with factorization property such that every inclusion in C splits. If f and g are composable morphisms in C, we have*

$$(fg)^{\circ} = f^{\circ}(\jmath_f g)^{\circ} \tag{4a}$$

where $\jmath_f$ is the inclusion of f. Further,

$$\operatorname{im} fg = g(\operatorname{im} f) \tag{4b}$$

where $g(\operatorname{im} f)$ is defined by Equation (3).

Proof We have

$$fg = f^{\circ}\jmath_f g = f^{\circ}(\jmath_f g) = f^{\circ}(\jmath_f g)^{\circ}\jmath_h$$

where $h = \jmath_f g$. The right-hand side of the equation above gives a canonical factorization of fg since $f^{\circ}(\jmath_f g)^{\circ}$ is an epimorphism. The desired equalities follow from the proposition above. □

2.2 Normal factorizations

Let C denote a category with factorization property. As in [16], a morphism $\varrho\colon B \to A$ is called a *retraction* if $A \subseteq B$ and ϱ is the right inverse of $\jmath_A^B$. We shall say that A is a *retract* of B if $A \subseteq B$ and there is a retraction $\varrho\colon B \to A$. Notice that when A is a retract of B, the retraction $\varrho\colon B \to A$ is not unique in general.

Definition 3 *A* normal factorization *of a morphism $f \in C$ is a factorization of the form $f = \varrho u \jmath$ where ϱ is a retraction, u is an isomorphism and $\jmath$ is an inclusion.*

We note that in categories **Set** and **Vct**$_{\boldsymbol{K}}$, every morphism has a normal factorization. In **Grp** a morphism $f\colon G \to H$ has a normal factorization if and only if the kernel of f is a direct summond of G. See [9] for other examples of morphisms having normal factorization.

If $f = \varrho u \jmath$ is an normal factorization of f, then ϱu is an epimorphism. This implies that there is a canonical factorization associated with every normal factorization. It is clear (from the examples above) that normal factorization of a morphism is not unique in general. However, we have the following.

Proposition 5 *Let $f = \varrho u \jmath = \varrho' v \jmath'$ be two normal factorizations of f in C. Then $\jmath = \jmath'$ and $\varrho u = \varrho' v$. Moreover, f has image with $f^\circ = \varrho u$ and $\jmath_f = \jmath$.*

Proof Since ϱ is a retraction, there exists an inclusion $\imath$ such that $\imath\varrho = 1_A$ where $A = \operatorname{cod}\varrho$. Since u is an isomorphism, $q = u^{-1}\imath$ is a left inverse of the epimorphism ϱu. Hence $\jmath = q\varrho' v \jmath'$. Similarly, if q' is a left inverse of $\varrho' v$, we have $\jmath' = q'\varrho v \jmath$. Therefore $\jmath \sim \jmath'$ and so, $\jmath = \jmath'$. Then $\varrho u \jmath = \varrho' v \jmath$ and since $\jmath$ is a monomorphism it follows that $\varrho u = \varrho' v$.

Suppose that $f = x\jmath'$ is a canonical factorization of f and that $f = \varrho u \jmath$ is a normal factorization. If q is a left inverse of ϱu, then $\jmath = q\varrho u \jmath = (qx)\jmath'$. It follows by Definition 1(c) that $qx = \jmath''$ is an inclusion and $x\jmath' = \varrho u \jmath'' \jmath'$. Since $\jmath'$ is a monomorphism, we have $x = \varrho u \jmath''$. It follows from Definition 2 that f has image and that $f^\circ = \varrho u$ and $\jmath_f = \jmath$. □

If the morphism f has normal factorization, then it is convenient to use the notation $f = \varrho_f u_f \jmath_f$ to denote one specific normal factorization for f, where ϱ_f is a retraction of f, u_f is the isomorphism component corresponding to ϱ_f and

$\jmath_f$, as before denote the inclusion of f. If ϱ is any other retraction equivalent to ϱ_f, then $\varrho_f = \varrho v$ for some isomorphism v and so $f = \varrho v u \jmath_f$ gives a normal factorization of f. This together with the proposition above gives:

COROLLARY 6 *Let $f \in C$ be a morphism with a normal factorization $f = \varrho_f u_f \jmath_f$. Then there exist a bijection between the set of all normal factorizations of f and the set of all retractions equivalent to ϱ_f.* □

Given a morphism f with normal factorization, we set

$$\mathcal{K}(f) = \{\, \varrho : \varrho \text{ is a retraction with } \varrho \sim \varrho_f \,\}. \tag{5}$$

We call $\mathcal{K}(f)$ as the *abstract kernel* of f. If $\varrho \in \mathcal{K}(f)$, then $\operatorname{cod} \varrho$ is called a *coimage* of f. It is clear that coimage of f, in this sense, is not unique; however two coimages of f are always isomorphic. It is easy to see that there exists a bijection between coimages of f and retractions in $\mathcal{K}(f)$ and hence by the corollary above, every coimage of f determines a unique normal factorization of f and conversely.

The following observations will be useful later.

PROPOSITION 7 *Let C be a category having factorization property. Then we have the following:*

(a) *Let f is an epimorphism. If $\operatorname{im} f$ exists and is a retract of $\operatorname{cod} f$, then $\operatorname{im} f = \operatorname{cod} f$. Moreover, every normal factorization of f (if exists) is of the form $f = \varrho u$ where ϱ is a retraction and u is an isomorphism.*

(b) *Let f is monomorphism. If $f = x\jmath$ is a canonical factorization of f where x is a split epimorphism, then x is an isomorphism. In particular, if f has normal factorization, it is unique.*

PROOF (*a*): If $f = f^\circ \jmath_f$ is an epimorphism, then $\jmath_f$ is an epimorphism and hence balanced. The hypothesis implies that $\jmath_f$ is split and so by Lemma 2, $\jmath_f$ is the identity on $\operatorname{cod} f$. Hence $\operatorname{im} f = \operatorname{cod} f$. If f has normal factorization, by Proposition 5, the inclusion of any normal factorization of f is equal to $\jmath_f$ and so the last statement follows.

(*b*): If $f = x\jmath$ is monomorphism, then x is a monomorphism and so x is balanced. If x is split, this implies that x is an isomorphism. If $f = \varrho u \jmath$ is any normal factorization of f, then as before ϱ is a monomorphism. Hence by

Lemma 2, ϱ is identity on $\operatorname{dom} f$. Hence the normal factorization of f reduces to $f = uj$ which, by Proposition 5, is unique. □

It may be noted that for an epimorphism f with image, the equality $\operatorname{im} f = \operatorname{cod} f$ is not valid without the hypothesis that the image is a retract of the codomain. For in **Top** the inclusion of a dense subspace Y of the space X into X is an epimorphism with image Y and codomain X.

Corollary 8 *Let $\mathcal{C}$ be a category such that*

(1) *every inclusion splits; and*
(2) *every morphism has normal factorization.*

Then every balanced morphism in $\mathcal{C}$ is an isomorphism.

Proof If $f = \varrho uj$ is balanced, then by Proposition 7(a) and (b), j is the identity on $\operatorname{cod} f$ and ϱ is identity on $\operatorname{dom} f$. Hence $f = u$. □

3 REGULAR CATEGORIES

Recall that an element x in a semigroup S is regular if there is $x' \in S$ such that $xx'x = x$. Following this, we shall say that a morphism $f \in \mathcal{C}(A, B)$ (where the category $\mathcal{C}$ has factorization property) is a *regular morphism* if there is $f' \in \mathcal{C}(B, A)$ such that $ff'f = f$. If we also have $f'ff' = f'$, then f' is called an *generalized inverse* (or *ginverse* for short) of f. If f is regular and if f' is a morphism with $ff'f = f$, then it is easy to check that $f'' = f'ff'$ is a ginverse of f. Again, as in the case of semigroups, a regular morphism may have more than one ginverse; we denote the set of all ginverses of f by $\mathcal{V}(f)$.

For the remainder of this chapter we assume that all categories considered are categories with subobjects and factorization.

3.1 Regularity and normal factorization

The following result gives a characterization of regularity of a morphism in terms of normal factorizations.

Proposition 9 *A morphism f in the category $\mathcal{C}$ is regular if and only if f has normal factorization and $\operatorname{im} f$ is a retract of $\operatorname{cod} f$.*

Proof Let $f \in \mathcal{C}$ be regular and let $f' \in \mathcal{C}$ with $ff'f = f$. Assume that $f = xj$ and $f' = yj'$ be canonical factorizations. Then we have

$$xjyj'xj = xj \qquad \text{and} \qquad yj'xjyj' = yj'.$$

Since x and y are epimorphisms and j and j' are monomorphisms, we have

$$jyj'x = 1_A \qquad \text{and} \qquad j'xjy = 1_B$$

where $A = \operatorname{cod} x$ and $B = \operatorname{cod} y$. These imply that $\varrho = xjy$ is a retraction which is a right inverse of j' and $\varrho' = yj'x$ is a right inverse of j. Further $u = j'x$ is an isomorphism with inverse $v = jy$. Moreover

$$f = xj = xjyj'xj = \varrho uj$$

which is a normal factorization for f.

Conversely if $f = \varrho uj$ is a normal factorization for f such that j splits, then $f' = \varrho' u^{-1} j'$ where ϱ' is a right inverse of j and j' is a left inverse of ϱ, then it is easy to see that $ff'f = f$. □

It is clear that a split monomorphism or split epimorphism is regular. Conversely if a monomorphism f is regular, then by the above it has a normal factorization. By Proposition 7, f has a unique normal factorization of the form $f = uj$ where u is an isomorphism and j is a split inclusion. If ϱ is a right inverse of j, then $g = \varrho u^{-1}$ is a right inverse of f. Similarly if f is a regular epimorphism, by Proposition 7 and the result above, $f = \varrho u$ where ϱ is a retraction and u is an isomorphism. If j is a left inverse of ϱ, then $u^{-1}j$ is a left inverse of f. Thus

Corollary 10 *Suppose that $f \in \mathcal{C}$ is either a monomorphism or an epimorphism. Then f is regular if and only if f is split.* □

From the discussion preceding the corollary above, we see that f is a split monomorphism if and only if $f = uj$ where u is an isomorphism and j is a split inclusion. This says that f is equivalent to a split inclusion. This together with a similar argument for epimorphisms gives the following.

Corollary 11 *A morphism f is a split monomorphism if and only if f is equivalent to a split inclusion. Dually f is a split epimorphism if and only if it is equivalent to a retraction.* □

Proposition 3 says that every morphism in C has unique canonical factorization if every inclusion in C split. Under the assumption of regularity, we can weaken this condition.

Proposition 12 *Every regular morphism in C has a unique canonical factorization.*

Proof Let f in C be regular. Then by Proposition 9, f has normal factorization, say, $f = \varrho uj$ where j is split. Let $f = xj'$ be a canonical factorization. Then $j = qxj'$ where q is a left inverse of ϱu. Also, if ϱ' is a right inverse of j, then $xj'\varrho'j = \varrho uj = xj'$ and since x is an epimorphism, we have $(j'\varrho')j = j'$. Hence $j \sim j'$ and so $j = j'$ and $\varrho u = x$. The result follows from Proposition 4. □

3.2 Regularity of categories

We are now in a position to define regular categories.

Definition 4 *Let C be a category with subobject and factorization. We say that a C is regular if*

(a) *C has factorization property;*
(b) *every morphism in C is regular.*

Remark 4 Natural examples of regular categories include **Set**, $\mathbf{Vct}_K$, the category of all semisimple (completely reducible) modules over a ring, **Frd** the category whose objects are locally convex spaces and morphisms are Fredholm operators, etc. Observe that our definition applies to only categories with factorization property.

We say that a category C is *balanced* if every balanced morphism in C is an isomorphism.

Theorem 13 *A category $\mathcal{C}$ is regular if and only if $\mathcal{C}$ has the following properties:*

(1) *Every inclusion in $\mathcal{C}$ splits.*
(2) *Every morphism in $\mathcal{C}$ has normal factorization.*

In particular, a regular category is balanced.

Proof This follows immediately from Proposition 9. The last statement follows from Corollary 8. □

CHAPTER III

Normal Categories

In this chapter we introduce and study the important notion of a normal category and see how this concept is related to the structure of regular semigroups. We show that with every small regular category $\mathcal{C}$, we can associate a semigroup $T\mathcal{C}$ of normal cones and this leads to the definition of normal categories in §1. The definition of normality is equivalent to the condition that the semigroup $T\mathcal{C}$ is regular. In §2, we study Green's relations on the semigroup $T\mathcal{C}$; when $\mathcal{C}$ is normal. We show that the partially ordered set of $\mathcal{L}$–classes of $T\mathcal{C}$ is order isomorphic to the partially ordered set $v\mathcal{C}$ of vertices of $\mathcal{C}$ and the partially ordered set of $\mathcal{R}$–classes is order isomorphic to a partially ordered set of set-valued functors on $\mathcal{C}$. It is shown that these functors are representable and that the $\mathcal{R}$–classes of $T\mathcal{C}$ can be identified with the set of universal elements of these functors. In §3, we show that with every regular semigroup S, we can associate two normal categories $\mathcal{L}(S)$ of principal left ideals and $\mathcal{R}(S)$ of all principal right ideals. Finally, in §4 we show that when $\mathcal{C}$ is normal, there is a natural isomorphism of $\mathcal{C}$ onto $\mathcal{L}(T\mathcal{C})$ and there is a category embedding of $\mathcal{R}(T\mathcal{C})$ into $\mathcal{C}^*$. We conclude the section (and the chapter) with the definition of the normal dual of a normal category.

1 THE SEMIGROUP OF NORMAL CONES

In the following, we assume that $\mathcal{C}$ stands for a small regular category. This implies in particular that $\mathcal{C}$ has subobjects, images and every morphism in $\mathcal{C}$ has normal factorizations in which the inclusion splits (see Proposition II.3, Definition II.4 and Theorem II.13). In particular every morphism in $\mathcal{C}$ has a unique canonical factorization. These facts will be used in the sequel, often with out further comments.

1.1 Normal cones

Recall that a cone from $\boldsymbol{\sigma}^*\mathcal{C}$ to the vertex $d \in \boldsymbol{v}\mathcal{C}$ is a map $\gamma\colon a \mapsto \gamma(a) \in \mathcal{C}(a,d)$ of $\boldsymbol{v}\mathcal{C}$ to $\mathcal{C}$ such that whenever $a \subseteq b$, $j_a^b\gamma(b) = \gamma(a)$ (see I.3.3).

Definition 1 *By a* normal cone *γ in the category $\mathcal{C}$ with vertex $d \in \mathcal{C}$, we mean a map $\gamma\colon \boldsymbol{v}\mathcal{C} \to \mathcal{C}$ such that*

(i) $\gamma\colon \boldsymbol{\sigma}^*\mathcal{C} \stackrel{\cdot}{\to} d$;
(ii) *there exists at least one $c \in \boldsymbol{v}\mathcal{C}$ such that $\gamma(c)\colon c \to d$ is an isomorphism.*

In the following we denote by $T\mathcal{C}$ the set of all normal cones in $\mathcal{C}$. For $\gamma \in T\mathcal{C}$ we denote by c_γ the vertex of γ and by $M\gamma$, the set defined by

(1) $$M\gamma = \{\, c \in \boldsymbol{v}\mathcal{C} : \gamma(c) \text{ is an isomorphism} \,\}$$

We shall refer to this set as the M-*set* of the normal cone γ. Axiom (ii) in Definition 1 above implies that $M\gamma$ is not empty for any $\gamma \in T\mathcal{C}$.

Remark 1 Notice that $T\mathcal{C}$ may be empty; for the trivial category on a set X is clearly a small regular category and there exist no cone from $\boldsymbol{\sigma}^*X = X$ to any $x \in X$ so that $TX = \emptyset$. Hence the condition $T\mathcal{C} \neq \emptyset$ is a strong connectivity condition on the category $\mathcal{C}$. Therefore, in the following, we shall assume that $T\mathcal{C} \neq \emptyset$ to avoid trivial cases. In case $T\mathcal{C} = \emptyset$, the results can still be interpreted as vacuously true.

1.2 The semigroup $T\mathcal{C}$

We proceed to show that $T\mathcal{C}$ is a semigroup. We need the following lemma which will be of constant use in what follows.

Lemma 1 *Let $\gamma \in T\mathcal{C}$ and $f\colon c_\gamma \to d$ be an epimorphism. Then the map*

(2) $$\gamma \star f \;:\; a \mapsto \gamma(a)f$$

from $\boldsymbol{v}\mathcal{C}$ to $\mathcal{C}$ is a normal cone with vertex d. Moreover, if $f \in \mathcal{C}(c_\gamma, c)$ and $g \in \mathcal{C}(c,d)$, then we have

$$(\gamma \star f^\circ) \star (j_{c_1}^c g)^\circ = \gamma \star (fg)^\circ$$

where $c_1 = \operatorname{im} f = \operatorname{im} f^\circ$.

Proof For each $a \in v\mathcal{C}$, clearly $\gamma(a)f \in \mathcal{C}(a,d)$. If $a \subseteq b$, then $j_a^b(\gamma \star f)(b) = j_a^b\gamma(b)f = \gamma(a)f = (\gamma \star f)_a$ since $\gamma \in T\mathcal{C}$. Hence $\gamma \star f \colon \sigma^*\mathcal{C} \rightharpoonup d$ is a cone. Now let $c \in M\gamma$. Then $\gamma(c)f$ is an epimorphism because $\gamma(c)$ is an isomorphism. Since $\mathcal{C}$ is regular, by Propositions II.7(a) and II.9, $\gamma_c f$ has a normal factorization of the form $\gamma(c)f = \varrho u$ where $\varrho \colon c \to c'$ is a retraction and $u \colon c' \to d$ is an isomorphism. So

$$\begin{aligned}(\gamma \star f)(c') &= \gamma(c')f = j_{c'}^c\gamma(c)f \\ &= j_{c'}^c \varrho u \\ &= 1_{c'}u = u\end{aligned}$$

and hence $(\gamma \star f)(c')$ is an isomorphism. To prove the last statement, note that $\gamma' = \gamma \star f^\circ$ is a normal cone with $c_{\gamma'} = \operatorname{cod} f^\circ = \operatorname{im} f = c_1$. Hence for any $a \in v\mathcal{C}$,

$$\begin{aligned}\left((\gamma \star f^\circ) \star j_{c_1}^c g^\circ\right)(a) &= \left(\gamma' \star (j_{c_1}^c g)^\circ\right)(a) \\ &= \gamma(a) f^\circ (j_{c_1}^c g)^\circ \\ &= (\gamma \star (fg)^\circ)(a)\end{aligned}$$

by Equation (II.4a) (see Corollary II.4). This proves the required equality. □

For $\gamma^1, \gamma^2 \in T\mathcal{C}$, define

$$\gamma^1 \cdot \gamma^2 = \gamma^1 \star \left(\gamma^2(c_{\gamma^1})\right)^\circ . \tag{3}$$

It follows from the Lemma 1 that the equation above defines a single-valued binary operation in $T\mathcal{C}$.

Theorem 2 *Let $\mathcal{C}$ be a small regular category. Then $T\mathcal{C}$ is a semigroup with binary operation defined by Equation (3) such that:*

(a) *$\gamma \in T\mathcal{C}$ is an idempotent if and only if $\gamma(c_\gamma) = 1_{c_\gamma}$;*
(b) *$\gamma \in T\mathcal{C}$ is a regular element in $T\mathcal{C}$ if and only if , $\gamma'(c_\gamma)$ is a monomorphismfor some $\gamma' \in T\mathcal{C}$.*

In particular, the set of regular elements of $T\mathcal{C}$ is a subsemigroup of $T\mathcal{C}$.

PROOF To show that TC is a semigroup, it is sufficient to show that the binary operation defined by Equation (3) is associative. Let $\gamma^i \in TC$ and let $c_i = c_{\gamma^i}$ for $i = 1,2,3$. By Equations (2) and (3), we have $c_{\gamma^1 \cdot \gamma^2} = \operatorname{im}\gamma^2(c_1)$ and for convenience we write this as c_{12}. Similarly we write $c_{\gamma^2 \cdot \gamma^3} = c_{23}$ and $c_{123} = \operatorname{im}\gamma^3(c_{12})$. Now it follows from definitions that for any $a \in \boldsymbol{v}\mathcal{C}$,

$$(\gamma^1 \cdot \gamma^2)(a) = \gamma^1(a)\left(\gamma^2(c_1)\right)^\circ.$$

Therefore

$$\begin{aligned}\left((\gamma^1 \cdot \gamma^2)\cdot\gamma^3\right)(a) &= (\gamma^1 \cdot \gamma^2)(a)\left(\gamma^3(c_{12})\right)^\circ \\ &= \gamma^1(a)(\gamma^2(c_1))^\circ\left(\gamma^3(c_{12})\right)^\circ;\end{aligned}$$

and

$$\left(\gamma^1 \cdot (\gamma^2 \cdot \gamma^3)\right)(a) = \gamma^1(a)\left((\gamma^2 \cdot \gamma^3)(c_1)\right)^\circ.$$

By Proposition II.12, every morphism in $\mathcal{C}$ has a unique canonical factorization. Using the definition of the binary operation in TC, the definition of normal cones and the uniqueness of canonical factorization, we obtain

$$\begin{aligned}\left((\gamma^2 \cdot \gamma^3)(c_1)\right)^\circ &= \left(\gamma^2(c_1)(\gamma^3(c_2))^\circ\right)^\circ \\ &= \left((\gamma^2(c_1))^\circ j^{c_2}_{\operatorname{im}\gamma^2(c_1)}(\gamma^3(c_2))^\circ\right)^\circ \\ &= \left((\gamma^2(c_1))^\circ(\gamma^3(\operatorname{im}\gamma^2(c_1))^\circ j^{c_{23}}_{c_{123}}\right)^\circ \\ &= (g^2(c_1))^\circ(\gamma^3(c_{12}))^\circ\end{aligned}$$

This proves that

$$\left((\gamma^1 \cdot \gamma^2)\cdot\gamma^3\right)(a) = \left(\gamma^1 \cdot (\gamma^2 \cdot \gamma^3)\right)(a)$$

for all $a \in \boldsymbol{v}\mathcal{C}$ and hence TC is a semigroup.

Suppose that γ is an idempotent and let $c \in M\gamma$. Then by Equation (3), we have

$$\gamma(c)\left(\gamma(c_\gamma)\right)^\circ = (\gamma \cdot \gamma)(c) = \gamma(c).$$

Since $\gamma(c)$ is an isomorphism, we have $(\gamma(c_\gamma))^\circ = 1_{c_\gamma}$. Clearly $\gamma(c_\gamma) \in \mathcal{C}(c_\gamma, c_\gamma)$ and so $\gamma(c_\gamma) = 1_{c_\gamma}$. Conversely, if $\gamma(c_\gamma) = 1_{c_\gamma}$ then for every $a \in v\mathcal{C}$

$$(\gamma \cdot \gamma)(a) = \gamma(a)\,(\gamma(c_\gamma))^\circ = \gamma(a)1_{c_\gamma} = \gamma(a)$$

by Equation (3). Hence γ is an idempotent. This proves (a).

Suppose that γ is regular. Then, by definition, there is $\bar{\gamma}$ such that $\gamma\cdot\bar{\gamma}\cdot\gamma = \gamma$. Let $\bar{c} = c_{\bar{\gamma}}$, $\epsilon = \bar{\gamma}\cdot\gamma$. Then, for any $c \in M\gamma$, we have $\gamma(c)\,(\epsilon(c_\gamma))^\circ = \gamma(c)$ and so $(\epsilon(c_\gamma))^\circ = 1_{c_\gamma}$. Hence $c_\gamma \subseteq c_\epsilon$. This together with the equality $\epsilon = \bar{\gamma}\cdot\gamma$ gives $c_\epsilon = c_\gamma$. Therefore ϵ is a normal cone such that $\epsilon(c_\gamma)$ is, in particular, a monomorphism. Conversely assume that $\gamma' \in T\mathcal{C}$ such that $\gamma'(c_\gamma)$ is monomorphism. Since $\mathcal{C}$ is regular, by Corollary II.10, $\gamma'(c_\gamma)$ is a split monomorphism. Let $\epsilon = \gamma' \star f$ where f is a right inverse of $\gamma'(c_\gamma)$. It follows from Lemma 1 and Equation (3) that $\epsilon(c_\gamma) = 1_{c_\gamma}$ and hence by (a), ϵ is an idempotent with $c_\epsilon = c_\gamma$. Let $c \in M\gamma$ and

$$\tilde{\gamma} = \epsilon \star (\gamma(c))^{-1}. \tag{4}$$

Since $\gamma(c)$ is an isomorphism, $\tilde{\gamma} \in T\mathcal{C}$ with $c_{\tilde{\gamma}} = c$ by Lemma 1. Also

$$\begin{aligned}
(\tilde{\gamma}\cdot\gamma)(c_\gamma) &= \tilde{\gamma}(c_\gamma)\,(\gamma(c))^\circ && \text{by Equation (3)}\\
&= \tilde{\gamma}(c_\gamma)\gamma(c) && \text{since } \gamma(c) \text{ is an isomorphism}\\
&= (\gamma(c))^{-1}\gamma(c) = 1_{c_\gamma} && \text{by the definition of } \tilde{\gamma}.
\end{aligned}$$

Hence, for any $a \in v\mathcal{C}$, we have, by Equation (3)

$$(\gamma\cdot\tilde{\gamma}\cdot\gamma)(a) = \gamma(a)\,(\tilde{\gamma}\cdot\gamma)^\circ = \gamma(a)1_{c_\gamma} = \gamma(a)$$

and so γ is regular.

Finally assume that γ^1 and γ^2 are regular elements in $T\mathcal{C}$. Then it follows from Equation (3) that $c_{\gamma^1.\gamma^2} = \operatorname{im}\gamma^2(c_{\gamma^1})$ and so $c_{\gamma^1.\gamma^2} \subseteq c_{\gamma^2}$. Since γ^2 is regular, there exists as above, an idempotent cone ϵ with $c_\epsilon = c_{\gamma^2}$ and it follows from (a) that $\epsilon(c_{\gamma^1.\gamma^2}) = j_{c_{\gamma^1.\gamma^2}}^{c_{\gamma^2}}$. Hence by (b), it follows that $\gamma^1\cdot\gamma^2$ is regular.

□

1.3 Normality

We are now ready to define normal categories.

Definition 2 *A small regular category $\mathcal{C}$ is said to be a* normal category *if for every $a \in \boldsymbol{v}\mathcal{C}$ there exists some $\gamma \in T\mathcal{C}$ such that $\gamma(a)$ is a monomorphism. Two normal categories are* isomorphic *if there exists a category isomorphism between them that preserves inclusions.*

Note that a functor $F: \mathcal{C} \to \mathcal{D}$ from a normal category $\mathcal{C}$ to a normal category $\mathcal{D}$ ia an isomorphism of normal categories if F is an isomorphism of categories with subobjects (see §II.1). Also it follows immediately from Theorem 2 that, if $\mathcal{C}$ is normal, then the semigroup $T\mathcal{C}$ is regular.

Remark 2 Note that the definition of normal categorys here is different from the definition given in [16]. A small regular category is normal in the sense of [16] and a category $\mathcal{C}$ is normal in the sense of the definition above if and only if it is normal and reductive in the sense of [16].

Remark 3 It is useful to observe that the condition for regularity of $\gamma \in T\mathcal{C}$ stated above is equivalent to the fact that there is an idempotent normal cone with vertex c_γ. This is part of the proof above (of Theorem 2). In particular, $\mathcal{C}$ is normal if and only if for every $a \in \boldsymbol{v}\mathcal{C}$ there exists an idempotent $\epsilon \in T\mathcal{C}$ such that $c_\epsilon = a$. However, the form in which the condition is given in Theorem 2(b) is often simpler to apply as was demonstrated in the concluding part of the proof. Another application of this condition shows that any small regular category with a largest object is normal. Here by the largest object, we mean $c \in \boldsymbol{v}\mathcal{C}$ such that $a \subseteq c$ for all $a \in \boldsymbol{v}\mathcal{C}$.

Remark 4 Let $\mathcal{C}$ be a small category with subobjects. For $a \in \boldsymbol{v}\mathcal{C}$, we denote by $\langle a \rangle_\mathcal{C}$ (or by $\langle a \rangle$ if the category $\mathcal{C}$ is clear from the context) the full subcategory of $\mathcal{C}$ whose objects are subobjects of a in $\mathcal{C}$. A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is called an *ideal* if $\mathcal{C}'$ is a full subcategory of $\mathcal{C}$ such that $\langle a \rangle \subseteq \mathcal{C}'$ for all $a \in \boldsymbol{v}\mathcal{C}'$. Note that for all $a \in \boldsymbol{v}\mathcal{C}$ $\langle a \rangle_\mathcal{C}$ is an ideal of $\mathcal{C}$; $\langle a \rangle_\mathcal{C}$ is called the *principal ideal* generated by a.

Now call an object $a \in \boldsymbol{v}\mathcal{C}$ to be normal if there is $\gamma \in T\mathcal{C}$ such that $\gamma(a)$ is a monomorphism. Now the full subcategory generated by the collection of all normal objects is an ideal which we denote by $\mathcal{C}^{(n)}$. Evidently $\mathcal{C}^{(n)}$ is normal and the mapping which sends $\gamma \in T\mathcal{C}$ to the restriction of γ to $\mathcal{C}^{(n)}$ is a homomorphism of the regular semigroup of regular elements of $T\mathcal{C}$ on to $T\mathcal{C}^{(n)}$.

2 GREEN'S RELATIONS ON THE SEMIGROUP OF NORMAL CONES

In this section we discuss some properties of the Green's relations on the semigroup TC. We refer the reader to [1] for the definitions of the relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{D}$ and $\mathcal{H}$ on a semigroup S; we shall also use the notations of [1] associated with these.

2.1 The M–set of a normal cone

Recall that for any subset X of a semigroup S, the set of idempotents of S contained in X is denoted by $E(X)$.

Lemma 3 *Let $\gamma \in TC$ and $\epsilon \in E(TC)$. Then $\epsilon \cdot \gamma = \gamma$ if and only if there exists a unique epimorphism $f\colon c_\epsilon \to c_\gamma$ such that $\gamma = \epsilon \star f$.*

Proof Let $\epsilon \cdot \gamma = \gamma$. Then by Equation (3),

$$\gamma(a) = (\epsilon \cdot \gamma)(a) = \epsilon(a)\,(\gamma(c_\epsilon))^\circ$$

for each $a \in vC$. Taking $a = c_\epsilon$, we get $\gamma(c_\epsilon) = (\gamma(c_\epsilon))^\circ$ which implies that $f = \gamma(c_\epsilon)$ is an epimorphism. By Equation (2) we see that $\gamma = \epsilon{\star}f$. Conversely assume that $\gamma = \epsilon{\star}f$ where $f \in C(c_\epsilon, c_\gamma)$ is an epimorphism. Then by Lemma 1 and Theorem 2, $\gamma(c_\epsilon) = f$ and using Equations (2) and (3) we get

$$(\epsilon \cdot \gamma)(a) = \epsilon(a)f = \gamma(a)$$

for all $a \in vC$. If $\gamma = \epsilon{\star}f = \epsilon{\star}g$ for epimorphisms f and g, then $f = \gamma(c_\epsilon) = g$ which proves the uniqueness. □

The M–set $M\gamma$ of $\gamma \in TC$ (see Equation (1)) plays an important role in sequel; they are also related to the Green's relations on TC. Next proposition lists a few basic properties of the set $M\gamma$ associated with a normal cone γ in C.

Proposition 4 *For $\gamma \in TC$ we have the following.*

(a) *For each $c \in M\gamma$ there is a unique idempotent normal cone ϵ^c with $c_{\epsilon^c} = c$ and an isomorphism $f: c \to c_\gamma$ (both ϵ^c and f depending on c) such that $\gamma = \epsilon^c \star f$. Conversely for $\epsilon \in E(TC)$, if there exists an isomorphism $f: c_\epsilon \to c_\gamma$ with $\gamma = \epsilon \star f$ then $c_\epsilon \in M\gamma$.*

(b) *Let $c \in M\gamma$. Then $E(R_{\epsilon^c}) = \{ \epsilon^{c'} : c' \in M\gamma \}$.*

(c) *If $\gamma' \in TC$ and if $\gamma \mathcal{R} \gamma'$ then $M\gamma = M\gamma'$.*

In particular, γ is regular in TC if and only if $\epsilon^c \mathcal{R} \gamma$ for some $c \in M\gamma$.

Proof (a) Let $c \in M\gamma$. Then $\gamma(c)$ is an isomorphism and by Lemma 1 and Theorem 2(a), we have $\epsilon^c = \gamma \star (\gamma(c))^{-1}$ is an idempotent normal cone with $c_{\epsilon^c} = c$ such that $\epsilon^c \star \gamma(c) = \gamma$. If ϵ is another idempotent with $c_\epsilon = c$ and such that $\gamma = \epsilon \star f$ for some isomorphism, then

$$(\epsilon^c \star \gamma(c))(c) = \gamma(c) = (\epsilon \star f)(c) = f$$

and hence $\epsilon^c = \gamma \star (\gamma(c))^{-1} = \gamma \star f^{-1} = \epsilon$. This proves that both ϵ^c and f are unique. Conversely suppose that ϵ is an idempotent such that $\gamma = \epsilon \star f$ for some isomorphism f. Then $\gamma(c_\epsilon) = f$ and so $c_\epsilon \in M\gamma$.

(b) Let ϵ be an idempotent with $c_\epsilon = c'$. First assume that $\epsilon \mathcal{R} \epsilon^c$. Then by Lemma 3 there exist epimorphisms $h: c \to c'$ and $k: c' \to c$ such that $\epsilon = \epsilon^c \star h$ and $\epsilon^c = \epsilon \star k$. Hence $\epsilon = \epsilon^c \star (kh)$ and $\epsilon^c = \epsilon \star (hk)$ so that by Equation (2), $hk = 1_c$ and $kh = 1'_c$ and so h and k are isomorphisms. Now from the proof of (a),

$$\gamma = \epsilon^c \star \gamma(c) = (\epsilon \star k) \star \gamma(c) = \epsilon \star k\gamma(c).$$

Since $k\gamma(c)$ is an isomorphism, $c_\epsilon \in M\gamma$ by (a). Conversely if $c_\epsilon = c' \in M\gamma$, then $h = \gamma(c)(\gamma(c'))^{-1}: c \to c'$ and $k = \gamma(c')(\gamma(c))^{-1}: c' \to c$ are isomorphisms such that $\epsilon = \epsilon^c \star h$ and $\epsilon^c = \epsilon \star k$ by Equation (2). Hence by Lemma 3, $\epsilon \cdot \epsilon^c = \epsilon^c$ and $\epsilon^c \cdot \epsilon = \epsilon$; that is, $\epsilon^c \mathcal{R} \epsilon$.

(c) If $\gamma \mathcal{R} \gamma'$ there exist $\sigma, \tau \in TC$ such that $\gamma' = \gamma \cdot \sigma$ and $\gamma = \gamma' \cdot \tau$. Using Equation (3) we deduce that $\gamma' = \gamma \star h$ and $\gamma = \gamma' \star k$ where $h: c_\gamma \to c_{\gamma'}$ and $k: c_{\gamma'} \to c_\gamma$ are epimorphisms. It follows that $hk = 1_{c_\gamma}$ and $kh = 1_{c_{\gamma'}}$ and hence these are isomorphisms. Now if $c \in M\gamma$, then by (a), we have $\gamma = \epsilon^c \star \gamma(c)$ and hence $\gamma' = \epsilon^c \star (\gamma(c)h)$. Since $\gamma(c)h$ is an isomorphism, $c \in M\gamma'$ by (a). Thus $M\gamma \subseteq M\gamma'$; similarly $M\gamma' \subseteq M\gamma$. This proves (c).

If $\epsilon^c \mathcal{R} \gamma$, then γ is obviously regular (see [1]). Conversely, if γ is regular, then by the proof of Theorem 2(a), there is $\epsilon \in E(TC)$ such that $c_\gamma = c_\epsilon$. Let $c \in M\gamma$ and let $\tilde{\gamma}$ be defined by Equation (4). Then it is easy to see that

$$\epsilon^c = \gamma \star (\gamma(c))^{-1} = \gamma \cdot \tilde{\gamma}.$$

Moreover, since $\tilde{\gamma}$ is a ginverse of γ, it is well known that $\gamma \cdot \tilde{\gamma} = \epsilon^c$ is an idempotent in the $\mathcal{R}$-class of γ. □

2.2 The partially ordered set of left ideals of TC

Observe that the set $S/\mathcal{L}$ of all $\mathcal{L}$-classes of S is order isomorphic with the partially ordered set of all principal left ideals of the semigroup S under inclusion when $S/\mathcal{L}$ is partially ordered with respect to the relation $\leq$ defined by:

$$(5) \qquad L_x \leq L_y \iff S^1x \subseteq S^1y.$$

for L_x, $L_y \in S/\mathcal{L}$. In view of this we shall, in the following, identify the partially ordered set of all principal left ideals with the set $S/\mathcal{L}$ (with partial order defined above). Dually the inclusion among the principal right ideals induces a partial order on the set $S/\mathcal{R}$ of all $\mathcal{R}$-classes of S and $S/\mathcal{R}$ is order isomorphic with the partially ordered set of principal right ideals under inclusion.

The following proposition shows that the partially ordered set of left ideals of TC is order-isomorphic with vC when the category C is normal. This will, in turn, give a characterization of the Green's relation $\mathcal{L}$ on TC in this case.

PROPOSITION 5 *Let C be a small regular category. If $\gamma, \gamma' \in TC$ and if $L_\gamma \leq L_{\gamma'}$ then $c_\gamma \subseteq c'_\gamma$. The converse also holds if γ' is regular. In particular, if C is normal then the map $L_\gamma \mapsto c_\gamma$ is an order isomorphism of the partially ordered set of $\mathcal{L}$-classes onto the partially ordered set vC.*

PROOF If $L_\gamma \leq L_{\gamma'}$, then $\gamma = \tau \cdot \gamma'$ for some $\tau \in TC$ and so $\gamma = \tau \star f^\circ$ for some $f \in C(c_\tau, c_{\gamma'})$ by Equation (3). This implies that $c_\gamma = \operatorname{im} f \subseteq c_{\gamma'}$.

Conversely suppose that $c_\gamma = c \subseteq c' = c_{\gamma'}$. If γ' is regular then by Theorem 2 (see also Remark 3), there exists an idempotent ϵ with $c_\epsilon = c'$. Let $\tilde{\gamma}'$ be defined by Equation (4). Then, as in the proof of Theorem 2, we can show that $\tilde{\gamma}'$ is a ginverse of γ' such that $\tilde{\gamma}' \cdot \gamma' = \epsilon$; in particular, $\gamma' \mathcal{L} \epsilon$.

Hence to show that $L_\gamma \leq L'_\gamma$, it is sufficient to show that $\gamma \cdot \epsilon = \gamma$. But since $\epsilon(c) = j_c^{c'}$, we have, by Equation (3) and Lemma 1

$$\gamma \cdot \epsilon = \gamma \star (\epsilon(c_\gamma))^\circ = \gamma \star 1_{c_\gamma} = \gamma.$$

The last statement is now clear. □

2.3 The partially ordered set of H-functors

We proceed to obtain a representation of the partially ordered set of right ideals of $T\mathcal{C}$ as a partially ordered set of certain set-valued functors, called *H-functors*, where $\gamma \in T\mathcal{C}$. This in particular yields a characterization of the Green's relation $\mathcal{R}$ on $T\mathcal{C}$.

For each $\gamma \in T\mathcal{C}$, define $H(\gamma; -)$ on objects and morphisms of $\mathcal{C}$ as follows:for each $c \in v\mathcal{C}$ and for each morphism $g\colon c \to c'$ in $\mathcal{C}$, let

$$\begin{aligned} H(\gamma; c) &= \{\gamma \star f^\circ : f \in \mathcal{C}(c_\gamma, c)\} \\ H(\gamma; g)&\colon \gamma \star f^\circ \mapsto \gamma \star (fg)^\circ. \end{aligned} \tag{6}$$

The uniqueness of epimorphic components of morphisms imply that $H(\gamma; g)$ is a single valued map of $H(\gamma; c)$ to $H(\gamma; c)'$. The following lemma shows that this defines a set-valued functor on $\mathcal{C}$ for every $\gamma \in T\mathcal{C}$.

Lemma 6 *For each $\gamma \in T\mathcal{C}$, $H(\gamma; -)\colon \mathcal{C} \to$ **Set** is a functor such that γ is a universal element for $H(\gamma; -)$ in $H(\gamma; c_\gamma)$. In particular, there exists a natural isomorphism $\eta_\gamma\colon H(\gamma; -) \to \mathcal{C}(c_\gamma, -)$ such that $\eta_\gamma(c_\gamma)(\gamma) = 1_{c_\gamma}$.*

Proof It is clear that $H(\gamma; c)$ defined by the first Equation of (6) is a set for each $c \in v\mathcal{C}$ and we have noted above that $H(\gamma; g)$ defined by the second Equation of (6) is a map for each $g\colon c \to c'$. To see that this is functorial, consider $h\colon c' \to c''$. Then, using the definition of the map $H(\gamma; g)$, we obtain

$$\begin{aligned} H(\gamma; g)H(\gamma; h)(\gamma \star f^\circ) &= H(\gamma; h)\,(H(\gamma; g)(\gamma \star f^\circ)) \\ &= H(\gamma; h)\,(\gamma \star (fg)^\circ) \\ &= \gamma \star ((fg)h)^\circ = \gamma \star (fgh)^\circ \\ &= H(\gamma; gh)(\gamma \star f^\circ). \end{aligned}$$

Evidently $H(\gamma; 1_c) = 1_{H(\gamma;c)}$ and hence $H(\gamma; -)$ is a functor. If $\tau \in H(\gamma; c)$ where $\tau = \gamma \star f^\circ$ we have $H(\gamma; f)(\gamma) = \tau$ by the definition of the map $H(\gamma; f)$. Moreover, f is also unique. For if $H(\gamma; k)(\gamma) = \tau$, then for any $c \in M\gamma$, we have $\gamma(c)f^\circ = \gamma(c)k^\circ$ by Equations (2) and (6). Since $\gamma(c)$ is an isomorphism, by uniqueness of canonical factorization, we get $f = k$. Therefore $\gamma \in H(\gamma; c_\gamma)$ is a universal element for $H(\gamma; -)$ (see I.3.1). The last statement follows from Yoneda lemma (see also §I.3). □

For any $\gamma \in T\mathcal{C}$, the functor $H(\gamma; -)$ will be called the *hom-functor* determined by the cone γ. If τ is any universal element for $H(\gamma; -)$, by Yoneda lemma and the discussion of representable functors in §I.3, there is a unique representing object $c \in v\mathcal{C}$ and a natural isomorphism of $H(\gamma; -)$ to $\mathcal{C}(c, -)$. In the following, this natural isomorphism, uniquely determined by τ, will be denoted by η_τ.

If F and G are two functors from a category $\mathcal{C}$ to a category $\mathcal{D}$ with subobjects, we say that F is a *subfunctor* of G if for all $c \in v\mathcal{C}$,

(7i) $$F(c) \subseteq G(c)$$

and the map

(7ii) $$j_F^G : c \mapsto j_{F(c)}^{G(c)}$$

of $v\mathcal{C}$ into **Set** is a natural transformation. In this case we write $F \subseteq G$. Observe that the natural transformation j_F^G above is a monomorphism in the category $[\mathcal{C}, \mathcal{D}]$ and it is easy to see that the functorial inclusion determines a choice of subobjects for the category $[\mathcal{C}, \mathcal{D}]$ (cf. Definition II.1). In particular, if $\mathcal{C}$ is a small regular category, then $\mathcal{C}^*$ is a category with subobjects. We use this subobject relation on $\mathcal{C}^*$ in the proposition below.

Proposition 7 *For $\gamma, \gamma' \in T\mathcal{C}$, we have the following.*

(a) $H(\gamma; -) \subseteq H(\gamma'; -)$ *if and only if there exists a unique epimorphism h from $c_{\gamma'}$ to c_γ such that $\gamma = \gamma' \star h$. Moreover, $H(\gamma; -) = H(\gamma'; -)$ if and only if there is a unique isomorphism $h: c_{\gamma'} \to c_\gamma$ such that $\gamma = \gamma' \star h$.*

(b) *If $R_\gamma \leq R_{\gamma'}$ then $H(\gamma; -) \subseteq H(\gamma'; -)$. The converse also holds if γ' is regular.*

In particular, when $\mathcal{C}$ is a normal category, the map $R_\gamma \mapsto H(\gamma; -)$ is an order embedding of the partially ordered set $T\mathcal{C}/\mathcal{R}$ into the partially ordered class $v\mathcal{C}^*$.

Proof (a): Let $\gamma = \gamma' \star h$ where $h\colon c_{\gamma'} \to c_\gamma$ is an epimorphism and let $c \in v\mathcal{C}$. If $\gamma \star f^\circ \in H(\gamma; c)$, by Equation (2), we have $\gamma \star f^\circ = \gamma' \star h f^\circ$. Since h is an epimorphism, by the uniqueness of the canonical factorization (see Proposition II.12), $hf^\circ = (hf)^\circ$ and so $\gamma \star f^\circ \in H(\gamma'; c)$. Thus $H(\gamma; c) \subseteq H(\gamma'; c)$ for all $c \in v\mathcal{C}$. Let $g\colon c \to c'$ be a morphism. Then $H(\gamma; g)(\gamma \star f^\circ) = \gamma \star (fg)^\circ$ and

$$\begin{aligned} H(\gamma'; g)(\gamma \star f^\circ) &= H(\gamma'; g)\left(\gamma' \star (hf)^\circ\right) \\ &= \gamma' \star (hfg)^\circ = \gamma \star (fg)^\circ \end{aligned}$$

and this shows that the following diagram commutes.

$$\begin{array}{ccc} H(\gamma'; c) & \xrightarrow{H(\gamma'; g)} & H(\gamma'; c') \\ {\scriptstyle j_{H(\gamma;c)}^{H(\gamma';c)}}\uparrow & & \uparrow{\scriptstyle j_{H(\gamma;c')}^{H(\gamma';c')}} \\ H(\gamma; c) & \xrightarrow{H(\gamma; g)} & H(\gamma; c') \end{array}$$

Hence $H(\gamma; -) \subseteq H(\gamma'; -)$. Conversely suppose that $H(\gamma; -) \subseteq H(\gamma'; -)$. Then $\gamma \in H(\gamma; c_\gamma) \subseteq H(\gamma'; c_\gamma)$ and so $\gamma = \gamma' \star f^\circ$ for some $f \in \mathcal{C}(c_{\gamma'}, c_\gamma)$. Now by Lemma 1, $c_{\gamma' \star f^\circ} = \operatorname{im} f$ and so, by the above equality, $\operatorname{im} f = c_\gamma = \operatorname{cod} f$. Hence it follows that f is an epimorphism and $\gamma = \gamma' \star f$. If $\gamma' \star h = \gamma' \star k$, then for any $c \in M\gamma'$, we have $\gamma'(c)h = \gamma'(c)k$ using Equation (2) and since $\gamma'(c)$ is an isomorphism, we conclude that $h = k$. This proves the uniqueness. If $H(\gamma; -) = H(\gamma'; -)$, then by the above there exist epimorphisms h and k such that $\gamma = \gamma' \star h$ and $\gamma' = \gamma \star k$ and so, $\gamma = \gamma \star kh$. It follows that $kh = 1_{c_\gamma}$. Similarly, $hk = 1_{c_{\gamma'}}$ and hence h is an isomorphism. On the other hand if $\gamma = \gamma' \star h$ for some isomorphism, then $\gamma' = \gamma \star h^{-1}$ and so $H(\gamma; -) = H(\gamma'; -)$. Uniqueness follows from the first part of the statement.

