Theory of Regular Semigroups

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Theory of Regular Semigroups

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semigroup!regular semigroup
semigroup

Preface

As with every branch of mathematics, the "algebraic theory of semigroup theory" has also grown to such an extent that some degree of specialization is inavitable in any reasonable work on on the subject. This book is an attempt to write a account of the modern theory of "regular semigroups" with special emphesis on structure theory. A justification for this choice, apart from the research interests of the authors, is the fact that "regular semigroups" froms one of the most important subclass of the class of "semgroups" for which a well-knit theory is possible. Moreover a significant part of the existing theory of semigroups deals with regular semigroups. The book is aimed at "graduate students" and research workers in this or related area. The prerequisite for the material is a