(b): If $R_\gamma \leq R_{\gamma'}$, then $\gamma = \gamma' \cdot \tau$ for some $\tau \in T\mathcal{C}$ and so, using Equation (3) we see that $\gamma = \gamma' \star h$ for some epimorphism. Hence by (a), $H(\gamma; -) \subseteq H(\gamma'; -)$. Conversely, if $H(\gamma; -) \subseteq H(\gamma'; -)$ then by (a), $\gamma = \gamma' \star h$ for some epimorphism h. If γ' is regular and $c \in M\gamma'$, by Proposition 4, there exists an isomorphism f such that $\gamma' = \epsilon^c \star f$. Hence $\gamma = \epsilon^c \star fh$ and fh is an epimorphism. Hence by Lemma 3, $\epsilon^c \cdot \gamma = \gamma$ which implies that $R_\gamma \leq R_{\epsilon^c} = R_{\gamma'}$.

The remaining statement follows immediately from (b). □

The following observations will be useful in the sequel.

Corollary 8 *Let $\gamma, \gamma' \in T\mathcal{C}$. If $H(\gamma; -) = H(\gamma'; -)$ then $M\gamma = M\gamma'$.*

Proof If $c \in M\gamma$, then $\gamma = \epsilon^c \star \gamma(c)$ by Proposition 4(a) and there is an isomorphism h with $\gamma' = \gamma \star h$ by (a) above. Hence $\gamma' = \epsilon^c \star \gamma(c)h$ and so $\gamma'(c) = \gamma(c)h$ by Equation (2). Therefore $\gamma'(c)$ is an isomorphism and hence $c \in M\gamma'$; thus $M\gamma \subseteq M\gamma'$. Similarly, $M\gamma' \subseteq M\gamma$ and so, $M\gamma = M\gamma'$. □

In view of the above result, we may write $MH(\gamma; -)$ for $M\gamma$ which, by Proposition 7, depends only on the $\mathcal{R}$–class R_γ of γ in $T\mathcal{C}$.

Corollary 9 *The normal cone γ' in $T\mathcal{C}$ is a universal element of $H(\gamma; -)$ if and only if $\gamma' = \gamma \star h$ for some isomorphism h. In particular, if $\mathcal{C}$ is normal, the set of all universal elements of $H(\gamma; -)$ is R_γ.*

Proof If $\gamma' = \gamma \star h$ for some isomorphism h, then by Proposition 7(a), $H(\gamma; -) = H(\gamma'; -)$ and so γ' is a universal element of $H(\gamma; -)$ by Lemma 6. Conversely, if $\gamma' \in H(\gamma; c)$ is a universal element of $H(\gamma; -)$, then by the definition of universal elements (see §I.3), there exists unique morphisms $k \in \mathcal{C}(c, c_\gamma)$ and $h \in \mathcal{C}(c_\gamma, c)$ such that

$$\gamma = H(\gamma; k)(\gamma') = \gamma' \star k^\circ \quad \text{and} \quad \gamma' = H(\gamma'; h)(\gamma) = \gamma \star h^\circ.$$

Hence by Equation (6) we have $\gamma = \gamma \star (hk)^\circ$ and $\gamma' = H(\gamma; kh)(\gamma')$ and so $hk = 1_{c_\gamma}$ and $kh = 1_c$. This proves that h is an isomorphism.

The last statement is an immediate consequence of the result proved above and Proposition 7. □

Corollary 10 *Let $H(\gamma; -) = H(\gamma'; -)$. If $h\colon c_{\gamma'} \to c_\gamma$ is the unique isomorphism such that $\gamma = \gamma' \star h$, then the following diagram of functors and natural transformations commute.*

$$\text{(8)} \qquad \begin{array}{ccc} H(\gamma; -) & \xrightarrow{\eta_\gamma} & \mathcal{C}(c_\gamma, -) \\ \| & & \downarrow{\scriptstyle \mathcal{C}(h,-)} \\ H(\gamma'; -) & \xrightarrow[\eta_{\gamma'}]{} & \mathcal{C}(c_{\gamma'}, -) \end{array}$$

Proof By the given condition, we have $\gamma' = \gamma \star h^{-1}$. Hence $\gamma' \in H(\gamma; c_{\gamma'})$ and so,

$$\eta_\gamma(c_{\gamma'})\mathcal{C}(h, c_{\gamma'})(\gamma') = \mathcal{C}(h, c_{\gamma'})(h^{-1}) = hh^{-1} = 1_{c_{\gamma'}}.$$

Hence the $c_{\gamma'}$ component of the natural transformation $\eta_\gamma \mathcal{C}(h, -)$ maps γ' to $1_{c_{\gamma'}}$. Since the $c_{\gamma'}$ component of $\eta_{\gamma'}$ also does the same (by Lemma 6) we have

$$\eta_\gamma \mathcal{C}(h, -) = \eta_{\gamma'}$$

by Yoneda lemma. Hence the given diagram commutes. □

2.4 Green's relations

Propositions 5 and 7 characterizes Green's relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{D}$ on the semigroup $T\mathcal{C}$ when $\mathcal{C}$ is normal. For convenience of reference, we state the result as a proposition. In the following, we write $c \cong c'$ (for $c, c' \in v\mathcal{C}$) to mean that there is an isomorphism $h\colon c \to c'$.

Theorem 11 *Let $\mathcal{C}$ be a normal category and $\gamma, \gamma' \in T\mathcal{C}$. Then*

(a) $\gamma \,\mathcal{L}\, \gamma' \iff c_\gamma = c_{\gamma'}$.
(b) $\gamma \,\mathcal{R}\, \gamma' \iff H(\gamma; -) = H(\gamma'; -)$.
(c) $\gamma \,\mathcal{D}\, \gamma' \iff c_\gamma \cong c_{\gamma'}$.

Proof Statement (a) follows from Proposition 5 and (b) follows from Proposition 7. To prove (c), first assume that $h\colon c \to c'$ is an isomorphism where $c = c_\gamma$ and $c' = c_{\gamma'}$. Let $\tau = \gamma \star h$. Then by Proposition 5, we have, $\tau \,\mathcal{L}\, \gamma'$ and by Proposition 7, $\tau \,\mathcal{R}\, \gamma$ and hence $\gamma \,\mathcal{D}\, \gamma'$. Conversely, if $\gamma \,\mathcal{D}\, \gamma'$, there exists $\tau \in T\mathcal{C}$ such that $\gamma \,\mathcal{R}\, \tau \,\mathcal{L}\, \gamma'$. Then $c_\tau = c_{\gamma'}$ by Proposition 5 and there exists an isomorphism h such that $\tau = \gamma \star h$ by Proposition 7. It follows that $h\colon c_\gamma \to c_{\gamma'}$ is an isomorphism. □

3 CATEGORIES OF LEFT AND RIGHT IDEALS

Let S be a regular semigroup. As in [1], for each $a \in S$ we denote the map $x \mapsto xa$ by ρ_a; it is called the *right translation* determined by a and the map $x \mapsto ax$ is denoted by λ_a (the *left translation* determined by a). By a *translation* we mean a map which is either a left or a right translation.

3.1 Definition and basic properties

In the following, by a *partial right translation* of the regular semigroup S, we mean a map $\rho\colon Se \to Sf$ where $\rho = \rho_u|Se$ for some $u \in S$; in this case ρ is a partial right translation from Se into Sf. Note that, in this case, we can always choose u to be in Sf. For, if $v = uf$ one easily see that $\rho = \rho_u|Se = \rho_v|Se$. Thus if ρ is any partial right translation from Se to Sf, then we may always assume that $\rho = \rho_u|Se$ for some $u \in Sf$ (see 12(a) below). A *partial left translation* is defined dually.

Definition 3 *Let $\mathcal{L}(S)$, called the category of left ideals, denote the category defined as follows:*

$$(9)\qquad \begin{aligned} v\mathcal{L}(S) &= \{\, Se : e \in E(S)\,\} \\ \mathcal{L}(S)(Se, Sf) &= \{\, \rho\colon Se \to Sf : (st)\rho = s(t\rho) \quad \text{for all } s,\, t \in Se\,\} \end{aligned}$$

Dually, the category of right ideals, $\mathcal{R}(S)$, is defined as follows:

$$(9^*)\qquad \begin{aligned} v\mathcal{R}(S) &= \{\, eS : e \in E(S)\,\} \\ \mathcal{R}(S)(eS, fS) &= \{\, \lambda\colon eS \to fS : \lambda(st) = (\lambda s)t \quad \text{for all } s,\, t \in eS\,\} \end{aligned}$$

It follows that $\mathcal{L}(S)$ is the category whose *vertex set* is the set of all principal left ideals and whose *morphism set* is the set of all partial right translations. Similarly, $\mathcal{R}(S)$ is the category whose *vertex set* is the set of all principal right ideals and *morphisms set* is the set of all partial left translations.

The next lemma gives some basic properties of $\mathcal{L}(S)$.

Lemma 12 *$\mathcal{L}(S)$, with vertex set defined by the first equation in (9) and for each pair of vertices Se and Sf, the hom-set defined by the second equation of (9) is a category with factorization property in which inclusions are the usual* set-inclusions. *Moreover let*

$$\rho(e, u, f) = \rho_u | eS \quad \textit{where} \quad e, f \in E(S);\ u \in eSf.$$

Then we have the following:

(a) *For every $e, f \in E(S)$ and $u \in eSf$, $\rho(e, u, f) \in \mathcal{L}(S)(Se, Sf)$. Moreover, the map $\rho(e, u, f) \mapsto u$ is a bijection of $\mathcal{L}(S)(Se, Sf)$ onto eSf.*

(b) *$\rho(e, u, f) = \rho(e', v, f')$ if and only if $e \mathcal{L} e'$, $f \mathcal{L} f'$, $u \in eSf$, $v \in e'Sf'$ and $v = e'u$.*

(c) *If $\rho(e, u, f)$ and $\rho(g, v, h)$ are composable morphisms in $\mathcal{L}(S)$ (so that $f \mathcal{L} g$, $u \in eSf$ and $v \in gSh$), then*

$$\rho(e, u, f)\rho(g, v, h) = \rho(e, uv, h).$$

In particular, $\mathcal{L}(S)$ is a regular category.

Proof It is clear that identity mapping and set-inclusions of left ideals are morphisms in the sense of the definition above (cf. Equation (9)). It is therefore clear that $\mathcal{L}(S)$ is a category with subobjects. Let $\rho \in \mathcal{L}(S)(Se, Sf)$. If $u = e\rho$, then it easy to see that $\rho = \rho_u | Se$ and that $Su = \rho_u(Se)$. If ρ' denote the morphism of Se to Su determined by restricting ρ_u to Se, then ρ' is surjective and hence an epimorphism in $\mathcal{L}(S)$ and $\rho = \rho' j^{Se}_{Su}$. Hence $\mathcal{L}(S)$ is a category with factorization property.

It is clear that for any $u \in eSf$, $\rho(e, u, f) = \rho_u | Se$ is a morphism from Se to Sf. Conversely, the proof above shows that every morphism in $\mathcal{L}(S)$ is of this form. Now if $\rho(e, u, f) = \rho(e, v, f)$, then

$$u = eu = e\rho(e, u, f) = e\rho(e, v, f) = ev = v$$

and so the map $\rho(e, u, f) \mapsto u$ is a bijection. This proves (a).

To prove (b), we note that if $\rho(e, u, f) = \rho(e', v, f')$ then domains (and codomains) of $\rho(e, u, f)$ and $\rho(e', v, f')$ coincide . Hence $e \mathcal{L} e'$ and $f \mathcal{L} f'$. Also we have

$$v = e'v = e'\rho(e', v, f') = e'\rho(e, u, f) = e'u$$

by (a). Conversely $e'u \in e'Sf'$ if $u \in eSf$, $e \mathcal{L} e'$ and $f \mathcal{L} f'$. Also, for $s \in Se = Se'$, $s\rho(e, e'u, f') = se'u = su = s\rho(e, u, f)$.

If $\rho(e, u, f)$ and $\rho(g, v, h)$ are composable, then $Sf = Sg$ and hence $f \mathcal{L} g$. Now, $uv \in eSh$ and for $s \in eS$, we have

$$s\rho(e, u, f)\rho(g, v, h) = suv = s\rho(e, uv, h).$$

Hence (c) follows.

We have already noted that $\mathcal{L}(S)$ is a category with factorization property. Let $\rho(e, u, f)$ be a morphism in $\mathcal{L}(S)$ so that $u \in eSf$. Since S is regular, it is easy to see that u has an inverse $u' \in fSe$ so that $\rho(f, u', e)$ is a morphism such that the compositions $\rho(e, u, f)\rho(f, u', e)$ and $\rho(f, u', e)\rho(e, u, f)$ exists. Moreover,

$$\begin{aligned} \rho(e, u, f)\rho(f, u', e)\rho(e, u, f) &= \rho(e, uu'u, f) && \text{by (c)} \\ &= \rho(e, u, f) && \text{since } u' \text{ is an inverse of } u. \end{aligned}$$

This proves that $\mathcal{L}(S)$ is a regular category. □

Remark 5 Statement (a) above says that each morphism $\rho\colon Se \to Sf$ in $\mathcal{L}(S)$ may be uniquely represented in the form $\rho(e, u, f)$ with $u \in eSf$. However, this representation is not natural in the sense that for each $e' \in E(L_e)$, there is a $u_1 \in e'Sf$ such that $\rho = \rho(e', u_1, f)$ (cf. (b)). This implies that it is impossible to identify the hom-set $\mathcal{L}(S)(Se, Sf)$ with the set eSf in a natural fashion.

Remark 6 Note that if S^{op} denote the semigroup on the set S with binary operation $\circ$ defined by $x \circ y = yx$ for all $x, y \in S$, then it is easy to see that

(10) $$\mathcal{R}(S) = \mathcal{L}(S^{op}) \quad \text{and} \quad \mathcal{L}(S) = \mathcal{R}(S^{op})$$

It follows that to every statement which holds for $\mathcal{L}(S)$ (or $\mathcal{R}(S)$) there is a corresponding dual statement which holds for $\mathcal{R}(S)$ (respectively $\mathcal{L}(S)$). Thus dual of Lemma 12 holds for the category $\mathcal{R}(S)$. In fact, let

$$\lambda(e, u, f) = \lambda_u | eS \quad \text{where} \quad e, f \in E(S);\ u \in fSe$$

Then $\mathcal{R}(S)$ is a regular category satisfying the following dual properties:

(a)* For every $e, f \in E(S)$ and $u \in fSe$, $\lambda(e,u,f) \in \mathcal{L}(S)(Se, Sf)$. Moreover, the map $\lambda(e,u,f) \mapsto u$ is a bijection of $\mathcal{R}(S)(eS, fS)$ onto eSf.

(b)* $\lambda(e,u,f) = \lambda(e',v,f')$ if and only if $e \,\mathcal{R}\, e'$, $f \,\mathcal{R}\, f'$, $u \in fSe$, $v \in f'Se'$ and $v = ue'$.

(c)* If $\lambda(e,u,f)$ and $\lambda(g,v,h)$ are composable morphisms, then

$$\lambda(e,u,f)\lambda(g,v,h) = \lambda(e,vu,h).$$

Normally we shall omit the explicit statements of the duals since they can be derived from the original using Equation (10).

In the following, to avoid repetitions, we shall assume that whenever a morphism $\rho \in \mathcal{L}(S)$ is represented as $\rho(e,u,f)$, we shall assume that $e, f \in E(S)$ and $u \in eSf$ as in the statement (a) of Lemma 12. Dually when $\lambda \in \mathcal{R}(S)$ is represented as $\lambda(e,u,f)$ we will assume that e, f and u are as in statement (a)* of the remark above. Next proposition gives some properties of morphisms of $\mathcal{L}(S)$.

Proposition 13 *Let $\rho = \rho(e,u,f)\colon Se \to Sf$ be a morphism in $\mathcal{L}(S)$. We have the following.*

(a) *The morphism $\rho(e,u,f)$ is a monomorphism if and only if $\rho(e,u,f)$ is injective and this is true if and only if $e \,\mathcal{R}\, u$. In this case $x \,\mathcal{R}\, x\rho$ for all $x \in Se$.*

(b) *$\rho(e,u,f)$ is an epimorphism if and only if it is surjective and this is true if and only if $u \,\mathcal{L}\, f$.*

(c) *Se and Sf are isomorphic in $\mathcal{L}(S)$ if and only if $e \,\mathcal{D}\, f$. In this case, there is a bijection between the set of all isomorphisms of Se onto Sf and the $\mathcal{H}$-class* ~~$L_e \cap R_f$.~~ $R_e \cap L_f$

(d) *If $Se \subseteq Sf$, then $j_{Se}^{Sf} = \rho(e,e,f)$ and $\rho\colon Sf \to Se$ is a retraction if and only if $\rho = \rho(f,g,e)$ for some $g \in E(L_e) \cup \omega(f)$. In particular, $\rho(f,fe,e)\colon Sf \to Se$ is a retraction.*

Proof (a) Suppose that $\rho = \rho(e,u,f)$ is monomorphism. Since $\mathcal{L}(S)$ is regular by Corollary II.10, ρ has a right inverse $\rho(f,u',e)\colon Sf \to Se$ so that $\rho\rho(f,u',e) = 1_{Se}$. This implies that ρ is injective and by Lemma 12(a) and (c), $e = uu'$. Since $eu = u$ this implies that $e \,\mathcal{R}\, u$. Since $\mathcal{R}$ is a left congruence it follows that $x \mathcal{R} xu = x\rho$ for all $x \in Se$. Conversely, if $e \mathcal{R} u$, then there is an inverse $u' \in fSe$ of u such that $e = uu'$. Then it follows as before that $\rho\rho(f, u', e) = 1_{Se}$ from which we conclude that ρ is injective and is a monomorphism. This proves (a).

(b) Suppose that ρ is an epimorphism Since $\mathcal{L}(S)$ is regular by Corollary II.10, ρ has a left inverse $\rho(f, u', e)\colon Sf \to Se$ so that $\rho(f, u', e)\rho = 1_{Sf}$. This implies that ρ is surjective and by Lemma 12(a) and (c), $f = u'u$. Since $uf = u$ this implies that $e \mathcal{L} u$. Conversely, if $f \mathcal{L} u$, then there is an inverse $u' \in fSe$ of u such that $f = u'u$. Then it follows as before that $\rho(f, u', e)\rho = 1_{Sf}$ from which we conclude that ρ is surjective and is an epimorphism. This proves (b).

(c) If $\rho(e, u, f)$ is an isomorphism, then $u \in R_e \cap L_f$ by (a) and (b) and hence $e \mathcal{D} f$ On the other hand, if $e \mathcal{D} f$, then $R_e \cap L_f \neq \emptyset$ and for any $u \in R_e \cap L_f$, $\rho(e, u, f)\colon Se \to Sf$ is an isomorphism by (a) and (b). It is also clear that the bijection of Lemma 12(a) maps the set of isomorphisms of $\mathcal{L}(S)(Se, Sf)$ onto the $\mathcal{H}$–class $R_e \cap L_f$.

(d) If $Se \subseteq Sf$ then it is clear that $\rho(e, e, f)$ acts as identity on elements of Se and so $\rho(e, e, f) = j^{Sf}_{Se}$ Let $\rho\colon Sf \to Se$ be a retraction. Then by Lemma 12, $\rho = \rho(f, u, e)$ for some $u \in fSe$ and $\rho(e, e, f)\rho(f, u, e) = 1_{Se}$. Since $u \in Se$, we therefore have

$$u = u\rho(e, e, f)\rho(f, u, e) = u\rho(e, eu, e) = u(eu) = u^2$$

by Lemma 12(c) and by Lemma 12(a) we obtain $eu = e$. This together with the fact that $u \in fSe$ implies that $u \in E(L_e) \cap \omega(f)$. Conversely if $g \in E(L_e) \cap \omega(f)$, then, using Lemma 12(c), we obtain $\rho(e, e, f)\rho(f, g, e) = \rho(e, e, e) = 1_{Se}$. Since $e\ \omega^l\ f$, from the biorder properties of $E(S)$, we have $e \mathcal{L} fe\ \omega\ f$ and hence $\rho(f, fe, e)\colon Sf \to Se$ is a retraction by the above. □

Since $\mathcal{L}(S)$ is regular, by Proposition II.9 every morphism has a normal factorization. The proposition above can be used to compute all the normal factorizations of a given morphism as follows.

Corollary 14 *Let $\rho = \rho(e, u, f)$ be a morphismin $\mathcal{L}(S)$ and fix $h \in E(L_u)$. Then the set $E(R_u) \cap \omega(e) \neq \emptyset$ and for each $g \in E(R_u) \cap \omega(f)$, ρ has a normal factorization in $\mathcal{L}(S)$ given by the equation*

$$\rho = \rho(e, g, g)\rho(g, u, h)\rho(h, h, f).$$

Moreover, every normal factorization of ρ has a representation of this form.

Proof Let $g' \in E(R_u)$. Then $g' \in eS$ and so $g' \,\omega^r\, e$. Therefore $g' \,\mathcal{R}\, g'e \,\omega\, e$ and so $E(R_u) \cap \omega(f) \neq \emptyset$. Now let $g \in E(R_u) \cap \omega(f)$. Then by Proposition 13(c) and (d), $\rho(e,g,g)\colon Se \to Sg$ is a retraction, $\rho(g,u,h)\colon Sg \to Sh$ is an isomorphism and $\rho(h,h,f)\colon Sh \to Sf$ is the inclusion. Moreover, by Lemma 12(c),

$$\rho(e,g,g)\rho(g,u,h)\rho(h,h,f) = \rho(e,guh,f) = \rho(e,u,f).$$

Suppose now that $\rho = \varrho\rho j$ is any normal factorization. Then by Proposition 13(d), we can represent $\varrho = \rho(e,g,g')$ where $g' \,\mathcal{L}\, g \,\omega\, e$; also by Lemma 12(b), $\rho(e,g,g') = \rho(e,g,g)$. Since ϱ and ρ are composable and since ρ is an isomorphism, we can represent $\rho = \rho(g,v,h)$ with $v \in R_g \cap L_h$ by Lemma 12(c) and Proposition 13(c). Similarly the composite ρj exists and so, $j = \rho(h,h.f)$. Thus the given normal factorization can be written as

$$\rho = \rho(e,g,g)\rho(g,v,h)\rho(h,h,f).$$

Now using Lemma 12(a) and (c) we get

$$u = e\rho(e,u.f) = e\rho(e,g,g)\rho(g,v,h)\rho(h,h,f) = egvh = v.$$

Hence the given normal factorization is in the required form. □

Next lemma describes certain normal cones in $\mathcal{L}(S)$.

Lemma 15 *Let $a \in S$ and $f \in E(L_a)$. Then the map*

$$\rho^a(Se) \;=\; \rho(e,ea,f)$$

is a normal cone in $\mathcal{L}(S)$ with vertex Sa and such that

$$M\rho^a = \{\, Se : e \in E(R_a)\,\}.$$

Moreover, ρ^a is an idempotent in $T\mathcal{L}(S)$ if and only if $a \in E(S)$.

Proof We first observe that ρ^a defined in the statement is a well-defined map. For if $e' \,\mathcal{L}\, e$, then by Lemma 12(b),

$$\rho(e,ea,f) = \rho(e',e'ea,f) = \rho(e',e'a,f).$$

If $Se' \subseteq Se$, then we have

$$\begin{aligned} j_{Se'}^{Se}\rho^a(Se) &= \rho(e', e', e)\rho(e, ea, f) && \text{by Proposition 13(d)} \\ &= \rho(e', e'ea, f) && \text{by Lemma 12(c)} \\ &= \rho(e', e'a, f) = \rho^a(Se'). \end{aligned}$$

Hence $\rho^a \colon \boldsymbol{\sigma}^*C \xrightarrow{\cdot} Sa = Sf$. Therefore to prove that ρ^a is a normal cone, we must show that $\rho^a(Se)$ is an isomorphism for some $Se \in \boldsymbol{v}\mathcal{L}(S)$. Since S is regular, $E(R_a) \neq \emptyset$. If $e \in E(R_a)$, then by Lemma 12(c), $\rho^a(Se) = \rho(e, ea, f) = \rho(e, a, f)$ is an isomorphism. It follows that ρ^a is a normal cone and that $Se \in M\rho^a$ for all $e \in E(R_a)$. Suppose that $Se' \in M\rho^a$. Then $\rho(e', e'a, f)$ is an isomorphism and so $e' \ \mathcal{R} \ e'a \ \mathcal{L} \ f \ \mathcal{L} \ a$. Hence by Clifford-Miller theorem $L_{e'} \cap R_a$ contains an idempotent e (See [1]). This proves that a principal left ideal belongs to $M\rho^a$ if and only if it is generated by an idempotent in R_a. Suppose that $a = e \in E(S)$. Then the component of ρ^e at $Se = Sf$ is $\rho(e, e, f) = 1_{Se}$ and so by Theorem 2(a), ρ^e is an idempotent. Conversely, if ρ^a is an idempotent then $\rho^a(Sf) = 1_{Sf}$ and so $\rho(f, fa, f) = \rho(f, f, f)$ which implies by Lemma 12(a) that $fa = f$. Since $f \ \mathcal{L} \ a$, this gives $a^2 = (af)a = a(fa) = af = a$. □

3.2 Representations of S by normal cones

Recall that the *right-regular representation* of the semigroup S is the homomorphism $\rho \colon a \mapsto \rho_a$ of S into the *full transformation semigroup* $\mathcal{T}_S$ and we write S_ρ for the image of ρ so that $\rho \colon S \to S_\rho$ is a surjective homomorphism. S is said to be *right reductive* if ρ is injective (cf. [1]). Dually the *left-regular antirepresentation* is the antihomomorphism $\lambda \colon a \mapsto \lambda_a$ of S into $\mathcal{T}_S$ and S_λ denotes the image of this homomorphism. S is said to be *left-reductive* if λ is injective. It is useful to recall further that S is *reductive* if both ρ and λ are faithful (injective) and *weakly reductive* if the map $(\rho, \lambda) \colon a \mapsto (\rho_a, \lambda_a)$ of S into $\mathcal{T}_S \times \mathcal{T}_S^{op}$ is faithful.

We now prove the principal result of this section.

Theorem 16 *Let S be a regular semigroup. Then $\mathcal{L}(S)$ is a normal category. Moreover there exists a homomorphism $\bar{\rho} \colon S \to T\mathcal{L}(S)$ and an injective* *homomorphism* $\phi\colon S_\rho \to T\mathcal{L}(S)$ *such that the following diagram commutes:*

$$(11)\qquad \begin{array}{ccc} S & \xrightarrow{\ \rho\ } & S_\rho \\ \| & & \downarrow \phi \\ S & \xrightarrow[\bar{\rho}]{} & T\mathcal{L}(S) \end{array}$$

In particular S *is isomorphic to a subsemigroup of* $T\mathcal{L}(S)$ *if and only if* S *is right reductive.*

Proof The last statement in Lemma 15 shows that for each $Se \in \boldsymbol{v}\mathcal{L}(S)$, ρ^e is a normal cone with vertex Se. Hence $\mathcal{L}(S)$ is normal by Definition 2 (see also Theorem 2(b) and Remark 3).

For $a \in S$ define
$$\phi(\rho_a) = \rho^a.$$
If $\rho_a = \rho_b$, then $Sa = Sb$ and so the vertices of ρ^a and ρ^b are equal. Let $Se \in \boldsymbol{v}\mathcal{L}(S)$. If $x \in Se$, then
$$x\rho^a(Se) = x\rho(e, ea, f) = xea = xa = x\rho_a = x\rho_b = x\rho^b(Se)$$
and so $\rho^a = \rho^b$. Similarly, if $\rho^a = \rho^b$ it is easy to see that for all $x \in S$, $xa = xb$ and so $\rho_a = \rho_b$. Hence $\phi\colon S_\rho \to \mathcal{L}(S)$ is injective. Now for $a, b \in S$ and $Se \in \boldsymbol{v}\mathcal{L}(S)$, by Equation (3), we have
$$\rho^a\rho^b(Se) = \rho^a(Se)\rho^b(Sa)^\circ = \rho(e, ea, f)\rho(f, fb, g)^\circ$$
if $f \in E(L_a)$ and $g \in E(L_b)$. Now, if $h \in E(L_{fb})$, then by Corollary 14, we have, $\rho(f, fb, g)^\circ = \rho(f, fb, h)$. Since $fb\,\mathcal{L}\,ab$, using Lemma 12(c), we obtain
$$\rho^a\rho^b(Se) = \rho(e, eafb, h) = \rho(e, eab, h) = \rho^{ab}(Se).$$
This proves that $\phi(\rho_a)\phi(\rho_b) = \phi(\rho_{ab})$ and hence ϕ is an injective homomorphism. If we set $\bar{\rho} = \rho\phi$ then $\bar{\rho}\colon S \to \mathcal{L}(S)$ is a homomorphism making the given diagram commute. The last statement follows from the fact that $\bar{\rho}$ is injective if and only if ρ is injective. $\square$

Remark 7 As already remarked (see Remark 6) dual of the result above holds for the category $\mathcal{R}(S)$. Thus $\mathcal{R}(S)$ is a normal category and there exists homomorphisms $\bar{\lambda}: S \to T\mathcal{R}(S)$ and $\phi^*: S_\lambda \to T\mathcal{R}(S)$ such that ϕ^* is injective and $\bar{\lambda} = \lambda\phi^*$. In particular S is anti-isomorphic to a subsemigroup of $\mathcal{R}(S)$ if and only if S is left reductive.

If S is a regular monoid then it is right (and left) reductive and so $\bar{\rho}$ is an injective homomorphism of S into $T\mathcal{L}(S)$. If γ is any normal cone in $T\mathcal{L}(S)$, then it is easy to show that γ is completely determined by the component of $\gamma(S1) = \gamma(S)$ and if this component is $\rho(1, 1a, f) = \rho(1, a, f) = \rho_a$ then $\gamma = \rho^a$. Thus $\bar{\rho}$ is an isomorphism.

Corollary 17 *If S is a regular semigroup with identity then S is isomorphic to $T\mathcal{L}(S)$.* □

If S does not have identity, then $S \subseteq S^1$ and by the corollary above, S^1 is isomorphic to $T\mathcal{L}(S^1)$ and hence S is isomorphic to a subsemigroup of $T\mathcal{L}(S^1)$. We can therefore state

Corollary 18 *Every regular semigroup S is isomorphic to a subsemigroup of TC for a suitable normal category C.* □

3.3 Isomorphism of C with $\mathcal{L}(TC)$

Recall (cf.Definition 2) that two normal categories are isomorphic (as categories with subobjects) if there is an isomorphism that preserves inclusions. Notice that in this case the isomorphism will also reflect inclusions. If S is a regular semigroup we have seen that $\mathcal{L}(S)$ is a normal category (cf. Theorem 16). We proceed to show that every normal category arises in this way.

Theorem 19 *Let C be a normal category. Define $\boldsymbol{F}$ on objects and morphisms of C as follows. For $c \in vC$, let*

$$\boldsymbol{vF}(c) = TC\epsilon$$

where $\epsilon \in E(TC)$ with $c_\epsilon = c$; and for $f \in C(c, d)$, let

$$\boldsymbol{F}(f) = \rho(\epsilon, \epsilon \star f^\circ, \epsilon') \tag{12}$$

where ϵ, $\epsilon' \in E(T\mathcal{C})$ *with* $c_\epsilon = c$ *and* $c_{\epsilon'} = d$. *Then* $\boldsymbol{F}\colon \mathcal{C} \to \mathcal{L}(T\mathcal{C})$ *is an isomorphism of normal categories.*

Proof By Proposition 5, the map $\boldsymbol{vF}$ defined above is an order isomorphism of $\boldsymbol{v}\mathcal{C}$ with the partially ordered set $\boldsymbol{v}\mathcal{L}(T\mathcal{C})$ of all principal left ideals of $T\mathcal{C}$. Since $\mathcal{C}$ is normal, by Remark 3, given any $f \in \mathcal{C}(c,d)$, there exist ϵ, $\epsilon' \in E(T\mathcal{C})$ with $c = c_\epsilon$ and $d = c_{\epsilon'}$. Also by Proposition 5 and Lemma 12(b), the morphism $\rho(\epsilon, \epsilon \star f^\circ, \epsilon')$ is independent of the choice of ϵ and ϵ' satisfying the conditions above. Hence the mapping $\boldsymbol{F}$ is well-defined by (12). To show that $\boldsymbol{F}$ is a functor, first observe that for $f, g \in \mathcal{C}$ if $\boldsymbol{F}(f) = \rho(\epsilon, \epsilon \star f^\circ, \epsilon')$ and $\boldsymbol{F}(g) = (\epsilon_1, \epsilon_1 \star g^\circ, \epsilon'')$ then fg exists in $\mathcal{C}$ if and only if $\epsilon' \mathcal{L} \epsilon_1$ and hence by Lemma 12(b), we may represent $\boldsymbol{F}(g)$ as $\rho(\epsilon', \epsilon' \star g^\circ, \epsilon'')$. Therefore by Lemma 12(c) we have

$$\boldsymbol{F}(f)\boldsymbol{F}(g) = \rho(\epsilon, (\epsilon \star f^\circ) \cdot (\epsilon' \star g^\circ), \epsilon'').$$

Now if $c_1 = \operatorname{im} f = \operatorname{im} f^\circ$, we have

$$\begin{aligned}
(\epsilon \star f^\circ) \cdot (\epsilon' \star g^\circ) &= (\epsilon \star f^\circ) \star \big((\epsilon' \star g^\circ)(c_1)\big)^\circ, && \text{by (3)} \\
&= (\epsilon \star f^\circ) \star (j_{c_1}^{c} g^\circ)^\circ && \text{since } c_1 \subseteq c = c_{\epsilon'} \\
&= (\epsilon \star f^\circ) \star (j_{c_1}^{c} g)^\circ \\
&= \epsilon \star (fg)^\circ. && \text{by Lemma 1}
\end{aligned}$$

Hence $\boldsymbol{F}(f)\boldsymbol{F}(g) = \boldsymbol{F}(fg)$. From the definition of $\boldsymbol{F}$, it is clear that if $c' \subseteq c$, $c_\epsilon = c$ and $c_{\epsilon'} = c'$, then

$$\boldsymbol{F}(j_{c'}^{c}) = \rho(\epsilon', \epsilon', \epsilon) = j_{(T\mathcal{C})\epsilon'}^{(T\mathcal{C})\epsilon}$$

by Proposition 13(d). In particular, $\boldsymbol{F}$ preserves identities. Hence $\boldsymbol{F}$ is an inclusion preserving functor.

To show that $\boldsymbol{F}$ is an isomorphism, it remains to show that $\boldsymbol{F}$ is fully-faithful. Since every morphism in $\mathcal{C}$ has a unique canonical factorization (cf. Proposition II.12), it follows that the map $f \mapsto f^\circ$ is a bijection of $\mathcal{C}(c,d)$ onto the set of epimorphisms h with $\operatorname{dom} h = c$ and $\operatorname{cod} h \subseteq d$. Since every cone in $\epsilon(T\mathcal{C})\epsilon'$ is uniquely representable in the form $\epsilon \star f^\circ$ with $f \in \mathcal{C}(c_\epsilon, c_{\epsilon'})$ (which is an immediate consequence of Lemma 3), there is a bijection between $\mathcal{C}(c_\epsilon, c_{\epsilon'})$ and $\epsilon T\mathcal{C}\epsilon'$. Hence by Lemma 12(a), the map $f \mapsto \rho(\epsilon, \epsilon \star f^\circ, \epsilon')$ is a bijection of $\mathcal{C}(c_\epsilon, c_{\epsilon'})$ onto $\boldsymbol{\mathcal{L}}(T\mathcal{C})(T\mathcal{C}\epsilon, T\mathcal{C}\epsilon')$ and this proves that $\boldsymbol{F}$ is fully-faithful. □

The following corollary is an immediate consequence of Theorems 16 and 19.

Corollary 20 *A small regular category $\mathcal{C}$ is normal if and only if $\mathcal{C}$ is isomorphic to $\mathcal{L}(S)$ for some regular semigroup S.* □

Since $\mathcal{R}(S) = \mathcal{L}(S^{op})$, the corollary remains valid if we replace $\mathcal{L}(S)$ in the statement by $\mathcal{R}(S)$.

Remark 8 Since $\mathcal{L}(S)$ is a subcategory of **Set** and since $\boldsymbol{F}$ is inclusion preserving, the Corollary 20 above implies that every normal category possess an inclusion-preserving functor $U: \mathcal{C} \to \mathbf{Set}$. Recall that a category is concrete if it has a faithful set-valued functor U. We shall say that a category $\mathcal{C}$ with subobjects is a *concrete category with subobjects* if U is, in addition, inclusion preserving so that $U: \mathcal{C} \to \mathbf{Set}$ is an embedding of categories with subobjects. Thus every normal category is concrete category with subobjects. Thus we may assume, without loss of generality, that a normal category $\mathcal{C}$ is a small subcategory of sets. Also, since **Set** is complete and co-complete, $X = \varprojlim \boldsymbol{\sigma}^* U$ exists. Adjoining X as the largest object in $\mathcal{C}$ (with inclusions of objects in X as the corresponding components of the limiting cone), we obtain a representation of $T\mathcal{C}$ as a subsemigroup of $\mathcal{T}_X$.

4 NORMAL DUAL

If $\mathcal{C}$ is a normal category then (by Theorem 2) $T\mathcal{C}$ is a regular semigroup. If for $\gamma \in T\mathcal{C}$, $H(\gamma; -)$ is defined by Equation (6), then by Proposition 7, the partially ordered set $T\mathcal{C}/\ \mathcal{R}$ is order-isomorphic to the set of functors $\{ H(\gamma; -) : \gamma \in T\mathcal{C} \}$ ordered by functorial inclusion (see Equations (7i) and (7ii)). Since $H(\gamma; -) = H(\gamma'; -)$ if and only if $\gamma \mathcal{R} \gamma'$ (again by Proposition 7 since $\mathcal{C}$ is normal) and since $T\mathcal{C}$ is regular, for every $\gamma \in T\mathcal{C}$ we have $H(\gamma; -) = H(\epsilon; -)$ for some $\epsilon \in E(T\mathcal{C})$.

4.1 Definition of the normal dual

Definition 4 *If $\mathcal{C}$ is a normal category, then the* normal dual *of $\mathcal{C}$, denoted by $\boldsymbol{N}^*\mathcal{C}$, is the full subcategory of $\mathcal{C}^*$ with*

(13a) $$\boldsymbol{v}\boldsymbol{N}^*\mathcal{C} = \{ H(\epsilon; -) : \epsilon \in E(T\mathcal{C}) \}.$$

We have already noted that $\mathcal{C}^*$ is a category with subobjects with respect to functorial inclusion and so, with the subobject relation induced from $\mathcal{C}^*$,

$\mathcal{N}^*\mathcal{C}$ also becomes a category with subobjects. The following lemma describes morphisms of $\mathcal{N}^*\mathcal{C}$ in terms of those of $\mathcal{C}$.

Lemma 21 *To every morphism $\sigma\colon H(\epsilon;-) \to H(\epsilon';-)$ morphism in $\mathcal{N}^*\mathcal{C}$, there is a unique $\widehat{\sigma}\colon c_{\epsilon'} \to c_\epsilon$ in $\mathcal{C}$ such that the following diagram commutes.*

$$\begin{array}{ccc} H(\epsilon;-) & \xrightarrow{\eta_\epsilon} & \mathcal{C}(c_\epsilon,-) \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\mathcal{C}(\widehat{\sigma},-)} \\ H(\epsilon';-) & \xrightarrow[\eta_{\epsilon'}]{} & \mathcal{C}(c_{\epsilon'},-) \end{array}$$

In this case, the component of the natural transformation σ at $c \in v\mathcal{C}$ is the map given by

$$\text{(13b)} \qquad \sigma(c)\colon \epsilon \star f^\circ \mapsto \epsilon' \star (\widehat{\sigma} f)^\circ.$$

In particular, σ is the inclusion $H(\epsilon;-) \subseteq H(\epsilon';-)$ if and only if

$$\epsilon = \epsilon' \star \widehat{\sigma}.$$

Moreover, the map

$$\sigma \mapsto \widehat{\sigma}$$

is a bijection of $\mathcal{N}^*\mathcal{C}(H(\epsilon;-), H(\epsilon';-))$ onto $\mathcal{C}(c_{\epsilon'}, c_\epsilon)$.

Proof Clearly $\eta_\epsilon^{-1}\sigma\eta_{\epsilon'}$ is a natural transformation of $\mathcal{C}(c_\epsilon,-)$ to $\mathcal{C}(c_{\epsilon'},-)$. Since the contravariant Yoneda embedding $H^{\mathcal{C}}$ is fully-faithful, there is a unique morphism $\widehat{\sigma}\colon c_{\epsilon'} \to c_\epsilon$ making the diagram above commutes. The expression for components of σ is obtained by chasing the commutative diagram for σ. If $\epsilon = \epsilon' \star \widehat{\sigma}$ then it is easy to check that the map $\sigma(c)$ defined by Equation (13b) is an inclusion for each $c \in v\mathcal{C}$. Conversely, if σ is the inclusion, then $\epsilon \in H(\epsilon'; c_\epsilon)$ and so $\epsilon = \epsilon' \star h$ for some epimorphism $h\colon c_{\epsilon'} \to c_\epsilon$ and it is easy to see that $\eta_\epsilon \mathcal{C}(h,-)\eta_{\epsilon'}^{-1}$ is the inclusion $H(\epsilon;-) \subseteq H(\epsilon';-)$. Comparing this with the commutative diagram above, we see that $\mathcal{C}(\widehat{\sigma},-) = \mathcal{C}(h,-)$. Since $H^{\mathcal{C}}$ is fully-faithful, this implies that $\widehat{\sigma} = h$. The last statement is also an immediate consequence of the fact that $H^{\mathcal{C}}$ is fully-faithful. □

4.2 Normality of $N^*\mathcal{C}$

We proceed to show that $N^*\mathcal{C}$ is a normal category. This will be done by showing that $N^*\mathcal{C}$ is isomorphic to $\mathcal{R}(T\mathcal{C})$ which is normal by the dual of Theorem 16. We need the following lemmas.

Lemma 22 *Let ϵ, $\epsilon' \in E(T\mathcal{C})$. Then the map*

$$\lambda(\epsilon, \gamma, \epsilon') \mapsto \tilde{\gamma}$$

where $\gamma \in \epsilon'(T\mathcal{C})\epsilon$ and

$$\tilde{\gamma} = \gamma(c_{\epsilon'})j_{c_\gamma}^{c_\epsilon} \tag{14}$$

is a bijection of $\mathcal{R}(T\mathcal{C})(\epsilon(T\mathcal{C}), \epsilon'(T\mathcal{C}))$ onto $\mathcal{C}(c_{\epsilon'}, c_\epsilon)$.

Proof By the dual of Lemma 12(a) (see Remark 6(a)*), there is a bijection between $\mathcal{R}(T\mathcal{C})(\epsilon(T\mathcal{C}), \epsilon'(T\mathcal{C}))$ and $\epsilon' T\mathcal{C}\epsilon$ and so it is sufficient to show that the mapping $\gamma \mapsto \tilde{\gamma}$ is a bijection of $\epsilon'(T\mathcal{C})\epsilon$ onto $\mathcal{C}(c_{\epsilon'}, c_\epsilon)$. First, note that for $\gamma \in \epsilon'(T\mathcal{C})\epsilon$, $\gamma \cdot \epsilon = \gamma$ and so by Proposition 5, $c_\gamma \subseteq c_\epsilon$ and hence $\tilde{\gamma}$ is well-defined by (14). Also, $\epsilon' \cdot \gamma = \gamma$ and so, by Lemma 3, there is a unique epimorphism f with $\gamma = \epsilon' \star f$. Then $\gamma(c_{\epsilon'}) = f$ and hence $\gamma(c_{\epsilon'})$ is an epimorphism; in particular, the mapping $\gamma \mapsto \tilde{\gamma}$ is injective. It is surjective because for any $f \in \mathcal{C}(c_{\epsilon'}, c_\epsilon)$, we have, $\gamma = \epsilon' \star f^\circ \in \epsilon'(T\mathcal{C})\epsilon$ and $\tilde{\gamma} = f$. □

Note that the morphism $\tilde{\gamma}$ defined by Equation (14) above depend not only on γ but also on the idempotents ϵ and ϵ'. The next lemma deals with this dependance.

Lemma 23 *If $\epsilon \mathrel{\mathcal{R}} \epsilon_1$ and $\epsilon' \mathrel{\mathcal{R}} \epsilon'_1$ and if $\tilde{\gamma} \in \mathcal{C}(c_{\epsilon'}, c_\epsilon)$ and $\tilde{\gamma}_1 \in \mathcal{C}(c_{\epsilon'_1}, c_{\epsilon_1})$ are morphisms defined by Equation (14) corresponding to the representations $\lambda = \lambda(\epsilon, \gamma, \epsilon') = \lambda(\epsilon_1, \gamma_1, \epsilon'_1)$ of the morphism $\lambda \in \mathcal{R}(T\mathcal{C})(\epsilon T\mathcal{C}, \epsilon' T\mathcal{C})$, then we have the following commutative diagram:*

$$\begin{array}{ccc} c_{\epsilon'_1} & \xrightarrow{\tilde{\gamma}_1} & c_{\epsilon_1} \\ {\scriptstyle \epsilon'_1(c_{\epsilon'})}\uparrow & & \uparrow{\scriptstyle \epsilon_1(c_\epsilon)} \\ c_{\epsilon'} & \xrightarrow[\tilde{\gamma}]{} & c_\epsilon \end{array} \tag{15}$$

Proof By Remark 6(a)* and (b)*, $\gamma \in \epsilon'(TC)\epsilon$, $\gamma_1 \in \epsilon'_1(TC)\epsilon_1$ and $\gamma_1 = \gamma \cdot \epsilon_1$. Since $\gamma_1 \in \epsilon' TC \epsilon_1$, we have $\epsilon' \star \gamma_1(c_{\epsilon'}) = \gamma_1 = \epsilon'_1 \star \gamma_1(c_{\epsilon'_1})$ and so

$$\begin{aligned}
\tilde{\gamma}_1 &= \gamma_1(c_{\epsilon'_1}) j^{c_{\epsilon_1}}_{c_{\gamma_1}} && \text{byEquation (14)}\\
&= \epsilon'(c_{\epsilon'_1})\gamma_1(c_{\epsilon'}) j^{c_{\epsilon_1}}_{c_{\gamma_1}} && \text{by Equation (2)}\\
&= \epsilon'(c_{\epsilon'_1})\gamma(c_{\epsilon'})(\epsilon_1(c_\gamma))^{\circ} j^{c_{\epsilon_1}}_{c_{\gamma_1}} && \text{using Equation (3)}\\
&= \epsilon'(c_{\epsilon'_1})\gamma(c_{\epsilon'})\epsilon_1(c_\gamma) && \text{since } im\, \epsilon_1(c_\gamma) = c_{\gamma_1}\\
&= \epsilon'(c_{\epsilon'_1})\gamma(c_{\epsilon'}) j^{c_\epsilon}_{c_\gamma} \epsilon_1(c_\epsilon)\\
&= \epsilon'(c_{\epsilon'_1})\tilde{\gamma}\epsilon_1(c_\epsilon).
\end{aligned}$$

Again from $\epsilon' \mathcal{R} \epsilon'_1$ it follows that $\epsilon'_1(c_{\epsilon'}) = \left(\epsilon'(c_{\epsilon'_1})\right)^{-1}$. Hence the given diagram commutes. □

Lemma 24 *Let $\gamma \in \epsilon' TC \epsilon$ and $\gamma' \in \epsilon'' TC \epsilon'$. Assume that $\tilde{\gamma}$, $\tilde{\gamma}'$ and $\widetilde{\gamma' \cdot \gamma}$ are morphisms defined by Equation (14). Then $\widetilde{\gamma' \cdot \gamma} = \tilde{\gamma}'\tilde{\gamma}$.*

Proof Clearly $c_{\gamma' \cdot \gamma} \subseteq c_\gamma \subseteq c_\epsilon$. Also

$$\begin{aligned}
\widetilde{\gamma' \cdot \gamma} &= (\gamma' \cdot \gamma)(c_{\epsilon''}) j^{c_\epsilon}_{c_{\gamma' \cdot \gamma}} && \text{by Equation (14)}\\
&= \gamma'(c_{\epsilon''})\left(\gamma(c_{\gamma'})\right)^{\circ} j^{c_\gamma}_{c_{\gamma' \cdot \gamma}} j^{c_\epsilon}_{c_\gamma} && \text{by Equation (3)}\\
&= \gamma'(c_{\epsilon''})\gamma(c_{\gamma'}) j^{c_\epsilon}_{c_\gamma}\\
&= \gamma'(c_{\epsilon''}) j^{c_{\epsilon'}}_{c_{\gamma'}} \gamma(c_{\epsilon'}) j^{c_\epsilon}_{c_\gamma}.
\end{aligned}$$

By Equation (14), this proves the required equality. □

Theorem 25 *Let C be a normal category. Define $\boldsymbol{G}$ on objects and morphisms of $\mathcal{R}(TC)$ as follows.*

$$v\boldsymbol{G}(\epsilon TC) = H(\epsilon; -)$$

and for $\lambda = \lambda(\epsilon, \gamma, \epsilon')\colon \epsilon TC \to \epsilon' TC$, *let* $G(\lambda)$ *be the natural transformation making the following diagram commutative.*

$$\begin{array}{ccc} H(\epsilon; -) & \xrightarrow{\eta_\epsilon} & \mathcal{C}(c_\epsilon, -) \\ {\scriptstyle G(\lambda)}\downarrow & & \downarrow{\scriptstyle \mathcal{C}(\tilde{\gamma}, -)} \\ H(\epsilon'; -) & \xrightarrow[\eta_{\epsilon'}]{} & \mathcal{C}(c_{\epsilon'}, -) \end{array} \tag{16}$$

Then $G\colon \mathcal{R}(TC) \to N^*\mathcal{C}$ *is an isomorphism of normal categories.*

Proof Observe that the definition of $G(\lambda)$ is equivalent to

$$G(\lambda) = \eta_\epsilon \mathcal{C}(\tilde{\gamma}, -)\eta_{\epsilon'}^{-1} \tag{17}$$

We first show that $G(\lambda)$ is independent of the representation for λ chosen to define it. Assume that $\lambda = \lambda(\epsilon, \gamma, \epsilon') = \lambda(\epsilon_1, \gamma_1, \epsilon_1')$. Then by Lemma 23 we get

$$\begin{aligned} \eta_{\epsilon_1}\mathcal{C}(\tilde{\gamma}_1, -)\eta_{\epsilon_1'}^{-1} &= \eta_{\epsilon_1}\mathcal{C}(\epsilon'(c_{\epsilon_1'}\tilde{\gamma}\epsilon_1(c_\epsilon), -)\eta_{\epsilon_1'}^{-1} \\ &= \eta_{\epsilon_1}\mathcal{C}(\epsilon_1(c_\epsilon), -)\mathcal{C}(\tilde{\gamma}, -)\mathcal{C}(\epsilon'(c_{\epsilon_1'}), -)\eta_{\epsilon_1'}^{-1} \\ &= \eta_\epsilon \mathcal{C}(\tilde{\gamma}, -)\eta_{\epsilon'}^{-1}. \end{aligned}$$

The last equality is a consequence of the fact that $\eta_{\epsilon_1}\mathcal{C}(\epsilon_1(c_\epsilon), -) = \eta_\epsilon$ by Corollary 9 since $\epsilon_1 = \epsilon \star \epsilon_1(c_\epsilon)$. Also since $\epsilon_1'(c_{\epsilon'}) = \epsilon'(c_{\epsilon_1'})^{-1}$, we similarly have $\eta_{\epsilon_1'}\mathcal{C}(\epsilon_1'(c_{\epsilon'}), -) = \eta_{\epsilon'}$. Hence for each morphism $\lambda\colon \epsilon TC \to \epsilon' TC$ in $\mathcal{R}(TC)$, $G(\lambda)$ is a well-defined natural transformation of $G(\epsilon TC) = H(\epsilon; -)$ to $G(\epsilon' TC) = H(\epsilon'; -)$.

To prove that G is a functor consider $\lambda = \lambda(\epsilon, \gamma, \epsilon')$ and $\lambda' = \lambda(\epsilon', \gamma', \epsilon'')$ so that $\lambda\lambda'$ exists. Then $\lambda\lambda' = \lambda(\epsilon, \gamma' \cdot \gamma, \epsilon'')$ by Remark 6(c)*. Hence by Lemma 24, we get

$$\begin{aligned} G(\lambda\lambda') &= \eta_\epsilon \mathcal{C}(\widetilde{\gamma' \cdot \gamma}, -)\eta_{\epsilon''}^{-1} \\ &= \eta_\epsilon \mathcal{C}(\tilde{\gamma}'\tilde{\gamma}, -)\eta_{\epsilon''}^{-1} \\ &= \eta_\epsilon \mathcal{C}(\tilde{\gamma}, -)\eta_{\epsilon'}^{-1}\eta_{\epsilon'}\mathcal{C}(\tilde{\gamma}', -)\eta_{\epsilon''}^{-1} \\ &= G(\lambda)G(\lambda'). \end{aligned}$$

Suppose that $\epsilon TC \subseteq \epsilon' TC$. Then by the dual of Proposition 13, we have, $\lambda = \lambda(\epsilon, \epsilon, \epsilon') = \jmath_{\epsilon TC}^{\epsilon' TC}$. Also, by Equation (14), the morphism $\tilde{\epsilon} = \varrho \colon c_{\epsilon'} \to c_\epsilon$ is a retraction. Now $H(\epsilon; -) \subseteq H(\epsilon'; -)$ and it is easy to see that the component of the natural transformation $\jmath_{H(\epsilon;-)}^{H(\epsilon';-)} \eta_{\epsilon'}(c_\epsilon)$ maps ϵ to ϱ. Similarly $\eta_\epsilon \mathcal{C}(\varrho, -)$ also maps ϵ to ϱ and hence by the Yoneda lemma, the following diagram commutes.

$$\begin{array}{ccc} H(\epsilon'; -) & \xrightarrow{\eta_{\epsilon'}} & \mathcal{C}(c_{\epsilon'}, -) \\ {\scriptstyle \jmath_{H(\epsilon;-)}^{H(\epsilon';-)}} \uparrow & & \uparrow {\scriptstyle \mathcal{C}(\varrho,-)} \\ H(\epsilon; -) & \xrightarrow[\eta_\epsilon]{} & \mathcal{C}(c_\epsilon, -) \end{array}$$

Comparing this with (16), we get

$$G(\jmath_{\epsilon TC}^{\epsilon' TC}) = \jmath_{H(\epsilon;-)}^{H(\epsilon';-)}$$

and so G is inclusion preserving. In particular G preserves identities and hence it is an inclusion preserving functor.

Suppose that $\sigma \colon H(\epsilon; -) \to H(\epsilon; -)$ be a natural transformation. Then by Lemma 21, there is a unique morphism $f \colon c_{\epsilon'} \to c_\epsilon$ such that the diagram (13b) commutes (f is the $\widehat{\sigma}$ of Lemma 21). By Lemma 22, there is a unique $\lambda(\epsilon, \gamma, \epsilon') \colon \epsilon TC \to \epsilon' TC$ such that $\tilde{\gamma} = f$. Hence by (16), we get $G(\lambda(\epsilon, \gamma, \epsilon')) = \sigma$ and so, G is full. If $\lambda = \lambda(\epsilon, \gamma, \epsilon')$ and $\lambda' = \lambda(\epsilon, \gamma', \epsilon')$ are morphisms such that $G(\lambda) = G(\lambda')$, then from (16) we see that $\mathcal{C}(\tilde{\gamma}, -) = \mathcal{C}(\tilde{\gamma'}, -)$ and again since the contra-variant representation $H_{\mathcal{C}}$ is fully-faithful, we conclude that $\tilde{\gamma} = \tilde{\gamma'}$. Then by Lemma 22, $\lambda = \lambda'$. Therefore G is fully-faithful. It follows from Proposition 7 that vG defined in the statement is an order-isomorphism of the partially ordered set of principal right ideals of TC onto the partially ordered set of functors $\{H(\epsilon; -) : \epsilon \in E(TC)\}$. This proves that G is an isomorphism of $\mathcal{R}(TC)$ onto $N^*\mathcal{C}$. □

CHAPTER IV

Cross-connections

If S is a regular semigroup, it follows from §III.3 that the categories $\mathcal{L}(S)$ and $\mathcal{R}(S)$ of left and right ideals of S are normal categories. In this chapter we consider the converse question: given two normal categories $\mathcal{C}$ and $\mathcal{D}$, under what conditions can we assert the existence of a regular semigroup S so that $\mathcal{C}$ and $\mathcal{D}$ are isomorphic (as normal categories) to $\mathcal{L}(S)$ and $\mathcal{R}(S)$ respectively. It follows from Theorems III.19 and III.25 that any normal category $\mathcal{C}$ and its dual $N^*\mathcal{C}$ are related in this way. We shall describe the relation that must exist between $\mathcal{C}$ and $\mathcal{D}$, in order that they are respectively isomorphic to the categories of left and right ideals of a regular semigroup, in terms of a functor from $\mathcal{D}$ to $N^*\mathcal{C}$. Such a functor (to be defined below) is called a cross-connection between these categories .

In §1, we study the relation between the categories of left and right ideals of a regular semigroup S: we show that there is a local isomorphism (see Definition 1) $\boldsymbol{\Gamma} S\colon \mathcal{R}(S) \to N^*\mathcal{L}(S)$, called the connection of S and a functor $\Delta S\colon \mathcal{L}(S) \to N^*\mathcal{R}(S)$, called the dual connection such that the set-valued bifunctor from $\mathcal{L}(S) \times \mathcal{R}(S)$ associated with $\boldsymbol{\Gamma} S$ and ΔS respectively (as described in §I.2) are naturally isomorphic (Theorems 2 and 4). In §2, we study the connection and the dual connection of the semigroup $T\mathcal{C}$ of normal cones in a normal category $\mathcal{C}$. Given any local isomorphism $\Gamma\colon \mathcal{D} \to N^*\mathcal{C}$ from a normal category $\mathcal{D}$ to the dual of $\mathcal{C}$, we show in §3 that there exists a local isomorphism $\Gamma^*\colon \mathcal{C} \to N^*\mathcal{D}$, called the dual of Γ, such that the associated bifunctors $\Gamma(-,-)$, $\Gamma^*(-,-)$ from $\mathcal{C} \times \mathcal{D}$ to **Set** are naturally isomorphic (Theorems 15 and 16). This leads to the definition of cross-connections (see Definition 5). Moreover, the connection $\boldsymbol{\Gamma} S$ of a regular semigroup S is a cross-connection and the dual connection ΔS of S is the dual of $\boldsymbol{\Gamma} S$ (Theorem 17). A cross-connection of a normal category $\mathcal{D}$ with $\mathcal{C}$ imply certain relations between objects and morphisms of $\mathcal{C}$ and $\mathcal{D}$. These relations are studied in §4. These investigations leads us, in particular, to the conclusion that the double dual of a cross-connection is itself which implies that the relation between a cross-connection and its dual is completely symmetric. Finally in §5 we show that with every cross-connection $\Gamma\colon \mathcal{D} \to N^*\mathcal{C}$ we can associate, in a natural fashion, a regular semigroup $\tilde{S}\Gamma$ such that $\mathcal{L}(\tilde{S}\Gamma)$ is isomorphic to $\mathcal{C}$ and $\mathcal{R}(\tilde{S}\Gamma)$ is isomorphic to $\mathcal{D}$ (Theorems 32 and 35). Moreover, every regular semigroup is isomorphic to one obtained in this way (Theorem 38).

1 THE CONNECTION OF A REGULAR SEMIGROUP

In this chapter, we shall assume that $\mathcal{C}$, $\mathcal{D}$, etc., stand for normal categories, unless indicated otherwise. Recall that for any category with subobjects and $c \in v\mathcal{C}$, $\langle c \rangle$ denotes the principal ideal of $\mathcal{C}$ generated by c (see Remark III.4).

1.1 The local isomorphisms F_ρ and F_λ

We begin by defining the concept of local isomorphisms.

Definition 1 *Let $\mathcal{C}$ and $\mathcal{D}$ be categories with subobjects (not necessarily normal). A functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be a* local isomorphism *if F is inclusion-preserving, fully-faithful and for each $c \in v\mathcal{C}$, $F|\langle c \rangle$ is an isomorphism of the ideal $\langle c \rangle$ onto $\langle F(c) \rangle$.*

Remark 1 It is clear from the definition above that, to check whether a given functor $F\colon \mathcal{C} \to \mathcal{D}$ is a local isomorphism, it is sufficient to check whether F has the following properties:

- F is fully-faithful;
- F is inclusion-preserving and
- vF induces an order isomorphism on each principal order ideal of $v\mathcal{C}$

Also, when F is a local isomorphism, the image of F in $\mathcal{D}$ (in the usual sense) is an ideal in $\mathcal{D}$ and hence, a subcategory of $\mathcal{D}$ with subobjects. In particular, when $\mathcal{C}$ and $\mathcal{D}$ are normal, the image of a local isomorphism of $\mathcal{C}$ into $\mathcal{D}$ is a normal subcategory of $\mathcal{D}$.

Let S be a regular semigroup. Then by Theorem III.16, there is a homomorphism $\bar{\rho}\colon a \mapsto \rho^a$ of S into $T\mathcal{L}(S)$ such that $\bar{\rho}(a) = \bar{\rho}(b)$ if and only if $\rho_a = \rho_b$.

Define FS_ρ from $\mathcal{R}(S)$ to $\mathcal{R}(T\mathcal{L}(S))$ on objects and morphisms by

$$FS_\rho(eS) = \rho^e(T\mathcal{L}(S)) \quad \text{and}$$
$$(1) \qquad FS_\rho(\lambda(e,u,f)) = \lambda(\rho^e, \rho^u, \rho^f).$$

Similarly, corresponding to the anti-homomorphism $\bar{\lambda}\colon S \to T\mathcal{R}(S)$, we define FS_λ from $\mathcal{L}(S)$ to $\mathcal{R}(T\mathcal{R}(S))$ as follows:

$$FS_\lambda(Se) = \lambda^e(T\mathcal{R}(S)) \quad \text{and}$$
$$(1^*) \qquad FS_\lambda(\rho(e,u,f)) = \lambda(\lambda^e, \lambda^u, \lambda^f).$$

In the following, for convenience, we shall write F_ρ and F_λ for FS_ρ and FS_λ respectively if the semigroup S is clear from the context. We have the following.

Proposition 1 *For any regular semigroup S, FS_ρ defined by Equation* (1) *is a local isomorphism. Dually, FS_λ defined by Equation* (1*) *is also a local isomorphism.*

Proof Note that by Equation (III.10), $FS_\lambda = FS_\rho^{op}$ and so the dual statement follows from the first. Hence it is sufficient to show that $FS_\rho = F_\rho$ is a local isomorphism. If $eS = fS$, then $e \mathcal{R} f$. Since $\bar{\rho}\colon S \to T\mathcal{L}(S)$ is a homomorphism, we have $\bar{\rho}(e) = \rho^e \mathcal{R} \bar{\rho}(f) = \rho^f$ and so $\rho^e T\mathcal{L}(S) = \rho^f T\mathcal{L}(S)$. Thus F_ρ is well-defined on vertices of the category $\mathcal{R}(S)$. If $\lambda(e,u,f)$ is any morphism in $\mathcal{R}(S)$, then by Remark III.6(a)*, $u \in fSe$ so that $\rho^u \in \rho^f T\mathcal{L}(S)\rho^e$. Hence $F_\rho(\lambda(e,u,f))$ represents a unique morphism in $\mathcal{R}(T\mathcal{L}(S))$. Suppose that $\lambda(e,u,f)$ and $\lambda(e',u',f')$ are representations of the same morphism $\lambda\colon eS \to fS$. From Remark III.6(b)* we infer that $e \mathcal{R} e'$, $f \mathcal{R} f'$ and $u' = ue'$. But then $\rho^e \mathcal{R} \rho^{e'}$, $\rho^f \mathcal{R} \rho^{f'}$ and $\rho^{u'} = \rho^u \rho^{e'}$ and it follows that $\lambda(\rho^e, \rho^u, \rho^f) = \lambda(\rho^{e'}, \rho^{u'}, \rho^{f'})$ by Remark III.6(b)*. Hence F_ρ is well-defined on objects and morphisms of $\mathcal{R}(S)$ by Equation (1). Since $\bar{\rho}$ is a homomorphism, using Remark III.6(c)*, we deduce that F_ρ is a functor. It follows from the dual of Proposition III.13(d) that F_ρ is inclusion-preserving.

We now show that F_ρ is fully-faithful. Let $\lambda = \lambda(e,u,f)$ and $\lambda' = \lambda(e,v,f)$ be two morphisms in $\mathcal{R}(S)(eS, fS)$ such that $F_\rho(\lambda) = F_\rho(\lambda')$. Then $\lambda(\rho^e, \rho^u, \rho^f) = \lambda(\rho^e, \rho^v, \rho^f)$. Hence $\rho^u = \rho^v$ by Remark III.6(a)*. Now, $u, v \in fSe$ and so, by Lemmas III.12(a) and III.15, we obtain

$$u = fu = f\rho(f,u,e) = f\rho^u(Sf) = f\rho^v(Sf) = fv = v.$$

Therefore $\lambda = \lambda'$; thus F_ρ is faithful. To prove that F_ρ is full, choose a morphism $\lambda\colon \rho^e T\mathcal{L}(S) \to \rho^f T\mathcal{L}(S)$ in $\mathcal{R}(T\mathcal{L}(S))$ so that $\lambda = \lambda(\rho^e, \gamma, \rho^f)$ for some $\gamma \in \rho^f T\mathcal{L}(S)\rho^e$. Then $\rho^f \cdot \gamma = \gamma = \gamma \cdot \rho^e$ and so, $\gamma = \rho^f \star \gamma(Sf)$ and $c_\gamma \subseteq c_e = Se$ by Lemma III.3 and Proposition III.5 respectively. Also by III.3, $\gamma(Sf)\colon Sf \to Sh = c_\gamma$ is an epimorphism and so, we may assume, by Proposition III.13(b), that $\gamma(Sf) = \rho(f, u, h)$ where $u \in fS$ and $u \mathrel{\mathcal{L}} h \mathrel{\omega^l} e$. Now by the definition of product in $T\dot{\mathcal{L}}(S)$ (see Equation (III.3)), we have

$$\rho^u = \rho^f \cdot \rho^u = \rho^f \star \rho(f, u, h) = \rho^f \star \gamma(Sf) = \gamma.$$

Thus $F_\rho(\lambda(e, u, f)) = \lambda(\rho^e, \rho^u, \rho^f) = \lambda$. Hence F_ρ is fully-faithful.

In order to show that F_ρ is a local isomorphism it remains to show that $\boldsymbol{v}F_\rho$ induces an order isomorphism on the vertex set of any principal ideal of $\mathcal{R}(S)$. To this end, suppose that eS, $e'S \subseteq fS$ and $F_\rho(eS) = F_\rho(e'S)$. Then $\rho^e T\mathcal{L}(S) = \rho^{e'} T\mathcal{L}(S)$ and so, $\rho^e \mathrel{\mathcal{R}} \rho^{e'}$. Since $\bar{\rho}$ is a homomorphism, $\bar{\rho}$ weakly reflects the quasi-orders ω^r and ω^l (see Proposition 2.14 of [15]). Therefore there exists $e_1 \mathrel{\omega^r} e'$ such that $\rho^{e_1} = \rho^e$. Then $e_1 e' \mathrel{\omega} e'$ and

$$\rho^{e_1 e'} = \rho^{e_1} \rho^{e'} = \rho^e \rho^{e'} = \rho^{e'}.$$

Hence $Se_1e' = Se'$ or $e_1 e' \mathrel{\mathcal{L}} e'$. This implies that $e_1 e' = e'$ so that $e_1 \mathrel{\mathcal{R}} e'$. Since e, $e_1 \in \omega^r(f)$ and $\rho^e = \rho^{e_1}$, we obtain

$$e = fe = f\rho^e(Sf) = f\rho^{e_1}(Sf) = fe_1 = e_1.$$

Thus $e \mathrel{\mathcal{R}} e'$ and so, $eS = e'S$. This proves that F_ρ is one-to-one on $\boldsymbol{v}\langle f\rangle S$. To prove that it is onto, consider $\epsilon \in E(T\mathcal{L}(S))$ such that $\epsilon T\mathcal{L}(S) \subseteq \rho^f T\mathcal{L}(S)$. Then $\epsilon \mathrel{\omega^r} \rho^f$ and so $\epsilon \mathrel{\mathcal{R}} \epsilon\rho^f \mathrel{\omega} \rho^f$. Replacing ϵ by $\epsilon\rho^f$ if necessary, we may assume that $\epsilon \mathrel{\omega} \rho^f$. If $c_\epsilon = Sg$, then $Sg = c_\epsilon \subseteq c_{\rho^f} = Sf$ and so $g \mathrel{\omega^l} f$. Again $g \mathrel{\mathcal{L}} fg \mathrel{\omega} f$ and so we may assume that $g \mathrel{\omega} f$. Since $\rho^f \cdot \epsilon = \epsilon$, there exists a unique epimorphism $\varrho\colon Sf \to Sg$ such that $\epsilon = \rho^f \star \varrho$ by Lemma III.3. Then we have

$$1_{Sg} = \epsilon(Sg) = \rho^f(Sg)\varrho = \rho(g, g, f)\varrho$$

and so ϱ is a retraction. By Proposition III.13(d), there is $g' \mathrel{\omega} f$ such that $g' \mathrel{\mathcal{L}} g$ and $\varrho = \rho(f, g', g)$. Since $\rho(f, g', g) = \rho^{g'}(Sf)$ by Lemma III.15, we have

$$\epsilon = \rho^f \star \rho(f, g', g) = \rho^f \star \rho^{g'}(Sf) = \rho^f \cdot \rho^{g'} = \rho^{fg'} = \rho^{g'}$$

by Equation (III.3) and Theorem III.16. Hence $F_\rho(g'S) = \epsilon T\mathcal{L}(S)$ with $g'S \subseteq fS$ and so $\boldsymbol{v}F_\rho$ is surjective on $\boldsymbol{v}\langle fS\rangle$. This also proves that this map is an order isomorphism on $\boldsymbol{v}\langle f\rangle S$. □

1.2 The connection

The following theorem clarifies the relation between the normal categories $\mathcal{L}(S)$ and $\mathcal{R}(S)$ associated with a regular semigroup S.

Theorem 2 *Let S be a regular semigroup. For $fS \in \boldsymbol{v}\mathcal{R}(S)$ and $\lambda = \lambda(e, u, f)$ in $\mathcal{R}(S)$, let $\boldsymbol{\Gamma}S$ be defined on objects and morphisms of $\mathcal{R}(S)$ by:*

(2) $\qquad \boldsymbol{v}\boldsymbol{\Gamma}S(fS) = H(\rho^f; -), \qquad \boldsymbol{\Gamma}S(\lambda) = \eta_{\rho^e}\mathcal{L}(S)(\rho(f,u,e), -)\eta_{\rho^f}^{-1}.$

Then $\boldsymbol{\Gamma}S$ is a local isomorphism from $\mathcal{R}(S)$ to $\boldsymbol{N}^*\mathcal{L}(S)$. Dually, ΔS, defined on objects and morphisms of $\mathcal{L}(S)$ by

(2*) $\qquad \boldsymbol{v}\Delta S(Se) = H(\lambda^e; -), \qquad \Delta S(\rho) = \eta_{\lambda^e}\mathcal{R}(S)(\lambda(f,u,e), -)\eta_{\lambda^f}^{-1}$

for all $Se \in \boldsymbol{v}\mathcal{L}(S)$ and $\rho = \rho(e,u,f) \in \mathcal{L}(S)$, defines a local isomorphism.

Proof Comparing the definition of $\boldsymbol{\Gamma}S$ with Equation (1) and the definition of $\boldsymbol{G}$ in Theorem III.25 (see also §III.4), we observe that

(3) $$\boldsymbol{\Gamma}S = FS_\rho \circ \bar{\boldsymbol{G}}$$

where $FS_\rho\colon \mathcal{R}(S) \to \mathcal{R}(T\mathcal{L}(S))$ is the local isomorphism defined by Equation (1) and $\bar{\boldsymbol{G}}\colon \mathcal{R}(T\mathcal{L}(S)) \to \boldsymbol{N}^*\mathcal{L}(S)$ is the isomorphism defined in Theorem III.25. It follows that $\boldsymbol{\Gamma}S\colon \mathcal{R}(S) \to \boldsymbol{N}^*\mathcal{L}(S)$ is a local isomorphism. Dully we have

(3*) $$\Delta S = FS_\lambda \circ \tilde{\boldsymbol{G}}$$

where $FS_\lambda\colon \mathcal{L}(S) \to \mathcal{R}(T\mathcal{R}(S))$ is defined by Equation (1*) and $\tilde{\boldsymbol{G}}$ is the isomorphism of $\mathcal{R}(T\mathcal{R}(S))$ onto $\boldsymbol{N}^*\mathcal{R}(S)$ defined in Theorem III.25. Again it is clear that ΔS is a local isomorphism. □

Definition 2 *Let S be a regular semigroup. By the* connection *of S we shall mean the local isomorphism $\boldsymbol{\Gamma} S\colon \mathcal{R}(S) \to \boldsymbol{N}^*\mathcal{L}(S)$. The local isomorphism $\Delta S\colon \mathcal{L}(S) \to \boldsymbol{N}^*\mathcal{R}(S)$ will be called the* dual connection *of S.*

A natural question that arises in this context is: under what conditions can we assert that the connection (or the dual connection) of the semigroup S is an embedding. Since $\boldsymbol{\Gamma} S$ is a local isomorphism, it is an embedding if and only if $\boldsymbol{v\Gamma} S$ is injective. By Equation (3), this is true if and only if $\boldsymbol{v}FS_\rho$ is injective since $\bar{G}$ is an isomorphism. If $FS_{eS} = FS_\rho(e'S)$ then by Equation (1), it follows that $\rho^e \;\mathcal{R}\; \rho^{e'}$ and as in the proof of Proposition 1 we see that there exists $e \;\mathcal{L}\; e_1 \;\mathcal{R}\; e'$ such that $\rho^e = \rho^{e_1}$ which implies (by Theorem III.16) that $\rho_e = \rho_{e_1}$. Now if S is right reductive, we have $e_1 = e$ and so $eS = e_1S = e'S$. Thus in this case FS_ρ and hence $\boldsymbol{\Gamma} S$ is an embedding. Conversely if S is not right reductive, then it is easy to see that there exist distinct $\mathcal{L}$-related idempotents e and e' such that $\rho_e = \rho_{e'}$. Then by Theorem III.16, $\rho^e = \rho^{e'}$ and so by Equation (1), $FS_\rho(eS) = FS_\rho(e'S)$ and $eS \neq e'S$. Therefore FS_ρ and hence $\boldsymbol{\Gamma} S$ is not an embedding. We thus have:

Corollary 3 *The connection of a regular semigroup S is an embedding if and only if S is right reductive. Dually the dual connection of S is an embedding if and only if S is left reductive.* □

1.3 The duality χ_S

Since $\boldsymbol{\Gamma} S\colon \mathcal{R}(S) \to \boldsymbol{N}^*\mathcal{L}(S) \subseteq \mathcal{L}(S)^*$, by the category isomorphisms given by Equation (I.6) there is a unique bifunctor $\boldsymbol{\Gamma} S(-,-)\colon \mathcal{L}(S) \times \mathcal{R}(S) \to \mathbf{Set}$. By Equations (I.7a) and (I.7b), $\boldsymbol{\Gamma} S(-,-)$ is defined on objects and morphisms as follows:

$$\begin{aligned} \boldsymbol{\Gamma} S(Se, fS) &= \boldsymbol{\Gamma} S(fS)(Se); \\ \boldsymbol{\Gamma} S(\rho,\lambda) &= \boldsymbol{\Gamma} S(fS)(\rho)\boldsymbol{\Gamma} S(\lambda)(Se') = \boldsymbol{\Gamma} S(\lambda)(Se)\boldsymbol{\Gamma} S(f'S)(\rho) \end{aligned} \tag{4}$$

for all $(Se, fS) \in \boldsymbol{v}\mathcal{L}(S) \times \boldsymbol{v}\mathcal{R}(S)$ and $(\rho,\lambda)\colon (Se, fS) \to (Se', f'S)$. Similarly corresponding to the functor $\Delta S\colon \mathcal{L}(S) \to \boldsymbol{N}^*\mathcal{R}(S)$, there is a unique bifunctor $\Delta S(-,-)$ defined by

$$\begin{aligned} \Delta S(Se, fS) &= \Delta S(Se)(fS); \\ \Delta S(\rho,\lambda) &= \Delta S(Se)(\lambda)\Delta S(\rho)(f'S) = \Delta S(\rho)(fS)\Delta S(Se')(\lambda) \end{aligned} \tag{4*}$$

In the following, for brevity the bifunctors $\boldsymbol{\Gamma}S(-,-)$ and $\Delta S(-,-)$ [the functors $\boldsymbol{\Gamma}S$ and ΔS] will be often denoted respectively by $\boldsymbol{\Gamma}(-,-)$ and $\Delta(-,-)$ [by $\boldsymbol{\Gamma}$ and Δ] if the semigroup S is clear from the context. These bifunctors are related as follows.

Theorem 4 *Let S be a regular semigroup. Then there is a natural isomorphism χ_S from $\boldsymbol{\Gamma}S(-,-)$ to $\Delta S(-,-)$ whose components are defined by*

$$(5) \qquad \chi_S(Se, fS) \colon \rho^f \star \rho(f,u,e)^\circ \mapsto \lambda^e \star \lambda(e,u,f)^\circ$$

for each $(Se, fS) \in \boldsymbol{v}(\mathcal{L}(S) \times \mathcal{R}(S))$.

Proof We first show that for each $(Se, fS) \in \boldsymbol{v}(\mathcal{L}(S) \times \mathcal{R}(S))$, $\chi_S(Se, fS)$ defined by Equation (5) in the statement is a bijection of the set $\boldsymbol{\Gamma}S(Se, fS)$ onto $\Delta S(Se, fS)$. By Equations (2) and (4), we have $\boldsymbol{\Gamma}S(Se, fS) = H(\rho^f; Se)$ and hence by Equation (III.6) every element of the set $\boldsymbol{\Gamma}S(Se, fS)$ is a normal cone of the form $\rho^f \star \rho(f,u,e)^\circ$ where $u \in fSe$. Dually every element of $\Delta S(Se, fS)$ is a normal cone of the form $\lambda^e \star \lambda(e,u,f)^\circ$ with $u \in fSe$. Now by Corollary III.9, ρ^f is a universal element for the functor $H(\rho^f; -)$. By Yoneda lemma, the component $\eta_{\rho^f}(Se)$ of the corresponding natural isomorphism of $H(\rho^f; -)$ to $\mathcal{L}(S)(Sf, -)$ is the map $\rho^f \star \rho(f,u,e)^\circ$ to $\rho(f,u,e)$. Also, by Lemma III.12(a) the map $\rho(f,u,e) \mapsto u$ is a bijection of $\mathcal{L}(S)(Sf, Se)$ onto fSe. It follows that the map $\rho^f \star \rho(f,u,e)^\circ \mapsto u$ is a bijection of $\boldsymbol{\Gamma}S(Se, fS)$ onto fSe. Similarly we see that $\lambda^e \star \lambda(e,u,f)^\circ \mapsto u$ is a bijection of $\Delta S(Se, fS)$ onto fSe. Consequently the map $\chi_S(Se, fS)$ defined by (5) is a bijection of $\boldsymbol{\Gamma}S(Se, fS)$ onto $\Delta S(Se, fS)$.

We now show that χ_S is a natural isomorphism. Let $(\rho, \lambda) \in \mathcal{L}(S) \times \mathcal{R}(S)$ with $\rho \colon Se \to Se'$ and $\lambda \colon fS \to f'S$. We show that the following diagrams commute:

$$(*) \qquad \begin{array}{ccc} \boldsymbol{\Gamma}(Se, fS) & \xrightarrow{\chi_S(Se,fS)} & \Delta(Se, fS) \\ {\scriptstyle \boldsymbol{\Gamma}(\rho,fS)}\downarrow & & \downarrow{\scriptstyle \Delta(\rho,fS)} \\ \boldsymbol{\Gamma}(Se', fS) & \xrightarrow[\chi_S(Se',fS)]{} & \Delta(Se', fS) \end{array}$$

$$(**)\qquad \begin{array}{ccc} \boldsymbol{\Gamma}(Se', fS) & \xrightarrow{\chi_S(Se', fS)} & \Delta(Se', fS) \\ {\scriptstyle \boldsymbol{\Gamma}(Se',\lambda)}\downarrow & & \downarrow{\scriptstyle \Delta(Se',\lambda)} \\ \boldsymbol{\Gamma}(Se', f'S) & \xrightarrow[\chi_S(Se', f'S)]{} & \Delta(Se', f'S) \end{array}$$

To prove that the first square $(*)$ above commutes, assume that $\rho = \rho(e, v, e')$. Let $\lambda' = \lambda(e', u, e)$. Consider $x = \rho^f \star \rho(f, u, e)^\circ \in \boldsymbol{\Gamma}(Se, fS)$. Let

$$X = \chi_S(Se, fS)\Delta(\rho, fS)(x).$$

Then we have

$$X = \Delta(\rho, fS)(\lambda^e \star \lambda(e, u, f)^\circ)$$

by (5). Hence using Equations (2*) and (4*) we obtain

$$\begin{aligned} X &= \eta_{\lambda^e}(fS)\mathcal{R}(S)(\lambda(e', v, e), fS)\eta^{-1}_{\lambda^{e'}}(fS)(\lambda^e \star \lambda(e, u, f)^\circ) \\ &= \eta^{-1}_{\lambda^{e'}}(fS)\left(\mathcal{R}(S)(\lambda(e', v, e), fS)\left(\eta_{\lambda^e}(fS)(\lambda^e \star \lambda(e, u, f)^\circ)\right)\right) \\ &= \eta^{-1}_{\lambda^{e'}}(fS)\left(\lambda(e', v, e)\lambda(e, u, f)\right) \\ &= \eta^{-1}_{\lambda^{e'}}(fS)\left(\lambda(e', uv, f)\right) \\ &= \lambda^{e'} \star \lambda(e', uv, f)^\circ. \end{aligned}$$

Thus $\chi_S(Se, fS)\Delta(\rho, fS)(x) = \lambda^{e'} \star \lambda(e', uv, f)^\circ$. Further,

$$\boldsymbol{\Gamma}(\rho, fS)\chi_S(Se', fS)(x) = \chi_S(Se', fS)\left(H(\rho^f; \rho)(x)\right)$$

by Equations (2) and (4). By Equation (III.6), we have

$$\begin{aligned} &= \chi_S(Se', fS)\left(\rho^f \star (\rho(f, u, e)\rho(e, v, e'))^\circ\right) \\ &= \chi_S(Se', fS)\left(\rho^f \star (\rho(f, uv, e'))^\circ\right) \\ &= \lambda^{e'} \star \lambda(e', uv, f)^\circ \quad \text{by Equation (5).} \end{aligned}$$

This proves that the diagram (*) above commutes. The proof for the second square (**) is similar. Let $(\rho, \lambda) \in \mathcal{L}(S) \times \mathcal{R}(S)$ with $\rho \in \mathcal{L}(S)(Se, Se')$ and $\lambda \in \mathcal{R}(S)(fS, f'S)$. It follows from Equations (4*) and (4) and commutativity of diagrams (*) and (**) above that

$$\begin{aligned}\chi_S(Se, fS)\Delta(\rho, \lambda) &= \chi_S(Se, fS)\Delta(\rho, fS)\Delta(Se', \lambda)\\ &= \Gamma(\rho, fS)\chi_S(Se', fS)\Delta(Se', \lambda)\\ &= \Gamma(\rho, fS)\Gamma(Se', \lambda)\chi_S(Se', f'S)\\ &= \Gamma(\rho, \lambda)\chi_S(Se', f'S).\end{aligned}$$

Therefore χ_S is a natural isomorphism. □

The natural isomorphism χ_S will be called the *duality associated with S*.

2 CONNECTION AND DUAL CONNECTION OF TC

If C is a normal category then we have seen that $S = TC$ is a regular semigroup. In this section we prove some results on $\Gamma = \Gamma S$, the connection of S and $\Delta = \Delta S$, the dual connection which will be of interest in their own right and will also be of use later.

2.1 Isomorphisms of dual categories

We begin with a useful construction of an isomorphism of normal duals which will be needed often in the sequel. Let $F: C \to \mathcal{D}$ be an isomorphism of normal categories (so that F is, in particular, inclusion preserving). If γ is a normal cone in C, since F is an isomorphism, the equation

(6) $$F(\gamma)(F(c)) = F(\gamma(c))$$

clearly defines a normal cone in $\mathcal{D}$; we denote this cone by $F(\gamma)$. Hence for each $H(\epsilon; -) \in vN^*C$, $H(F(\epsilon); -)$ is an object in $N^*\mathcal{D}$ and it is easy to see that map $H(\epsilon; -) \mapsto H(F(\epsilon); -)$ is an order isomorphism of vN^*C on to $vN^*\mathcal{D}$. Now by Lemma III.21, for each morphism $\eta: H(\epsilon; -) \to H(\epsilon'; -)$ in N^*C, there is a unique morphism $f = \widehat{\eta}: c_{\epsilon'} \to c_\epsilon$ in C such that the diagram (•) of Lemma III.22 commutes so that $\eta = \eta_\epsilon C(f, -)\eta_{\epsilon'}^{-1}$. Again it is easy to show that $\eta_{F(\epsilon)}\mathcal{D}(F(f),-)\eta^{-1}_{F(\epsilon')}$ is a morphism in $\boldsymbol{N}^*\mathcal{D}$ from $H(F(\epsilon);-)$ to $H(F(\epsilon');-)$ and that the map

$$\eta_\epsilon\mathcal{C}(f,-)\eta^{-1}_{\epsilon'} \mapsto \eta_{F(\epsilon)}\mathcal{D}(F(f),-)\eta^{-1}_{F(\epsilon')}$$

defines the morphism map of an isomorphism of $\boldsymbol{N}^*\mathcal{C}$ onto $\boldsymbol{N}^*\mathcal{D}$ for which the object map is the one defined above. We thus have the following:

Lemma 5 *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an isomorphism of normal categories. For each $H(\epsilon;-) \in \boldsymbol{vN}^*\mathcal{C}$ and $\eta = \eta_\epsilon\mathcal{C}(f,-)\eta^{-1}_{\epsilon'} \in \boldsymbol{N}^*\mathcal{C}$ define*

$$(7)\qquad \begin{aligned} F_*(H(\epsilon;-)) &= H(F(\epsilon);-);\\ F_*(\eta) &= \eta_{F(\epsilon)}\mathcal{D}(F(f),-)\eta^{-1}_{F(\epsilon')}. \end{aligned}$$

Then F_* is an isomorphism of $\boldsymbol{N}^*\mathcal{C}$ onto $\boldsymbol{N}^*\mathcal{D}$. □

2.2 The connection of $T\mathcal{C}$

We now use the Lemma 5 above to study the connection of the semigroup $T\mathcal{C}$. In the following we denote by $\boldsymbol{F}$ the isomorphism of $\mathcal{C}$ to $\mathcal{L}(T\mathcal{C})$ defined by Equation (III.12) (see Theorem III.19) and by $\boldsymbol{G}$, the isomorphism of $\mathcal{R}(T\mathcal{C})$ onto $\boldsymbol{N}^*\mathcal{C}$ defined in Theorem III.25.

Theorem 6 *Let $\mathcal{C}$ be a normal category. If $\boldsymbol{\Gamma}$ denote the connection of $T\mathcal{C}$, then we have*

$$(8)\qquad \boldsymbol{\Gamma} = \boldsymbol{G} \circ \boldsymbol{F}_*.$$

In particular, $\boldsymbol{\Gamma}\colon \mathcal{R}(T\mathcal{C}) \to \boldsymbol{N}^*\mathcal{L}(T\mathcal{C})$ is an isomorphism.

Proof Let $S = T\mathcal{C}$. By the definition of $\boldsymbol{G}$ (see Theorem III.25) and Equation (7), we have

$$\boldsymbol{G} \circ \boldsymbol{F}_*(\epsilon S) = \boldsymbol{F}_*\left(\boldsymbol{G}(\epsilon S)\right) = \boldsymbol{F}_*(H(\epsilon;-)) = H(\boldsymbol{F}(\epsilon);-)$$

for any $\epsilon \in E(S)$. Now by Equation (III.12) and the definition of the cone $\boldsymbol{F}(\epsilon)$, we obtain

$$\begin{aligned} \boldsymbol{F}(\epsilon)(S\epsilon') &= \boldsymbol{F}(\epsilon(c_{\epsilon'}))\\ &= \rho(\epsilon', \epsilon' \star \epsilon(c_{\epsilon'})^\circ, \epsilon)\\ &= \rho(\epsilon', \epsilon' \cdot \epsilon, \epsilon) = \rho^\epsilon(S\epsilon') \quad \text{by Lemma III.15.} \end{aligned}$$

Hence $\boldsymbol{F}(\epsilon) = \rho^\epsilon$ and so, $\boldsymbol{G} \circ \boldsymbol{F}_*(\epsilon S) = H(\rho^\epsilon; -) = \boldsymbol{\Gamma}(\epsilon S)$. Now let $\lambda \colon \epsilon S \to \epsilon' S$ be a morphism in $\mathcal{R}(S)$. Then it follows from Lemma III.22 that there is a unique $f \colon c_{\epsilon'} \to c_\epsilon$ such that $\lambda = \lambda(\epsilon, \epsilon' \star f^\circ, \epsilon')$. Also by Equation (III.14), $\widetilde{\epsilon' \star f^\circ} = f$. Hence it follows from Theorem III.25 that $\boldsymbol{G}(\lambda) = \eta_\epsilon \mathcal{C}(f, -)\eta_{\epsilon'}^{-1}$ (see commutative diagram III.4). Therefore using Equations (III.12) and (7), we deduce

$$\begin{aligned} \boldsymbol{G} \circ \boldsymbol{F}_*(\lambda) &= \boldsymbol{F}_*(\eta_\epsilon \mathcal{C}(f, -)\eta_{\epsilon'}^{-1}) \\ &= \eta_{\boldsymbol{F}(\epsilon)} \mathcal{L}(S)(\boldsymbol{F}(f), -)\eta_{\boldsymbol{F}(\epsilon')}^{-1} \\ &= \eta_{\rho^{\epsilon'}} \mathcal{L}(S)(\rho(\epsilon', \epsilon' \star f^\circ, \epsilon)\eta_{\rho^\epsilon}^{-1} \\ &= \boldsymbol{\Gamma}(\lambda) \quad \text{by Equation (2).} \end{aligned}$$

This proves the required equality. □

It follows from this result and Corollary 3 that for any normal category, the regular semigroup TC is right reductive.

2.3 The local isomorphism $\theta_{\mathcal{C}}$

The statement for the dual connection Δ of $S = TC$ corresponding the above proposition (that Δ is an isomorphism) is not true in general. However, we can use the dual connection of S to obtain a canonical local isomorphism of $\mathcal{C}$ into its *double dual* $\boldsymbol{N}^{**}\mathcal{C}$ where $\boldsymbol{N}^{**}\mathcal{C} = \boldsymbol{N}^*\boldsymbol{N}^*\mathcal{C}$.

For any normal category $\mathcal{C}$ define the functor $\theta_{\mathcal{C}} \colon \mathcal{C} \to \boldsymbol{N}^{**}\mathcal{C}$ by the following commutative diagram:

(9)
$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ \boldsymbol{F}\ } & \mathcal{L}(TC) \\ {\scriptstyle \theta_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle \Delta} \\ \boldsymbol{N}^{**}\mathcal{C} & \xleftarrow[\ \boldsymbol{G}_*\ ]{} & \boldsymbol{N}^*\mathcal{R}(TC) \end{array}$$

Since $\boldsymbol{F}$ and $\boldsymbol{G}_*$ are isomorphisms and Δ is a local isomorphism (by Theorem 2 and Lemma 5), $\theta_{\mathcal{C}}$ is a local isomorphism. As usual, we denote by $\theta_{\mathcal{C}}(-,-)$, the bifunctor from $\mathcal{C} \times \boldsymbol{N}^*\mathcal{C}$ to **Set** determined by $\theta_{\mathcal{C}}$ (defined by Equations (I.7a) and (I.7b)).

Recall that the evaluation functor of the category $\mathcal{C}$ is the bifunctor $\boldsymbol{E}_{\mathcal{C}}: \mathcal{C} \times \mathcal{C}^* \to \mathbf{Set}$ defined by Equation (I.12). Since $\boldsymbol{N}^*\mathcal{C} \subseteq \mathcal{C}^*$, this induces a bifunctor of $\mathcal{C} \times \boldsymbol{N}^*\mathcal{C}$ to $\mathbf{Set}$. Since no confusion is likely, when $\mathcal{C}$ is a normal category, we shall denote by $\boldsymbol{E}_{\mathcal{C}}$ the restriction of the bifunctor defined by Equation (I.12) to $\mathcal{C} \times \boldsymbol{N}^*\mathcal{C}$. We proceed to show that the bifunctor $\theta_{\mathcal{C}}(-,-)$ is naturally isomorphic to $\boldsymbol{E}_{\mathcal{C}}$. We divide the proof into a sequence of lemmas. In the following lemmas, S denotes the regular semigroup $T\mathcal{C}$.

Lemma 7 *For any $c \in v\mathcal{C}$ and $g : c \to c'$ in $\mathcal{C}$, we have*

$$\begin{aligned} \theta_{\mathcal{C}}(c) &= H(\boldsymbol{G}(\lambda^{\epsilon}); -); \\ \theta_{\mathcal{C}}(g) &= \eta_{\boldsymbol{G}(\lambda^{\epsilon})} \boldsymbol{N}^*\mathcal{C}(\eta_{\epsilon'}\mathcal{C}(g,-)\eta_{\epsilon}^{-1}, -)\eta^{-1}_{\boldsymbol{G}(\lambda^{\epsilon'})} \end{aligned} \tag{10}$$

for any ϵ with $c_{\epsilon} = c$ and ϵ' with $c_{\epsilon'} = c'$.

Proof Fix ϵ with $c_{\epsilon} = c$. By the definition of $\theta_{\mathcal{C}}$, we then have

$$\begin{aligned} \theta_{\mathcal{C}}(c) &= \boldsymbol{G}_*(\Delta(\boldsymbol{F}(c))) && \text{by Diagram(9)} \\ &= \boldsymbol{G}_*(\Delta(S\epsilon)) && \text{by Equation (III.12)} \\ &= \boldsymbol{G}_*(H(\lambda^{\epsilon}; -)) && \text{by Equation (2*)} \\ &= H(\boldsymbol{G}(\lambda^{\epsilon}); -) && \text{by Equation (7).} \end{aligned}$$

Now given $g: c \to c'$, fix ϵ and ϵ' with $c_{\epsilon} = c$ and $c_{\epsilon'} = c'$. Then as before, using definitions of functors $\theta_{\mathcal{C}}$, $\boldsymbol{F}$, Δ and $\boldsymbol{G}_*$ (see Diagram(9), Equation (III.12), Equation (2*) and Equation (7) respectively) we obtain

$$\begin{aligned} \theta_{\mathcal{C}}(g) &= \boldsymbol{G}_*(\Delta(\boldsymbol{F}(g))) \\ &= \boldsymbol{G}_*\left(\Delta\left(\rho(\epsilon, \epsilon \star g^{\circ}, \epsilon')\right)\right) \\ &= \boldsymbol{G}_*\left(\eta_{\lambda^{\epsilon}} \mathcal{R}(S)(\lambda(\epsilon', \epsilon \star g^{\circ}, \epsilon), -)\eta^{-1}_{\lambda^{\epsilon'}}\right) \\ &= \eta_{\boldsymbol{G}(\lambda^{\epsilon})} \boldsymbol{N}^*\mathcal{C}(\boldsymbol{G}(\lambda(\epsilon', \epsilon \star g^{\circ}, \epsilon)), -)\eta^{-1}_{\boldsymbol{G}(\lambda^{\epsilon'})} \\ &= \eta_{\boldsymbol{G}(\lambda^{\epsilon})} \boldsymbol{N}^*\mathcal{C}(\eta_{\epsilon'}\mathcal{C}(g,-)\eta_{\epsilon}^{-1}, -)\eta^{-1}_{\boldsymbol{G}(\lambda^{\epsilon'})}. \end{aligned}$$

The last equality above follows from Theorem III.25 (definition of $\boldsymbol{G}$) since, by Equation (III.14), $\widetilde{\eta \star g^{\circ}} = g$. $\square$

Lemma 8 *Let* ϵ, $\epsilon' \in E(S)$ *with* $c_\epsilon = c_{\epsilon'}$. *Then*

$$G(\lambda^{\epsilon'}) = G(\lambda^{\epsilon}) \star (\eta_\epsilon \eta_{\epsilon'}^{-1}) \tag{11}$$

In particular $G(\lambda^{\epsilon'})\ \mathcal{R}\ G(\lambda^{\epsilon})$.

Proof Since $c_\epsilon = c_{\epsilon'}$, $\epsilon\ \mathcal{L}\ \epsilon'$ in S. Since $\bar{\lambda}$ is an anti-homomorphism, we have $\lambda^{\epsilon'}\ \mathcal{R}\ \lambda^{\epsilon}$. Hence by Lemma III.15, $\epsilon S \in M\epsilon'$ and so $\lambda^{\epsilon'}(\epsilon S)$ is an isomorphism. By the dual of Lemma III.15, $\lambda^{\epsilon'}(\epsilon S) = \lambda(\epsilon, \epsilon' \cdot \epsilon, \epsilon') = \lambda(\epsilon, \epsilon', \epsilon')$ and so, by Proposition III.7(a), $\lambda^{\epsilon'} = \lambda^{\epsilon} \star \lambda(\epsilon, \epsilon', \epsilon')$. Hence for any $H(\epsilon''; -) \in \boldsymbol{v}\boldsymbol{N}^*\mathcal{C}$, by the definition of the cone $G(\lambda^{\epsilon'})$, we have

$$\begin{aligned} G(\lambda^{\epsilon'})(H(\epsilon''; -)) &= G(\lambda^{\epsilon'}(\epsilon'' S)) \\ &= G(\lambda^{\epsilon} \star \lambda(\epsilon, \epsilon', \epsilon')(\epsilon'' S)) \\ &= G(\lambda^{\epsilon}(\epsilon'' S))G(\lambda(\epsilon, \epsilon', \epsilon')) \\ &= G(\lambda^{\epsilon} \star \lambda(\epsilon, \epsilon', \epsilon'))(H(\epsilon''; -)) \end{aligned}$$

and so $G(\lambda^{\epsilon'}) = G(\lambda^{\epsilon}) \star G(\lambda(\epsilon, \epsilon', \epsilon'))$. Now $\tilde{\epsilon'} = 1_{c_{\epsilon'}}$ and so by the definition of the functor G, we have $G(\lambda(\epsilon, \epsilon', \epsilon')) = \eta_\epsilon \eta_{\epsilon'}^{-1}$. □

Lemma 9 *Let* $c \in \boldsymbol{v}\mathcal{C}$ *and* $H(\epsilon'; -) \in \boldsymbol{v}\boldsymbol{N}^*\mathcal{C}$. *Fix* $\epsilon \in E(S)$ *with* $c_\epsilon = c$. *Then the map*

$$\theta_{\mathcal{C}}\ \Phi(c, H(\epsilon'; -))(\epsilon' \star f^\circ) = G(\lambda^{\epsilon}) \star (\eta_\epsilon \mathcal{C}(f, -)\eta_{\epsilon'}^{-1})^\circ \tag{12}$$

for all $\epsilon' \star f^\circ \in H(\epsilon'; c)$ *is a bijection of* $\boldsymbol{E}_{\mathcal{C}}(c, H(\epsilon'; -))$ *onto* $\theta_{\mathcal{C}}(c, H(\epsilon'; -))$ *which is independent of the choice of* ϵ.

Proof We first observe that $G(\lambda^{\epsilon})$ is a cone in $\boldsymbol{N}^*\mathcal{C}$ and by Lemma 7, we have

$$G(\lambda^{\epsilon}) \star (\eta_\epsilon \mathcal{C}(f, -)\eta_{\epsilon'}^{-1})^\circ \in H(G(\lambda^{\epsilon}); H(\epsilon'; -)) = \theta_{\mathcal{C}}(c, H(\epsilon'; -))$$

where $H(G(\lambda^{\epsilon}); -)$ denotes the set-valued functor on $\boldsymbol{N}^*\mathcal{C}$ defined by Equation (III.6). Hence for a given ϵ, Equation (12) defines a map of $\boldsymbol{E}_{\mathcal{C}}(c, H(\epsilon'; -)) = H(\epsilon'; c)$ into $\theta_{\mathcal{C}}(c, H(\epsilon'; -))$. Clearly $\epsilon' \star f^\circ \mapsto f$ is a bijection of $H(\epsilon'; c)$ onto $\mathcal{C}(c_{\epsilon'}, c)$ and by Lemma III.21, the map

$$f \mapsto \eta_\epsilon \mathcal{C}(f, -)\eta_{\epsilon'}^{-1} : \mathcal{C}(c_{\epsilon'}, c) \to \boldsymbol{N}^*\mathcal{C}(H(\epsilon; -), H(\epsilon'; -))$$

is a bijection. Hence $f \mapsto G(\lambda^{\epsilon}) \star (\eta_{\epsilon}\mathcal{C}(f,-)\eta_{\epsilon'}^{-1})^{\circ}$ is a bijection. Therefore ϖ defined by Equation (12) is a bijection. To show that ϖ is independent of ϵ, let $\bar{\epsilon} \in E(S)$ with $c_{\bar{\epsilon}} = c = c_{\epsilon}$. Then by Lemma 8, we have $G(\lambda^{\bar{\epsilon}}) = G(\lambda^{\epsilon}) \star (\eta_{\epsilon}\eta_{\bar{\epsilon}}^{-1})$. Hence, using the fact that $\eta_{\epsilon}\eta_{\bar{\epsilon}}^{-1}$ is an isomorphism, we obtain

$$\begin{aligned} G(\lambda^{\bar{\epsilon}}) \star (\eta_{\bar{\epsilon}}\mathcal{C}(f,-)\eta_{\epsilon'}^{-1})^{\circ} &= G(\lambda^{\epsilon}) \star (\eta_{\epsilon}\eta_{\bar{\epsilon}}^{-1})(\eta_{\bar{\epsilon}}\mathcal{C}(f,-)\eta_{\epsilon'}^{-1})^{\circ} \\ &= G(\lambda^{\epsilon}) \star ((\eta_{\epsilon}\eta_{\bar{\epsilon}}^{-1})(\eta_{\bar{\epsilon}}\mathcal{C}(f,-)\eta_{\epsilon'}^{-1}))^{\circ} \\ &= G(\lambda^{\epsilon}) \star (\eta_{\epsilon}\mathcal{C}(f,-)\eta_{\epsilon'}^{-1})^{\circ}. \end{aligned}$$

This proves that $\varpi(c, H(\epsilon'; -))$ is independent of ϵ. □

LEMMA 10 *There is a natural isomorphism $\varpi\colon \boldsymbol{E}_{\mathcal{C}} \to \theta_{\mathcal{C}}(-,-)$ such that the component at each $(c, H(\epsilon'; -)) \in \boldsymbol{v}\mathcal{C} \times \boldsymbol{v}N^{*}\mathcal{C}$ is the map $\varpi(c, H(\epsilon'; -))$ defined by Equation* (12).

PROOF Let $g\colon c \to c''$ be in $\mathcal{C}$ and fix $\epsilon, \epsilon'' \in E(S)$ with $c_{\epsilon} = c$ and $c_{\epsilon''} = c''$. We show that the following diagram commutes for all $H(\epsilon'; -) \in \boldsymbol{v}N^{*}\mathcal{C}$.

$$(*) \qquad \begin{array}{ccc} \boldsymbol{E}_{\mathcal{C}}(c, H(\epsilon'; -)) & \xrightarrow{\varpi(c, H(\epsilon'; -))} & \theta_{\mathcal{C}}(c, H(\epsilon'; -)) \\ {\scriptstyle \boldsymbol{E}_{\mathcal{C}}(g, H(\epsilon'; -))}\downarrow & & \downarrow{\scriptstyle \theta_{\mathcal{C}}(g, H(\epsilon'; -))} \\ \boldsymbol{E}_{\mathcal{C}}(c'', H(\epsilon'; -)) & \xrightarrow{\varpi(c'', H(\epsilon'; -))} & \theta_{\mathcal{C}}(c'', H(\epsilon'; -)) \end{array}$$

If $y = G(\lambda^{\epsilon}) \star (\eta_{\epsilon}\mathcal{C}(f,-)\eta_{\epsilon'}^{-1})^{\circ} \in H(G(\lambda^{\epsilon}); H(\epsilon'; -)) = \theta_{\mathcal{C}}(c, H(\epsilon'; -))$, where $f \in \mathcal{C}(c', c)$, then

$$\begin{aligned} y' &= \eta_{G(\lambda^{\epsilon})}(H(\epsilon'; -))(y) \\ &= \eta_{\epsilon}\mathcal{C}(f,-)\eta_{\epsilon'}^{-1} \qquad \text{and} \\ y'' &= N^{*}\mathcal{C}(\eta_{\epsilon''}\mathcal{C}(g,-)\eta_{\epsilon}^{-1}, H(\epsilon'; -))(y') \\ &= \eta_{\epsilon''}\mathcal{C}(fg,-)\eta_{\epsilon'}^{-1}. \end{aligned}$$

Hence by Equation (10) we have

$$\begin{aligned} \theta_{\mathcal{C}}(g, H(\epsilon'; -))(y) &= \eta_{G(\lambda^{\epsilon''})}^{-1}(H(\epsilon'; -))(y'') \\ &= G(\lambda^{\epsilon''}) \star \eta_{\epsilon''}\mathcal{C}(fg,-)\eta_{\epsilon'}^{-1}. \end{aligned}$$

Therefore if $x = \epsilon' \star f^\circ \in \boldsymbol{E}_{\mathcal{C}}(c, H(\epsilon'; -)) = H(\epsilon'; c)$, then by the definition of $\varpi(c, H(\epsilon'; -))$–see Equation (12),

$$\begin{aligned}\theta_{\mathcal{C}}(g, H(\epsilon'; -))\left(\varpi(c, H(\epsilon'; -))(x)\right) &= \theta_{\mathcal{C}}(g, H(\epsilon'; -))(\boldsymbol{G}(\lambda^\epsilon) \star (\eta_\epsilon \mathcal{C}(f, -)\eta_{\epsilon'}^{-1})^\circ \\ &= \theta_{\mathcal{C}}(g, H(\epsilon'; -))(y) \\ &= \boldsymbol{G}(\lambda^{\epsilon''}) \star \eta_{\epsilon''}\mathcal{C}(fg, -)\eta_{\epsilon'}^{-1}.\end{aligned}$$

Similarly using the definitions of $\boldsymbol{E}_{\mathcal{C}}$ (cf. Equation (I.12)) and ϖ we obtain

$$\begin{aligned}\varpi(c'', H(\epsilon'; -))\left(\boldsymbol{E}_{\mathcal{C}}(g, H(\epsilon'; -))(x)\right) &= \varpi(c'', H(\epsilon'; -))\left(H(\epsilon'; g))(\epsilon' \star f^\circ)\right) \\ &= \varpi(c'', H(\epsilon'; -))(\epsilon' \star fg^\circ) \\ &= \boldsymbol{G}(\lambda^{\epsilon''}) \star \eta_{\epsilon''}\mathcal{C}(fg, -)\eta_{\epsilon'}^{-1}\end{aligned}$$

and so the diagram $(*)$ commutes.

Now suppose that $\eta \colon H(\epsilon'; -) \to H(\epsilon''; -)$ be a morphism in $\boldsymbol{N}^*\mathcal{C}$. Then by Lemma III.21, there is $g \colon c_{\epsilon''} \to c_{\epsilon'}$ such that $\eta = \eta_{\epsilon'}\mathcal{C}(g, -)\eta_{\epsilon''}^{-1}$. Now for any $\epsilon \in E(S)$ with $c_\epsilon = c$, by Lemma 7, we have

$$\theta_{\mathcal{C}}(c, \eta) = \theta_{\mathcal{C}}(c)(\eta) = H(\boldsymbol{G}(\lambda^\epsilon); \eta)$$

and by Equation (III.6), $H(\boldsymbol{G}(\lambda^\epsilon); \eta)(\boldsymbol{G}(\lambda^\epsilon) \star (\eta')^\circ = \boldsymbol{G}(\lambda^\epsilon) \star (\eta'\eta)^\circ$. Hence if $x = \epsilon' \star f^\circ \in \boldsymbol{E}_{\mathcal{C}}(c, H(\epsilon'; -)) = H(\epsilon'; c)$, using the definition of $\varpi(c, H(\epsilon'; -))$ and the equation above, we obtain

$$\begin{aligned}\theta_{\mathcal{C}}\left(\varpi(c, H(\epsilon'; -))(x)\right) &= \theta_{\mathcal{C}}(c, \eta)\left(\boldsymbol{G}(\lambda^\epsilon) \star (\eta_\epsilon \mathcal{C}(f, -)\eta_{\epsilon'}^{-1})^\circ\right) \\ &= \boldsymbol{G}(\lambda^\epsilon) \star \left((\eta_\epsilon \mathcal{C}(f, -)\eta_{\epsilon'}^{-1})\eta\right)^\circ \\ &= \boldsymbol{G}(\lambda^\epsilon) \star (\eta_\epsilon \mathcal{C}(gf, -)\eta_{\epsilon''}^{-1})^\circ.\end{aligned}$$

By Equation (I.12),

$$\boldsymbol{E}_{\mathcal{C}}(c, \eta) = \eta(c) = \eta_{\epsilon'}(c)\mathcal{C}(g, c)\eta_{\epsilon''}^{-1}(c)$$

and so $\boldsymbol{E}_{\mathcal{C}}(c, \eta)(x) = \epsilon'' \star gf^\circ$. Hence

$$\begin{aligned}\varpi(c, H(\epsilon''; -))\left(\boldsymbol{E}_{\mathcal{C}}(c, \eta)(x)\right) &= \varpi(c, H(\epsilon''; -))(\epsilon'' \star gf^\circ) \\ &= \boldsymbol{G}(\lambda^\epsilon) \star (\eta_\epsilon \mathcal{C}(gf, -)\eta_{\epsilon''}^{-1})^\circ\end{aligned}$$

This proves that the following diagram commutes.

$$(**)\qquad \begin{array}{ccc} \boldsymbol{E}_{\mathcal{C}}(c, H(\epsilon'; -)) & \xrightarrow{\varpi(c,H(\epsilon';-))} & \theta_{\mathcal{C}}(c, H(\epsilon'; -)) \\ {\scriptstyle \boldsymbol{E}_{\mathcal{C}}(c,\eta)}\big\downarrow & & \big\downarrow{\scriptstyle \theta_{\mathcal{C}}(c,\eta)} \\ \boldsymbol{E}_{\mathcal{C}}(c, H(\epsilon''; -)) & \xrightarrow[\varpi(c,H(\epsilon'';-))]{} & \theta_{\mathcal{C}}(c, H(\epsilon''; -)) \end{array}$$

It now follows from the bifunctor criterion that for any morphism $(g, \eta) \in \mathcal{C} \times \boldsymbol{N}^*\mathcal{C}$ with $g\colon c \to c''$ and $\eta\colon H(\epsilon'; -) \to H(\epsilon''; -)$, we have

$$(\bullet)\qquad \theta_{\mathcal{C}}(g, \eta)\,\big(\varpi(c, H(\epsilon'; -))(x)\big) = \varpi(c'', H(\epsilon''; -))\,(\boldsymbol{E}_{\mathcal{C}}(g, \eta)(x))$$

for all $x = \epsilon' \star f^\circ \in \boldsymbol{E}_{\mathcal{C}}(c, H(\epsilon'; -))$. For by Equation (I.7b) the left-hand side of the equation above is

$$\begin{aligned} &= \theta_{\mathcal{C}}(g, H(\epsilon''; -))\,\big(\theta_{\mathcal{C}}(c, \eta)\,\big(\varpi(c, H(\epsilon'; -))(x)\big)\big) \\ &= \theta_{\mathcal{C}}(g, H(\epsilon''; -))\,\big(\varpi(c, H(\epsilon''; -))\,(\boldsymbol{E}_{\mathcal{C}}(c, \eta)(x))\big) \quad \text{by } (**) \\ &= \varpi(c'', H(\epsilon''; -))\,\big(\boldsymbol{E}_{\mathcal{C}}(g, H(\epsilon''; -))\,(\boldsymbol{E}_{\mathcal{C}}(c, \eta)(x))\big) \quad \text{by } (*) \\ &= \varpi(c'', H(\epsilon''; -))\,(\boldsymbol{E}_{\mathcal{C}}(g, \eta)(x)) \end{aligned}$$

by Equation (I.7b), which is equal to the right-hand side of $(\bullet)$. Hence ϖ is a natural isomorphism as required. □

We have thus proved the following.

Theorem 11 *Let $\mathcal{C}$ be a normal category. Then there exists a local isomorphism $\theta_{\mathcal{C}}\colon \mathcal{C} \to \boldsymbol{N}^{**}\mathcal{C}$ such that the bifunctor $\theta_{\mathcal{C}}(-,-)$ from $\mathcal{C} \times \boldsymbol{N}^*\mathcal{C}$ to* **Set** *corresponding to $\theta_{\mathcal{C}}$ is naturally isomorphic to $\boldsymbol{E}_{\mathcal{C}}$.* □

By Corollary 3 the dual connection of $S = T\mathcal{C}$ is an embedding if and only if S is left reductive. From the definition of the functor $\theta_{\mathcal{C}}$ (see Diagram(9)), it follows that $\theta_{\mathcal{C}}$ is an embedding of $\mathcal{C}$ into $\boldsymbol{N}^{**}\mathcal{C}$ if and only if the dual connection of S is an embedding. Thus $\theta_{\mathcal{C}}$ is an embedding if and only if S is left reductive. This is clearly a condition on the category $\mathcal{C}$ which can be directly stated as a property of normal cones on $\mathcal{C}$.

3 DUALS OF CONNECTIONS

In this section, we show that given a local isomorphism $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ where $\mathcal{C}$ and $\mathcal{D}$ are normal categories, there exists an ideal $\mathcal{C}'$ of $\mathcal{C}$ and a local isomorphism $\Gamma^* : \mathcal{C} \to N^*\mathcal{D}$. The functor Γ^* will be called the *dual* of Γ.

3.1 Connections of normal categories

A local isomorphism of the type mentioned above will be called a *connection* of normal categories; more specifically, if $\mathcal{D}$ with $\mathcal{C}$ are normal categories, a local isomorphism $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ will be called a connection of $\mathcal{D}$ with $\mathcal{C}$. Note that the connection of a regular semigroup S (see Theorem 2 and Definition 2) is a connection of $\mathbb{R}(S)$ with $\mathbb{L}(S)$ in this sense.

Proposition 12 *Let $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ be a connection between normal categories. Then we have the following.*

(a) Let $\mathcal{C}_\Gamma$ denote the subcategory of $\mathcal{C}$ such that

(13) $$v\mathcal{C}_\Gamma = \{c \in \mathcal{C} : c \in M\Gamma(d) \text{ for some } d \in v\mathcal{D}\}.$$

Then $\mathcal{C}_\Gamma$ is an ideal in $\mathcal{C}$.

(b) Suppose that

(14a) $$\mathcal{T}_\Gamma = \{\tau \in E(TN^*\mathcal{C}) : \exists\ d \in v\mathcal{D} \text{ with } c_\tau = \Gamma(d)\}.$$

Let $\tilde{g}(\mathcal{D})$ denote the full subcategory of $N^{**}\mathcal{C}$ with

$$\text{(14b)} \qquad v\tilde{g}(\mathcal{D}) = \{ H(\tau; -) : \tau \in T_\Gamma \}.$$

Then $\tilde{g}(\mathcal{D})$ is an ideal in $N^{**}\mathcal{C}$.

Proof (a): It is sufficient to show that $c \in v\mathcal{C}_\Gamma$ and $c' \subseteq c$ implies $c' \in v\mathcal{C}_\Gamma$. Let $c \in M\Gamma(d)$ with $d \in v\mathcal{D}$. Then by Proposition III.4(i), there is $\epsilon \in E(T\mathcal{C})$ with $c_\epsilon = c$ and $\Gamma(d) = H(\epsilon; -)$. Let $h: c \to c'$ be a retraction and $\epsilon' = \epsilon \star h^\circ = \epsilon \star h$. Then $\epsilon' \,\omega\, \epsilon$ and $c' = c_{\epsilon'} \in MH(\epsilon'; -)$. Also, by Proposition III.7(a), we have $H(\epsilon'; -) \subseteq H(\epsilon; -) = \Gamma(d)$. Since Γ is a local isomorphism there is a unique $d' \subseteq d$ such that $\Gamma(d') = H(\epsilon'; -)$. Thus $c' \in M\Gamma(d')$ and so $c' \in v\mathcal{C}_\Gamma$ by the definition of $\mathcal{C}_\Gamma$.

(b): Let $H(\tau; -) \in v\tilde{g}(\mathcal{D})$ and $H(\tau'; -) \subseteq H(\tau; -)$. An application of Proposition III.7 shows that we can choose the cone τ' to be an idempotent so that $\tau' \,\omega\, \tau$ in $E(TN^*\mathcal{C})$. Since $N^*\mathcal{C}$ is normal, again by Proposition III.7, there is an idempotent $\epsilon' \in E(T\mathcal{C})$ such that $H(\epsilon'; -) = c_{\tau'}$. Hence

$$H(\epsilon'; -) = c_{\tau'} \subseteq c_\tau = \Gamma(d)$$

for some $d \in v\mathcal{D}$. Since Γ is a local isomorphism there is a unique $d' \subseteq d$ such that $\Gamma(d') = H(\epsilon'; -) = c_{\tau'}$. This implies, by the definition of $v\tilde{g}(\mathcal{D})$ that $H(\tau'; -) \in v\tilde{g}(\mathcal{D})$. Hence $\tilde{g}(\mathcal{D})$ is an ideal in $N^{**}\mathcal{C}$. □

In particular, it follows that $\mathcal{C}_\Gamma$ is a normal subcategory of $\mathcal{C}$ and $\tilde{g}(\mathcal{D})$ is a normal subcategory of $N^{**}\mathcal{C}$.

3.2 Dual of a connection

In the following we use the notations introduced in Proposition 12 above.

Lemma 13 *Let $\tau \in T_\Gamma$ where T_Γ denote the set of normal cones in $N^*\mathcal{C}$ defined by Equation* (14a). *Then for each $d \in v\mathcal{D}$ such that $c_\tau = \Gamma(d)$ there exists a unique normal cone $\tilde{\tau}$ in $\mathcal{D}$ with vertex d such that*

$$\text{(15)} \qquad \Gamma\left(\tilde{\tau}(d')\right) = \tau\left(\Gamma(d')\right)$$

for all $d' \in v\mathcal{D}$.

Proof Let $\tau \in \mathcal{T}_\Gamma$. Then by Equation (14a) (see Proposition 12(b)), there is $d \in v\mathcal{D}$ such that $c_\tau = \Gamma(d)$. Since Γ is fully-faithful, for each $d' \in v\mathcal{D}$, there is a unique morphism $\tilde{\tau}(d'): d' \to d$ such that $\Gamma(\tilde{\tau}(d')) = \tau(\Gamma(d'))$. Since Γ is a local isomorphism it preserves inclusions and preserves and reflects isomorphisms. From this it follows immediately that the map $\tilde{\tau}: d' \to \tilde{\tau}(d')$ is a normal cone in $\mathcal{D}$ to the vertex d. Given the cone τ in $N^*\mathcal{C}$ and $d \in v\mathcal{D}$ such that $c_\tau = \Gamma(d)$, the cone $\tilde{\tau}$ in $\mathcal{D}$ with $c_{\tilde{\tau}} = d$ constructed above is clearly unique since Γ is fully-faithful. □

We use the construction described in the lemma above in the proof of the following.

Proposition 14 *Let $\Gamma: \mathcal{D} \to N^*\mathcal{C}$ be a connection of $\mathcal{D}$ with $\mathcal{C}$. Then there exists a local isomorphism $\tilde{\Gamma}$ of $\tilde{g}(\mathcal{D})$ into $N^*\mathcal{D}$.*

Proof Let $\tau: vN^*\mathcal{C} \xrightarrow{\cdot} \Gamma(d)$ be a normal cone in $N^*\mathcal{C}$ so that $H(\tau; -) \in v\tilde{g}(\mathcal{D})$. Define

$$v\tilde{\Gamma}(H(\tau; -)) = H(\tilde{\tau}; -) \tag{16a}$$

where $\tilde{\tau}$ is the normal cone in $\mathcal{D}$ satisfying Equation (15). We first show that $v\tilde{\Gamma}$ is an order preserving map of $v\tilde{g}(\mathcal{D})$ into $vN^*\mathcal{D}$ which induces isomorphisms on every principal (order) ideals of $v\tilde{g}(\mathcal{D})$. If $\Gamma(d) = c_\tau = \Gamma(d')$ and $\tilde{\tau}: v\mathcal{D} \xrightarrow{\cdot} d$, $\tilde{\tilde{\tau}}: v\mathcal{D} \xrightarrow{\cdot} d'$ are two normal cones in $\mathcal{D}$ constructed as in the lemma above (see Equation (15)), then $\Gamma(d) = \Gamma(d')$ and there exists a unique isomorphism $f: d \to d'$ such that $\Gamma(f) = 1_{\Gamma(d)}$. Then $\Gamma(\tilde{\tau}(d'')f) = \Gamma(\tilde{\tilde{\tau}}(d''))$ for all $d'' \in v\mathcal{D}$ and since Γ is fully-faithful, we have $\tilde{\tau}(d'')f = \tilde{\tilde{\tau}}(d'')$ for all $d'' \in v\mathcal{D}$. Therefore $\tilde{\tau} \star f = \tilde{\tilde{\tau}}$ by Equation (III.2). It follows from Proposition III.7(a) that $H(\tilde{\tau}; -) = H(\tilde{\tilde{\tau}}; -)$. Suppose that $H(\tau; -), H(\tau'; -) \in v\tilde{g}(\mathcal{D})$ and $H(\tau; -) \subseteq H(\tau'; -)$. Let $c_\tau = \Gamma(d)$ and $c_{\tau'} = \Gamma(d')$. Then by Proposition III.7(a), there is a unique epimorphism $\bar{h}: \Gamma(d') \to \Gamma(d)$ such that $\tau = \tau' \star \bar{h}$. Let $h: d' \to d$ such that $\Gamma(h) = \bar{h}$. Then it is easy to see that $\tilde{\tau} = \tilde{\tau}' \star h$ and so,

$$\tilde{\Gamma}(H(\tau; -)) = H(\tilde{\tau}; -) \subseteq H(\tilde{\tau}'; -) = \tilde{\Gamma}(H(\tau'; -)).$$

In particular, $H(\tau; -) = H(\tau'; -)$ implies $H(\tilde{\tau}; -) = H(\tilde{\tau}'; -)$. Hence $v\tilde{\Gamma}$ is a single-valued, order-preserving map.

To show that $\boldsymbol{v}\tilde{\Gamma}$ is injective on each principal ideal of $\boldsymbol{v}\tilde{g}(\mathcal{D})$, suppose that $H(\tau_i; -), H(\tau; -) \in \boldsymbol{v}\tilde{g}(\mathcal{D})$ with vertices $\Gamma(d_i)$, $\Gamma(d)$ respectively and $H(\tau_i; -) \subseteq H(\tau; -)$ for $i = 1, 2$. Then, as above, there are epimorphisms $h_i \colon d \to d_i$ such that $\tau_i = \tau \star \Gamma(h_i)$ and so, $\tilde{\tau}_i = \tilde{\tau} \star h_i$ for $i = 1, 2$. If $H(\tilde{\tau}_1; -) = H(\tilde{\tau}_2; -)$, by Proposition III.7(a), there exists an isomorphism $g \colon d_1 \to d_2$ such that $\tilde{\tau}_2 = \tilde{\tau}_1 \star g$ from which we find that $h_2 = \tilde{\tau}_2(d) = \tilde{\tau}_1(d)g = h_1 g$. Hence

$$\begin{aligned}\tau_2 = \tau \star \Gamma(h_2) &= \tau \star \Gamma(h_1 g)\\ &= (\tau \star \Gamma(h_1)) \star \Gamma(g)\\ &= \tau_1 \star \Gamma(g).\end{aligned}$$

Since $\Gamma(g)$ is an isomorphism it follows, by Proposition III.7(a), that

$$H(\tau_1; -) = H(\tau_2; -).$$

So $\boldsymbol{v}\tilde{\Gamma}$ is an isomorphism on each principal ideal of $\boldsymbol{v}\tilde{g}(\mathcal{D})$. To show that $\boldsymbol{v}\tilde{\Gamma}$ is surjective on principal ideals, let $H(\sigma; -) \subseteq H(\tilde{\tau}; -)$. Then there exists an epimorphism $h \colon d \to d'$ where $d = c_{\tilde{\tau}}$ and $d' = c_\sigma$ such that $\sigma = \tilde{\tau} \star h$. Hence $1_{d'} = \sigma(d') = \tilde{\tau}(d')h$ and so, we have $1_{\Gamma(d')} = \tau(\Gamma(d'))\Gamma(h)$. Thus $\tau' = \tau \star \Gamma(h)$ is an idempotent normal cone in $N^{**}\mathcal{C}$ with vertex $\Gamma(d')$. Also for any $d'' \in \boldsymbol{v}\mathcal{D}$,

$$\begin{aligned}\Gamma(\sigma(d'')) &= \Gamma(\tilde{\tau}(d''))\Gamma(h)\\ &= \tau(\Gamma(d''))\Gamma(h)\\ &= \tau'(\Gamma(d''))\\ &= \Gamma(\tilde{\tau}'(d''))\end{aligned}$$

which shows that $\sigma = \tilde{\tau}'$ and so, we have $\boldsymbol{v}\tilde{\Gamma}(H(\tau'; -)) = H(\sigma; -)$. This proves that $\boldsymbol{v}\tilde{\Gamma}$ is surjective on principal ideals. Therefore $\boldsymbol{v}\tilde{\Gamma}$ is an isomorphism on each principal ideals.

Let $\eta \colon H(\tau; -) \to H(\tau'; -)$ be a morphism in $\tilde{g}(\mathcal{D})$. Let $c_\tau = \Gamma(d)$ and $c_{\tau'} = \Gamma(d')$. Since η is a morphism in $N^{**}\mathcal{C}$, by Lemma III.21 there exists a unique morphism $\widehat{\eta} \colon \Gamma(d') \to \Gamma(d)$ such that $\eta = \eta_\tau N^*\mathcal{C}(\widehat{\eta}, -)\eta_{\tau'}^{-1}$ and since Γ is fully-faithful there exists a unique $f \colon d' \to d$ such that $\Gamma(f) = \widehat{\eta}$. Define

$$\tilde{\Gamma}(\eta) = \eta_{\tilde{\tau}}\mathcal{D}(f, -)\eta_{\tilde{\tau}'}^{-1} \tag{16b}$$

where $\tilde{\tau}$ and $\tilde{\tau}'$ are normal cones in $\mathcal{D}$ satisfying Equation (15) (with respect to τ and τ' respectively). If $\tilde{\tau}_1$ and $\tilde{\tau}_1'$ are also normal cones with vertices d_1 and d_1' constructed from τ and τ' respectively by Lemma 13, then we see as above that $\tilde{\tau} \star g = \tilde{\tau}_1$ and $\tilde{\tau}' \star h = \tilde{\tau}_1'$ for some isomorphisms g and h in $\mathcal{D}$ and it follows using Corollary III.9 that there exists a morphism $f_1 \colon d_1' \to d_1$ such that

$$\eta_{\tilde{\tau}} \mathcal{D}(f, -) \eta_{\tilde{\tau}'}^{-1} = \eta_{\tilde{\tau}_1} \mathcal{D}(f_1, -) \eta_{\tilde{\tau}_1'}^{-1}.$$

Hence the morphism map of $\tilde{\Gamma}$ defined by (16b) is single-valued. Let

$$\eta = \eta_\tau \boldsymbol{N}^*\mathcal{C}(\Gamma(f), -)\eta_{\tau'}^{-1} \quad \text{and} \quad \eta' = \eta_{\tau'} \boldsymbol{N}^*\mathcal{C}(\Gamma(g), -)\eta_{\tau''}^{-1}$$

be composable morphisms in $\tilde{g}(\mathcal{D})$. Then clearly $\eta\eta' = \eta_\tau \boldsymbol{N}^*\mathcal{C}(\Gamma(gf), -)\eta_{\tau''}^{-1}$ and so

$$\begin{aligned} \tilde{\Gamma}(\eta\eta') &= \eta_{\tilde{\tau}} \mathcal{D}(gf, -) \eta_{\tilde{\tau}''}^{-1} \\ &= \eta_{\tilde{\tau}} \mathcal{D}(f, -) \eta_{\tilde{\tau}'}^{-1} \eta_{\tilde{\tau}'} \mathcal{D}(g, -) \eta_{\tilde{\tau}''}^{-1} \\ &= \tilde{\Gamma}(\eta)\tilde{\Gamma}(\eta'). \end{aligned}$$

Since $\tilde{\Gamma}$ clearly preserves identities, it is a functor from $\tilde{g}(\mathcal{D})$ to $\boldsymbol{N}^*\mathcal{D}$. Now by Lemma III.21, the map $\eta \mapsto \widehat{\eta}$ is a bijection of $\boldsymbol{N}^{**}\mathcal{C}(H(\tau; -), H(\tau'; -))$ with $\boldsymbol{N}^*\mathcal{C}(\Gamma(d'), \Gamma(d'))$. Similarly, the map $f \mapsto \eta_{\tilde{\tau}}\mathcal{D}(f, -)\eta_{\tilde{\tau}'}^{-1}$ is a bijection of $\mathcal{D}(d', d)$ with $\boldsymbol{N}^*\mathcal{D}(H(\tilde{\tau}; -), H(\tilde{\tau}, ;))$. Since Γ is fully-faithful, it induces a bijection of $\mathcal{D}(d', d)$ with $\boldsymbol{N}^*\mathcal{C}(\Gamma(d'), \Gamma(d))$. Hence it follows from (16b) that $\tilde{\Gamma}$ is a bijection of $\tilde{g}(\mathcal{D})(H(\tau; -), H(\tau'; -))$ onto $\boldsymbol{N}^*\mathcal{D}(H(\tilde{\tau}; -), H(\tilde{\tau}'; -))$ and so $\tilde{\Gamma}$ is fully-faithful. Finally, it follows from Lemma III.21 that if η is an inclusion so is $\Gamma(\eta)$. Therefore $\tilde{\Gamma}$ is a local isomorphism. □

Given a connection $\Gamma \colon \mathcal{D} \to \boldsymbol{N}^*\mathcal{C}$, we now construct a connection of $\mathcal{C}_\Gamma$ with $\mathcal{D}$ using the proposition above, where $\mathcal{C}_\Gamma$ is the ideal of $\mathcal{C}$ defined in Proposition 12.

Theorem 15 *Let $\Gamma \colon \mathcal{D} \to \boldsymbol{N}^*\mathcal{C}$ be a connection. Then there exists a connection $\Gamma^* \colon \mathcal{C}_\Gamma \to \boldsymbol{N}^*\mathcal{D}$ such that, for $c \in \boldsymbol{v}\mathcal{C}_\Gamma$ and $d \in \boldsymbol{v}\mathcal{D}$, $c \in M\Gamma(d)$ if and only if $d \in M\Gamma^*(c)$.*

Proof In the following proof we shall denote the semigroup $T\mathcal{C}$ by S. Let $\theta = \theta_\mathcal{C}$ be the local isomorphism defined by the commutative diagram (9).

We first show that $\theta|\mathcal{C}_\Gamma$ is a local isomorphism of $\mathcal{C}_\Gamma$ to $\tilde{g}(\mathcal{D})$. Let $c \in \boldsymbol{v}\mathcal{C}_\Gamma$. Then by Proposition 12(a), there is $d \in \boldsymbol{v}\mathcal{D}$ such that $c \in M\Gamma(d)$. Choose $\epsilon \in E(S)$ such that $c_\epsilon = c$ and $\Gamma(d) = H(\epsilon; -)$. Then by Equation (10), $\theta(c) = H(\boldsymbol{G}(\lambda^\epsilon); -)$ where $\boldsymbol{G}(\lambda^\epsilon)$ is a cone in $\boldsymbol{N}^*\mathcal{C}$ with vertex $H(\epsilon; -) = \Gamma(d)$. Hence by Proposition 12(b), $\theta(c) \in \boldsymbol{v}\tilde{g}(\mathcal{D})$. Since $\tilde{g}(\mathcal{D})$ is a full subcategory of $\boldsymbol{N}^{**}\mathcal{C}$, this implies that $\theta|\mathcal{C}_\Gamma$ is a local isomorphism of $\mathcal{C}_\Gamma$ to $\tilde{g}(\mathcal{D})$. Hence

$$\Gamma^* = (\theta|\mathcal{C}_\Gamma) \circ \tilde{\Gamma} \tag{17}$$

is a local isomorphism of $\mathcal{C}_\Gamma$ to $\boldsymbol{N}^*\mathcal{D}$. Let $c \in M\Gamma(d)$ with $c \in \boldsymbol{v}\mathcal{C}_\Gamma$ and $d \in \boldsymbol{v}\mathcal{D}$. We can find $\epsilon \in E(S)$ such that $c_\epsilon = c$ and $\Gamma(d) = H(\epsilon; -)$. Then $\boldsymbol{G}(\lambda^\epsilon)$ is a cone in $\boldsymbol{N}^*\mathcal{C}$ with vertex $\Gamma(d)$ and by Equation (10), $\theta(c) = H(\boldsymbol{G}(\lambda^\epsilon); -)$. If $\tilde{\boldsymbol{G}}(\lambda^\epsilon)$ denotes the cone in $\mathcal{D}$ constructed in Lemma 13, then $\tilde{\boldsymbol{G}}(\lambda^\epsilon)$ has vertex d. By Equation (16a) and the definition of Γ^* above, we have $\Gamma^*(c) = H(\tilde{\boldsymbol{G}}(\lambda^\epsilon); -)$. Hence $d \in M\Gamma^*(c)$.

Conversely, let $d \in M\Gamma^*(c)$. If $\epsilon \in E(S)$ with $c_\epsilon = c$, then we have $\Gamma^*(c) = H(\tilde{\boldsymbol{G}}(\lambda^\epsilon); -)$ where $\tilde{\boldsymbol{G}}(\lambda^\epsilon)$ denotes the cone in $\mathcal{D}$ constructed in Lemma 13. Therefore the component of the cone $\tilde{\boldsymbol{G}}(\lambda^\epsilon)$ at d is an isomorphism and so the component of the cone $\boldsymbol{G}(\lambda^\epsilon)$ in $\boldsymbol{N}^*\mathcal{C}$ at $\Gamma(d)$ is an isomorphism by Equation (15). Hence $\Gamma(d) \in M\boldsymbol{G}(\lambda^\epsilon)$. Since $\boldsymbol{G}\colon \mathcal{R}(S) \to \boldsymbol{N}^*\mathcal{C}$ is an isomorphism (by Theorem III.25), there is a unique $\epsilon''S \in \boldsymbol{v}\mathcal{R}(S)$ such that $\epsilon''S \in M\lambda^\epsilon$ and $\boldsymbol{G}(\epsilon''S) = \Gamma(d)$. By the dual of Lemma III.15 we may assume that $\epsilon''\mathcal{L}\epsilon$ and so, $c_{\epsilon''} = c_\epsilon = c$. Also, by Theorem III.25, $\boldsymbol{G}(\epsilon''S) = H(\epsilon''; -)$. Hence $c \in M\boldsymbol{G}(\epsilon''S) = M\Gamma(d)$ □

Definition 4 *Let $\Gamma\colon \mathcal{D} \to \boldsymbol{N}^*\mathcal{C}$ be a connection. The functor $\Gamma^*\colon \mathcal{C}_\Gamma \to \boldsymbol{N}^*\mathcal{D}$ defined by Equation* (17) *is called the* dual *of the connection $\Gamma\colon \mathcal{D} \to \boldsymbol{N}^*\mathcal{C}$.*

For the rest of this section, Γ will denote a connection (of $\mathcal{D}$ with $\mathcal{C}$) and Γ^*, its dual.

3.3 The duality between a connection and its dual

It is clear that Γ and Γ^* determine bifunctors from $\mathcal{C}_\Gamma \times \mathcal{D}$ to **Set** (defined by Equation (I.12)). Since no confusion is likely we denote them by Γ and Γ^* itself respectively. We proceed to show that the bifunctors Γ and Γ^* are naturally equivalent.

Given the connection $\Gamma: \mathcal{D} \to N^*\mathcal{C}$, define the bifunctor Θ from $\mathcal{C}_\Gamma \times \mathcal{D}$ to **Set** as follows:

$$(\star) \qquad \Theta(c,d) = \mathcal{O}_\mathcal{C}(c, \Gamma(d)); \qquad \Theta(f,g) = \mathcal{O}_\mathcal{C}(f, \Gamma(g)).$$

Observe that $\Theta = (1_{\mathcal{C}_\Gamma} \times \Gamma) \circ \mathcal{O}_\mathcal{C}$ so that Θ is clearly a bifunctor. Also, for each $(c,d) \in \boldsymbol{v}\mathcal{C} \times \boldsymbol{v}\mathcal{D}$, define

$$(\star\star) \qquad \tilde{\omega}(c,d) = \varpi(c, \Gamma(d))$$

where ϖ is the map defined by Equation (12). Then for each (c,d), $\tilde{\omega}(c,d)$ is a bijection of $\boldsymbol{E}_\mathcal{C}(c,\Gamma(d)) = \Gamma(d)(c) = \Gamma(c,d)$ into $\mathcal{O}_\mathcal{C}(c,\Gamma(d)) = \Theta(c,d)$ by Lemma 9. It follows from Lemma 10 that the map $(c,d) \mapsto \tilde{\omega}(c,d)$ is a natural isomorphism $\tilde{\omega}: \Gamma(-,-) \to \Theta$.

Next, let $(c,d) \in \boldsymbol{v}\mathcal{C}_\Gamma \times \boldsymbol{v}\mathcal{D}$. It follows from Lemma 13 and $(\star)$ that $G(\lambda^{\epsilon'}) \star (\eta_{\epsilon'}C(f,-)\eta_\epsilon^{-1})^\circ \in \Theta(c,d)$ if $\epsilon' \in E(S)$ with $c_{\epsilon'} = c$ and $H(\epsilon';-) = \Gamma(d')$ for some $d' \in \boldsymbol{v}\mathcal{D}$ (such d' exists whenever $c = c_{\epsilon'} \in \boldsymbol{v}\mathcal{C}_\Gamma$), $H(\epsilon;-) = \Gamma(d)$ and $f: c_\epsilon \to c$. Since Γ is a local isomorphism there is a unique $g: d' \to d$ such that $\Gamma(g) = \eta_{\epsilon'}C(f,-)\eta_\epsilon^{-1}$. Therefore choosing ϵ, ϵ' and d' as above we can represent any element in $\Theta(c,d)$ in the form $G(\lambda^{\epsilon'}) \star \Gamma(g)^\circ$ for $g \in \mathcal{D}(d',d)$. Also, in this case, if $\tilde{G}(\lambda^{\epsilon'})$ is the cone to the vertex d' constructed as in Lemma 13 from the cone $G(\lambda^{\epsilon'})$, then from the definitions of $\tilde{\Gamma}$ and Γ^* (cf. Equations (16a) and (17)) it follows that $\tilde{G}(\lambda^{\epsilon'}) \star g^\circ \in \Gamma^*(c,d)$ for all $g: d' \to d$. Define

$$(\bullet) \qquad \phi(c,d)(G(\lambda^{\epsilon'}) \star \Gamma(g)^\circ) = \tilde{G}(\lambda^{\epsilon'}) \star g^\circ$$

where $\epsilon' \in E(S)$ with $c_{\epsilon'} = c$ and $\Gamma(d') = H(\epsilon';-)$ for some $d' \in \boldsymbol{v}\mathcal{D}$ and $g: d' \to d \in \mathcal{D}$. To show that $\phi(c,d)$ is well-defined, we must show that $\phi(c,d)$ is independent of the representation chosen for the cones $G(\lambda^{\epsilon'}) \star \Gamma(g)^\circ \in \Theta(c,d)$ and $\tilde{G}(\lambda^{\epsilon'}) \star g^\circ \in \Gamma^*(c,d)$; that is the definition above does not depend on the choice of ϵ' and d'. Accordingly, suppose that ϵ', $\epsilon'' \in E(S)$ with $c_{\epsilon'} =$

$c_{\epsilon''} = c$ and fix $d', d'' \in \boldsymbol{v}\mathcal{D}$ such that $H(\epsilon';-) = \Gamma(d')$, $H(\epsilon'';-) = \Gamma(d'')$. Then by Lemma 8

$$G(\lambda^{\epsilon''})(c_{G(\lambda^{\epsilon'})}) = G(\lambda^{\epsilon''})(\Gamma(d')) = \eta_{\epsilon'}\eta_{\epsilon''}^{-1}.$$

Hence $G(\lambda^{\epsilon'}) = G(\lambda^{\epsilon''}) \star \eta_{\epsilon''}\eta_{\epsilon'}^{-1}$. Moreover, $\Gamma(d'') = c_{G(\lambda^{\epsilon''})}$ and by Equation (15), there is an isomorphism $k\colon d'' \to d'$ such that $\Gamma(k) = \eta_{\epsilon''}\eta_{\epsilon'}^{-1}$ and $\tilde{G}(\lambda^{\epsilon'}) = \tilde{G}(\lambda^{\epsilon''}) \star k$. Hence

$$\tilde{G}(\lambda^{\epsilon'}) \star g^\circ = \tilde{G}(\lambda^{\epsilon''}) \star kg^\circ$$

and

$$\begin{aligned} G(\lambda^{\epsilon'}) \star \Gamma(g) &= G(\lambda^{\epsilon''}) \star \eta_{\epsilon''}\eta_{\epsilon'}^{-1}\Gamma(g) \\ &= G(\lambda^{\epsilon''}) \star \Gamma(k)\Gamma(g) \\ &= G(\lambda^{\epsilon''}) \star \Gamma(kg) \end{aligned}$$

This proves that $\phi(c,d)$ is independent of the choice of ϵ'. Finally assume that $d', d'' \in \boldsymbol{v}\mathcal{D}$ such that $\Gamma(d') = \Gamma(d'') = H(\epsilon';-)$. Let $\tilde{G}(\lambda^{\epsilon'})$ and $\tilde{\tilde{G}}(\lambda^{\epsilon})$ be cones constructed by Lemma 13 using $G(\lambda^{\epsilon'})$ to the vertices d' and d'' respectively. If $t\colon d'' \to d'$ is the isomorphism such that $\Gamma(t) = 1_{\Gamma(d')}$, then it follows as in the proof of Proposition 14 that $\tilde{\tilde{G}}(\lambda^{\epsilon'}) \star t = \tilde{G}(\lambda^{\epsilon'})$. Therefore we get $\tilde{G}(\lambda^{\epsilon'}) \star g^\circ = \tilde{\tilde{G}}(\lambda^{\epsilon'}) \star tg^\circ$ and $\Gamma(g) = \Gamma(tg)$. This proves that $\phi(c,d)$ is well-defined. Moreover

$$\begin{aligned} G(\lambda^{\epsilon'}) \star (\Gamma(g))^\circ &\leftrightarrow \Gamma(g); \\ \tilde{G}(\lambda^{\epsilon'}) \star g^\circ &\leftrightarrow g \end{aligned}$$

are bijections of $\Theta(c,d)$ onto $\boldsymbol{N}^*\mathcal{C}(\Gamma(d'),\Gamma(d))$ and $\Gamma^*(c,d)$ onto $\mathcal{D}(d',d)$ respectively since they are components of the natural isomorphisms $\eta_{G(\lambda^{\epsilon'})}$ and $\eta_{\tilde{G}(\lambda^{\epsilon'})}$ respectively. Since Γ is a local isomorphism, the map $g \leftrightarrow \Gamma(g)$ is a bijection of $\mathcal{D}(d',d)$ onto $\boldsymbol{N}^*\mathcal{C}(\Gamma(d'),\Gamma(d)) = \boldsymbol{N}^*\mathcal{C}(H(\epsilon';-),H(\epsilon;-))$. It follows that the map $\phi(c,d)$ defined by $(\bullet)$ is a bijection.

We now prove that the map $(c,d) \mapsto \phi(c,d)$ is a natural isomorphism $\phi\colon \Theta \to \Gamma^*$. Let $f \in \mathcal{C}(c,c')$. Fix ϵ, ϵ' and ϵ'_1 in $E(S)$ such that $H(\epsilon;-) = \Gamma(d)$, $c_{\epsilon'} = c$, $H(\epsilon';-) = \Gamma(d')$, $c_{\epsilon'_1} = c'$ and $H(\epsilon'_1;-) = \Gamma(d'')$ for some $d', d'' \in v\mathcal{D}$. Since Γ is full, there is a unique $h\colon d'' \to d'$ such that $\Gamma(h) = \eta_{\epsilon'_1}\mathcal{C}(f,-)\eta_{\epsilon'}^{-1}$. Note that, since Γ preserves epimorphisms, we have $\Gamma(g)^\circ = \Gamma(g^\circ)$ for all morphism g in $\mathcal{D}$. Now by by the definition of Θ and Equation (10),

$$\Theta(f,-) = \eta_{G(\lambda^{\epsilon'})}N^*\mathcal{C}(\Gamma(h),-)\eta^{-1}_{G(\lambda^{\epsilon'_1})}$$

and hence for any $G(\lambda^{\epsilon'}) \star \Gamma(g)^\circ \in \Theta(c,d)$, by Equation (III.13b), we have

$$\begin{aligned}\Theta(f,d)\left(G(\lambda^{\epsilon'}) \star \Gamma(g)^\circ\right) &= G(\lambda^{\epsilon_1}) \star (\Gamma(h)\Gamma(g))^\circ \\ &= G(\lambda^{\epsilon_1}) \star \Gamma((hg)^\circ)\end{aligned}$$

Similarly, by the definition of Γ^* (cf. Equation (16b)), we get

$$\Gamma^*(f,-) = \eta_{\tilde{G}(\lambda^{\epsilon'})}\mathcal{D}(h,-)\eta^{-1}_{\tilde{G}(\lambda^{\epsilon'_1})}$$

and so, again by Lemma III.21 (cf. Equation (III.13b)),

$$\Gamma^*(f,d)(\tilde{G}(\lambda^{\epsilon'}) \star g^\circ) = \tilde{G}(\lambda^{\epsilon-1'}) \star (hg)^\circ.$$

Therefore, it follows from the definition of ϕ that

$$(*) \qquad \Theta(f,d)\phi(c',d) = \phi(c,d)\Gamma^*(f,d).$$

Now let $h \in \mathcal{D}(d,d_1)$. If $\epsilon' \in E(S)$ with $c_{\epsilon'} = c$ and $\Gamma(d') = H(\epsilon';-)$, then for any $g\colon d' \to d$, by Equation (10) and the definition of Θ, we have

$$\begin{aligned}\Theta(c,\Gamma(h))(G(\lambda^{\epsilon'}) \star (\Gamma(g))^\circ) &= H(G(\lambda^{\epsilon'});\Gamma(h))(G(\lambda^{\epsilon'}) \star (\Gamma(g))^\circ) \\ &= G(\lambda^{\epsilon'}) \star (\Gamma(g)\Gamma(h))^\circ \\ &= G(\lambda^{\epsilon'}) \star \Gamma((gh)^\circ)\end{aligned}$$

and so,

$$\phi(c,d_1)\left(\Theta(c,\Gamma(h))(G(\lambda^{\epsilon'}) \star (\Gamma(g))^\circ)\right) = \tilde{G}(\lambda^{\epsilon'}) \star (gh)^\circ.$$

Similarly, using the definitions of Γ^* and ϕ, we obtain

$$\begin{aligned}\Gamma^*(c,h)\left(\phi(c,d)(G(\lambda^{\epsilon'})\star(\Gamma(g))^\circ)\right) &= \Gamma^*(c,h)\left(\tilde{G}(\lambda^{\epsilon'})\star g^\circ\right)\\ &= H(\tilde{G}(\lambda^{\epsilon'});h)\left(\tilde{G}(\lambda^{\epsilon'})\star g^\circ\right)\\ &= \tilde{G}(\lambda^{\epsilon'})\star(gh)^\circ.\end{aligned}$$

This proves that

$$(**)\qquad \Theta(c,\Gamma(h))\phi(c,d_1) = \phi(c,d)\Gamma^*(c,h).$$

It follows from Equations $(*)$, $(**)$ and the bifunctor criterion that ϕ is a natural isomorphism. The foregoing discussion leads to:

Theorem 16 *For each $(c,d)\in \boldsymbol{v}\mathcal{C}\times\boldsymbol{v}\mathcal{D}$ and for all $\epsilon\star f^\circ\in\Gamma(c,d)$, define*

$$(18)\qquad \chi_\Gamma(c,d)(\epsilon\star f^\circ) = \tilde{G}(\lambda^{\epsilon'})\star g^\circ$$

where $\epsilon\in E(S)$ with $H(\epsilon;-)=\Gamma(d)$, $\epsilon'\in E(S)$ with $c_{\epsilon'}=c$ and $c_{G(\lambda^{\epsilon'})} = H(\epsilon';-)=\Gamma(d')$ for some $d'\in\boldsymbol{v}\mathcal{D}$ and $g\colon d'\to d$ with $\Gamma(g)=\eta_{\epsilon'}\mathcal{C}(f,-)\eta_\epsilon^{-1}$. Then $\chi_\Gamma(c,d)\colon\Gamma(c,d)\to\Gamma^*(c,d)$ is a bijection. Moreover, if χ_Γ denote the map $(c,d)\mapsto\chi_\Gamma(c,d)$ then $\chi_\Gamma\colon\Gamma\to\Gamma^*$ is a natural isomorphism.

Proof It follows from Equations $(\star\star)$ and $(\bullet)$ that $\chi_\Gamma(c,d)=\tilde{\omega}(c,d)\circ\phi(c,d)$ for all $(c,d)\in\mathcal{C}_\Gamma\times\mathcal{D}$ and hence

$$(19)\qquad \chi_\Gamma = \tilde{\omega}\circ\phi$$

and it follows that χ_Γ is a natural isomorphism as required. □

Definition 5 *Let $\mathcal{C}$ and $\mathcal{D}$ be normal categories. A* cross-connection *is a triplet $(\mathcal{D},\mathcal{C};\Gamma)$ where $\Gamma\colon\mathcal{D}\to\boldsymbol{N}^*\mathcal{C}$ is a local isomorphism such that for every $c\in\boldsymbol{v}\mathcal{C}$ there is some $d\in\boldsymbol{v}\mathcal{D}$ such that $c\in M\Gamma(d)$.*

Remark 2 An ideal $\mathcal{I}$ in $N^*\mathcal{C}$ (where $\mathcal{C}$ is a normal category) is said to be a *total ideal* in $N^*\mathcal{C}$ if for all $c \in v\mathcal{C}$ there is some $\epsilon \in E(T\mathcal{C})$ such that $c \in MH(\epsilon; -)$ (or equivalently $c \in M\epsilon$). By Remark 1, image of any local isomorphism is an ideal and so the image of any connection Γ of $\mathcal{D}$ with $\mathcal{C}$ is an ideal in $N^*\mathcal{C}$. Hence $(\mathcal{D}, \mathcal{C}, \Gamma)$ is a cross-connection in the sense of the definition above if and only if the image of Γ is total in $N^*\mathcal{C}$. Again by Proposition 12(a), the condition that the image of Γ is total is equivalent to the condition $\mathcal{C}_\Gamma = \mathcal{C}$. This implies that if $(\mathcal{D}, \mathcal{C}; \Gamma)$ is a cross-connection then $(\mathcal{C}, \mathcal{D}; \Gamma^*)$ is also a cross-connection and it is called the *dual* of $(\mathcal{D}, \mathcal{C}, \Gamma)$. In fact, it can be seen that given any connection $\Gamma \colon \mathcal{D} \to N^*\mathcal{C}$, $(\mathcal{D}, \mathcal{C}_\Gamma; \Gamma^*)$ is a cross-connection. It is clear that every result for a cross-connection $(\mathcal{D}, \mathcal{C}, \Gamma)$ gives a corresponding result for the dual cross-connection $(\mathcal{C}, \mathcal{D}, \Gamma^*)$ and the latter result will be called the dual of the former.

The definition of a cross-connection as a triplet is intended to emphasize the fact that they will be treated as an object; in fact we will show in the next chapter that they will form the object class of a "category of cross-connections". However, in the following, to simplify the notation we shall often say that the connection Γ of $\mathcal{D}$ with $\mathcal{C}$ is a "cross-connection" whenever $(\mathcal{D}, \mathcal{C}, \Gamma)$ satisfies the conditions of the definition above.

3.4 Cross-connections of regular semigroups

We end this section with the following theorem which clarifies the relation between cross-connections defined above and connections of regular semigroups. In particular, it shows that the connection of a regular semigroup S is a cross-connection of $\mathcal{R}(S)$ with $\mathcal{L}(S)$ and the dual connection of S is the dual of its connection in the sense of Theorem 15.

Theorem 17 *Let S be a regular semigroup. Then the connection $\boldsymbol{\Gamma} S$ of S is a cross-connection of $\mathcal{R}(S)$ with $\mathcal{L}(S)$. Moreover we have $\Delta S = \boldsymbol{\Gamma} S^*$ and $\chi_S = \chi_{\boldsymbol{\Gamma} S}$.*

Proof It is clear from the definition of $\boldsymbol{\Gamma} S$ (see Equation (2)) that for any $Se \in \boldsymbol{v}\mathcal{L}(S)$, we have $Se \in M\boldsymbol{\Gamma} S(eS)$ and so the image of $\boldsymbol{\Gamma} S$ is total in $N^*\mathcal{L}(S)$. Hence $\boldsymbol{\Gamma} S \colon \mathcal{R}(S) \to N^*\mathcal{L}(S)$ is a cross-connection.

We now show that $\Delta S = \boldsymbol{\Gamma} S^*$. For convenience, we write $T = T\mathcal{L}(S)$ and $\boldsymbol{\Gamma} S = \Gamma$. Also let $\boldsymbol{G}$ denote the isomorphism of $\mathcal{R}(T)$ with $N^*\mathcal{L}(S)$

defined in Theorem III.25. Given any morphism $\lambda = \lambda(\rho^e, \rho^a, \rho^f)\colon \rho^e T \to \rho^f T$ in $\mathcal{R}(T)$ where $a \in fSe$ with $e, f \in E(S)$, if $\tilde{\rho^a}$ denote the morphism defined by Equation (III.14), then $\tilde{\rho^a} = \rho(f, a, e)$. Hence by the definition of $\boldsymbol{G}$, we have

$$\begin{aligned} \boldsymbol{G}(\lambda) &= \eta_{\rho^e} \mathcal{L}(S)(\rho(f,a,e), -)\eta^{-1}_{\rho^f} \\ \text{(A)} \qquad &= \Gamma(\lambda(e,a,f)) \quad \text{by Equation (2).} \end{aligned}$$

Let $Se \in \boldsymbol{v}\mathcal{L}(S)$. By the definition of Γ^* (see Equation (17)),

$$\text{(B)} \qquad \Gamma^*(Se) = H(\tilde{\boldsymbol{G}}(\lambda^{\rho^e}); -).$$

By the dual of Lemma III.15 we have

$$\begin{aligned} \lambda^{\rho^e}(\rho^f T) &= \lambda(\rho^f, \rho^e \cdot \rho^f, \rho^e) \\ &= \lambda(\rho^f, \rho^e \star (\rho(e, ef, f))^\circ, \rho^e) \end{aligned}$$

so that by Equation (III.14), $\widetilde{\rho^e \cdot \rho^f} = \rho(e, ef, f)$. Hence , for all $\rho^f T \in \boldsymbol{v}\mathcal{R}(T)$,

$$\boldsymbol{G}(\lambda^{\rho^e}(\rho^f T)) = \Gamma(\lambda(f, ef, e)) = \Gamma(\lambda^e(fS))$$

by Equation (A) and Lemma III.15. Now by the definition of the cone $\boldsymbol{G}(\lambda^{\rho^e})$ (see Equation (7a)) and the equation above, we have $\boldsymbol{G}(\lambda^{\rho^e})(\boldsymbol{G}(\rho^f T)) = \Gamma(\lambda(f, ef, e))$ and it follows from Lemma 13 that

$$\text{(C)} \qquad \tilde{\boldsymbol{G}}(\lambda^{\rho^e}) = \lambda^e.$$

Therefore by (B), $\Gamma^*(Se) = H(\lambda^e; -) = \Delta S(Se)$.

Next, let $\rho(e, a, f)$ be any morphism in $\mathcal{L}(S)$. By the definition of $\boldsymbol{\Gamma} S = \Gamma$ (see Equation (2)), $\Gamma(\lambda(f, a, e)) = \eta_{\rho^f} \mathcal{L}(S)(\rho(e, a, f), -)\eta^{-1}_{\rho^e}$. Hence if $\theta = \theta_{\mathcal{L}(S)}$, by Lemma 13,

$$\theta(\rho(e,a,f)) = \eta_{\boldsymbol{G}(\lambda^{\rho^e})} \boldsymbol{N}^* \mathcal{L}(S)(\Gamma(\lambda(f,a,e), -)\eta^{-1}_{\boldsymbol{G}(\lambda^{\rho^f})}.$$

Therefore, by the definition of Γ^* (see Equations (16b) and (17))

$$\begin{aligned}\Gamma^*(\rho(e,a,f)) &= \tilde{\Gamma}(\theta(\rho(e,a,f)))\\ &= \eta_{\tilde{G}(\lambda^{\rho^e})}\mathcal{R}(S)(\lambda(f,a,e),-)\eta^{-1}_{\tilde{G}(\lambda^{\rho^e})}\\ &= \eta_{\lambda^e}\mathcal{R}(S)(\lambda(f,a,e),-)\eta^{-1}_{\lambda^f} \quad \text{by (C)}\\ &= \Delta S(\rho(e,a,f)) \quad \text{by Equation (I.7*)}.\end{aligned}$$

This proves that $\Gamma S^* = \Gamma^* = \Delta S$.

Now by Equation (18), we have

$$\chi_\Gamma(Se, fS)(\rho^f \star \rho(f,a,e)^\circ) = \tilde{G}(\lambda^{\rho^e}) \star g^\circ$$

where $g\colon eS \to fS$ is a morphism in $\mathcal{R}(S)$ such that

$$\Gamma(g) = \eta_{\lambda^e}\mathcal{L}(S)(\rho(f,a,e),-)\eta^{-1}_{\lambda^f} = \Gamma(\lambda(e,a,f))$$

by Equation (18) and the definition of Γ. Since Γ is faithful, we have $g = \lambda(e,a,f)$. Hence by (C),

$$\begin{aligned}\chi_\Gamma(Se, fS)(\rho^f \star \rho(f,a,e)^\circ) &= \lambda^e \star \lambda(e,a,f)^\circ\\ &= \chi_S(Se, fS)(\rho^f \star \rho(f,a,e)^\circ) \quad \text{by (5)}.\end{aligned}$$

It follows that $\chi_\Gamma = \chi_S$.

□

4 TRANSPOSE OF MORPHISMS

Throughout this section we shall assume that $\Gamma\colon \mathcal{D} \to N^*\mathcal{C}$ is a connection of normal categories. Also we shall write:

(20) $$E_\Gamma = \{\, (c,d) : c \in v\mathcal{C}_\Gamma,\ d \in v\mathcal{D} \text{ and } c \in M\Gamma(d)\,\}.$$

Note that by Theorem 15, $(c,d) \in E_\Gamma$ if and only if $c \in v\mathcal{C}_\Gamma$, $d \in v\mathcal{D}$ and $d \in M\Gamma^*(c)$; that is, if we define the set E_{Γ^*} as above (using Γ^* instead of Γ), $(c,d) \in E_\Gamma$ if and only if $(d,c) \in E_{\Gamma^*}$. We shall see later that the set E_Γ defined above is a biordered set; however, for the present we shall use it as a device for simplifying the notation.

4.1 The cones $\gamma(c, d)$ and $\gamma^*(c, d)$

Suppose that $(c, d) \in E_\Gamma$. If $\Gamma(d) = H(\epsilon; -)$, then $c \in MH(\epsilon; -)$ and so there is a unique idempotent cone ϵ' with $c_{\epsilon'} = c$ and $H(\epsilon'; -) = \Gamma(d)$ (by the definition of $MH(\epsilon; -)$, Propositions III.4(i), III.7(a) and Corollary III.8). We shall denote this cone by $\gamma(c, d)$. Thus for each $(c, d) \in E_\Gamma$, $\gamma(c, d)$ denotes the unique cone in $\mathcal{C}$ such that

$$c_{\gamma(c,d)} = c \quad \text{and} \quad \Gamma(d) = H(\gamma(c, d); -). \tag{21}$$

In a similar way, for each $(c, d) \in E_\Gamma$, we see that there is a unique cone $\gamma^*(c.d)$ in $\mathcal{D}$ such that

$$c_{\gamma^*(c,d)} = d \quad \text{and} \quad \Gamma^*(c) = H(\gamma^*(c, d); -). \tag{21*}$$

Now if $\tilde{G}(\lambda^{\gamma(c,d)})$ is the unique cone in $\mathcal{D}$ with vertex d constructed using $G(\lambda^{\gamma(c,d)})$ as in Lemma 13, it follows from the definition of Γ^* that

$$\Gamma^*(c) = H(\tilde{G}(\lambda^{\gamma(c,d)}); -).$$

By the uniqueness, we have

$$\gamma^*(c, d) = \tilde{G}(\lambda^{\gamma(c,d)}) \tag{22}$$

for all $(c, d) \in E_\Gamma$. Components of the cones $\gamma(c, d)$ and $\gamma^*(c, d)$ are related as follows:

Lemma 18 *For (c, d), $(c', d') \in E_\Gamma$, the following diagram*

$$\begin{array}{ccc} \Gamma(d') & \xrightarrow{\eta_{\gamma(c',d')}} & \mathcal{C}(c', -) \\ {\scriptstyle \Gamma(\gamma^*(c,d)(d'))}\downarrow & & \downarrow{\scriptstyle \mathcal{C}(\gamma(c',d')(c),-)} \\ \Gamma(d) & \xrightarrow[\eta_{\gamma(c,d)}]{} & \mathcal{C}(c, -) \end{array}$$

is commutative.

Proof By Equation (15) and by the definition of the cone $G(\lambda^{\gamma(c,d)})$ (see Equation (7a)), we have

$$\Gamma\left(\tilde{G}(\lambda^{\gamma(c,d)})(d')\right) = G(\lambda^{\gamma(c,d)})(\Gamma(d')) = G\left(\lambda^{\gamma(c,d)}(\gamma(c', d')T)\right).$$

By the dual of Lemma III.15,

$$\lambda^{\gamma(c,d)}(\gamma(c',d')T) = \lambda(\gamma(c',d'), \gamma(c,d)\cdot\gamma(c',d'), \gamma(c,d))$$

and by Equation (III.14)

$$\gamma(c,d)\cdot\widetilde{\gamma(c',d')} = \gamma(c',d')(c).$$

Hence, by the definition of G (see Theorem III.25), we have

$$\Gamma\left(\tilde{G}(\lambda^{\gamma(c,d)})(d')\right) = \eta_{\gamma(c',d')}\mathcal{C}(\gamma(c',d')(c), -)\eta^{-1}_{\gamma(c,d)}.$$

Hence the given diagram commutes. □

4.2 The transpose

Using the notations introduced above, we may rewrite the equation defining the natural isomorphism χ_Γ as follows. Let $(c,d) \in v\mathcal{C}_\Gamma \times v\mathcal{D}$ and choose $c' \in \mathcal{C}_\Gamma$ and $d' \in v\mathcal{D}$ such that $(c,d'), (c',d) \in E_\Gamma$. Then every cone in $\Gamma(c,d)$ can be represented as $\gamma(c',d) \star f^\circ$ with $f \in \mathcal{C}(c',c)$ and every element of $\Gamma^*(c,d)$ can be written as $\gamma^*(c,d') \star g^\circ$ with $g \in \mathcal{D}(d',d)$. Hence for every $(c,d) \in v\mathcal{C}_\Gamma \times v\mathcal{D}$ and $\gamma(c',d) \star f^\circ \in \Gamma(c,d)$, we have

(23) $$\chi_\Gamma(c,d)(\gamma(c',d) \star f^\circ) = \gamma^*(c,d') \star (g)^\circ$$

where $(c,d), (c',d') \in E_\Gamma$ and $f \in \mathcal{C}(c,c')$, $g \in \mathcal{D}(d',d)$ are such that the following diagram commutes.

(D) $$\begin{array}{ccc} \Gamma(d') & \xrightarrow{\eta_{\gamma(c,d')}} & \mathcal{C}(c,-) \\ {\scriptstyle \Gamma(g)}\downarrow & & \downarrow{\scriptstyle \mathcal{C}(f,-)} \\ \Gamma(d) & \xrightarrow[\eta_{\gamma(c',d)}]{} & \mathcal{C}(c',-) \end{array}$$

We proceed to show that the relation between f and g is "dual". To simplify the proof we shall first prove a lemma which will contain some of the computations needed.

Lemma 19 *Let $f \in \mathcal{C}(c, c')$ and $g \in \mathcal{D}(d', d)$ where $(c, d), (c', d') \in E_\Gamma$. If the equation* (23) *holds then we have:*

$$\Gamma(c', g)(\gamma(c', d')) = \gamma(c, d) \star f^\circ = \Gamma(f, d)(\gamma(c, d)); \tag{24}$$
$$\Gamma^*(c', g)(\gamma^*(c', d')) = \gamma^*(c', d') \star g^\circ = \Gamma^*(f, d)(\gamma^*(c, d)). \tag{24*}$$

Proof " Since Equation (23) holds, the diagram (D) commutes. Hence we have

$$\eta_{\gamma(c', d')}(c')\mathcal{C}(f, c') = \Gamma(c', g)\eta_{\gamma(c,d)}(c').$$

Now

$$\mathcal{C}(f, c')\left(\eta_{\gamma(c', d')}(c')\left(\gamma(c', d')\right)\right) = \mathcal{C}(f, c')(1_{c'}) = f.$$

On the other hand, since $\Gamma(c', g)\left(\gamma(c', d')\right) \in \Gamma(c', d)$, there is a $k \in \mathcal{C}(c, c')$ such that $\Gamma(c', g)\left(\gamma(c', d')\right) = \gamma(c, d) \star k^\circ$. Then, by the above,

$$f = \eta_{\gamma(c,d)}(c')\left(\Gamma(c', d)\left(\gamma(c', d')\right)\right) = \eta_{\gamma(c,d)}(\gamma(c, d) \star k^\circ) = k.$$

Since by Equation (21), $\Gamma(f, d) = H(\gamma(c, d); f)$, we have $\Gamma(f, d)(\gamma(c, d)) = \gamma(c, d) \star f^\circ$ by the definition of the functor $H(\gamma(c, d); -)$ (see Equation (III.6)). This proves (24).

Now $\chi_\Gamma(-, d)\colon \Gamma(d) \to \Gamma^*(d)$ and $\eta_{\gamma(c,d)}\colon \Gamma(d) \to \mathcal{C}(c, -)$ are natural isomorphisms for every $(c, d) \in E_\Gamma$ and so,

$$\varphi = \eta^{-1}_{\gamma(c,d)}\chi_\Gamma(-, d)\colon \mathcal{C}(c, -) \to \Gamma^*(-, d) \tag{$*$}$$

is a natural isomorphism. Therefore $\varphi(c)\Gamma^*(f, d) = \mathcal{C}(c, f)\varphi(c')$. Also

$$\varphi(c)(1_c) = \chi_\Gamma(c, d)\left(\eta_{\gamma(c,d)}(1_c)\right) = \chi_\Gamma(c, d)\left(\gamma(c, d)\right) = \gamma^*(c, d).$$

Hence

$$\begin{aligned}
\Gamma^*(f, d)\left(\gamma^*(c, d)\right) &= \Gamma^*(f, d)\left(\varphi(c)(1_c)\right) \\
&= \varphi(c')\left(\mathcal{C}(c, f)(1_c)\right) \\
&= \varphi(c')(f) \\
&= \chi_\Gamma(c', d)\left(\gamma(c, d) \star f^\circ\right) \quad \text{using } (*) \\
&= \gamma^*(c', d') \star g^\circ \\
&= \Gamma^*(c', g)\left(\gamma^*(c', d')\right)
\end{aligned}$$

using Equations (21*) and (III.6). This proves (24*). □

Theorem 20 *Let* $\Gamma: \mathcal{D} \to \mathcal{N}^*\mathcal{C}$ *be a connection,* $(c,d), (c',d') \in E_\Gamma$*,* $f \in \mathcal{C}(c,c')$ *and* $g \in \mathcal{D}(d',d)$*. Then* f *and* g *make the diagram (D) commute if and only if they make the dual diagram*

$$(\mathrm{D}^*) \qquad \begin{array}{ccc} \Gamma^*(c) & \xrightarrow{\eta_{\gamma^*(c,d)}} & \mathcal{D}(d,-) \\ {\scriptstyle \Gamma^*(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{D}(g,-)} \\ \Gamma^*(c') & \xrightarrow[\eta_{\gamma^*(c',d')}]{} & \mathcal{D}(d',-) \end{array}$$

commute. Moreover, there is a bijection of $\mathcal{C}(c',c)$ *with* $\mathcal{D}(d',d)$ *which assigns to each* $f \in \mathcal{C}(c',c)$ *a morphism* $g \in \mathcal{D}(d',d)$ *making the diagrams (D) and (D*) commute.*

Proof Suppose that f and g make the diagram (D) commute. Now $\mathcal{D}(g,-)$ and $\psi = \eta_{\gamma^*(c,d)}^{-1}\Gamma^*(f)\eta_{\gamma^*(c',d')}$ are natural transformations from $\mathcal{D}(d,-)$ to $\mathcal{D}(d',-)$ and we have

$$\begin{aligned} \psi(d)(1_d) &= \eta_{\gamma^*(c',d')}(d)\,(\Gamma^*(f,d)\,(\gamma^*(c,d))) \\ &= \eta_{\gamma^*(c',d')}(d)(\gamma^*(c',d') \star g^\circ) = g \quad \text{by (24*)} \\ &= g = \mathcal{D}(g,d)(1_d). \end{aligned}$$

Hence by Yoneda lemma $\psi = \mathcal{D}(g,-)$. This proves that the diagram (D*) commutes.

Conversely assume that (D*) commutes. Now, since $\chi_\Gamma(c',d)$ is a bijection of $\Gamma(c',d)$ onto $\Gamma^*(c',d)$, there exists $\bar{g} \in \mathcal{D}(d',d)$ such that

$$\chi_\Gamma(c',d)(\gamma(c,d) \star f^\circ) = \gamma^*(c',d') \star \bar{g}^\circ.$$

Then by Equation (23), f and $\bar{g}$ make the diagram (D) commute and so, by the above, (D*) also commutes. Hence it follows that $\mathcal{D}(g,-) = \mathcal{D}(\bar{g},-)$ from which we conclude that $g = \bar{g}$. Therefore f and g make diagram (D) commute. Since χ_Γ is a natural isomorphism, it follows immediately that the correspondence $f \leftrightarrow g$ is a bijection of $\mathcal{C}(c,c')$ onto $\mathcal{D}(d',d)$. □

Given $f \in \mathcal{C}(c,c')$, we shall say that $g \in \mathcal{D}(d',d)$ is the *transpose* of f from d' to d (relative to Γ) and f is called the transpose of g from c to c' if $(c,d), (c',d') \in E_\Gamma$ and the pair f and g make the diagrams (D) and (D*) commute. We shall use the notation f^* $[g^*]$ to denote a transpose of f $[g]$. Note that given $f \in \mathcal{C}(c.c')$ there is a *unique* transpose $f^*: d' \to d$ for each choice of $d \in M\Gamma^*(c)$ and d' im $M\Gamma^*(c')$ [and similarly for g]. We also note that in terms of transposes, Lemma 18 may be expressed as

$$(\gamma(c,d)(c'))^* = \gamma^*(c',d')(d) \tag{25}$$

for all (c,d), $(c',d') \in E_\Gamma$. The uniqueness of transpose (*for given domain and codomain*) has the following consequence:

Corollary 21 *Let $f^*: d' \to d$ be a transpose of $f: c \to c'$ and $f^{**}: c \to c'$ be a transpose of f^*. Then $f^{**} = f$.* □

Observe that arbitrary transposes of two composable morphisms $f_1 \in \mathcal{C}(c,c')$ and $f_2 \in \mathcal{C}(c',c'')$ need not be composable. However it is equally obvious that we can find transposes f_1^* and f_2^* such that $f_2^* f_1^*$ exists.

Corollary 22 *Suppose that f_1^* and f_2^* are transposes of $f_1 \in \mathcal{C}(c,c')$ and $f_2 \in \mathcal{C}(c',c'')$ respectively such that $f_2^* f_1^*$ exists. Then $(f_1 f_2)^* = f_2^* f_1^*$.* □

As an immediate consequence of the corollary above we obtain

Corollary 23 *A morphism $f: c \to c'$ is a monomorphism [epimorphism] if and only if its transpose is an epimorphism [monomorphism].* □

More specifically, when the morphism considered is either an inclusion or a retraction we have:

Proposition 24 *Let $c \subseteq c'$ and $d' \in M\Gamma^*(c')$. Then we have the following.*

(a) *For each $d \in M\Gamma^*(c)$ with $d \subseteq d'$, $(j_c^{c'})^*: d' \to d$ is a retraction.*
(b) *If $\varrho: c' \to c$ is a retraction, there exists $d \in M\Gamma^*(c)$ with $d \subseteq d'$ such that $\varrho^*: d \to d' = j_d^{d'}$.*

Proof Let $d \in M\Gamma^*(c)$ with $d \subseteq d'$. Since $c \subseteq c'$, we have $j_c^{c'} = \gamma(c',d')(c)$ and so by Equation (25), $(j_c^{c'})^* = \gamma^*(c,d)(d')$. Now $\Gamma^*(c) \subseteq \Gamma^*(c')$ and it follows from Equation (21*) and Proposition III.7 that $\gamma^*(c,d) = \gamma^*(c',d') \star \gamma^*(c,d)(d')$. Since $\gamma^*(c',d')(d) = \jmath_d^{d'}$, it follows that

$$1_d = \gamma^*(c',d')(d)\gamma^*(c,d)(d') = \jmath_d^{d'}\gamma^*(c,d)(d').$$

Hence $(\jmath_c^{c'})^* = \gamma^*(c,d)(d')$ is a retraction. This proves (a).

To prove (b), note that by Proposition III.7, $\epsilon = \gamma(c',d') \star \varrho$ is an idempotent normal cone such that $H(\epsilon;-) \subseteq \Gamma(d')$. Since Γ is a local isomorphism, there is $d \subseteq d'$ such that $\Gamma(d) = H(\epsilon;-)$. Since $c_\epsilon = \operatorname{cod} \varrho = c$, it follows that $\epsilon = \gamma(c,d)$ and so $\varrho = \gamma(c,d)(c')$. Hence by Equation (25), $\varrho^* = \gamma^*(c',d')(d) = \jmath_d^{d'}$. □

Those morphisms for which identity is a transpose, is of particular interest. By a *perspectivity* we shall mean a morphism $f\colon c \to c'$ such that there is a transpose $f^*\colon d \to d = 1_d$.

Proposition 25 *A morphism $f\colon c \to c'$ is a perspectivity if and only if there is $d \in v\mathcal{D}$ such that c, $c' \in M\Gamma(d)$ and $f = \gamma(c',d)(c)$.*

Proof If $f = \gamma(c',d)(c)$, then by Equation (25), $f^* = \gamma^*(c,d)(d)$ and since $\gamma^*(c,d)$ is an idempotent cone with vertex d, it follows that $f^* = \gamma^*(c,d)(d) = 1_d$. Hence f is a perspectivity. Conversely let $f^*\colon d \to d = 1_d$. Now by the above, $\gamma(c',d)(c)^* = 1_d = f^*$ and so, by Corollary 21, we have $f = \gamma(c',d)(c)$. □

4.3 The duality of dual cross-connection

The results above show that the relation between a morphism $f \in \mathcal{C}$ and its transpose is completely symmetric (dual). In particular, we could define the natural isomorphism χ_Γ alternatively by requiring that the diagram (D*) (instead of (D)) commute. In fact, we have

Theorem 26 *Let $\Gamma\colon \mathcal{D} \to N^*\mathcal{C}$ be a cross-connection. If Γ^{**} denote the dual of Γ^*, then $\Gamma^{**} = \Gamma$ and $\chi_\Gamma^{-1} = \chi_{\Gamma^*}$.*

Proof By Theorem 15, $c \in M\Gamma(d)$ if and only if $d \in M\Gamma^*(c)$ which is true if and only if $c \in M\Gamma^{**}(d)$. Thus $M\Gamma(d) = M\Gamma^{**}(d)$ for all $d \in v\mathcal{D}$. Fix $c \in M\Gamma(d)$. By Equation (21), $\Gamma(d) = H(\gamma(c,d);-)$ and let $\gamma^{**}(c,d)$ denote the idempotent cone such that $\Gamma^{**}(d) = H(\gamma^{**}(c,d); -)$. If $c' \in v\mathcal{C}$ then there is $d' \in v\mathcal{D}$ such that $(c', d') \in E_\Gamma$ since Γ is a cross-connection. Then it follows from Lemma 18, Theorem 20 and the definition of transpose that

$$(\gamma(c,d)(c'))^* = \gamma^*(c', d')(d) = (\gamma^{**}(c,d)(c'))^*.$$

Since $\gamma(c,d)(c'), \gamma^{**}(c,d)(c') \in \mathcal{C}(c', c)$, by the uniqueness of transposes, we have $\gamma(c,d)(c') = \gamma^{**}(c,d)(c')$. Since c' is arbitrary, it follows that $\gamma(c,d) = \gamma^{**}(c,d)$. Therefore $\Gamma(d) = \Gamma^{**}(d)$ for all $d \in v\mathcal{D}$. If $g\colon d' \to d$ is any morphism in $\mathcal{D}$ and if we fix $c \in \Gamma(d)$ and $c' \in \Gamma(d')$, from the diagram (D) we have

$$\Gamma(g) = \eta_{\gamma(c',d')}\mathcal{C}(g^*, -)\eta^{-1}_{\gamma(c,d)} = \Gamma^{**}(g).$$

Hence $\Gamma = \Gamma^{**}$. If $(c', d) \in v\mathcal{C} \times v\mathcal{D}$, then for any $\gamma^*(c', d') \star g^\circ \in \Gamma^*(c', d)$, by Equation (23)

$$\begin{aligned}\chi_{\Gamma^*}(c', d)(\gamma^*(c', d') \star g^\circ) &= \gamma(c,d) \star (g^*)^\circ \\ &= \chi_\Gamma(c', d)^{-1}(\gamma^*(c', d') \star g^\circ)\end{aligned}$$

from which we conclude that $\chi_{\Gamma^*} = \chi_\Gamma^{-1}$. □

We have noted that if $\Gamma\colon \mathcal{D} \to N^*\mathcal{C}$ is any connection then $\Gamma^*\colon \mathcal{C}_\Gamma \to \mathcal{D}$ is a cross-connection. Hence from the above we have

Corollary 27 *If $\Gamma\colon \mathcal{D} \to N^*\mathcal{C}$ is any connection then $\Gamma^{***} = \Gamma^*$.* □

5 THE CROSS-CONNECTION SEMIGROUP

As already observed (see Remark 2), with every statement regarding the cross-connection $\Gamma\colon \mathcal{D} \to N^*\mathcal{C}$ one obtains a corresponding statement regarding the dual cross-connection $\Gamma^*\colon \mathcal{C} \to N^*\mathcal{D}$ for which no separate proof is needed. In the following we will use this observation with out any further comment.

In this section we shall show that with every cross-connection we can associate a regular semigroup, called the *cross-connection semigroup* in a natural fashion and that every regular semigroup is isomorphic to the cross-connection semigroup determined by its connection.

5.1 The semigroup $U\Gamma$

Let Γ be a cross-connection of $\mathcal{D}$ with $\mathcal{C}$. Define

$$U\Gamma = \bigcup\{\Gamma(c,d) : (c,d) \in v\mathcal{C} \times v\mathcal{D}\}; \tag{26}$$

$$U\Gamma^* = \bigcup\{\Gamma^*(c,d) : (c,d) \in v\mathcal{C} \times v\mathcal{D}\}. \tag{26*}$$

The following lemmas give certain properties of normal cones in $U\Gamma$ and will be useful in the proof of the main theorem of this section. Dual statements hold for normal cones in $U\Gamma^*$.

Lemma 28 *A normal cone $\gamma \in U\Gamma$ if and only if $\gamma = \gamma(c,d) \star f$ for some $(c,d) \in E_\Gamma$ and some isomorphism $f: c \to c'$.*

Proof Let $\gamma = \gamma(c,d) \star f$ where $f: c \to c'$ is an isomorphism. Then we have $\gamma \in H(\gamma(c,d); c') = \Gamma(c',d)$ and so $\gamma \in U\Gamma$. Conversely let $\gamma \in U\Gamma$. Then by the definition of $U\Gamma$, $\gamma \in \Gamma(c,d)$ for some $(c,d) \in v\mathcal{C} \times v\mathcal{D}$. Hence $\gamma = \gamma(c',d) \star f^\circ$ where $f \in \mathcal{C}(c',c)$. If $f = \varrho u j$ is a normal factorization of f, then $\gamma = \gamma(c',d) \star \varrho u$. As in the proof of Proposition 24, we see that $\gamma(c',d) \star \varrho = \gamma(c'',d')$ where $c'' = \operatorname{im} \varrho$ and $d' \subseteq d$. It follows that $\gamma = \gamma(c'',d') \star u$. □

Lemma 29 *Let $\gamma = \gamma(c,d) \star f^\circ \in U\Gamma$ where $f: c \to c'$. If $u: c_1 \to c_1'$ is the isomorphism component in a normal factorization of f, then there exists $\gamma(c_1,d_1)\ \omega\ \gamma(c,d)$ such that*

$$\gamma = \gamma(c_1,d_1) \star u \qquad \text{and} \qquad \chi_\Gamma(c',d)(\gamma) = \chi_\Gamma(c_1',d_1)(\gamma). \tag{27}$$

Conversely if $\gamma(c_1,d_1)\ \omega\ \gamma(c,d)$ and if $u: c_1 \to c_1'$ is an isomorphism satisfying the equations above, then u is the isomorphism component in a normal factorization of f.

Proof Suppose that $f = \varrho u j_{c_1'}^{c'}$ is a normal factorization of f. Then $\epsilon = \gamma(c,d) \star \varrho$ is an idempotent cone and by Proposition III.7,

$$H(\epsilon; -) \subseteq H(\gamma(c,d); -) = \Gamma(d)$$

by Equation (21). Since Γ is a local isomorphism, there is a unique $d_1 \subseteq d$ such that $\Gamma(d_1) = H(\epsilon; -)$. Hence by Equation (21), $\gamma(c_1,d_1) = \epsilon$ if $c_1 = c_\epsilon = \operatorname{im} \varrho$.

Since $c_1 \subseteq c$ and $\Gamma(d_1) \subseteq \Gamma(d)$, it follows from Propositions III.5 and III.7 that $\gamma(c_1, d_1)\ \omega\ \gamma(c, d)$. Further,

$$\gamma = \gamma(c,d) \star (\varrho u \jmath_{c_1'}^{c'})^\circ = \gamma(c,d) \star \varrho u = \gamma(c_1, d_1) \star u.$$

Now, let $d' \in M\Gamma^*(c')$ and $d_1' \in M\Gamma^*(c_1')$. Then we have $\varrho = \gamma(c_1, d_1)(c)$ and so, by Equation (25), the transpose of ϱ from d_1 to d is $\gamma^*(c,d)(d_1) = \jmath_{d_1}^{d}$. By Proposition 24, the transpose of $\jmath_{c_1'}^{c'}$ from d' to d_1' is a retraction ϱ'. Hence if $u^*: d_1' \to d_1$ is the transpose of u, the transpose of f from d' to d is $f^* = \varrho' u^* \jmath_{d_1}^{d}$. Also $\gamma^*(c', d') \star \varrho' = \gamma^*(c_1', d_1')$. Therefore by Equation (23),

$$\begin{aligned} \chi_\Gamma(c', d)(\gamma) &= \gamma^*(c', d') \star (f^*)^\circ \\ &= \gamma^*(c', d') \star \varrho' u^* \\ &= \gamma^*(c_1', d_1') \star u^* \quad \text{and} \\ \chi_\Gamma(c_1', d_1)(\gamma) &= \gamma^*(c_1', d_1') \star u^*. \end{aligned}$$

Conversely if $\gamma = \gamma(c_1, d_1) \star u$ where $\gamma(c_1, d_1)\ \omega\ \gamma(c.d)$ and $u: c_1 \to c_1'$ is an isomorphism, then $\gamma(c,d) \star f^\circ u^{-1} = \gamma(c_1, d_1)$ and so $\varrho = f^\circ u^{-1}: c \to c_1$ is a retraction and $f = \varrho u \jmath_{c_1'}^{c'}$ is a normal factorization of f. □

Lemma 30 *Let $\gamma_i \in \Gamma(c_i', d_i)$, for $i = 1, 2$. Then $\gamma_2 \cdot \gamma_1 \in \Gamma(c_1', d_2)$ and*

(28) $$\chi_\Gamma(c_1', d_1)(\gamma_1) \cdot \chi_\Gamma(c_2', d_2)(\gamma_2) = \chi_\Gamma(c_1', d_2)(\gamma_2 \cdot \gamma_1).$$

Proof " Let $\gamma_i = \gamma(c_i, d_i) \star f_i^\circ$ where $f_i: c_i \to c_i'$ and $d_i \in \Gamma^*(c_i)$ for $i = 1, 2$. Then by the definition of product in TC (see Equation (III.3)) we have

$$\gamma_2 \cdot \gamma_1 = \gamma_2 \star (\gamma_1(c_{\gamma_2}))^\circ = \gamma(c_2, d_2) \star f_2^\circ\,(\gamma(c_1, d_1)(c_{\gamma_2}) f_1)^\circ.$$

If $\gamma(c_1, d_1)(c_{\gamma_2}) f_1 = (\gamma(c_1, d_1)(c_{\gamma_2}) f_1)^\circ j$, we have

$$\begin{aligned} f_2^\circ\,(\gamma(c_1, d_1)(c_{\gamma_2}) f_1)^\circ j &= f_2^\circ \gamma(c_1, d_1)(c_{\gamma_2}) f_1 \\ &= f_2^\circ \jmath_{c_{\gamma_2}}^{c_2'} \gamma(c_1, d_1)(c_2') f_1 \\ &= f_2 \gamma(c_1, d_1)(c_2') f_1 \end{aligned}$$

and so,

(1) $$\gamma_2 \cdot \gamma_1 = \gamma(c_2, d_2) \star (f_2 \gamma(c_1, d_1)(c_2') f_1)^\circ.$$

In particular, $\gamma_2 \cdot \gamma_1 \in \Gamma(c'_1, d_2)$. Now let $d'_i \in M\Gamma^*(c'_i)$. If $\gamma^*_i = \chi_\Gamma(c'_i, d_i)(\gamma)$ for $i = 1, 2$ then by Equation (23) we have $\gamma^*_i = \gamma^*(c'_i, d'_i) \star (f^*)^\circ$ and hence as above we obtain

$$\gamma^*_1 \cdot \gamma^*_2 = \gamma^*(c'_1, d'_1) \star (f^*_1 \gamma^*(c'_2, d'_2)(d_1) f^*_2)^\circ. \tag{2}$$

By Corollary 22 and Equation (25)

$$(f_2 \gamma(c_1, d_1)(c'_2) f_1)^* = f^*_1 \gamma^*(c'_2, d'_2)(d_1) f^*_2.$$

Hence applying Equation (23) to (1), it follows that

$$\chi_\Gamma(c'_1, d_2)(\gamma_2 \cdot \gamma_1) = \gamma^*(c'_1, d'_1) \star (f^*_1 \gamma^*(c'_2, d'_2)(d_1) f^*_2)^\circ$$

which, along with (2), gives the required equality. □

We proceed to show that $U\Gamma$ is a regular subsemigroup of $T\mathcal{C}$. We shall then construct the required cross-connection semigroup as a suitable subdirect product of $U\Gamma$ and the dual $U\Gamma^*$. The result is also of independent interest.

Proposition 31 *For any cross-connection $\Gamma \colon \mathcal{D} \to \mathbf{N}^*\mathcal{C}$, $U\Gamma$ is a regular subsemigroup of $T\mathcal{C}$ such that*

$$E(U\Gamma) = \{\, \gamma(c, d) : (c, d) \in E_\Gamma \,\} \tag{$*$}$$

Moreover, $\hat{F} \colon \mathcal{C} \to \mathcal{L}(U\Gamma)$ defined by

$$\begin{aligned} &\hat{F}(c) = U\Gamma\gamma(c, d); \\ &\hat{F}(f) = \rho\left(\gamma(c, d), \gamma(c, d) \star f^\circ, \gamma(c', d')\right) | U\Gamma \end{aligned} \tag{29}$$

for all $c \in v\mathcal{C}$ and $f \in \mathcal{C}(c, c')$, is an isomorphism.

Proof Let $\gamma_1 \in U\Gamma$ and $\gamma_2 \in T\mathcal{C}$. If $\gamma_1 = \gamma(c, d) \star f^\circ$ and $\gamma_2(c_{\gamma_1}) = f'$ then by Equation (III.3),

$$\gamma_1 \cdot \gamma_2 = \gamma(c, d) \star (f f')^\circ \in \Gamma(c_{\gamma_2}, d)$$

and so $\gamma_1 \cdot \gamma_2 \in U\Gamma$. Thus $U\Gamma$ is a right ideal in $T\mathcal{C}$ and hence, in particular, a subsemigroup. If $\gamma \in U\Gamma$, then by Lemma 28, $\gamma = \gamma(c,d) * f$ where $f\colon c \to c'$ is an isomorphism. Then for any $d' \in M\Gamma^*(c')$, $\gamma' = \gamma(c',d') * f^{-1} \in U\Gamma$ and

$$\gamma \cdot \gamma' \cdot \gamma = \gamma(c,d) * ff^{-1}f = \gamma(c,d) * f = \gamma.$$

Hence $U\Gamma$ is a regular subsemigroup of $T\mathcal{C}$. Clearly for every $(c,d) \in E_\Gamma$, $\gamma(c,d) \in E(U\Gamma)$ (see Equation (21)). On the other hand, if $\epsilon \in E(U\Gamma)$, then $\epsilon \in \Gamma(c,d)$ for some $c \in \boldsymbol{v}\mathcal{C}$ and $d \in \boldsymbol{v}\mathcal{D}$ and so by Lemma 28, there is $(c',d) \in E_\Gamma$ and an isomorphism$f\colon c' \to c$ such that $\epsilon = \gamma(c',d) * f$. Then by Proposition III.7, $H(\epsilon;-) = H(\gamma(c',d);-) = \Gamma(d)$ and it follows from Equation (21) that $(c,d) \in E_\Gamma$ and $\epsilon = \gamma(c,d)$. This proves the equality $(*)$.

Since Γ is a cross-connection, every $\mathcal{L}$–class of $T\mathcal{C}$ contains at least one idempotent of the form $\gamma(c,d)$ and so $T\mathcal{C}\gamma(c,d) \leftrightarrow U\Gamma\gamma(c,d)$ is a bijection. Since $\boldsymbol{F}$ defined by Equation (III.12) is an isomorphism, it follows that $\hat{F}$ is a $\boldsymbol{v}$-bijection. Also any morphism in $\mathbb{L}(U\Gamma)$ from $U\Gamma\gamma(c,d)$ to $U\Gamma\gamma(c',d')$ is of the form $\rho|U\Gamma$ where ρ is a morphism in $\mathbb{L}(T\mathcal{C})$. Since $U\Gamma$ is a right ideal, for all $\gamma(c,d), \gamma(c',d') \in E(U\Gamma)$, $\gamma(c,d)T\mathcal{C}\gamma(c',d') = \gamma(c,d)U\Gamma\gamma(c',d')$. Hence $\rho \leftrightarrow \rho|U\Gamma$ is a bijection of hom-sets of $\mathbb{L}(T\mathcal{C})$ with the corresponding hom-set of $\mathbb{L}(U\Gamma)$ and this correspondence is clearly functorial. Since $\boldsymbol{F}\colon \mathcal{C} \to \mathbb{L}(T\mathcal{C})$ is an isomorphism(by Theorem III.19), it follows that $\hat{F}\colon \mathcal{C} \to \mathbb{L}(U\Gamma)$ is an isomorphism.

□

By duality, it follows that $U\Gamma^*$ is a regular subsemigroup of $T\mathcal{D}$ such that $\bar{F}\colon \mathcal{D} \to \mathbb{L}(U\Gamma^*)$ defined by

(29*)
$$\bar{F}(d) = U\Gamma^*\gamma^*(c,d);$$
$$\bar{F}(g) = \rho\left(\gamma^*(c,d), \gamma^*(c,d) * g^\circ, \gamma^*(c',d')\right) |U\Gamma^*$$

for all $d \in \boldsymbol{v}\mathcal{D}$ and $g\colon d \to d' \in \mathcal{D}$ is an isomorphism.

5.2 Linked cones and the cross-connection semigroups

Given a cross-connection $\Gamma\colon \mathcal{D} \to N^*\mathcal{C}$, we shall say that $\gamma \in U\Gamma$ is *linked* to $\gamma^* \in U\Gamma^*$ or that (γ, γ^*) *is linked* relative to Γ if there is $(c,d) \in \boldsymbol{v}\mathcal{C} \times \boldsymbol{v}\mathcal{D}$ such that

(30)
$$\gamma \in \Gamma(c,d) \qquad \text{and} \qquad \gamma^* = \chi_\Gamma(c,d)(\gamma).$$

For brevity, we shall often omit explicit reference to the cross-connection Γ and say that γ is linked to γ^* if Γ is clear from the context. Also let

$$\tilde{S}\Gamma = \{\, (\gamma, \gamma^*) \in U\Gamma \times U\Gamma^* : (\gamma, \gamma^*) \text{ is linked}\,\}. \tag{31}$$

Observe that, by Theorem 26, γ is linked to γ^* relative to Γ if and only if γ^* is linked to γ relative to Γ^*. Now $\tilde{S}\Gamma$ is a relation from $U\Gamma$ to $U\Gamma^*$, and it follows from the observation above that the inverse of $\tilde{S}\Gamma$ is $\tilde{S}\Gamma^*$. Also it is clear that every $\gamma \in U\Gamma$ is linked to at least one $\gamma^* \in U\Gamma^*$ and similarly every $\gamma^* \in U\Gamma^*$ is linked to some $\gamma \in \Gamma$. Hence the projection maps $\pi\colon \tilde{S}\Gamma \to U\Gamma$ and $\pi^*\colon \tilde{S}\Gamma \to U\Gamma^*$ are surjective.

Theorem 32 *Let $\Gamma\colon \mathcal{D} \to \mathbf{N}^*\mathcal{C}$ be a cross-connection. Then $\tilde{S}\Gamma$ is a regular semigroup with binary operation defined by*

$$(\gamma, \gamma^*)(\delta, \delta^*) = (\gamma \cdot \delta, \delta^* \cdot \gamma^*) \tag{32}$$

for all $(\gamma, \gamma^*), (\delta, \delta^*) \in \tilde{S}\Gamma$. Moreover the projection $\pi\colon (\gamma, \gamma^*) \mapsto \gamma$ is homomorphism of $\tilde{S}\Gamma$ onto $U\Gamma$ and $\pi^*\colon (\gamma.\gamma^*) \mapsto \gamma^*$ is an anti-homomorphism of $\tilde{S}\Gamma$ onto $U\Gamma^*$. Consequently $\tilde{S}\Gamma$ is a subdirect product of $U\Gamma$ and $(U\Gamma^*)^{op}$.

Proof Suppose that $(\gamma, \gamma^*), (\delta, \delta^*) \in \tilde{S}\Gamma$. Then the pairs (γ, γ^*) and (δ, δ^*) are linked and so $\gamma^* = \chi_\Gamma(c, d)(\gamma)$ and $\delta^* = \chi_\Gamma(c', d')(\delta)$, if $\gamma \in \Gamma(c, d)$ and $\delta \in \Gamma(c', d')$. Hence by Lemma 30,

$$\delta^* \cdot \gamma^* = \chi_\Gamma(c', d)(\gamma \cdot \delta).$$

Thus $\gamma \cdot \delta$ is linked to $\delta^* \cdot \gamma^*$ and so, (32) defines a binary operation in $\tilde{S}\Gamma$ which is clearly associative. Moreover, it follows from (32) that the projection π is a homomorphism which, by the observation preceding the theorem, is surjective. Similarly $\pi^*\colon \tilde{S}\Gamma \to U\Gamma^*$ is a surjective anti-homomorphism.

It remains to prove that $\tilde{S}\Gamma$ is regular. Suppose that $(\gamma, \gamma^*) \in \tilde{S}\Gamma$ so that $\gamma \in \Gamma(c, d)$ and $\gamma^* = \chi_\Gamma(c, d)(\gamma)$ for some $(c, d) \in v\mathcal{C} \times v\mathcal{D}$. Then $\gamma = \gamma(c, d) \star f^\circ$ where $c \in M\Gamma$ and $f \in \mathcal{C}(c, c')$. By Lemma 29, there is $\gamma(c_1, d_1)\ \omega\ \gamma(c, d)$ and an isomorphism $u\colon c_1 \to c_1' \subseteq c'$ such that $\gamma = \gamma(c_1, d_1) \star u$ and $\gamma^* = \chi_\Gamma(c_1', d_1)(\gamma)$. If $d_1' \in M\Gamma^*(c_1')$, then it follows from Equation (23) that $\gamma^* = \gamma^*(c_1', d_1') \star u^*$ where $u^*\colon d_1' \to d_1$ is the transpose of u. Let $\delta = \gamma(c_1', d_1') \star u^{-1}$ and $\delta^* = \gamma^*(c_1, d_1) \star u^{*-1}$. Since the transpose of $u^{-1}\colon c_1' \to c_1$ from d_1

to d_1' is u^{*-1}, by Equation (23), $\delta^* = \chi_\Gamma(c_1, d_1')(\delta)$. Hence $(\delta, \delta^*) \in \tilde{S}\Gamma$. It is easy to see, using Equation (III.3) that $(\delta, \delta^*)(\gamma, \gamma^*)(\delta, \delta^*) = (\delta, \delta^*)$ and $(\gamma, \gamma^*)(\delta, \delta^*)(\gamma, \gamma^*) = (\gamma, \gamma^*)$. Therefore $\tilde{S}\Gamma$ is regular. □

Given a cross-connection $\Gamma\colon \mathcal{D} \to \mathbf{N}^*\mathcal{C}$, the semigroup $\tilde{S}\Gamma$ constructed in the theorem above will be called the *cross-connection semigroup* determined by vG. For brevity, we shall refer to these semigroups as CR-semigroups. In the following, for convenience, we shall use the notation σ, σ', etc., to represent elements of $\tilde{S}\Gamma$. Thus $\sigma = (\gamma, \gamma^*)$ where $\gamma \in \Gamma(c,d)$ and $\gamma^* = \chi_\Gamma(c,d)(\gamma)$, for some $(c,d) \in v\mathcal{C} \times v\mathcal{D}$. Clearly, using the projections π and π^* we can represent the linked pair of cones corresponding to σ as $(\pi\sigma, \pi^*\sigma)$.

Remark 3 We observe that for each $(c,d) \in E_\Gamma$, $(\gamma(c,d), \gamma^*(c,d))$ is an idempotent in $\tilde{S}\Gamma$ and the map α_Γ defined by

$$\alpha_\Gamma(c,d) = (\gamma(c,d), \gamma^*(c,d)) \tag{33}$$

is a bijection of E_Γ onto $E(\tilde{S}\Gamma)$. For, it follows from the equation $(*)$ of Proposition 31 that the map α_Γ is surjective. If $\alpha_\Gamma(c,d) = \alpha_\Gamma(c',d')$, then $c = c_{\pi\alpha_\Gamma(c,d)} = c_{\pi\alpha_\Gamma(c',d')} = c'$. Similarly $d = d'$.

5.3 The right regular representation of CR-semigroups

If $\phi\colon S \to T$ is a semigroup homomorphism we use the notation cong ϕ to denote the congruence on S determined by ϕ. Recall that S_ρ denotes the image of the right regular representation of a semigroup S. We have the following.

Proposition 33 *For any cross-connection $\Gamma\colon \mathcal{D} \to \mathbf{N}^*\mathcal{C}$, the map*

$$\psi(\rho_\sigma) = \pi\sigma \tag{34}$$

of $(\tilde{S}\Gamma)_\rho$ onto $U\Gamma$ is an isomorphism of semigroups such that $\rho \circ \psi = \pi$ where ρ denotes the right regular representation of $\tilde{S}\Gamma$.

Proof Let ρ denote the right regular representation of $\tilde{S}\Gamma$. Suppose that $(\gamma, \gamma_i^*) \in \tilde{S}\Gamma$ for $i = 1, 2$. If $c = c_\gamma$, then by Equation (30) there exist $d_i \in v\mathcal{D}$ such that $\gamma \in \Gamma(c, d_i)$ and $\gamma_i^* = \chi_\Gamma(c, d_i)(\gamma)$ for $i = 1, 2$. Let $(\delta, \delta^*) \in \tilde{S}\Gamma$ where $\delta \in \Gamma(c', d')$. Then by Lemma 30 we have

$$\gamma_1^* \cdot \delta^* = \chi_\Gamma(c, d')(\delta \cdot \gamma) = \gamma_2^* \cdot \delta^*.$$

Hence, by (32), $(\delta, \delta^*)(\gamma, \gamma_1^*) = (\delta, \delta^*)(\gamma, \gamma_2^*)$. Since this is true for all $(\delta, \delta^*) \in \tilde{S}\Gamma$, we have $((\gamma, \gamma_1^*), (\gamma, \gamma_2^*)) \in \operatorname{cong}\rho$. Thus $\operatorname{cong}\pi \subseteq \operatorname{cong}\rho$.

To prove the reverse inclusion, we first show that $\operatorname{cong}\rho$ is identity on $U\Gamma$. Since $\mathcal{H} \cap \operatorname{cong}\rho$ is identity on any regular semigroup, it is sufficient to show that no two distinct idempotents are $\operatorname{cong}\rho$–related in $U\Gamma$. Accordingly, assume that $(\gamma(c, d), \gamma(c', d')) \in \operatorname{cong}\rho$. Since $\operatorname{cong}\rho \subseteq \mathcal{L}$, it follows, by Proposition III.5, that $c = c'$. Let $c_1 \in \boldsymbol{v}\mathcal{C}$ and let $\gamma(c_1, d_1)$ be an idempotent in $U\Gamma$ with vertex c_1. Then $\gamma(c_1, d_1) \cdot \gamma(c, d) = \gamma(c_1, d_1) \cdot \gamma(c, d')$. Therefore

$$\begin{aligned}(\gamma(c, d)(c_1))^\circ &= (\gamma(c_1, d_1) \cdot \gamma(c, d))(c_1) \\ &= (\gamma(c_1, d_1) \cdot \gamma(c, d'))(c_1) \\ &= (\gamma(c, d')(c_1))^\circ\end{aligned}$$

which implies that $\gamma(c, d)(c_1) = \gamma(c, d')(c_1)$. Since this is true for all $c_1 \in \boldsymbol{v}\mathcal{C}$, we have $\gamma(c, d) = \gamma(c, d')$. Now if $((\gamma, \gamma^*), (\delta, \delta^*) \in \operatorname{cong}\rho$ in $\tilde{S}\Gamma$, then clearly $(\gamma, \delta) \in \operatorname{cong}\rho$ in $U\Gamma$ and by the above $\gamma = \delta$. Hence $((\gamma, \gamma^*), (\delta, \delta^*)) \in \operatorname{cong}\pi$ and so $\operatorname{cong}\rho \subseteq \operatorname{cong}\pi$.

It follows that $\operatorname{cong}\rho = \operatorname{cong}\pi$. Hence the map ψ defined in the statement is a bijection $\psi\colon (\tilde{S}\Gamma)_\rho \to U\Gamma$ such that $\rho \circ \psi = \pi$. Since $\pi(\sigma\sigma') = \pi\sigma\pi\sigma'$ by the definition of the binary operation in $\tilde{S}\Gamma$, it is clear that the map ψ is an isomorphism. □

5.4 Categories of left and right ideals of CR-semigroups

Let Γ be a cross-connection of $\mathcal{D}$ with $\mathcal{C}$. We use the proposition above to show that the categories of left and right ideals of $\tilde{S}\Gamma$ are isomorphic to $\mathcal{C}$ and $\mathcal{D}$ respectively. We need the following:

Lemma 34 *Let S be a regular semigroup. Then the assignments*

$$Se \mapsto (S_\rho)\rho_e; \qquad \rho(e, u, f) \mapsto \rho(\rho_e, \rho_u, \rho_f) \tag{35}$$

for all $Se \in \boldsymbol{v}\mathcal{L}(S)$ and for all morphism $\rho(e, u, f) \in \mathcal{L}(S)$, is an isomorphism of $\mathcal{L}(S)$ to $\mathcal{L}(S_\rho)$.

Proof Since $\operatorname{cong}\rho \subseteq \mathcal{L}$, it is clear that the first assignment above is a bijection. Also it is clear that the second assignment is surjective on each hom-set of $\mathcal{L}(S)$. If $\rho(e,u,f)$ and $\rho(e,v,f)$ are morphisms in the same home set in $\mathcal{L}(S)$ so that $u,\ v \in eSf$, and if $\rho(\rho_e,\rho_u,\rho_f) = \rho(\rho_e,\rho_v,\rho_f)$ then $\rho_u = \rho_v$ since $\rho_u,\ \rho_v \in \rho_e(S_\rho)\rho_f$. Therefore

$$u = eu = e\rho_u = e\rho_v = ev = v$$

which proves that the second assignment is a bijection on each hom-set. It is clearly functorial. Thus (35) defines an isomorphism. □

Since $\psi\colon (\tilde{\boldsymbol{S}}\Gamma)_\rho \to U\Gamma$ is an isomorphism by Proposition 33, there is an obvious isomorphism of $\mathcal{L}(U\Gamma)$ onto $\mathcal{L}((\tilde{\boldsymbol{S}}\Gamma)_\rho)$ induced by ψ. Hence it follows from Lemma 34 above that there is an isomorphism $F_\pi\colon \mathcal{L}(U\Gamma) \to \mathcal{L}(\tilde{\boldsymbol{S}}\Gamma)$. Therefore by Proposition 31, $F_\Gamma = \hat{F}\circ F_\pi$ is an isomorphism of $\mathcal{C}$ onto $\mathcal{L}(\tilde{\boldsymbol{S}}\Gamma)$. Combining Equations (29), (34) and (35), we see that F_Γ is defined on objects and morphisms of $\mathcal{C}$ as follows:

$$\begin{aligned} &F_\Gamma(c) = \tilde{\boldsymbol{S}}\Gamma\alpha_\Gamma(c,d) \quad \text{for some } d \in \Gamma^*(c);\\ &F_\Gamma(f) = \rho\left(\alpha_\Gamma(c,d),\sigma,\alpha_\Gamma(c',d')\right) \end{aligned} \tag{36}$$

where $f\colon c \to c' \in \mathcal{C}$, $d \in M\Gamma^*(c)$, $d' \in M\Gamma^*(c')$, $\sigma \in \tilde{\boldsymbol{S}}\Gamma$ with $\pi\sigma = \gamma(c,d)\star f^\circ$ and α_Γ denotes the bijection defined by Equation (33). Dually there is an isomorphism $F_{\Gamma^*}\colon \mathcal{D} \to \mathcal{L}(\tilde{\boldsymbol{S}}\Gamma^*)$ defined as above. Now, it follows from Theorem 32 and the discussion preceding it, that the map $(\gamma^*,\gamma) \mapsto (\gamma,\gamma^*)$ is an isomorphism of $\tilde{\boldsymbol{S}}\Gamma^*$ onto $\tilde{\boldsymbol{S}}\Gamma^{op}$ and hence induces an isomorphism of $\mathcal{L}(\tilde{\boldsymbol{S}}\Gamma^*)$ onto $\mathcal{R}(\tilde{\boldsymbol{S}}\Gamma)$. Thus there is an isomorphism G_Γ defined on objects and morphisms of $\mathcal{D}$ as follows:

$$\begin{aligned} &G_\Gamma(d) = \alpha_\Gamma(c,d)\tilde{\boldsymbol{S}}\Gamma \quad \text{for some } c \in \Gamma(d);\\ &G_\Gamma(g) = \lambda\left(\alpha_\Gamma(c,d),\sigma,\alpha_\Gamma(c',d')\right) \end{aligned} \tag{37}$$

where $g\colon d' \to d \in \mathcal{D}$, $c \in M\Gamma(d)$, $c' \in M\Gamma(d')$, $\sigma \in \tilde{\boldsymbol{S}}\Gamma$ with $\pi^*\sigma = \gamma^*(c',d')\star g^\circ$ and α_Γ denotes the bijection defined by Equation (33). We thus have:

Theorem 35 *For any cross-connection $\Gamma\colon \mathcal{D} \to \boldsymbol{N}^*\mathcal{C}$ there exist isomorphisms*

$$F_\Gamma\colon \mathcal{C} \to \boldsymbol{\mathcal{L}}(\tilde{\boldsymbol{S}}\Gamma) \quad \text{and} \quad G_\Gamma\colon \mathcal{D} \to \boldsymbol{\mathcal{R}}(\tilde{\boldsymbol{S}}\Gamma)$$

defined by Equations (36) *and* (37) *respectively.* □

We have seen that the connection of a regular semigroup is an embedding of categories if and only if it is right reductive (see Corollary 3). For arbitrary cross-connections we have the following.

Corollary 36 *The following statements are equivalent for any cross-connection $\Gamma\colon \mathcal{D} \to \boldsymbol{N}^*\mathcal{C}$:*

(a) *$\tilde{\boldsymbol{S}}\Gamma$ is right reductive.*
(b) *The projection $\pi\colon \tilde{\boldsymbol{S}}\Gamma \to U\Gamma$ is an isomorphism.*
(c) *$\Gamma\colon \mathcal{D} \to \boldsymbol{N}^*\mathcal{C}$ is an embedding.*

Proof Equivalence of (a) and (b) follows from Proposition 33. Suppose that (b) holds. If $\Gamma(d) = \Gamma(d')$, then for any $c \in M\Gamma(d)$, $\gamma(c,d) = \gamma(c,d')$ and so, $(\gamma(c,d), \gamma^*(c,d)), (\gamma(c,d'), \gamma^*(c,d')) \in \operatorname{cong}\pi$. It follows from (b) that

$$(\gamma(c,d), \gamma^*(c,d)) = (\gamma(c,d'), \gamma^*(c,d')).$$

Hence $\gamma^*(c,d) = \gamma^*(c,d')$. This implies that $d = c_{\gamma^*(c,d)} = c_{\gamma^*(c,d')} = d'$. Therefore $\boldsymbol{v}\Gamma$ is injective. Since Γ is a local isomorphism, this implies that Γ is an embedding. Now suppose that (c) holds and that $(\alpha_\Gamma(c,d), \alpha_\Gamma(c',d')) \in \operatorname{cong}\rho = \operatorname{cong}\pi$. Then $\gamma(c,d) = \pi\alpha_\Gamma(c,d) = \pi\alpha_\Gamma(c',d') = \gamma(c',d')$. Hence

$$c = c_{\gamma(c,d)} = c_{\gamma(c',d')} = c'$$

and
$$\Gamma(d) = H(\gamma(c,d); -) = H(\gamma(c,d'); -) = \Gamma(d')$$

by Equation (21). So $d = d'$ by (c). Therefore $\alpha_\Gamma(c,d) = \alpha_\Gamma(c',d')$ which implies that $\operatorname{cong}\rho$ is the identity congruence on $\tilde{\boldsymbol{S}}\Gamma$. □

5.5 Representations by CR-semigroups

We proceed to show that any regular semigroup is isomorphic to a cross-connection semigroup.

Recall that if S is a regular semigroup, by Theorem 17, its connection $\boldsymbol{\Gamma}S$ from $\boldsymbol{\mathcal{R}}(S)$ to $\boldsymbol{N}^*\boldsymbol{\mathcal{L}}(S)$ defined by Equation (2) is a cross-connection.

Proposition 37 *Let S be a regular semigroup. Then we have*

(38) $$U\boldsymbol{\Gamma}S = \{\,\rho^a : a \in S\,\};$$

(38*) $$U\Delta S = \{\,\lambda^a : a \in S\,\}.$$

Moreover $\rho^a \in U\boldsymbol{\Gamma}S$ is linked to $\lambda^b \in U\Delta S$ if and only if there is $c \in S$ with $\rho^a = \rho^c$ and $\lambda^b = \lambda^c$.

Proof For brevity, in the following, we write $\boldsymbol{\Gamma}$ for $\boldsymbol{\Gamma}S$ and Δ for ΔS. We first observe that any idempotent of $U\boldsymbol{\Gamma}$ is of the form ρ^e for $e \in E(S)$. Now, by Proposition 31, any idempotent of $U\boldsymbol{\Gamma}$ is of the form $\gamma(Se, fS)$ for $(Se, fS) \in E_{\boldsymbol{\Gamma}}$. By Equation (2), $(Se, fS) \in E_{\boldsymbol{\Gamma}}$ if and only if $Se \in MH(\rho^f; -)$; which is equivalent to $Se \in M\rho^f$. By Lemma III.15 this is true if and only if there is $g \in E(\mathcal{R}_f)$ such that $Se = Sg$. Thus $(Se, fS) \in E_{\boldsymbol{\Gamma}}$ if and only if $Se = Sg$ and $fS = gS$ for some $g \in E(S)$. Now, by Equations (2) and (21), we have $H(\gamma(Se, eS); -) = \boldsymbol{\Gamma}(Se) = H(\rho^e; -)$ and since vertices of $\gamma(Se, eS)$ and ρ^e are the same, we conclude that $\gamma(Se, eS) = \rho^e$. It follows that

(39) $$E(U\boldsymbol{\Gamma}) = \{\,\rho^e : e \in E(S)\,\}.$$

Clearly for any $a \in S$, $\rho^a = \rho^e \star \rho(e, a, f)$ where $e \in E(R_a)$ and $f \in E(L_a)$ and so by Lemma 28, $\rho^a \in U\boldsymbol{\Gamma}$. If $\gamma \in U\boldsymbol{\Gamma}$, then again by Lemma 28 and the observation above, $\gamma = \rho^e \star \rho$ where ρ is an isomorphism of Se onto the vertex Sf of γ. Hence by Proposition III.13, there is $a \in R_e \cap L_f$ such that $\rho = \rho(e, a, f)$ and it follows that

$$\gamma = \rho^e \star \rho(e, a, f) = \rho^a.$$

This proves the first equation. Since $\Delta = \boldsymbol{\Gamma}^*$ and $\chi_{\boldsymbol{\Gamma}} = \chi_S$ by Theorem 17, the second follows by duality.

Let $a \in R_e \cap L_f$ where $e, f \in E(S)$. Then by Proposition III.13 and its dual, $\rho(e, a, f)\colon Se \to Sf$ and $\lambda(f, a, e)\colon fS \to eS$ are isomorphisms and we have

$$\rho^a = \rho^e \star \rho(e, a, f) \quad \text{and} \quad \lambda^a = \lambda^f \star \lambda(f, a, e).$$

Hence, in view of the equality $\chi_{\boldsymbol{\Gamma}} = \chi_S$, Equation (5) shows that ρ^a and λ^a are linked. Conversely suppose that ρ^a is linked to λ^b. Then $\lambda^b = \chi_S(Se, fS)(\rho^a)$ for $Se \in \boldsymbol{v}\mathcal{L}(S)$ and $fS \in \boldsymbol{v}\mathcal{R}(S)$. Let Sg be the vertex of ρ^a. Since $Sf \in$

$M\Gamma(fS)$ we have $\rho^a = \rho^f \star \rho(f,a,e)^\circ$. Also, we can find $h \in E(R_a)$ with $h\ \omega\ f$ such that $\rho(h,a,g)$ is the isomorphism component in a normal factorization of $\rho(f,a,e)$ (see Corollary III.14). Hence by Lemma 29 and Equation (39), we have $\rho^a = \rho^h \star \rho(h,a,g)$ and

$$\chi_S(Se,fS)(\rho^a) = \chi_S(Sg,hS)(\rho^a) = \lambda^b.$$

Hence by Equation (5), $\lambda^b = \lambda^g \star \lambda(g,a,h) = \lambda^a$. □

We thus have the following theorem which shows that every regular semigroup is isomorphic to a cross-connection semigroup.

Theorem 38 *Let S be a regular semigroup. Then*

$$\tilde{S}\Gamma S = \{(\rho^a,\lambda^a) : a \in S\} \tag{40}$$

and the map $\varphi(S)\colon S \to \tilde{S}\Gamma S$ defined by

$$\varphi(S)(a) = (\rho^a,\lambda^a) \tag{41}$$

is an isomorphism of S onto $\tilde{S}\Gamma S$.

Proof The equality (40) is an immediate consequence of Proposition 37. Also from this, it is clear that $\varphi(S)$ is surjective. If a, $b \in S$, then by Theorem III.16 and its dual, $\rho^{ab} = \rho^a\rho^b$ and $\lambda^{ab} = \lambda^b\lambda^a$ and it follows from Equation (32) that $\varphi(S)(ab) = \varphi(S)(a)\varphi(S)(b)$ and hence $\varphi(S)$ is a homomorphism. Now if $\varphi(S)(a) = \varphi(S)(b)$, than $\rho^a = \rho^b$ and $\lambda^a = \lambda^b$. Since any regular semigroup is weakly reductive, by Theorem III.16 and its dual, it follows that $a = b$. □

Remark 4 It follows from Theorem 32 and 38 that one can construct a regular semigroup, namely, the cross-connection semigroup from the information provided by a cross-connection and that every regular semigroup is isomorphic to one such semigroup. Thus the data required to construct a regular semigroup is the same as that required to specify a cross-connection. Now a cross-connection $\Gamma\colon \mathcal{D} \to N^*\mathcal{C}$ is completely determined by the image of Γ which is a total ideal $\mathcal{D}_0$ in $N^*\mathcal{C}$ and the map $v\Gamma\colon v\mathcal{D} \to vN^*\mathcal{C}$. This follows from the observation that since Γ is fully-faithful, it induces a bijection on each hom-set $\mathcal{D}(d, d')$ onto $N^*\mathcal{C}(\Gamma(d), \Gamma(d'))$ and so these may be identified. In fact if $d,\ d' \in \boldsymbol{v}\mathcal{D}$ and $c \in M\Gamma(d),\ c' \in M\Gamma(d')$, by Lemma III.21 and Equation (21), we have

$$N^*\mathcal{C}(\Gamma(d), \Gamma(d')) = \{\, \eta_{\gamma(c,d)}\mathcal{C}(f, -)\eta^{-1}_{\gamma(c',d')} : f \in \mathcal{C}(c', c)\,\}.$$

This implies that $\mathcal{D}(d, d')$ is completely determined by $\Gamma(d)$, $\Gamma(d')$ and $\mathcal{C}(c', c)$. Since every normal category is isomorphic to a category of left [right] ideals of a suitable regular semigroup (see Corollary III.20), the vertex set of any normal category is a regular partially ordered set in the sense of Grillet (see [3,13]). In fact, given a normal category $\mathcal{C}$ and a regular partially ordered set D, it possible to specify a cross-connection essentially in terms of a certain order-preserving map of D into the partially ordered set $\boldsymbol{v}N^*\mathcal{C}$.

To see this, we first introduce some terminology. We shall say that an order-preserving map $\vartheta\colon D \to D'$ of partially ordered sets is a *local isomorphism* if the associated functor of pre-orders is a local isomorphism as defined earlier (see §II.1 and Definition 1); this is obviously equivalent to the fact that the map $\vartheta|D(d)$ is an order isomorphism for each principal order ideal $D(d)$ of D onto $D'(\vartheta(d))$. Note that the image of a local isomorphism $\vartheta\colon D \to D'$ is an order ideal of D'. Also an ideal J of $\boldsymbol{v}N^*\mathcal{C}$ is a *total ideal* if the full subcategory of $N^*\mathcal{C}$ generated by J is a total ideal of $N^*\mathcal{C}$ (or equivalently, if for each $c \in \boldsymbol{v}\mathcal{C}$ there is $H(\epsilon; -) \in J$ such that $c \in MH(\epsilon; -)$). Now if $\Gamma\colon \mathcal{D} \to N^*\mathcal{C}$ is a cross-connection, Γ is a local isomorphism and hence $\boldsymbol{v}\Gamma$ is a local isomorphism of partially ordered sets whose image is a total order ideal. Thus, up to an isomorphism, a cross-connection can be specified by (or equivalently, a regular semigroup can be constructed from) the following data:

(1) A normal category $\mathcal{C}$.
(2) A regular partially ordered set D.
(3) A local isomorphism $\vartheta\colon D \to \boldsymbol{v}N^*\mathcal{C}$ of D whose image is total in $\boldsymbol{v}N^*\mathcal{C}$.

Given these, we may construct the category $\mathcal{D}$ by setting:

$$\boldsymbol{v}\mathcal{D} = D; \quad \mathcal{D}(d.d') = \{\, \eta_\epsilon \mathcal{C}(f, -)\eta^{-1}_{\epsilon'} : f \in \mathcal{C}(c', c)\,\} \tag{42}$$

for each $d,\ d' \in D$. Here $c,\ c' \in \boldsymbol{v}\mathcal{C}$ are arbitrary, except for the condition that $c \in M\vartheta(d)$ and $c' \in M\vartheta(d')$ and $\epsilon,\ \epsilon'$ are unique cones in $\mathcal{C}$ such that $c_\epsilon = c$,

$c_{\epsilon'} = c'$, $\vartheta(d) = H(\epsilon; -)$ and $\vartheta(d') = H(\epsilon'; -)$. The cross-connection Γ can then be constructed by setting

$$(43) \qquad v\Gamma = \vartheta; \quad \Gamma|\mathcal{D}(d, d') = 1_{\mathcal{D}(d,d')}$$

for each d, $d' \in D$. Up to an isomorphism, every cross-connection can be constructed in this way. In this case, clearly,

$$\begin{aligned}\Gamma(c, d) &= \{\, \epsilon \star f^\circ : \vartheta(d) = H(\epsilon; -), \;\; f \in \mathcal{C}(c_\epsilon, c) \,\} \\ &= \{\, \gamma \in T\mathcal{C} : c_\gamma = c; \;\; H(\gamma; -) \subseteq \vartheta(d) \,\}\end{aligned}$$

by Proposition III.7(a). Hence

$$(44) \qquad U\Gamma = \{\, \gamma \in T\mathcal{C} : H(\gamma; -) \subseteq \vartheta(d) \text{ for some } d \in D \,\}.$$

Also, we note that given $\gamma \in U\Gamma$ and $d \in D$ such that $\gamma \in \Gamma(c_\gamma, d)$ the normal cone in $\mathcal{D}$ linked to γ (that is, $\chi_\Gamma(c_\gamma, d)(\gamma)$; see Equation (30)) is uniquely determined. Hence if

$$(46) \qquad \hat{S}\Gamma = \{\, (\gamma, d) \in T\mathcal{C} \times D : H(\gamma; -) \subseteq \vartheta(d) \,\}$$

and if we define multiplication in $\hat{S}\Gamma$ by

$$(46) \qquad (\gamma, d)(\delta, d') = (\gamma \cdot \delta, d)$$

then it can be seen that $\hat{S}\Gamma$ becomes a semigroup isomorphic to the cross-connection semigroup of Γ.

If our aim were to construct right reductive regular semigroups with a given normal category $\mathcal{C}$ as its category of left ideals, then the construction can be further simplified. For by Corollary 36, Γ is an isomorphism of $\mathcal{D}$ onto image of Γ in $\boldsymbol{N}^*\mathcal{C}$. Thus, in this case, we may take the category $\mathcal{D}$ as a total ideal in $\boldsymbol{N}^*\mathcal{C}$ and the cross-connection as the inclusion functor of the total ideal $\mathcal{D}$ in $\boldsymbol{N}^*\mathcal{C}$. Therefore there is one-to-one correspondence between right reductive regular semigroups whose category of left ideals is isomorphic to $\mathcal{C}$ and total ideals in $\boldsymbol{N}^*\mathcal{C}$. In view of the fact that ideals in a category are full subcategories (see Remark III.4), we observe that picking a total ideal in $\boldsymbol{N}^*\mathcal{C}$ only amounts to picking a total order ideal in $v\boldsymbol{N}^*\mathcal{C}$.

CHAPTER V

The Category of Cross-connections

In the Chapter IV, we have introduced the notion of cross-connections. Also, we showed that every cross-connection Γ determines a regular semigroup, viz, the cross-connection semigroup $\tilde{\boldsymbol{S}}\Gamma$ (see Theorem IV.32) and conversely, every regular semigroup is isomorphic to one such semigroup (see Theorem IV.38). Thus we have the assignments:

$$\Gamma \mapsto \tilde{\boldsymbol{S}}\Gamma \qquad \text{and} \qquad S \mapsto \boldsymbol{\Gamma} S$$

between the class of all cross-connections to the object class of the category $\boldsymbol{RS}$ of regular semigroups. Our aim in this chapter is to extend these correspondences to a category equivalence between the category $\boldsymbol{RS}$ and the (suitably defined) 'category of cross-connections' (see §2 for the definition of this category).

Observe that by Remark IV.3, given any cross-connection Γ between normal categories $\mathcal{D}$ and $\mathcal{C}$, the map $\alpha = \alpha_\Gamma$ defined by Equation (IV.33) is a bijection of E_Γ with the biordered set $E(\tilde{\boldsymbol{S}}\Gamma)$ of the CR-semigroup $\tilde{\boldsymbol{S}}\Gamma$. It is therefore obvious that a biorder structure can be defined on E_Γ in such a way that α becomes a biorder-isomorphism. In §1 we give an explicit computation of this biorder structure on E_Γ which will be useful in the sequel. In §2, we complete the definition of the 'category of cross-connections,' denoted by $\boldsymbol{Cr}$, by introducing the concept of a morphism of cross-connections. We also obtain some alternate descriptions of morphisms which will be convenient for applications. In §3 we show that every morphism of cross-connections determines a unique homomorphism of the corresponding CR-semigroups and conversely that every homomorphism of regular semigroups determines a unique morphism of their cross-connections. Moreover, these correspondences are functorial. In §4 we use these results to give an explicit construction of an adjoint equivalence between the category $\boldsymbol{Cr}$ of cross-connections and the category $\boldsymbol{RS}$ of regular semigroups.

1 THE BIORDERED SET OF A CROSS-CONNECTION

As noted above, given any cross-connection Γ between normal categories $\mathcal{D}$ and $\mathcal{C}$, the map $\alpha = \alpha_\Gamma$ defined by Equation (IV.33) is a bijection of E_Γ with the biordered set $E(\tilde{S}\Gamma)$ of the $C\mathcal{R}$-semigroup $\tilde{S}\Gamma$. In this section we shall give an explicit description of the biorder-structure induced on E_Γ by α.

1.1 Biordered sets and bimorphisms

We begin by briefly recalling the notion of *biordered sets* and related concepts. For details, we refer the reader to [15].

Recall that a biordered set is a partial algebra E whose partial binary operation, called the *basic product* of E, is determined by two quasi-orders ω^l and ω^r in the sense that the basic product ef of $e, f \in E$ exists if and only if e and f are related by ω^l, ω^r or their inverses; moreover, the partial algebra satisfies certain axioms (see [15],§1). Easdown has shown that the set of idempotents $E(S)$ of a semigroup is a biordered set whose biorder structure is induced by the binary operation in S and that any partial algebra E is a biordered set if and only if it is isomorphic (as a partial algebra) to the biordered set of some semigroup (see [2]).

Recall also that with every pair of elements e, f in a biordered set E, we can associate a quasiordered set $M(e, f)$ as follows:

$$M(e,f) = \left(\omega^l(e) \cap \omega^r(f), \prec\right)$$

where $\prec$ is the relation defined by

$$g \prec h \iff eg\,\omega^r\,eh, \quad gf\,\omega^l\,hf.$$

The relation $\prec$ is easily shown to be reflexive and transitive and hence, a quasi-order. The subset $\mathcal{S}(e, f)$ of $M(e, f)$ consisting of all *maximum* elements in $M(e, f)$, is called the *sandwich set* of e and f; thus

(1a) $$\mathcal{S}(e, f) = \{\, h \in M(e, f) : g \prec h \text{ for all } g \in M(e, f) \,\}$$

The biordered set E is regular if

(R) $$\mathcal{S}(e, f) \neq \emptyset$$

for every pair $e, f \in E$ (see [15], Definition 1.1). If S is a regular semigroup then $E(S)$ is a regular biordered set. In this case, the sandwich set of any pair $e, f \in E(S)$ can be computed as follows:

(1b) $$\mathcal{S}(e, f) = \{ h \in E(S) : he = fh = h, \quad ehf = ef \} = M(e, f) \cap \mathcal{V}(ef).$$

where $\mathcal{V}(ef)$ denotes the set of inverses of the element $ef \in S$ (see [15], Theorem 1.1). It is also true that every regular biordered set is isomorphic to the biordered set of a regular semigroup (see [15]).

A mapping $\theta \colon E \to E'$ of biordered sets is a *bimorphism* if it is a homomorphism of partial algebras and θ is a *regular bimorphism* if

(2) $$\theta(\mathcal{S}(e, f)) \subseteq \mathcal{S}(\theta(e), \theta(f))$$

for all $e, f \in E$. Again, if $h \colon S \to S'$ is a homomorphism of regular semigroups, then $h|E(S) \colon E(S) \to E(S')$ is a regular bimorphism and, conversely, every regular bimorphism arises in this way (see [15], Corollary 4.15). A *biorder isomorphism* is a partial algebra isomorphism of biordered sets. Note that a biorder isomorphism is always a regular bimorphism.

1.2 Biordered sets of cross-connections

Let $\Gamma \colon \mathcal{D} \to N^*\mathcal{C}$ be a cross-connection. If α denotes the bijection of E_Γ onto $E(\tilde{S}\Gamma)$ defined by Equation (IV.33) it is clear from the remarks above that there is a unique biorder structure on E_Γ making α, a biorder isomorphism. From now on, E_Γ will denote the biordered set with this biorder structure (induced by α) and will be called the biordered set of Γ. We now give an explicit description of the biorder structure of E_Γ in terms of Γ and the categories $\mathcal{C}$ and $\mathcal{D}$. For $(c, d), (c', d') \in E_\Gamma$, let

(3) $$(c', d') \, \omega^l \, (c, d) \iff c' \subseteq c; \quad (c', d') \, \omega^r \, (c, d) \iff d' \subseteq d.$$

We define basic products in E_Γ as follows:

(4) $$(c, d)(c', d') = \begin{cases} (c, d), & \text{if } c \subseteq c'; \\ (c', \operatorname{im} \gamma^*(c, d)(d')), & \text{if } c' \subseteq c; \\ (c', d'), & \text{if } d' \subseteq d; \\ (\operatorname{im} \gamma(c', d')(d)), & \text{if } d \subseteq d'. \end{cases}$$

It follows from Theorem IV.35 that $\alpha(c,d)\ \omega^l\ \alpha(c',d')$ if and only if $c \subseteq c'$. Hence by Equation (3), $(c,d)\ \omega^l\ (c',d')$ if and only if $\alpha(c,d)\ \omega^l\ \alpha(c',d')$ in $\tilde{\mathbf{S}}\Gamma$. Also in this case it is clear that $\alpha((c,d)(c',d')) = \alpha(c,d) = \alpha(c,d)\alpha(c',d')$. Now, by Equations (IV.32) and (IV.33),

$$\begin{aligned}\alpha(c',d')\alpha(c,d) &= \left(\gamma(c',d'),\gamma^*(c',d')\right)\left(\gamma(c,d),\gamma^*(c,d)\right)\\ &= \left(\gamma(c',d')\gamma(c,d);\gamma^*(c,d)\gamma^*(c',d')\right)\\ &= \left(\gamma(c',d')\star\gamma(c,d)(c')^\circ,\gamma^*(c,d)\star\gamma^*(c',d')(d)^\circ\right).\end{aligned}$$

Since $c \subseteq c'$, $\gamma(c,d)(c')\colon c' \to c$ is a retraction and it is easy to see that the vertex of $\gamma(c',d')\star\gamma(c,d)(c')^\circ$ is c and the vertex of $\gamma^*(c,d)\star\gamma^*(c',d')(d)^\circ$ is $d'' = \operatorname{im}\gamma^*(c',d')(d)$. Further, by Equation (IV.25), $\gamma^*(c',d')(d) = (\gamma(c,d)(c'))^*$ and so, $\gamma^*(c',d')(d)$ is a monomorphism. Hence $\gamma^*(c',d')(d)^\circ$ is an isomorphism. Therefore it follows that $d'' \in M\gamma^*(c,d) = M\Gamma^*(c)$. Thus $(c,d'') \in E_\Gamma$ and by uniqueness, we have

$$\alpha(c',d')\alpha(c,d) = \alpha(c,d'') = \alpha\left((c',d')(c,d)\right)$$

by the definition of basic products above. Similarly, it can be shown (dually) that $(c,d)\ \omega^r\ (c',d')$ if and only if $\alpha(c,d)\ \omega^r\ \alpha(c',d')$ and in this case, we have

$$\alpha(c',d')\alpha(c,d) = \alpha\left((c',d')(c,d)\right); \qquad \alpha(c,d)\alpha(c',d') = \alpha\left((c,d)(c',d')\right).$$

We have proved the following:

Theorem 1 *Let $\Gamma\colon \mathcal{D} \to \mathbf{N}^*\mathcal{C}$ be a cross-connection. Then*

$$E_\Gamma = \{\,(c,d) : c \in M\Gamma(d)\,\}$$

becomes a regular biordered set when quasi-orders and basic products are defined by Equations (3) *and* (4) *respectively. Also, in this case, the mapping $\alpha\colon E_\Gamma \to E(\tilde{\mathbf{S}}\Gamma)$ defined by Equation* (IV.33) *is a biorder isomorphism.* □

Given a cross-connection Γ, we shall refer to the biordered set E_Γ given by the theorem above as the *biordered set of the cross-connection* Γ. We now proceed to give a computation of sandwich sets in E_Γ. First notice that, by the well known property of biordered sets, the sandwich set $\mathcal{S}(e,\ f)$ of any two elements in a biordered set depends only on the $\mathcal{L}$–class of e and the $\mathcal{R}$–class of f (see [15],§2). By Equation (3) the $\mathcal{L}$–class of $(c,d') \in E_\Gamma$ is uniquely represented by c and the $\mathcal{R}$–class of $(c',d) \in E_\Gamma$ is represented by d. Therefore given any $(c,d) \in v\mathcal{C} \times v\mathcal{D}$ we can unambiguously denote by $\mathcal{S}(c,\, d)$ the sandwich set $\mathcal{S}((c,d'),\, (c',d))$ for any choice of $d' \in M\Gamma^*(c)$ and $c' \in M\Gamma(d)$; we shall call $\mathcal{S}(c,\, d)$ as the *sandwich set of the pair* (c,d).

Recall from §II.2 and II.3 that if $f = \varrho u j$ is any normal factorization of a morphism f in a normal category $\mathcal{C}$, then $\operatorname{im}\varrho = \operatorname{coim} f = \operatorname{dom} u$ and $\operatorname{dom} j = \operatorname{im} f = \operatorname{cod} u$. Recall also that while the image of a morphism is unique, the coimage of a morphism need not be unique.

Theorem 2 *Let $\Gamma\colon \mathcal{D} \to N^*\mathcal{C}$ be a cross-connection. Then we have:*

(5) $$\mathcal{S}(c,\, d) = \{\, (c'',d'') : c'' = \operatorname{coim}\gamma(c',d)(c),\;\; d'' = \operatorname{coim}\gamma^*(c,d')(d)\,\}$$

for all $(c,d) \in v\mathcal{C} \times v\mathcal{D}$, for any $d' \in M\Gamma^*(c)$ and $c' \in M\Gamma(d)$.

Proof First fix $d' \in M\Gamma^*(c)$ and $c' \in M\Gamma(d)$ and assume that $(c'',d'') \in \mathcal{S}(c,\, d) = \mathcal{S}((c,d'),\, (c',d))$. Then $(c'',d'') \in M((c,d'),(c',d))$ and so $c'' \subseteq c$ and $d'' \subseteq d$. Now by Theorems IV.32 and 1, the map

$$\alpha \circ E(\pi)\colon (\bar{c},\bar{d}) \in E_\Gamma \mapsto \gamma(\bar{c},\bar{d})$$

is a bimorphism of E_Γ onto $E(U\Gamma)$ and the map

$$\alpha \circ E(\pi^*)\colon (\bar{c},\bar{d}) \mapsto \gamma^*(\bar{c},\bar{d})$$

is a bimorphism of E_Γ onto $E\left((U\Gamma^*)^{op}\right)$. Hence

$$\gamma(c'',d'') \in \mathcal{S}(\gamma(c,d'),\, \gamma(c',d)) \quad\text{and}\quad \gamma^*(c'',d'') \in \mathcal{S}(\gamma^*(c',d),\, \gamma^*(c,d')).$$

Therefore $\gamma(c,d')\cdot\gamma(c'',d'')\cdot\gamma(c',d) = \gamma(c,d')\cdot\gamma(c',d)$. It follows that

$$\gamma(c',d)(c)^\circ = \gamma(c'',d'')(c)\gamma(c',d)(c'')^\circ$$

and so,

(∗) $$\gamma(c',d)(c) = \gamma(c'',d'')(c)\gamma(c',d)(c'').$$

Since $c'' \subseteq c$, $\gamma(c'',d'')(c)\colon c \to c''$ is a retraction. Similarly $\gamma^*(c'',d'')(d)$ is also a retraction since $d'' \subseteq d$ and by Equation (IV.25), $\gamma^*(c'',d'')(d) = (\gamma(c',d)(c''))^*$. Therefore, by duality, $\gamma(c',d)(c'')$ is a monomorphism. It follows from Equation $(*)$ that $c'' = \operatorname{coim}\gamma(c',d)(c)$. Similarly, from

$$\gamma^*(c'',d'') \in \mathcal{S}(\gamma^*(c',d),\ \gamma^*(c,d')),$$

we deduce that $d'' = \operatorname{coim}\gamma^*(c.d')(d)$.

Conversely suppose that $(c'',d'') \in \boldsymbol{v}\mathcal{C} \times \boldsymbol{v}\mathcal{D}$ such that

$$c'' = \operatorname{coim}\gamma(c',d)(c) \qquad d'' = \operatorname{coim}\gamma^*(c,d')(d)$$

where $c' \in M\Gamma(d)$ and $d' \in M\Gamma^*(c)$. By the definition of coimages there is a retraction $\varrho\colon c \to c''$ such that $\gamma(c',d)(c) = \varrho u j$ is a normal factorization for some isomorphism $u\colon c'' \to c'_1$ and inclusion $j = j^{c'}_{c'_1}$ so that $c'_1 = \operatorname{im}\gamma(c',d)(c)$. Then $\gamma(c',d)(c'') = j^c_{c''}\gamma(c',d)(c) = uj$; in particular, $\gamma(c',d)(c'')$ is a monomorphism. Also $\gamma(c,d') \star \varrho$ is an idempotent cone such that $H(\gamma(c,d')\star\varrho;-) \subseteq H(\gamma(c,d');-)$ by Proposition III.7. Hence there exists a unique $d'_1 \subseteq d'$ such that $\Gamma(d'_1) = H(\gamma(c,d')\star\varrho;-)$ and so, by Equation (IV.21), we have $c'' \in M\Gamma(d'_1)$ and $\gamma(c'',d'_1) = \gamma(c,d')\star\varrho$. Also there is a unique $d_1 \subseteq d$ such that $c'_1 \in M\Gamma(d_1)$ and $\gamma(c'_1,d_1)(c'') = \gamma(c',d)(c'')^\circ = u$. It follows that $\gamma(c'_1,d_1)(c'')$ is an isomorphism so that $c'' \in M\Gamma(d_1)$ and

$$\gamma(c'_1,d_1)(c'')^{-1} = \gamma(c'',d_1)(c'_1). \tag{1}$$

Moreover, $\gamma(c',d)(c'') = uj = \gamma(c'_1,d_1)(c'')\gamma(c',d)(c'_1)$ and

$$\begin{aligned}\gamma(c,d')\cdot\gamma(c',d) &= \gamma(c,d')\star\gamma(c',d)(c)^\circ\\ &= \gamma(c'',d'_1)\star\gamma(c'_1,d_1)(c'').\end{aligned} \tag{2}$$

This gives

$$\gamma(c',d)(c) = \gamma(c'',d'_1)(c)\gamma(c'_1,d_1)(c'')\gamma(c',d)(c'_1) \tag{3}$$

which is a normal factorization of $\gamma(c',d)(c)$. Now using Corollary IV.22 and Equation (IV.25) we obtain from the above that

$$\gamma^*(c,d')(d) = \gamma^*(c'_1,d_1)(d)\gamma^*(c'',d'_1)(d_1)\gamma^*(c,d')(d'_1). \tag{4}$$

Now since $\gamma(c', d)(c'_1) = \jmath^{c'}_{c'_1}$, by Proposition IV.24(a),

$$\gamma^*(c'_1, d_1)(d) = (\gamma(c', d)(c'_1))^* \colon d \to d_1$$

is a retraction. By Corollary IV.23, $\gamma^*(c'', d'_1)(d_1)$ is an isomorphism. It follows that (4) gives a normal factorization of $\gamma^*(c, d')(d)$ since $\gamma^*(c, d')(d'_1)$ is an inclusion. In particular, $d_1 = \operatorname{coim} \gamma^*(c, d')(d)$. Since $c'' \in M\Gamma(d_1)$, $\gamma(c'', d_1)$ is an idempotent such that $\gamma(c'', d_1) \in M(\gamma(c, d'), \gamma(c', d))$ in $U\Gamma$ (by Equation (3)). By (4), $\operatorname{im} \gamma^*(c, d')(d_1) = d'_1$ and hence by Equation (4), $\gamma(c, d') \cdot \gamma(c'', d_1) = \gamma(c'', d'_1)$. Therefore

$$\begin{aligned}\gamma(c, d') \cdot \gamma(c'', d_1) \cdot \gamma(c', d) &= \gamma(c'', d'_1) \cdot \gamma(c', d)\\ &= \gamma(c'', d'_1) \star \gamma(c', d)(c'')^\circ\\ &= \gamma(c, d') \cdot \gamma(c', d)\end{aligned}$$

by (2). Hence $\gamma(c'', d_1) \in \mathcal{S}(\gamma(c, d'), \gamma(c', d))$. Since d'' is also a coimage of $\gamma^*(c, d')(d)$, there is a retraction $\varrho'' \colon d \to d''$ such that $\gamma^*(c'_1, d_1)(d) = \varrho'' k$ for some isomorphism $k \colon d'' \to d_1$. Then $\gamma^*(c', d) \star \varrho''$ is an idempotent in $U\Gamma^*$, $\mathcal{R}$–equivalent to $\gamma^*(c', d) \star \gamma^*(c'_1, d_1)(d) = \gamma^*(c'_1, d_1)$. Hence by Proposition III.7 and Corollary III.8, $d'' \in M\Gamma^*(c'_1)$ and $\gamma^*(c', d) \star \varrho'' = \gamma^*(c'_1, d'')$. Now $\gamma(c'_1, d_1)$, $\gamma(c'_1, d'') \in \omega(\gamma(c', d))$ and $\gamma(c'', d_1)\omega^r(\gamma(c', d))$ in $E(U\Gamma)$. Since the map $(c, d) \mapsto \gamma(c, d)$ is a bimorphism, by Equations (4) and (3), we have $\gamma(c'', d_1) \cdot \gamma(c', d) = \gamma(c'_1, d_1)$. Hence, using axiom (B4') of [15], we conclude that there is an idempotent $\gamma(c'', d'') \in L_{\gamma(c'', d_1)} \cap R_{\gamma(c'_1, d_1)}$ such that $\gamma(c'', d'') \cdot \gamma(c', d) = \gamma(c'_1, d_1)$ (see Propositions 2.4 and 2.9 of [15]). Clearly, $\gamma(c'', d'') \in M(\gamma(c, d'), \gamma(c', d))$ and it follows that

$$\gamma(c'', d'') \in \mathcal{S}(\gamma(c, d'), \gamma(c', d))$$

in $E(U\Gamma)$. Similarly $\gamma^*(c'', d'') \in \mathcal{S}(\gamma^*(c.d'), \gamma^*(c', d))$ in $E(U\Gamma^*)$. Therefore $\alpha(c'', d'') \in \mathcal{S}(\alpha(c, d'), \alpha(c', d))$ in $E(\tilde{S}\Gamma)$. It follows by the definition of α that $(c'', d'') \in \mathcal{S}((c, d'), (c', d))$ in E_Γ. $\square$

Remark 1 The construction of sandwich sets in the biordered set E_Γ described in the theorem above, apart from its use later in this chapter, is also of independent interest. Observe that sandwich idempotents are characterized directly in terms of the normal categories $\mathcal{C}$ and $\mathcal{D}$ and the cross-connection $\Gamma\colon \mathcal{D} \to N^*\mathcal{C}$ and it applies to arbitrary regular semigroups. One can therefore regard this as a generalization of various known characterizations of sandwich sets such as Pastijn's characterization of sandwich sets in the biordered set of a complemented modular lattice (biordered sets of strongly regular Baer semigroups—see [18,17]), sandwich sets in Fredholm semigroups (see [9]), etc.

2 THE MORPHISMS OF CROSS-CONNECTIONS

In the following, for brevity, we shall use the symbols e, f, etc., to denote elements of the biordered set E_Γ. If $e = (c, d) \in E_\Gamma$, then it follows from Theorem 1 that c represents the $\mathcal{L}$–class of e in the biordered set E_Γ and d represents the $\mathcal{R}$–class. In the following we use the notation c^e to denote the element $c \in v\mathcal{C}$ which is the first coordinate of e and d^e to denote the element in $d \in v\mathcal{D}$ which is the second coordinate of e. Thus we can write $e = (c^e, d^e)$ for all $e \in E_\Gamma$. Moreover, for $e, e' \in E_\Gamma$, $e \mathcal{L} e'$ if and only if $c^e = c^{e'}$ and $e \mathcal{R} e'$if and only if $d^e = d^{e'}$.

2.1 Definition of morphisms

The following definition shows that a morphism $m\colon \Gamma \to \Gamma'$ of cross-connections is a bifunctor $m\colon \mathcal{C} \times \mathcal{D} \to \mathcal{C}' \times \mathcal{D}'$ satisfying axioms (M1) and (M2) below.

Definition 1 *Let $\Gamma\colon \mathcal{D} \to N^*\mathcal{C}$ and $\Gamma'\colon \mathcal{D}' \to N^*\mathcal{C}'$ be two cross-connections. A* morphism of cross-connections *$m\colon \Gamma \to \Gamma'$ is a pair $m = (F_m, G_m)$ of inclusion-preserving functors $F_m\colon \mathcal{C} \to \mathcal{C}'$ and $G_m\colon \mathcal{D} \to \mathcal{D}'$ satisfying the following axioms:*

M1: $(c, d) \in E_\Gamma \implies (F(c), G(d)) \in E_{\Gamma'}$ *and*

$$F(\gamma(c, d)(c')) = \gamma(F(c), G(d))(F(c'))$$

for all $c' \in v\mathcal{C}$.

M2: *Let* $(c, d), (c', d') \in E_\Gamma$. *If* $f^*: d' \to d$ *is the transpose of* $f: c \to c'$, *then* $G(f^*) = (F(f))^*$.

The following lemma shows that our definition of morphism of cross-connections is self-dual.

Lemma 3 *Let* $\Gamma: \mathcal{D} \to \mathbf{N}^*\mathcal{C}$ *and* $\Gamma': \mathcal{D}' \to \mathbf{N}^*\mathcal{C}'$ *be two cross-connections. Suppose that* $F: \mathcal{C} \to \mathcal{C}'$ *and* $G: \mathcal{D} \to \mathcal{D}'$ *are inclusion preserving functors. Consider the following statements:*

M1: $(c, d) \in E_\Gamma \implies (F(c), G(d)) \in E_{\Gamma'}$ *and*

$$F(\gamma(c, d)(c')) = \gamma(F(c), G(d))(F(c'))$$

for all $c' \in v\mathcal{C}$.

M2: *Let* $(c, d), (c', d') \in E_\Gamma$. *If* $f^*: d' \to d$ *is the transpose of* $f: c \to c'$, *then* $G(f^*) = (F(f))^*$.

M1*: $(c, d) \in E_\Gamma \implies (F(c), G(d)) \in E_{\Gamma'}$ *and*

$$G(\gamma^*(c, d)(d')) = \gamma^*(F(c), G(d))(F(d'))$$

for all $d' \in v\mathcal{D}$.

M2*: *Let* $(c, d), (c', d') \in E_\Gamma$. *If* $g^*: c \to c'$ *is the transpose of* $g: d' \to d$, *then* $(G(g))^* = F(g^*)$.

The functors F *and* G *satisfy axioms* M1 *and* M2 *if and only if they satisfy axioms* M1* *and* M2*.

Proof Let F and G satisfy axioms M1 and M2 and let $(c, d), (c', d') \in E_\Gamma$. If $g: d' \to d$ is given and if $g^*: c \to c'$, is the transpose of g, then we have

$$G(g)^* = G(g^{**})^* = F(g^*)^{**} = F(g^*)$$

by axiom M2 and Corollary IV.21. Hence axiom M2* holds. To prove M1*, assume that $d' \in v\mathcal{D}$. Choose $c' \in M\Gamma(d')$. Then by M1, $(F(c'), G(d')) \in E_{\Gamma'}$ and

$$\begin{aligned}
\gamma^*(F(c), G(d))(G(d')) &= \left(\gamma(F(c'), G(d'))(F(c))\right)^* && \text{by (IV.25)} \\
&= F\left(\gamma(c', d')(c)\right)^* && \text{by M1} \\
&= G\left(\gamma^*(c, d)(d')\right) && \text{by M2 and (IV.25).}
\end{aligned}$$

Hence MI* also holds. Similarly one can deduce axioms MI and M2 from MI* and M2*.

□

Notice that the proof above shows that axiom M2 is equivalent to its dual M2*. However, a pair (F, G) satisfying MI alone (or MI* alone) may not satisfy MI* (MI). Further, the Lemma above shows that a morphism of cross-connections could be defined equivalently as those pairs of inclusion-preserving functors satisfying axioms MI* and M2*. In particular, any morphism of cross-connections satisfies all the four statements MI, M2, MI* and M2* of Lemma 3.

Since morphisms of cross-connections are bifunctors of product categories, the natural definition of composition is the component-wise composition. Thus we define *composition of morphisms* $m\colon \Gamma \to \Gamma'$ and $n\colon \Gamma' \to \Gamma''$ by $mn = (F_{mn}, G_{mn})$ where

(6)
$$F_{mn} = F_m F_n \qquad G_{mn} = G_m G_n .$$

It is easy to see that the pair of functors mn also satisfies MI and M2 and so $mn\colon \Gamma \to \Gamma''$ is a morphism of cross-connections. It is also clear that composition of morphisms defined in this way is associative and that $(1_{\mathcal{C}}, 1_{\mathcal{D}})\colon \Gamma \to \Gamma$ is a morphism. Therefore we have a category $\boldsymbol{Cr}$ in which objects are cross-connections and morphisms are those defined above; $\boldsymbol{Cr}$ will be called the *category of cross-connections.*

2.2 Alternate characterizations for morphisms

We proceed to derive some alternate forms for the axioms for morphisms.

Lemma 4 *Let Γ and Γ' be cross-connections and let*

$$F_m\colon \mathcal{C} \to \mathcal{C}' \quad \textit{and} \quad G_m\colon \mathcal{D} \to \mathcal{D}'$$

be inclusion preserving functors. Assume that $m = (F_m, G_m)$ satisfies axioms MI *and* MI*. *For $(c, d) \in E_\Gamma$, define*

(7)
$$\theta_m(c, d) = (F_m(c), G_m(d)) .$$

Then θ_m is a regular bimorphism of E_Γ to $E_{\Gamma'}$. In particular, if $m\colon \Gamma \to \Gamma'$ is a morphism, then θ_m is a regular bimorphism.

Proof We first observe that if $F: \mathcal{C} \to \mathcal{C}'$ is any inclusion preserving functor of normal categories, then F preserves normal factorizations; that is, if $f = \varrho u j$ is any normal factorization of a morphism $f: c \to c'$ in $\mathcal{C}$, then $F(f) = F(\varrho)F(u)F(j)$ is a normal factorization of $F(f)$ in $\mathcal{C}'$. In particular, we have

$$(8) \qquad F(\operatorname{im} f) = \operatorname{im} F(f) \quad \text{and} \quad F(\operatorname{coim} f) = \operatorname{coim} F(f)$$

Suppose that m satisfies M1 and M1*. Then $\theta = \theta_m$ is a map of E_Γ to $E_{\Gamma'}$. Let $(c, d)\ \omega^l\ (c', d')$ in E_Γ. Since F_m is inclusion preserving, by Equation (3) and the definition of θ, $\theta(c, d)\ \omega^l\ \theta(c', d')$. Further, by Equation (4),

$$\begin{aligned}\theta(c', d')\theta(c.d) &= \left(F_m(c'), G_m(d')\right)\left(F_m(c), G_m(d)\right) \\ &= \left(F_m(c), \operatorname{im}\gamma^*(F_m(c'), G_m(d'))(G_m(d))\right).\end{aligned}$$

Now

$$\begin{aligned}\operatorname{im}\left(\gamma^*(F_m(c'), G_m(d'))(G_m(d))\right) &= \operatorname{im} G_m\left(\gamma^*(c', d')(d)\right) && \text{by M1*} \\ &= G_m\left(\operatorname{im}\gamma^*(c', d')(d)\right) && \text{by (8)} \\ &= G_m(d'')\end{aligned}$$

where $d'' = \operatorname{im}\gamma^*(c', d')(d)$. By Equation (4), we have $(c', d')(c, d) = (c, d'')$ and hence

$$\begin{aligned}\theta(c', d')\theta(c, d) &= \left(F_m(c), G_m(d'')\right) \\ &= \theta(c, d'') = \theta\left((c', d')(c, d)\right).\end{aligned}$$

In a similar way, it can be shown that θ preserves ω^r and the corresponding basic products (defined by Equation (4)). Also, it follows immediately from Theorem 2 and Equation (8) that θ preserves sandwich sets. So, θ is a regular bimorphism. The last statement now follows from Lemma 3. □

To save repetition we shall assume, for the rest of this section, that $\Gamma: \mathcal{D} \to N^*\mathcal{C}$ and $\Gamma': \mathcal{D}' \to N^*\mathcal{C}'$ are cross-connections and that $m = (F, G): \Gamma \to \Gamma'$ is a morphism.

Theorem 5 *Let $F\colon \mathcal{C} \to \mathcal{C}'$ and $G\colon \mathcal{D} \to \mathcal{D}'$ be a pair of inclusion preserving functors satisfying axioms* M1 *and* M1*. *Then $m = (F, G)\colon \Gamma \to \Gamma'$ is a morphism of cross-connections if and only if it satisfies the following:*

M2′: *There is a natural transformation $\zeta_m\colon \Gamma(-,-) \to (F \times G)\Gamma'(-,-)$ such that for all $(c,d) \in E_\Gamma$, $\zeta_m(c,d)(\gamma(c,d)) = \gamma(F(c), G(d))$.*

M2′*: *There is a natural transformation $\zeta^*_m\colon \Gamma^*(-,-) \to (F \times G)\Gamma'^*(-,-)$ such that for all $(c,d) \in E_\Gamma$, $\zeta^*_m(c,d)(\gamma^*(c,d)) = \gamma^*(F(c), G(d))$.*

Conversely if $m = (F, G)$ satisfies M1, M1* *and either* M2′ *or* M2′*, *then m is a morphism of cross-connections.*

Proof Assume that m is a morphism of cross-connections. For each $(c,d) \in v\mathcal{C} \times v\mathcal{D}$ define the map $\zeta_m(c,d)$ as follows: For each $\gamma = \gamma(e) \star f^\circ \in \Gamma(c,d)$ where $e \in E_\Gamma$ with $d^e = g$ and $f\colon c^e \to c$, let

$$\zeta_m(c,d)(\gamma) = \gamma(\theta e) \star F(f)^\circ \tag{9}$$

where $\theta = \theta_m$ is defined by Equation (7). The right-hand side of the equation above clearly represents an element in $\Gamma'(F(c), G(d))$. To show that this does not depend on the representation chosen for γ above, assume that $\gamma = \gamma(e') \star f'^\circ$ is another representation of γ. Then $d^{e'} = d = d^e$ and hence $e\mathcal{R}e'$ in E_Γ. Also comparing the components of γ at $c^{e'}$, we obtain $f' = \gamma(e)(c^{e'})f$. Since $\gamma(e)(c^{e'})\colon c^{e'} \to c^e$ is an isomorphism, so is $F(\gamma(e)(c^{e'}))$ and we have

$$\begin{aligned}
\gamma(\theta e') \star F(f')^\circ &= \gamma(\theta e') \star F(\gamma(e)(c^{e'}))F(f)^\circ \\
&= \gamma(\theta e') \star \gamma(\theta e)(F(c^{e'}))F(f)^\circ && \text{by M1} \\
&= \gamma(\theta e') \star \gamma(\theta e)(c^{\theta e}F(f)^\circ && \text{by (7)} \\
&= \gamma(\theta e) \star F(f)^\circ && \text{since } \theta e\mathcal{R}\theta e'.
\end{aligned}$$

This proves that $\zeta(c,d) = \zeta_m(c,d)\colon \Gamma(c,d) \to \Gamma'(F(c), G(d))$ is a single-valued map. To show that the map $\zeta\colon (c,d) \mapsto \zeta(c,d)$ is a natural transformation, first consider $f\colon c' \to c'' \in \mathcal{C}$. If $\gamma = \gamma(e) \star \gamma(c) \in \Gamma(c', d)$, so that $e = (c,d) \in E_\Gamma$, then using Equation (IV.24), we get

$$\begin{aligned}
\Gamma(f,d)(\gamma) &= \Gamma(f,d)(\Gamma(\gamma(c),d),d)(\gamma(e)) \\
&= \Gamma(\gamma(c)f, d)(\gamma(e)) \\
&= \gamma(e) \star (\gamma(c)f)^\circ.
\end{aligned}$$

Therefore using the definition of ζ and the above, for any $\gamma = \gamma(e) \star \gamma(c) \in \Gamma(c', d)$, we have,

$$\begin{aligned}
\left(\Gamma(f,d)\zeta(c'',d)\right)(\gamma) &= \zeta(c'',d)\left(\Gamma(f,d)(\gamma)\right) \\
&= \zeta(c'',d)(\gamma(e) \star (\gamma(c)f)^{\circ} \\
&= \gamma(\theta e) \star (F(\gamma(c))F(f))^{\circ}\,; \qquad \textit{and} \\
\zeta(c',d)\Gamma'(F(f),G(d))(\gamma) &= \Gamma'(F(f),G(d))\left(\gamma(\theta e) \star F(\gamma(c))\right) \\
&= \gamma(\theta e)\left(F(\gamma(c))F(f)\right)^{\circ}.
\end{aligned}$$

Hence $\Gamma(f,d)\zeta(c'',d) = \zeta(c',d)\Gamma'\left(F(f),G(d)\right)$.

Let $g\colon d \to d'$ and let $g^*\colon c' \to c$ be a transpose of g so that, in particular, $e = (c,d)$, $e' = (c',d') \in E_\Gamma$. Then by Equation (IV.24), we have

(1) $$\Gamma(c,g)(\gamma(e)) = \gamma(e) \star (g^*)^{\circ} = \Gamma(g^*,d')(\gamma(e'));$$

(2) $$\Gamma'(F(c),G(g))(\gamma(\theta e)) = \gamma(\theta e)) \star (G(g)^*)^{\circ} = \Gamma'(F(\gamma^*),G(d'))(\gamma(\theta e')).$$

If $\gamma = \gamma(e) \star \gamma(c) \in \Gamma(c'',d)$ then by (1) above it follows that

(3) $$\gamma = \Gamma(\gamma(c),d)(\gamma(e))$$

and so,

$$\Gamma(c'',g)(\gamma) = \Gamma(c'',g)\left(\Gamma(\gamma(c),d)(\gamma(e))\right).$$

Now, since Γ is a bifunctor, by bifunctor criterion, the following diagram commutes:

(*) $$\begin{array}{ccc}
\Gamma(c'',d) & \xrightarrow{\Gamma(c'',g)} & \Gamma(c'',d') \\
{\scriptstyle \Gamma(\gamma(c),d)}\big\downarrow & & \big\downarrow{\scriptstyle \Gamma(\gamma(c),d')} \\
\Gamma(c,d) & \xrightarrow[\Gamma(c,g)]{} & \Gamma(c,d)
\end{array}$$

Hence we have

$$\begin{aligned}
\Gamma(c'',g)(\gamma) &= \Gamma(\gamma(c),d)\left(\Gamma(c,g)(\gamma(e))\right) && \text{by (3)} \\
&= \Gamma(\gamma(c),d)\left(\Gamma(g^*,d')(\gamma(e'))\right) && \text{by (1)} \\
&= \Gamma(g^*\gamma(c),d')(\gamma(e')) \\
&= \gamma(e') \star (g^*\gamma(e))^{\circ} && \text{by (1).}
\end{aligned}$$

Therefore, using the definition of ζ, we get

$$(\Gamma(c'',g)\zeta(c'',d))(\gamma) = \zeta(c'',d)\,(\Gamma(c'',g)(\gamma)) = \gamma(\theta e) \star (F(g^*\gamma(e)))^\circ.$$

Using similar arguments and the fact that $(F \times G)\Gamma'$ is a bifunctor (so that, by bifunctor criterion, the diagram $(*)$ commutes for this functor) we obtain

$$\begin{aligned}(\zeta(c'',d)\Gamma'\,(F(c''),G(g)))\,(\gamma) &= \Gamma'(\,F(c''),G(g)\,)\,(\zeta(c'',d)(\gamma))\\ &= \Gamma'(\,F(c''),G(g)\,)\,(\gamma(\theta e) \star (F(\gamma(e)))^\circ)\\ &= \Gamma'(\,F(c''),G(g)\,)\,(\Gamma'(\,F(\gamma(c)),G(d)\,)(\gamma(\theta e))\\ &= \Gamma'(\,F(\gamma(c)),G(d')\,)\,(\Gamma'(\,F(c),G(g)\,)(\gamma(\theta e))\\ &= \Gamma'(\,F(\gamma(c)),G(d')\,)\,(\Gamma'(\,G(g)^*,G(d')\,)(\gamma(\theta e))\\ &= \Gamma'\,(F(g^*)F(\gamma(c)),G(d'))\,(\,\gamma(\theta e)\,)\\ &= \gamma(\theta e) \star F(\,g^*\gamma(c)\,)^\circ.\end{aligned}$$

Thus $\Gamma(c'',g)\zeta(c'',d') = \zeta(c'',d)\Gamma'(\,F(c''),G(g)\,)$. It follows from the bifunctor criterion that ζ is a natural transformation. Dually for each $\gamma^*(e)\star g^\circ \in \Gamma^*(c,d)$ where $e \in E_\Gamma$, define

(9*) $$\zeta^*(c,d)(\gamma^*(e) \star g^\circ) = \gamma^*(\theta e) \star G(g)^\circ.$$

Then it can be shown dually that the map $(c,d) \mapsto \zeta^*(c,d)$ is a natural transformation ζ^* of $\Gamma^*(-,-)$ to $(F \times G)\Gamma'^*(-,-)$ satisfying (M2′*).

Conversely assume that (F,G) satisfies (M2). Since ζ is a natural transformation, for $\gamma = \gamma(e) \star f^\circ \in \Gamma(c,d)$ (so that $e \in E_\Gamma$ with $d^e = d$ and $f\colon c^e \to c$), we have

$$\begin{aligned}\zeta(c,d)(\gamma) &= \zeta(c,d)\,(\Gamma(f,d)(\,\gamma(e)\,)) && \text{by } (3)\\ &= \Gamma'(\,F(f),G(d)\,)(\,\gamma(\theta e)\,) && \text{by } (M2')\\ &= \gamma(\theta e) \star F(f)^\circ && \text{by } (2).\end{aligned}$$

Therefore for each $(c,d) \in v\mathcal{C} \times v\mathcal{D}$, $\zeta(c,d)$ is defined by the Equation (9). To show that (F,G) satisfies (M2), consider $f\colon c \to c'$ and let $f^*\colon d' \to d$ be a transpose of f. Then using Equation (IV.24), (or taking $g = f^*$ in (1)) we deduce that

$$\Gamma(c',f^*)(\,\gamma(e')\,) = \gamma(e) \star f^\circ = \Gamma(f,d)(\,\gamma(e)\,);$$

where $e = (c, d)$, $e' = (c', d') \in E_\Gamma$. Hence, using the fact that ζ satisfies Equation (9), we deduce from the equation above that

$$\zeta(c', d)\left(\Gamma(c', f^*)(\gamma(e'))\right) = \zeta(c', d)(\gamma(e) \star f^\circ) = \gamma(\theta e) \star F(f)^\circ.$$

Similarly using (2) with $g = f^*$, we have

$$\Gamma'(F(c'), G(f^*))(\gamma(\theta e')) = \gamma(\theta e) \star (G(f^*)^*)^\circ = \Gamma'(G(f^*)^*, G(d))(\gamma(\theta e)).$$

Hence from M2$'$and the above equation we get

$$\begin{aligned} \Gamma'(F(c'), G(f^*))\left(\zeta(c', d')(\gamma(e'))\right) &= \Gamma'(F(c'), G(f^*))(\gamma(\theta e')) \\ &= \Gamma(G(f^*)^*, G(d))(\gamma(\theta e)) \\ &= \gamma(\theta e) \star (G(f^*)^*)^\circ. \end{aligned}$$

Since ζ is a natural transformation, we have

$$\zeta(c', d)\left(\Gamma(c', f^*)(\gamma(e'))\right) = \Gamma'(F(c'), G(f^*))\left(\zeta(c'd')(\gamma(e'))\right)$$

and so $\gamma(\theta e) \star F(f)^\circ = \gamma(\theta e) \star (G(f^*)^*)^\circ$. Hence we conclude that $F(f) = G(f^*)^*$; that is, $F(f)^* = G(f^*)$. Thus (F, G) satisfies (M2) which implies that (F, G) is morphism of cross-connections. Dually we can show that when (M2$'$*) holds, (F, G) satisfies (M2*) and so, by Lemma 3 (see the remarks following Definition 1), (F, G) is a morphism of cross-connections. □

Remark 2 Let Γ, Γ', F and G as in the theorem above. If there exists $\zeta\colon \Gamma \to (F \times G)\Gamma'$ satisfying (M2$'$) (or $\zeta^*\colon \Gamma^* \to (F \times G)\Gamma'^*$ satisfying (M2$'$*)) then it follows from Equation (9) (from (9*)) that ζ (ζ^*) is uniquely determined by F and G. Moreover, if any pair of functors (F, G) satisfying (M1) and (M1*) satisfies any one of axioms (M2), (M2*), (M2$'$) and (M2$'$*) then it satisfies all of them and (F, G) is a morphism of cross-connections.

As a consequence of the Theorem 5, we have

Proposition 6 *Let $(F,G)\colon \Gamma \to \Gamma'$ be a morphism of cross-connections. Then the following diagram of functors and natural transformations commute:*

$$(*)\qquad \begin{array}{ccc} \Gamma & \xrightarrow{\ \chi_\Gamma\ } & \Gamma^* \\ {\scriptstyle\zeta}\big\downarrow & & \big\downarrow{\scriptstyle\zeta^*} \\ (F\times G)\Gamma' & \xrightarrow[(F\times G)\chi_{\Gamma'}]{} & (F\times G)\Gamma'^* \end{array}$$

Here $(F\times G)\chi_{\Gamma'}$ denotes the natural transformation whose component at (c,d) is $\chi_{\Gamma'}(F(c),G(d))$.

Proof Let $(c,d) \in \boldsymbol{v}\mathcal{C} \times \boldsymbol{v}\mathcal{D}$ and $\gamma = \gamma(e) \star f^\circ \in \Gamma(c,d)$ so that $e \in E_\Gamma$ with $d^e = d$ and $f\colon c^e \to c$. Let $e' \in E_\Gamma$ with $c^{e'} = c$. Then using Equations (IV.23) and (9*), we obtain

$$\begin{aligned}(\chi_\Gamma(c,d)\zeta^*(c,d))\,(\gamma) &= \zeta^*(c,d)\,(\chi_\Gamma(c,d)(\gamma(e)\star f^\circ))\\ &= \zeta^*(c,d)(\gamma^*(e')\star(f^*)^\circ)\\ &= \gamma^*(\theta e')\star(\,G(f^*)\,)^\circ\end{aligned}$$

where $\theta = \theta_{(F,G)}$ is defined by Equation (7). Similarly, using Equations (9) and (IV.23) we have

$$\begin{aligned}(\zeta(c,d)(F\times G)\Gamma'(c,d))\,(\gamma) &= \chi_{\Gamma'}(F(c),G(d))(\,\zeta(c,d)(\gamma(e)\star f^\circ)\,)\\ &= \chi_{\Gamma'}(F(c),G(d))(\gamma(\theta e)\star F(f)^\circ)\\ &= \gamma^*(\theta e')\star(F(f)^*)^\circ.\end{aligned}$$

Hence

$$(\chi_\Gamma(c,d)\zeta^*(c,d))\,(\gamma) = (\zeta(c,d)(F\times G)\Gamma'(c,d))\,(\gamma)$$

for all $(c,d) \in \boldsymbol{v}\mathcal{C} \times \boldsymbol{v}\mathcal{D}$ and all $\gamma \in \Gamma(c,d)$,as required. □

It follows from Remark IV.4 that a cross-connection $\Gamma\colon \mathcal{D} \to N^*\mathcal{C}$ is completely determined by $\mathcal{C}$ and the vertex map $\boldsymbol{v}\Gamma$ of Γ (or equivalently, $\mathcal{D}$ and $\boldsymbol{v}\Gamma^*$). Analogously, we show below that a morphism (F,G) of cross-connections is completely determined by one of the functors and the vertex map of the other. Note that by Lemma 4, vertex maps of F and G determine a unique regular bimorphism $\theta = \theta_{(F,G)}$. Hence specifying F and $\boldsymbol{v}G$ [$\boldsymbol{v}F$ and G] is equivalent to specifying θ and F [respectively, θ and G].

Theorem 7 *Let $\Gamma\colon \mathcal{D} \to N^*\mathcal{C}$ and $\Gamma'\colon \mathcal{D}' \to N^*\mathcal{C}'$ be cross-connections. Suppose that $F\colon \mathcal{C} \to \mathcal{C}'$ is an inclusion-preserving functor and $\theta\colon E_\Gamma \to E_{\Gamma'}$ is a regular bimorphism such that the pair (θ, F) satisfies the following condition: for all e, $e' \in E_\Gamma$*

$$\text{(M1}'\text{)} \qquad F(\,\gamma(e)(c^{e'})\,) = \gamma(\theta e)(c^{\theta e'}).$$

Then there exists a unique functor $G\colon \mathcal{D} \to \mathcal{D}'$ such that

(i) *$(F, G)\colon \Gamma \to \Gamma'$ is a morphism of cross-connections;*
(ii) *$\theta e = (\,F(c^e), G(d^e)\,)$ for all $e \in E_\Gamma$.*

Moreover, in this case, the pair (θ, G) satisfies the following condition:

$$\text{(M1}'^*\text{)} \qquad G(\,\gamma^*(e)(d^{e'})\,) = \gamma^*(\theta e)(d^{\theta e'})$$

for all e, $e' \in E_\Gamma$. Conversely, if $(F, G)\colon \Gamma \to \Gamma'$ is a morphism and if $\theta = \theta_{(F,G)}$ is the regular bimorphism defined in Lemma 4 (see Equation (7)), then the pair (θ, F) satisfies (M1$'$) and the pair (θ, G) satisfies (M1$'^*$).

Proof We first observe that the vertex map of F is defined by:

$$\text{(10)} \qquad F(c) = c^{\theta e}$$

for all $c \in v\mathcal{C}$ and any $e \in E_\Gamma$ with $c = c^e$. For if $c = c^e$ we have

$$1_{F(c)} = F(1_c) = F(\gamma(e)(c)) = \gamma(\theta e)(c^{\theta e}) = 1_{c^{\theta e}}$$

by (M1$'$) and so (10) holds. For any other $e' \in E_\Gamma$ with $c = c^{e'}$, by Theorem 1, $e\mathcal{L}e'$. Since θ is a bimorphism, $\theta e\mathcal{L}\theta e'$ and so, $c^{\theta e} = c^{\theta e'}$. Hence vF is well-defined by (10). Next for each $d \in v\mathcal{D}$ and $e \in E_\Gamma$ with $d = d^e$, define

$$\text{(11v)} \qquad G(d) = d^{\theta e}.$$

As before, we deduce from Theorem 5 and the fact that θ is a bimorphism, that the equation above defines a single-valued map from $v\mathcal{D}$ to $v\mathcal{D}'$. Let $g\colon d \to d'$ be a morphism in $\mathcal{D}$ and let $g^*\colon c' \to c$ is a transpose of g, where $e = (c, d)$, $e' = (c', d') \in E_\Gamma$. By (10) and (11v), we have $\theta e =$

$(F(c), G(d))$, $\theta e' = (F(c'), G(d')) \in E_{\Gamma'}$. Hence $F(g^*)\colon F(c') \to F(c)$ has a transpose $F(g^*)^*\colon G(d) \to G(d')$. We set

$$G(g) = F(g^*)^*\colon G(d) \to G(d'). \tag{11m}$$

To show that this is well-defined, consider another transpose $f\colon c_1' \to c_1$ of g. Then $e_1' = (c_1', d')$, $e_1 = (c_1, d) \in E_\Gamma$, $e_1 \mathcal{R} e$ and $e_1' \mathcal{R} e'$. Also, since $c, c_1 \in M\Gamma(d)$, it follows from Equation (IV.25) that the transpose of $\gamma(e)(c_1)$ from d to itself is 1_d. From this and similar arguments, we see that the following equations hold:

$$\begin{aligned} (\,\gamma(e)(c_1)\,)^* &= 1_d, & (\,\gamma(e_1')(c')\,)^* &= 1_{d'}; \\ (\,\gamma(\theta e)(F(c_1))\,)^* &= 1_{G(D)}, & (\,\gamma(\theta e_1')(F(c'))\,)^* &= 1_{G(D')}. \end{aligned}$$

It follows from Corollary IV.22 that $\gamma(e)(c_1)g^*\gamma(e_1')(c')$ is a transpose of g from c_1 to c_1'. Thus, by uniqueness of transpose, the following diagram commute:

$$\begin{array}{ccc} c & \xrightarrow{\;g^*\;} & c' \\ {\scriptstyle \gamma(e)(c_1)}\uparrow & & \downarrow{\scriptstyle \gamma(e_1')(c')} \\ c_1 & \xrightarrow[\;f\;]{} & c_1' \end{array}$$

Hence using Corollary IV.22, (M1′) and the diagram above, we obtain

$$\begin{aligned} F(f)^* &= (\,F(\gamma(e_1')(c'))\,)^* F(g^*)^* (\,F(\gamma(e)(c_1))\,)^* \\ &= (\,\gamma(\theta e_1')(F(c'))\,)^* F(g^*)^* (\,\gamma(the)(F(c_1))\,)^* \\ &= (\,F(g^*)\,)^* = G(g). \end{aligned}$$

This proves that G is well-defined on morphisms.

Suppose that $g\colon d \subseteq d'$. Then for any $e' \in E_\Gamma$ with $d^{e'} = d'$ we can find $e \in E_\Gamma$ such that $e \,\omega\, e'$ and $d^e = d$. Then $g = \gamma^*(e')(d)$ and hence, by Equation (IV.25), $g^*\colon c^{e'} \to c^e = \gamma(e)(c^{e'})$ which is a retraction. By (M1′), we have $F(g^*) = \gamma(\theta e)(F(c^{e'}))$ and so, again by Equation (IV.25), $F(g^*)^* = \gamma^*(\theta e')(d^{\theta e})$. Since $\theta e \,\omega\, \theta e'$, $d^{\theta e} \subseteq d^{\theta e'}$ and hence

$$G(g) = F(g^*)^* = j_{d^{\theta e}}^{d^{\theta e'}}.$$

Thus G is inclusion-preserving; in particular, $G(1_d) = 1_{G(d)}$ for all $d \in v\mathcal{D}$. Moreover, if g_i, $i = 1, 2$ are morphism in $\mathcal{D}$ for which $g_1 g_2$ exists, then using Corollary IV.22 and the definition of G we get

$$\begin{aligned} G(g_1 g_2) = F((g_1 g_2)^*)^* &= F(g_2^* g_1^*)^* \\ = F(g_1^*)^* F(g_2^*)^* &= G(g_1)G(g_2). \end{aligned}$$

This proves that G is an inclusion-preserving functor.

By (10) and (11v), it follows that the pair (F, G) satisfies axiom (M1); and axiom (M2) holds by (11m). Hence (F, G) is a morphism of cross-connections. Since for each $e \in E_\Gamma$, $\theta e = (F(c^e), G(d^e))$, it follows from Equation (7) that $\theta = \theta_{(F,G)}$. Also, the equation (M1$'$*) holds since it is equivalent to (M2*). Finally, uniqueness of G follows from the fact that, given θ and F, the vertex map and morphism map of G are uniquely defined by (11v) and (11m) respectively.

Conversely assume that (F, G) is a morphism of cross-connections and let $\theta = \theta_{(F,G)}$. Then by Lemma 4, $\theta \colon E_\Gamma \to E_{\Gamma'}$ is a regular bimorphism. By axiom (M1), (θ, F) satisfies (M1$'$) and by axiom (M1*), (θ, G) satisfies (M1$'$*) (see Lemma 3). □

Using the notations above, we observe that, given θ and G satisfying (M1$'$*), we can construct F in a completely analogous fashion (F is defined using equations dual to (11v) and (11m)) so that (θ, F) satisfies (M1$'$).

2.3 Properties of morphisms

Recall from Chapter I that a functor $F \colon \mathcal{C} \to \mathcal{C}'$ is injective [surjective] if F is injective [surjective] as a partial algebra homomorphism (see Chapter I, §1). We shall say that a morphism $m = (F, G) \colon \Gamma \to \Gamma'$ of cross-connections is *injective* [*surjective*] if both F and G are injective [surjective]. These properties can be characterized in terms the corresponding properties of the bimorphism θ_m and the functor F (or, analogously, by θ_m and G) as follows.

Proposition 8 *Let $m = (F, G) \colon \Gamma \to \Gamma'$ be a morphism and let $\theta = \theta_m$ be the regular bimorphism defined by Equation* (7). *Them*

(a) *m is injective if and only if both θ and F are injective.*
(b) *m is surjective if and only if both θ and F are surjective.*

Proof (a): If m is injective, then by definition, both F and G are injective and then by Equation (7), $\theta = \theta_m$ is injective. Conversely, assume that θ and F are injective. Then by Equation (11v), vG is injective. Hence, if $g_1, g_2 \in \mathcal{D}$ with $G(g_1) = G(g_2)$, then $g_1, g_2 \in \mathcal{D}(d, d')$ for some d, $d' \in v\mathcal{D}$. Let $e = (c, d)$, $e' = (c', d') \in E_\Gamma$ and let $g_i^*: c' \to c$ be the transpose of g_i, $i = 1, 2$. Then by Equation (11m),

$$F(g_1^*)^* = G(g_1) = G(g_2) = F(g_2^*)^*$$

and so, $F(g_1^*) = F(g_2^*)$. Since F is injective, this gives $g_1^* = g_2^*$ which implies that $g_1 = g_2$. Hence G is injective.

(b): Assume that m is surjective and $\bar{e} \in E_{\Gamma'}$. Since vF and vG are surjective, there exist $c \in v\mathcal{C}$ and $d \in v\mathcal{D}$ such that $\bar{e} = (F(c), G(d))$. Let e, $e' \in E_\Gamma$ with $c^e = c$ and $d^{e'} = d$. Then by (M1$'$),

$$c^{\theta e} = F(c) = c^{\bar{e}}, \quad d^{\theta e'} = G(d) = d^{\bar{e}'}.$$

Hence $\theta e \mathcal{L} \bar{e} \mathcal{R} \theta e'$. Let $e'' \in \mathcal{S}(e, e')$. Since θ is a regular bimorphism, $\theta e'' \in \mathcal{S}(\theta e, \theta e') = \{\bar{e}\}$. Therefore $\theta e'' = \bar{e}$ and so θ is surjective.

Conversely, let θ and F be surjective. Then vG is surjective by Equation (11v). Let $\bar{g}: \bar{d} \to \bar{d}_*$ be a morphism in $\mathcal{D}'$ and let $\bar{e} = (\bar{c}, \bar{d})$, $\bar{e}_* = (\bar{c}_*, \bar{d}_*)$ be idempotents in $E_{\Gamma'}$. Since θ is surjective, there exist e, $e_* \in E_\Gamma$ such that $\theta e = \bar{e}$ and $\theta e_* = \bar{e}_*$. Since F is surjective, there exists $f: c_1' \to c_1$ such that $F(f): \bar{c}_* \to \bar{c}$ is the transpose of $\bar{g}$. Let $e_1 = (c_1, d_1)$, $e_1' = (c_1', d_1') \in E_\Gamma$. Then

$$c^{\theta e_1} = F(c_1) = \bar{c}, \quad c^{\theta e_1'} = F(c_1') = \bar{c}_*.$$

Hence $\theta e_1 \mathcal{L} \bar{e}$ and $\theta e_1' \mathcal{L} \bar{e}_*$. Let $e' \in \mathcal{S}(e_1, e)$ and $e'' \in \mathcal{S}(e_1', e)_*$. Since θ is a regular bimorphism,

$$\theta e' \in \mathcal{S}(\theta e_1, \theta e) = \mathcal{S}(\theta e_1, \bar{e}) = \{\bar{e}\}.$$

Hence $\theta e' = \bar{e}$; similarly, $\theta e'' = \bar{e}_*$. Now $e'' \, \omega^l \, e_1'$ and so, $c^{e''} \subseteq c_1'$; similarly $c^{e'} \subseteq c_1$. Let

$$f' = j_{c^{e''}}^{c_1'} f \gamma(e')(c_1): c^{e''} \to c^{e'}.$$

Since $F(c^{e''}) = F(c_1') = \bar{c}_*$ and $F(c^{e'}) = F(c_1) = \bar{c}$, we have

$$\begin{aligned} F(f') &= F(\jmath^{c_1'}_{c^{e''}})F(f)F(\gamma(e')(c_1)) \\ &= 1_{\bar{c}_*}F(f)\gamma(\theta e')(\,F(c_1)\,) \\ &= F(f)\gamma(\bar{e})(\bar{c}) = F(f). \end{aligned}$$

Hence, if $g\colon d^{e'} \to d^{e''}$ is the transpose of f', then

$$G(g) = F(g^*)^* = F(f')^* = F(f)^* = \bar{g}.$$

This proves that G is surjective. □

3 HOMOMORPHISM OF CROSS-CONNECTION SEMIGROUPS

We have seen that every cross-connection $\Gamma\colon \mathcal{C} \to \mathcal{D}$ determine a unique regular semigroup $\tilde{S}\Gamma$ (see §IV.5, Theorem IV.32). In this section we shall extend this construction to morphisms.

3.1 The Green's relation $\mathcal{H}$ on $\tilde{S}\Gamma$

Let Γ be a cross-connection as above. For each $c \in \boldsymbol{v}\mathcal{C}$ let $L_c = L_{\alpha_\Gamma(e')}$, where $c^{e'} = c$. By Theorem IV.35, there is a bijection $c \mapsto L_c$ of $\boldsymbol{v}\mathcal{C}$ onto the set of $\mathcal{L}$–classes of $\tilde{S}$. Similarly, if $R_d = R_{\alpha_\Gamma(e)}$ with $d^e = d$, then there is a bijection $d \mapsto R_d$ of $\boldsymbol{v}\mathcal{D}$ with $\tilde{S}/\mathcal{R}$. Hence to each $\mathcal{H}$–class of $\tilde{S}$ there corresponds a unique pair (c, d) in $\boldsymbol{v}\mathcal{C} \times \boldsymbol{v}\mathcal{D}$. In the following, we denote by $H_{c,d}$ the $\mathcal{H}$–class $L_c \cap R_d$, if this set is non-empty; otherwise $H_{c,d}$ denotes the empty set. Also, we write $f\colon c \cong c'$ to mean that f is an isomorphism of c to c'.

Lemma 9 *For $(c,d) \in \boldsymbol{v}\mathcal{C} \times \boldsymbol{v}\mathcal{D}$, $H_{c,d} \neq \emptyset$ if and only if there is $c' \in M\Gamma(d)$ such that $c' \cong c$. In this case*

(12) $$H_{c,d} = \{\,(\rho, \lambda) : \rho \in H\Gamma(c,d), \quad \lambda = \chi_\Gamma(c,d)(\rho)\,\}$$

where $$H\Gamma(c,d) = \{\,\gamma(e) \star f : d^e = d \ \ and \ \ f\colon c^e \cong c\,\}.$$

Moreover, there is a bijection between the $\mathcal{H}$–class $H_{c,d}$ and the set of isomorphisms in $\mathcal{C}(c^e, c)$ for any $e \in E_\Gamma$ with $d^e = d$.

Proof Given $e \in E_\Gamma$ with $d^e = d$ and $f: c^e \cong c$, by the definition of $H\Gamma(c,d)$, we have $\rho = \gamma(e) \star f \in H\Gamma(c,d)$. Hence $\sigma = (\rho, \lambda)$ belongs to the right-hand side of Equation (12), where $\lambda = \chi_\Gamma(c,d)(\rho) \in \Gamma^*(c,d)$. Now since $c_\rho = c$, it follows from Theorem IV.32 that $\sigma \mathcal{L} \alpha_\Gamma(e')$ for any $e' \in E_\Gamma$ with $c^{e'} = c$. Thus $\sigma \in L_{\alpha_\Gamma(e')} = L_c$ (cf.Equation (IV.36)). Dually $\sigma \in R_d$ by Equation (IV.37). Hence $\sigma \in H_{c,d}$. Conversely suppose that $\sigma = (\rho, \lambda) \in H_{c,d}$. Then for any $e \in E_\Gamma$ with $d^e = d$, $\sigma \in R_d = R_{\alpha_\Gamma(e)}$ by Equation (IV.37). Since by Proposition IV.33, the projection $\pi: \tilde{S}\Gamma \to U\Gamma$ is equivalent to the right regular representation, it follows that $\rho \mathcal{R} \gamma(e)$. Hence by Proposition III.7, there is an isomorphism $f: c^e \to c$ such that $\rho = \gamma(e) \star f \in \Gamma(c,d)$. Similarly, $\lambda = \gamma^*(e') \star g \in \Gamma^*(c,d)$ where $c^{e'} = c$ and $g: d^{e'} \to d$ is an isomorphism. Since λ is linked to ρ, it follows that $\lambda = \chi_\Gamma(c,d)(\rho) = \gamma^*(e') \star f^*$. Hence $g = f^*$ and so σ belongs to the set on the right of Equation (12). It is clear from the above that every isomorphism $f: c^e \cong c$ uniquely determines an element in $H_{c,d}$. If f_1 and f_2 determine the same element, then we must have $\gamma(e) \star f_1 = \gamma(e) \star f_2$ which clearly implies that $f_1 = f_2$. □

Remark 3 Note that a dual description of $\mathcal{H}$–classes of $\tilde{S}\Gamma$ is possible; in fact we have

(12*) $$H_{c,d} = \{ (\rho, \lambda) : \lambda \in H\Gamma^*(c,d), \quad \rho = \chi_{\Gamma^*}(c,d)(\lambda) \}$$

where $$H\Gamma^*(c,d) = \{ \gamma^*(e') \star g : c^{e'} = c \text{ and } g: d^{e'} \cong d \}.$$

As before, there is a bijection of $H_{c,d}$ with the set of isomorphisms in $\mathcal{D}(d^{e'}, d)$ for any $e' \in E_\Gamma$ with $c^{e'} = c$. Also given e, $e' \in E_\Gamma$ with $d^e = d$, $c^{e'} = c$, every $\sigma \in H_{c,d}$ has a unique representation of the form

(13) $$\sigma = (\gamma(e) \star f, \gamma^*(e') \star f^*)$$

where $f: c^e \cong c$.

Lemma 10 *Let $m = (F, G): \Gamma \to \Gamma'$ be a morphism of cross-connections and let $\sigma = (\rho, \lambda) \in \tilde{S}\Gamma$. Assume that $\rho \in \Gamma(c,d)$ and $\lambda = \chi_\Gamma(c,d)(\rho)$. Also, let $c_1 = c_\rho$ and $d_1 = c_\lambda$. Then we have the following:*

(a) $\sigma \in H_{c_1,d_1}$.
(b) *If* $\rho' = \zeta_m(c,d)(\rho)$ *and* $\lambda' = \zeta_m^*(c,d)(\lambda)$, *then* $\sigma' = (\rho',\lambda') \in \tilde{\boldsymbol{S}}\Gamma'$
(c) $\rho' = \zeta_m(c_1,d_1)(\rho)$ *and* $\lambda' = \zeta_m^*(c_1,d_1)(\lambda)$.

In particular, $\sigma' \in H_{F(c_1),G(d_1)} \subseteq \tilde{\boldsymbol{S}}\Gamma'$.

Proof (a): Fix $e \in E_\Gamma$ with $d^e = d$. Since $\rho \in \Gamma(c,d)$, there exists $f\colon c^e \to c$ such that $\rho = \gamma(e) \star f^\circ$. Then $c_\rho = \operatorname{im} f = c_1 \subseteq c$. Let $f = \varrho u j$ be a normal factorization of f so that $f^\circ = \varrho u$. Hence $\rho = \gamma(e) \star \varrho u$. Now $\gamma(e) \star \varrho$ is an idempotent cone such that $\gamma(e) \star \varrho \;\omega\; \gamma(e)$. Since Γ is a local isomorphism, there is a unique $d_1 \subseteq d$ such that $\Gamma(d_1) = H(\gamma(e) \star \varrho; -) \subseteq H(\gamma(e); -)$. Then $e_1 = (\operatorname{im}\varrho, d_1) \in E_\Gamma$ and $\Gamma(d_1) = H(\gamma(e_1); -)$. Therefore we have $\gamma(e_1) = \gamma(e) \star \varrho$ and so $\rho = \gamma(e_1) \star u \in H\Gamma$. Also by Lemma IV.29 $\lambda = \chi_\Gamma(c,d)(\rho) = \chi_\Gamma(c_1,d_1)(\rho)$. Hence by Lemma 9, $\sigma = (\rho,\lambda) \in H_{c_1,d_1}$. Since $\chi_\Gamma(c_1,d_1)(\gamma(e_1) \star u) = \gamma^*(e_1') \star u^*$ for any $e_1' \in E_\Gamma$ with $c^{e_1'} = c_1$ and transpose $u^*\colon d^{e_1'} \cong d_1$ of u, it follows that $c_\lambda = d_1$. This proves (a).

(b): By Proposition 6 we have

$$\begin{aligned}\lambda' = \zeta_m^*(c,d)(\lambda) &= \zeta_m^*(c,d)\,(\chi_\Gamma(c,d)(\rho))\\ &= \chi'_\Gamma(\,F(c), G(d)\,)(\,\zeta_m(c,d)(\rho)\,) \quad \text{by 6}\\ &= \chi'_\Gamma(\,F(c), G(d)\,)(\rho')\end{aligned}$$

which shows that $\sigma' = (\rho',\lambda')$ is a linked pair and hence belongs to $\tilde{\boldsymbol{S}}\Gamma'$. Thus (b) follows.

(c): As observed above (in the para (a)), $\rho = \gamma(e) \star f^\circ = \gamma(e_1) \star u$ where $d^{e_1} = d_1$ and $u\colon c^{e_1} \cong c_1$. Now, since F is inclusion-preserving, $F(f) = F(\varrho)F(u)F(j)$ is a normal factorization of $F(f)$ and by Equation (9)

$$\rho' = \gamma(\theta e) \star F(f)^\circ = (\gamma(\theta e) \star F(\varrho)) \star F(u)$$

where $\theta = \theta_m$ defined by Equation (7). Since θ is a bimorphism (by Lemma 4) and $e_1 \;\omega\; e$ we have $\theta e_1 \;\omega\; \theta e$ and so $\gamma(\theta e_1) \;\omega\; \gamma(\theta e)$. Hence

$$\begin{aligned}\gamma(\theta e_1) &= \gamma(\theta e) \cdot \gamma(\theta e_1)\\ &= \gamma(\theta e) \star \gamma(\theta e_1)(\,F(c)\,)\\ &= \gamma(\theta e) \star F(\,\gamma(e_1)(c)\,) \quad \text{by } (M1')\\ &= \gamma(\theta e) \star F(\varrho).\end{aligned}$$

Therefore by Equation (9),

$$\zeta_m(c,d)(\rho) = \rho' = \gamma(\theta e_1) \star F(u) = \zeta_m(c_1,d_1)(\rho)$$

and it follows from Lemma IV.29 that

$$\chi'_\Gamma(\,F(c), G(d)\,)(\rho') = \chi'_\Gamma(\,F(c_1), G(d_1)\,)(\rho').$$

Therefore we have

$$\begin{aligned}
\zeta^*_m(c_1,d_1)(\lambda) &= \zeta^*_m(c_1,d_1)\,(\chi_\Gamma(c_1,d_1)(\rho)) \\
&= \chi'_\Gamma(\,F(c_1), G(d_1)\,)\,(\zeta_m(c_1,d_1)(\rho)) \quad \text{by Proposition 6} \\
&= \chi'_\Gamma(\,F(c), G(d)\,)\,(\zeta_m(c,d)(\rho)) \quad \text{by the above} \\
&= \zeta^*_m(c,d)\,(\chi_\Gamma(c,d)(\rho)) \quad \text{by Proposition 6} \\
&= \zeta^*_m(c,d)(\lambda) = \lambda'.
\end{aligned}$$

This proves (c).

Finally, since $d^{e_1} = d_1$, we have $d^{\theta e_1} = G(d_1)$. Therefore the equality $\rho' = \gamma(\theta e_1) \star F(u)$ implies that $\rho' \in H\Gamma'(\,F(c_1), G(d_1)\,)$. Moreover, by (c) above and Proposition 6, we have

$$\begin{aligned}
\chi'_\Gamma(\,F(c_1), G(d_1)\,)(\rho') &= \chi'_\Gamma(\,F(c_1), G(d_1)\,)\,(\zeta_m(c_1,d_1)(\rho)) \\
&= \zeta^*_m(c_1,d_1)\,(\chi_\Gamma(c_1,d_1)(\rho)) \\
&= \zeta^*_m(c_1,d_1)(\lambda) = \lambda'.
\end{aligned}$$

It follows from Lemma 9 that $\sigma' = (\rho', \lambda') \in H_{F(c_1),G(d_1)}$. □

Remark 4 Since $\mathcal{C}$ and $\mathcal{D}$ are categories with subobjects, $\mathcal{C}\times\mathcal{D}$ is a category with the obvious subobject relation. Since a cross-connection $\Gamma\colon \mathcal{D} \to N^*\mathcal{C}$ is inclusion preserving and since, for every $d \in v\mathcal{D}$, $\Gamma(d)\colon \mathcal{C} \to \mathbf{Set}$ is also inclusion preserving, it follows that $\Gamma(-,-)\colon \mathcal{C}\times\mathcal{D} \to \mathbf{Set}$ is an inclusion preserving bifunctor. Similarly, $\Gamma^*(-,-)$ is also inclusion preserving. If $m = (F,G)$ is a morphism then $F \times G$ is clearly inclusion preserving. It follows that the natural transformations ζ_m, $(F\times G)\chi'_\Gamma$, χ_Γ and ζ^*_m occurring in the commutative diagram $(*)$ of Proposition 6 are natural transformations between inclusion preserving functors. Now, using notations of the lemma above, we have $(c_1, d_1) \subseteq (c, d)$ in $\mathcal{C}\times\mathcal{D}$. In view of the discussion above the statement (c) as well as the concluding statement of Lemma 4 are immediate consequences of this fact.

3.2 Homomorphisms

Next theorem extends the construction of cross-connection semigroups to morphisms of cross-connections. We shall prove later that this construction is functorial (see Theorem 13).

Theorem 11 *Let $m = (F, G) \colon \Gamma \to \Gamma'$ be a morphism of cross-connections. For each $\sigma = (\rho, \lambda) \in \tilde{S}\Gamma$, define*

$$\tilde{S}m(\sigma) = (\zeta_m(c,d)(\rho), \zeta^*_m(c,d)(\lambda)) \tag{14}$$

where $\rho \in \Gamma(c,d)$ and $\lambda = \chi_\Gamma(c,d)(\rho)$. Then $\tilde{S}m \colon \tilde{S}\Gamma \to \tilde{S}\Gamma'$ is a homomorphism.

Proof By Lemma 10(b) and (c), $\tilde{S}m$ is a well-defined function of $\tilde{S}\Gamma$ into $\tilde{S}\Gamma'$. To prove that $\tilde{S}m$ is a homomorphism, consider $\sigma_i = (\rho_i, \lambda_i) \in \tilde{S}\Gamma$, $i = 1, 2$. Suppose that $\sigma \in H_{c_i,d_i}$ so that $\rho_i \in H\Gamma(c_i, d_i)$ and $\lambda_i = \chi_\Gamma(c_i, d_i)(\rho_i)$ by Lemma 9. If e_i, $e'_i \in E_\Gamma$ with $d^{e_i} = d_i$ and $c^{e'_i} = c_i$, then by the Remark 3, σ_i has a unique representation of the form

$$\sigma_i = (\gamma(e_i) \star u_i, \gamma^*(e'_i) \star u^*_i) \qquad i = 1, 2.$$

Using the definition of ζ_m (see Equation (9)), we have

$$\begin{aligned}
\zeta_m(c_1,d_1)(\rho_1)\zeta_m(c_2,d_2)(\rho_2) &= (\gamma(\theta e_1) \star F(u_1)) \cdot (\gamma(\theta e_2) \star F(u_2)) \\
&= \gamma(\theta e_1) \star (F(u_1)\gamma(\theta e_2)(F(c_1))F(u_2))^\circ \\
&= \gamma(\theta e_1) \star (F(u_1)F(\gamma(e_2)(c_1))F(u_2))^\circ \text{ by } (M1') \\
&= \gamma(\theta e_1) \star F(u_1\gamma(e_2)(c_1)u_2)^\circ .
\end{aligned}$$

Now $\rho_1 \cdot \rho_2 = \gamma(e_1) \star (u_1\gamma(e_2)(c_1)u_2)^\circ \in \Gamma(c_2, d_1)$ and so, by Equation (9),

$$\zeta_m(c_2,d_1)(\rho_1 \cdot \rho_2) = \gamma(\theta e_1) \star F(u_1\gamma(e_2)(c_1)u_2)^\circ$$

so that
$$\rho'_1 \cdot \rho'_2 = \zeta_m(c_2,d_1)(\rho_1 \cdot \rho_2)$$

where $\rho_i' = \zeta_m(c_i, d_i)(\rho_i)$, $i = 1, 2$. Therefore using Proposition 6 and Lemma IV.30 we get

$$\begin{aligned}
\zeta_m^*(c_2, d_2)(\lambda_2) \cdot \zeta_m^*(c_1, d_1)(\lambda_1) &= \chi'_{\Gamma}(F(c_2), G(d_2))(\rho_2') \cdot \chi'_{\Gamma}(F(c_1), G(d_1))(\rho_1') \\
&= \chi'_{\Gamma}(c_2, d_1)(\rho_1' \cdot \rho_2') \\
&= \chi'_{\Gamma}(c_2, d_1)\left(\zeta_m(c_2, d_1)(\rho_1 \cdot \rho_2)\right) \\
&= \zeta_m^*(c_2, d_1)\left(\chi_{\Gamma}(c_2, d_1)(\rho_1 \cdot \rho_2)\right) \\
&= \zeta_m^*(c_2, d_1)(\lambda_2 \cdot \lambda_1).
\end{aligned}$$

Hence it follows from Theorem IV.32 (see Equation (IV.32)) that

$$\begin{aligned}
\tilde{S}m(\sigma_1)\tilde{S}m(\sigma_2) &= (\zeta_m(c_2, d_1)(\rho_1 \cdot \rho_2), \zeta_m^*(c_2, d_2)(\lambda_2) \cdot \zeta_m^*(c_1, d_1)(\lambda_1)) \\
&= (\zeta_m(c_2, d_1)(\rho_1 \cdot \rho_2), \zeta_m^*(c_2, d_1)(\lambda_2 \cdot \lambda_1)) \\
&= \tilde{S}m(\sigma_1\sigma_2).
\end{aligned}$$

This proves that $\tilde{S}m \colon \tilde{S}\Gamma \to \tilde{S}\Gamma'$ is a homomorphism. □

3.3 Properties of the functor $\tilde{S}$

We next give some preservation properties of the construction described in the theorem above.

Suppose that $m = (F, G) \colon \Gamma \to \Gamma'$ is a morphism. By Theorem 1 the map $\alpha = \alpha_{\Gamma} \colon E_{\Gamma} \to E(\tilde{S}\Gamma)$ defined by $\alpha(e) = (\gamma(e), \gamma^*(e))$ (see Equation (IV.33)) is a biorder isomorphism. Now by the definition of $\tilde{S}m$ (see Equations (14) and (9)) (M2′) and (M2′*), it follows that $\tilde{S}m(\alpha(e)) = (\gamma(\theta e), \gamma^*(\theta e))$ where $\theta = \theta_m$ (see Equation (7)). On the other hand, from Equations (7) and (IV.33), it can be seen that $\alpha'(\theta e) = (\gamma(\theta e), \gamma^*(\theta e))$. We thus have the following commutative diagram:

(15)
$$\begin{array}{ccc}
E_{\Gamma} & \xrightarrow{\;\theta\;} & E_{\Gamma'} \\
{\scriptstyle \alpha}\big\downarrow & & \big\downarrow{\scriptstyle \alpha'} \\
E(\tilde{S}\Gamma) & \xrightarrow[E(\tilde{S}m)]{} & E(\tilde{S}\Gamma')
\end{array}$$

Therefore θ is injective or surjective if and only if $E(\tilde{S}m)$ has the corresponding property. Recall that a morphism $m = (F, G)$ is injective [surjective] if both F and G are injective [surjective] as partial algebra homomorphisms. We have:

Proposition 12 *Let $m = (F, G)\colon \Gamma \to \Gamma'$ be a morphism of cross-connections and let $\tilde{S}m\colon \tilde{S}\Gamma \to \tilde{S}\Gamma'$ be the homomorphism defined by Equation* (14). *Then*

(a) *m is injective if and only if $\tilde{S}m$ is injective.*
(b) *m is surjective if and only if $\tilde{S}m$ is surjective.*

Proof (a): Suppose that m is injective. Then by Equation (7), $\theta = \theta_m$ is injective and so, as observed above, $E(\tilde{S}m)$ is injective. Hence $\tilde{S}m$ is an idempotent separating homomorphism. Now suppose that σ, $\tau \in \tilde{S}\Gamma$ with $\tilde{S}m(\sigma) = \tilde{S}m(\tau)$. Then by the above, σ, $\tau \in H_{c,d}$ for some $(c, d) \in vC \times vD$. So for any e, $e' \in E_\Gamma$ with $d^e = d$ and $c^{e'} = c$, there exists isomorphisms $f, g\colon c^e \cong c$ such that

$$\sigma = (\gamma(e) \star f, \gamma^*(e') \star f^*), \qquad \tau = (\gamma(e) \star g, \gamma^*(e') \star g^*).$$

Hence by Equations (14) and (9), we have

$$\gamma(\theta e) \star F(f) = \gamma(\theta e) \star F(g)$$

from which we conclude that $F(f) = F(g)$. Since F is injective, we get $f = g$ and so, $\sigma = \tau$. Therefore $\tilde{S}m$ is injective.

Conversely, suppose that $\tilde{S}m$ is injective. Then, as above, θ is injective and by Proposition 8, it is sufficient to show that F is injective. Let $F(c) = F(c')$ for c, $c' \in vC$. If e, $e' \in E_\Gamma$ with $c^e = c$, $c^{e'} = c'$, then by Equation (7), $\theta e \mathcal{L} \theta e'$. Since θ is injective, we easily see that $e \mathcal{L} e'$ and so $c = c'$. Hence vF is injective. To show that F is faithful, let f, $g \in \mathcal{C}(c, c')$ such that $F(f) = F(g)$. Choose $e = (c, d)$, $e' = (c', d') \in E_\Gamma$. If f^*, $g^*\colon d' \to d$ denote the transposes of f and g, then by axiom (M2),

$$G(f^*) = F(f)^* = F(g)^* = G(g^*).$$

Let $\gamma = \gamma(e) \star f^\circ$, $\gamma' = \gamma(e) \star g^\circ$. Then $\gamma, \gamma' \in \Gamma(c', d)$ and by Equation (IV.23), we get

$$\lambda = \chi_\Gamma(c', d)(\gamma) = \gamma^*(e') \star (f^*)^\circ$$
$$\lambda' = \chi_\Gamma(c', d)(\gamma') = \gamma^*(e') \star (f^*)^\circ.$$

Then $\sigma = (\gamma, \lambda)$, $\sigma' = (\gamma', \lambda') \in \tilde{S}\Gamma$. Also by Equation (9), we see that

$$\zeta_m(c', d)(\gamma) = \gamma(\theta e) \star F(f)^\circ = \gamma(\theta e) \star F(g)^\circ = \zeta_m(c', d)(\gamma')$$

and similarly, $\zeta_m^*(c', d)(\lambda) = \zeta_m^*(c', d)(\lambda')$. By Equation (14) we have $\tilde{S}m(\sigma) = \tilde{S}m(\sigma')$ and by injectivity, we conclude that $\sigma = \sigma'$. In particular, $\gamma(e) \star f^\circ = \gamma(e) \star g^\circ$ from which we get $f = g$. Therefore F is injective.

(b): Let m be surjective Then it follows from the by commutative diagram (15) that $E(\tilde{S}m)$ is surjective. Assume that $\bar{\sigma} = (\bar{\gamma}, \bar{\lambda}) \in H_{\bar{c},\bar{d}} \subseteq \tilde{S}\Gamma'$ so that, $\bar{\gamma} = \gamma(\bar{e}) \star \bar{f}$ for some $\bar{e}$ with $d^{\bar{e}} = d$ and $\bar{f}\colon c^{\bar{e}} \cong \bar{c}$. Since θ and F are surjective (by Proposition 8), there is $e \in E_\Gamma$ with $\theta e = \bar{e}$ and $f\colon c' \to c$ with $F(f) = \bar{f}$. Then $F(c') = F(c^e) = c^{\bar{e}}$ and $F(c) = \bar{c}$. Choose $e' \in E_\Gamma$ with $c^{e'} = c'$. Then $\bar{e} = \theta e \,\mathcal{L}\, \theta e'$ and so, if $e'' \in \mathcal{S}(e', e)$, then $\theta e'' = \theta e = \bar{e}$. Since $c^{\theta e'} = c^{\bar{e}}$, using (M1′) (see Theorem 7), we have

$$\begin{aligned} F(\gamma(e')(c^{e''})) &= \gamma(\theta e')(F(c^{e''}) = \gamma(\theta e')(c^{\theta e''}) \\ &= \gamma(\theta e')(c^{\bar{e}}) = 1_{c^{\bar{e}}}. \end{aligned}$$

Therefore if $f_1 = \gamma(e')(c^{e''})f$, then by the above,

$$F(f_1) = F(\gamma(e')(c^{e''}))F(f) = f(f).$$

Let $\gamma = \gamma(e'') \star f_1^\circ$. Then $\gamma \in \Gamma(c, d)$ where $d = d^{e''}$ and by Equation (9),

$$\zeta_m(c, d)(\gamma) = \gamma(\theta e'') \star F(f_1)^\circ = \gamma(\bar{e}) \star \bar{f} = \bar{\gamma}$$

and if $\lambda = \chi_\Gamma(c, d)(\gamma)$, using Proposition 6 we get

$$\begin{aligned} \zeta_m^*(c, d)(\lambda) &= \chi'_\Gamma(F(c), G(d))\,(\zeta_m(c, d)(\gamma)) \\ &= \chi'_\Gamma(\bar{c}, \bar{d})(\bar{\gamma}) = \bar{\lambda}. \end{aligned}$$

Since $\sigma = (\gamma, \lambda) \in \tilde{S}\Gamma$, it follows from Equation (14) that $\tilde{S}m(\sigma) = \bar{\sigma}$. Hence $\tilde{S}m$ is surjective.

Conversely, assume that $\tilde{S}m$ is surjective. Again by the commutative diagram (15), θ is surjective and so, by Proposition 8, it is sufficient to show that F is surjective. Accordingly, let $\bar{f} \in \mathcal{C}'(\bar{c}', \bar{c})$. First assume that $\bar{f}$ is an epimorphism. Since θ is surjective, there exists $e \in E_\Gamma$ with $c^{\theta e} = \bar{c}'$. If $\bar{\gamma} = \gamma(\theta c) \star \bar{f}$ then $\bar{\gamma} \in \Gamma'(\bar{c}, d^{\theta e})$ and $(\bar{\gamma}, \bar{\lambda}) \in \tilde{S}\Gamma'$ where $\bar{\lambda} = \chi'_\Gamma(\bar{c}', \bar{c})(\bar{\gamma})$. Since $\tilde{S}m$ is surjective, it follows that there is $(\gamma, \lambda) \in \tilde{S}\Gamma$ such that $\tilde{S}m(\gamma, \lambda) = (\bar{\gamma}, \bar{\lambda})$. If $\gamma = \gamma(e_1) \star g$, then by Equations (9) and (14), $\bar{\gamma} = \gamma(\theta e_1) \star F(g)$ so that $\gamma(\theta e_1) \mathcal{R} \gamma(\theta e)$. Let $e' \in \mathcal{S}(e, e_1)$. Since the mapping $e \mapsto \gamma(\theta e)$ is a regular bimorphism, it follows that $\gamma(\theta e') = \gamma(\theta e)$ and so, $c^{\theta e'} = c^{\theta e} = \bar{c}'$. Thus $\bar{\gamma} = \gamma(\theta e') \star \bar{f} = \gamma(\theta e_1) \star F(g)$ and so, $\bar{f} = \left(\gamma(\theta e_1)(c^{\theta e'})F(g)\right)^\circ$. Hence if $g' = (\gamma(e_1)(c^{e'})g)^\circ$, by the condition (M1$'$) of Theorem 7 we have $F(g') = \bar{f}$. Now let $\bar{f} \in \mathcal{C}'(\bar{c}', \bar{c})$ be arbitrary and let $\operatorname{im} \bar{f} = \bar{c}_1$. Then by the above, there is $g' : c' \to c_1$ such that $F(g') = \bar{f}^\circ$. Let $e'' \in E_\Gamma$ with $c^{\theta e''} = \bar{c}$. Since $\bar{c}_1 \subseteq \bar{c}$, using (M1$'$), we get

$$F(\gamma(e'')(c_1)) = \gamma(\theta e'')(\bar{c}_1) = \jmath_{\bar{c}_1}^{\bar{c}}$$

and so, if $f = g'\gamma(e'')(c_1)$, then

$$F(f) = F(g')F(\gamma(e'')(c_1)) = \bar{f}^\circ \jmath_{\bar{c}_1}^{\bar{c}} = \bar{f}.$$

This proves that F is surjective. □

4 EQUIVALENCE OF THE CATEGORIES CR AND RS

We have seen that to each cross-connection Γ there corresponds a regular semigroup $\tilde{S}\Gamma$ (see Theorem IV.32) and to each morphism of cross-connections $m : \Gamma \to \Gamma'$ there corresponds a homomorphism $\tilde{S}m : \tilde{S}\Gamma \to \tilde{S}\Gamma'$ (by Theorem 11). In this section, we show that this correspondences yields a functor $\tilde{S}$ which is an equivalence of the categories***Cr*** and ***RS***, by explicitly constructing an adjoint inverse $\boldsymbol{\Gamma}$ to $\tilde{\boldsymbol{S}}$.

4.1 The functors $\tilde{S}$ and Γ

We begin by observing that the assignments $(\tilde{S})$ mentioned above is indeed a functor.

Theorem 13 *The assignments*

$$\Gamma \mapsto \tilde{S}\Gamma, \qquad m \mapsto \tilde{S}m$$

is a functor $\tilde{\boldsymbol{S}}\colon \boldsymbol{Cr} \to \boldsymbol{RS}$.

Proof Suppose that $m = (F, G)\colon \Gamma \to \Gamma'$ and $m' = (F', G')\colon \Gamma' \to \Gamma''$ are two morphisms of cross-connections. Then by Equation (6), $mm' = (FF', GG')$ and by Equation (7) $\theta_{mm'} = \theta\theta'$ where $\theta = \theta_m$ and $\theta' = \theta_{m'}$. Therefore from Equations (9) and (9*) we have

$$\zeta_m(c,d)\zeta_{m'}(F(c), G(d)) = \zeta_{mm'}(c,d);$$
$$\zeta^*_m(c,d)\zeta^*_{m'}(F(c), G(d)) = \zeta^*_{mm'}(c,d).$$

Hence it follows from Equation (14) that $\tilde{\boldsymbol{S}}m\tilde{\boldsymbol{S}}m' = \tilde{\boldsymbol{S}}mm'$. Clearly $\tilde{\boldsymbol{S}}1_\Gamma = 1_{\tilde{S}\Gamma}$. Therefore $\tilde{\boldsymbol{S}}\colon \boldsymbol{Cr} \to \boldsymbol{RS}$ is a functor. □

Recall that Theorem IV.2 constructs, for each regular semigroup S, a cross-connection $\boldsymbol{\Gamma} S\colon \mathcal{R}(S) \to \boldsymbol{N}^*\boldsymbol{\mathcal{L}}(S)$. We show below that this can be extended to a functor $\boldsymbol{\Gamma}\colon \boldsymbol{RS} \to \boldsymbol{Cr}$. It will be shown later that $\boldsymbol{\Gamma}$ is an adjoint inverse to the functor $\tilde{\boldsymbol{S}}$.

Theorem 14 *Let $h\colon S \to S'$ be a homomorphism of regular semigroups. For each $Sx \in \boldsymbol{v}\boldsymbol{\mathcal{L}}(S)$ and each morphism $\rho(e,u,f)\colon Se \to Sf \in \boldsymbol{\mathcal{L}}(S)$, define*

(16) $$F_h(Sx) = S'(hx), \qquad F_h(\rho(e,u,f)) = \rho(he, hu, hf).$$

Then $F_h\colon \boldsymbol{\mathcal{L}}(S) \to \boldsymbol{\mathcal{L}}(S')$ is an inclusion preserving functor. Dually

(16*) $$G_h(xS) = (hx)S', \qquad G_h(\lambda(e,u,f)) = \lambda(he, hu, hf)$$

for each $xS \in \boldsymbol{v}\mathcal{R}(S)$ and each morphism $\lambda(e,u,f)\colon eS \to fS \in \mathcal{R}(S)$, is an inclusion preserving functor $G_h\colon \mathcal{R}(S) \to \mathcal{R}(S')$. Moreover

$$\boldsymbol{\Gamma} h = (F_h, G_h)\colon \boldsymbol{\Gamma} S \to \boldsymbol{\Gamma} S'$$

is a morphism of cross-connections. Further, the assignments

$$S \mapsto \boldsymbol{\Gamma} S, \qquad h \mapsto \boldsymbol{\Gamma} h$$

is a functor $\boldsymbol{\Gamma}\colon \boldsymbol{RS} \to \boldsymbol{Cr}$.

Proof Since $h\colon S \to S'$ is a homomorphism, it is clear that the vertex map and morphism map of F_h defined by (16) are single-valued and that the vertex map is order-preserving. It follows from Lemma III.12(c) that F_h is a functor and by Proposition III.13(d), F_h is inclusion preserving. It follows dually that G_h is an inclusion preserving functor.

Now by Theorem IV.38, $e \mapsto (\rho^e, \lambda^e)$ is an isomorphism of $E(S)$ onto $E(\boldsymbol{\Gamma} S)$ (see Equation (IV.41)). Since by Theorem 1,

$$\alpha_{\boldsymbol{\Gamma} S}\colon (Se, fS) \mapsto (\gamma(Se, fS)), \gamma^*(Se, fS)$$

is an isomorphism of $E_{\boldsymbol{\Gamma} S}$ onto $E(\boldsymbol{\Gamma} S)$, it follows that $(Se, fS) \in E_{\boldsymbol{\Gamma} S}$ if and only if there is $g \in E(S)$ such that

$$\gamma(Se, fS) = \rho^g = \gamma(Sg, gS) \quad \text{and} \quad \gamma^*(Se, fS) = \lambda^g = \gamma^*(Sg, gS)$$

(see Proposition IV.37 and Equation (IV.39)); so that $Se = Sg$ and $fS = gS$. Hence it is clear from the definition of F_h and G_h that $(Se, fS) \in E_{\boldsymbol{\Gamma} S}$ if and only if $(S'he, hfS') \in E_{\boldsymbol{\Gamma} S'}$. Also, it follows from Lemma III.15 that

$$\begin{aligned} F_h(\gamma(Se, eS)(Sf)) &= F_h(\rho^e(Sf)) \\ &= F_h(\rho(f, fe, e)) \\ &= \rho(hf, (hf)(he), he) \\ &= \gamma(S'he, heS')(S'hf). \end{aligned}$$

This proves axiom (M1) of Definition 1 (see Lemma 3). To prove axiom (M2), observe that $\rho(e, u, f)^* = \lambda(f, u, e)$ (see Equations (IV.5) and (IV.23)). Hence

$$\begin{aligned} G_h(\rho(e, u, f)^*) &= G_h(\lambda(f, u, e)) \\ &= \lambda(hf, hu, he) = \rho(he, hu, hf)^* \\ &= F_h(\rho(e, u, f)^* \end{aligned}$$

which shows that $\boldsymbol{\Gamma} h = (F_h, G_h)$ is a morphism of cross-connections. It is immediate from the definition of $\boldsymbol{\Gamma} h$ that

$$\boldsymbol{\Gamma} hh' = \boldsymbol{\Gamma} h \boldsymbol{\Gamma} h' \quad \text{and} \quad \boldsymbol{\Gamma} 1_S = 1_{\boldsymbol{\Gamma} S}$$

and so $\boldsymbol{\Gamma}\colon \boldsymbol{RS} \to \boldsymbol{Cr}$ is a functor. □

4.2 The adjoint equivalence

We now show that the functors $\tilde{\boldsymbol{S}}$ and $\boldsymbol{\Gamma}$ are mutually inverse (adjoints of each other) which will imply that $\tilde{\boldsymbol{S}}$ [or $\boldsymbol{\Gamma}$] is an equivalence of categories. The next theorem shows that $\boldsymbol{\Gamma}$ is, up to a natural equivalence, the left-inverse of $\tilde{\boldsymbol{S}}$; that is, $\boldsymbol{\Gamma}\tilde{\boldsymbol{S}}$ is naturally equivalent to $1_{\boldsymbol{RS}}$.

Theorem 15 *For each regular semigroup S let $\varphi(S)\colon S \to \tilde{\boldsymbol{S}}(\boldsymbol{\Gamma}S)$ denote the isomorphism defined by Equation* (IV.41) *and let φ denote the map from $\boldsymbol{vRS}$ to $\boldsymbol{RS}$ which sends $S \in \boldsymbol{vRS}$ to $\varphi(S)$. Then $\varphi\colon 1_{\boldsymbol{RS}} \to \boldsymbol{\Gamma}\tilde{\boldsymbol{S}}$ is a natural isomorphism.*

Proof Let $h\colon S \to S'$ be a homomorphism of regular semigroups. We must show that the following diagram commutes:

$$\text{(17)} \qquad \begin{array}{ccc} S & \xrightarrow{\varphi(S)} & \tilde{\boldsymbol{S}}(\boldsymbol{\Gamma}S) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \tilde{\boldsymbol{S}}(\boldsymbol{\Gamma}h)} \\ S' & \xrightarrow[\varphi(S')]{} & \tilde{\boldsymbol{S}}(\boldsymbol{\Gamma}S') \end{array}$$

Let $x \in S$. Then by Equations (IV.41) and (14) and the definition of $\boldsymbol{\Gamma}h$ (see Theorem 14), we have

$$\tilde{\boldsymbol{S}}(\boldsymbol{\Gamma}h(\varphi(S)(x)) = \tilde{\boldsymbol{S}}(\boldsymbol{\Gamma}h(\rho^x, \lambda^x)) = (\zeta(\rho^x), \zeta^*(\lambda^x))$$

and

$$\varphi(S')(hx) = (\rho^{hx}, \lambda^{hx})$$

where we have written $\zeta = \zeta_{\boldsymbol{\Gamma}h}$ and $\zeta^* = \zeta^*_{\boldsymbol{\Gamma}h}$. Now if $e, f \in E(S)$ with $e\,\mathcal{R}\,x\,\mathcal{L}\,f$, we have

$$\rho^x = \rho^e \star \rho(e,x,f) = \gamma(Se, eS) \star \rho(e,x,f).$$

Therefore by Equation (9),

$$\zeta(\rho^x) = \gamma(S'he, heS') \star F_h(\rho(e,x,f)) = \rho^{he} \star \rho(he, hx, hf) = \rho^{hx}.$$

Now $\rho^x \in \boldsymbol{\Gamma}S(Sf, eS)$ and it follows from Theorems IV.38 and IV.17 that

$$\lambda^x = \chi_S(Sf, eS)(\rho^x) = \chi_{\boldsymbol{\Gamma}S}(Sf, eS)(\rho^x).$$

Hence by Proposition 6 and the definition of $\boldsymbol{\Gamma} S$ we have

$$\begin{aligned}\zeta^*(\lambda^x) &= \zeta^*(\chi_{\boldsymbol{\Gamma} S}(Sf, eS)(\rho^x)) \\ &= \chi_{\boldsymbol{\Gamma} S'}(S'hf, heS')(\zeta(\rho^x)) \\ &= \chi_{\boldsymbol{\Gamma} S'}(S'hf, heS')(\rho^{hx}) = \lambda^{hx}.\end{aligned}$$

This proves that the diagram above commutes and so φ is a natural transformation. Since each component of φ is an isomorphism by Theorem IV.38, it follows that φ is a natural isomorphism. □

It follows from the commutative diagram above that the homomorphism $h\, S \to S'$ is injective or surjective if and only if the homomorphism $\tilde{\boldsymbol{S}}(\boldsymbol{\Gamma} h)$ of $\tilde{\boldsymbol{S}}(\boldsymbol{\Gamma} S)$ to $\tilde{\boldsymbol{S}}(\boldsymbol{\Gamma} S')$ has the corresponding property since $\varphi(S)$ and $\varphi(S')$ are isomorphisms. By Proposition 12, this is true if and only if $\boldsymbol{\Gamma} h$ is injective or surjective. Thus we have:

Corollary 16 *Let $h\colon S \to S'$ be a homomorphism of regular semigroups. Then h is injective or surjective if and only if the morphism of cross-connections $\boldsymbol{\Gamma} h$ is injective or surjective.* □

To complete the proof that $\boldsymbol{\Gamma}$ is an adjoint inverse of $\tilde{\boldsymbol{S}}$, we must show that $1_{\boldsymbol{Cr}}$ is naturally equivalent to $\tilde{\boldsymbol{S}}\boldsymbol{\Gamma}$. Recall that for any cross-connection $\Gamma\colon \mathcal{D} \to \boldsymbol{N}^*\mathcal{C}$ there are isomorphisms $F_\Gamma\colon \mathcal{C} \to \mathcal{L}(\tilde{\boldsymbol{S}}\Gamma)$ and $G_\Gamma\colon \mathcal{D} \to \mathcal{R}(\tilde{\boldsymbol{S}}\Gamma)$ defined by Equations (IV.36) and (IV.37) respectively.

Theorem 17 *For each cross-connection $\Gamma\colon \mathcal{D} \to \boldsymbol{N}^*\mathcal{C}$, let*

$$\psi(\Gamma) = (F_\Gamma, G_\Gamma) \tag{18}$$

where $F_\Gamma\colon \mathcal{C} \to \mathcal{L}(\tilde{\boldsymbol{S}}\Gamma)$ and $G_\Gamma\colon \mathcal{D} \to \mathcal{R}(\tilde{\boldsymbol{S}}\Gamma)$ are isomorphisms of Theorem IV.35. *Then $\psi(\Gamma)$ is an isomorphism of cross-connections and the mapping*

$$\Gamma \mapsto \psi(\Gamma)$$

is a natural isomorphism $\psi\colon 1_{\boldsymbol{Cr}} \to \tilde{\boldsymbol{S}}\boldsymbol{\Gamma}$.

Proof We first show that $\psi = \psi(\Gamma)\colon \Gamma \to \boldsymbol{\Gamma}(\tilde{\boldsymbol{S}}\Gamma)$ is a morphism of cross-connections. For brevity let $S = \tilde{\boldsymbol{S}}\Gamma$ and let $\alpha\colon E_\Gamma \to E(S)$ and $\alpha'\colon E_{\boldsymbol{\Gamma} S} \to$

$E(\tilde{\boldsymbol{S}}(\boldsymbol{\Gamma} S)$ be the isomorphisms of Theorem 1. Let $e = (c, d) \in E_\Gamma$. Then by Equations (IV.36) and (IV.37)

$$F_\Gamma(c) = S\alpha(e) \qquad G_\Gamma(d) = \alpha(e)S.$$

Therefore $(F_\Gamma(c), G_\Gamma(d)) = (S\alpha(e), \alpha(e)S) \in E_{\boldsymbol{\Gamma} S}$. Also by Equation (IV.36),

$$F_\Gamma(\gamma(e)(c')) = \rho(\gamma(e'), \sigma, \gamma(e))$$

where,

$$\begin{aligned}\sigma &= (\gamma(e') \star \gamma(e)(c')^\circ, \gamma^*(e) \star \gamma^*(e')(d^e)^\circ \\ &= (\gamma(e') \cdot \gamma(e), \gamma^*(e) \cdot \gamma^*(e')) \\ &= (\gamma(e'), \gamma^*(e'))(\gamma(e), \gamma^*(e)) \quad \text{by (IV.32)} \\ &= \alpha(e')\alpha(e).\end{aligned}$$

Therefore

$$\begin{aligned}F_\Gamma(\gamma(e)(c')) &= \rho(\gamma(e'), \alpha(e')\alpha(e), \gamma(e)) = \rho^{\alpha(e)}_{S\alpha(e')} \\ &= \gamma(S\alpha(e), \alpha(e)S)(S\alpha(e')) = \gamma(F_\Gamma(c), G_\Gamma(d))(F_\Gamma(c')).\end{aligned}$$

This proves that ψ satisfies axiom (M1). Now let e, $e' \in E_\Gamma$ and $f\colon c = c^e \to c^{e'} = c'$. Then if $\sigma = (\gamma(e) \star f^\circ, \gamma^*(e) \star (f^*)^\circ)$, by Equations (IV.36) and (IV.37) we have

$$\begin{aligned}F_\Gamma(f)^* &= \rho\left(\alpha(e), \sigma, \alpha(e')\right)^* \\ &= \lambda\left(\alpha(e'), \sigma, \alpha(e)\right) \\ &= G_\Gamma(f^*)\end{aligned}$$

which proves that axiom (M2) also holds. Hence ψ is a morphism of cross-connections. Since F_Γ and G_Γ are isomorphisms, it follows that ψ is an isomorphism.

To prove that $\psi\colon \Gamma \to \psi(\Gamma)$ is a natural transformation, let $m = (F, G)$ be a morphism of cross-connections from Γ to Γ'. Let $S = \tilde{\boldsymbol{S}}\Gamma$, $S' = \tilde{\boldsymbol{S}}\Gamma'$ and $h = \tilde{\boldsymbol{S}}m\colon S \to S'$. We must show that $\psi(\Gamma)\boldsymbol{\Gamma} h = m\psi(\Gamma')$. By the definition of composition of morphism of cross-connections (see Equation (6)), this is equivalent to showing that

$$(\bullet) \qquad FF_{\Gamma'} = F_\Gamma F_h \quad \text{and} \quad GG_{\Gamma'} = G_\Gamma G_h.$$

Now for $c \in v\mathcal{C}$, we have by Equations (IV.36) and (16)

$$F_\Gamma F_h(c) = F_h(S\alpha(e)) = S'h\lambda(e)$$

where $c^e = c$. Also by Equations (IV.33) and (9)

$$h\alpha(e) = (\zeta_m(c, d^e)(\gamma(e)), \zeta_m^*(c, d^e)(\gamma^*(e))) = (\gamma(\theta e), \gamma^*(\theta e)) = \alpha(\theta e)$$

where $\theta = \theta_m$. Since $F_{\Gamma'}(F(c)) = S'\alpha(\theta e)$, it follows that $F_\Gamma F_h(c) = FF_{\Gamma'}(c)$ for all $c \in v\mathcal{C}$. Let $f: c \to c' \in \mathcal{C}$ and choose $e = (c, d)$, $e' = (c', d') \in E_\Gamma$. Then by Equation (IV.36),

$$F_\Gamma(f) = \rho(\alpha(e), (\gamma, \lambda), \alpha(e'))$$

where $\qquad \gamma = \gamma(e) \star f^\circ; \qquad \lambda = \chi_\Gamma(c', d)(\gamma) = \gamma^*(e') \star (f^*)^\circ$

by the definition of χ_Γ (see Equation (IV.23)). Hence if, as above, $\theta = \theta_m$, then by Equation (14),

$$\begin{aligned} F_h(F_\Gamma(f) &= \rho(\alpha(\theta e), h(\gamma, \lambda), \alpha(\theta e')) \\ &= \rho\left(\alpha(\theta e), (\zeta_m(c', d)(\rho), \zeta_m^*(c', d)(\lambda)), \alpha(\theta e')\right). \end{aligned}$$

Now, by Equations (9) and (9*),

$$\begin{aligned} \zeta_m(c', d)(\gamma) &= \gamma(\theta e) \star F(f)^\circ; \qquad \text{and} \\ \zeta_m^*(c', d)(\lambda) &= \gamma^*(\theta e') \star G(f^*)^\circ \\ &= \gamma^*(\theta e') \star (F(f)^*)^\circ \\ &= \chi_{\Gamma'}(F(c'), G(d))(\zeta_m(c', d)(\gamma)) \end{aligned}$$

by Equation (IV.23). It follows from Equation (IV.36) that

$$F_h(F_\Gamma(f)) = F_{\Gamma'}(F(f))$$

for all morphism $f \in \mathcal{C}$. This completes the proof of the first equation in $(\bullet)$. The second is proved in a similar fashion. Hence ψ is a natural transformation and since $\psi(\Gamma)$ is an isomorphism for each cross-connection Γ, $\psi: 1_{\mathbf{Cr}} \to \tilde{\mathbf{S}}\boldsymbol{\Gamma}$ is a natural isomorphism. □

We have thus proved the following:

Theorem 18 *The functor* $\tilde{S}: \boldsymbol{Cr} \to \boldsymbol{RS}$ *is an adjoint equivalence of the category* $\boldsymbol{Cr}$ *of cross-connections with the category* $\boldsymbol{RS}$ *of regular semigroups. Moreover, the functor* $\boldsymbol{\Gamma}: \boldsymbol{RS} \to \boldsymbol{Cr}$ *is an adjoint inverse of* $\tilde{S}$.

Proof By Theorems 13 and 14, $\tilde{S}: \boldsymbol{Cr} \to \boldsymbol{RS}$ and $\boldsymbol{\Gamma}: \boldsymbol{RS} \to \boldsymbol{Cr}$ are functors. Further, by Theorems 15 and 17, $\varphi: 1_{\boldsymbol{RS}} \to \boldsymbol{\Gamma}\tilde{S}$ and $\psi: 1_{\boldsymbol{Cr}} \to \tilde{S}\boldsymbol{\Gamma}$ are natural isomorphisms. Hence the theorem follows from the definition of adjoint equivalence (see § I.1.3). □

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List of Symbols

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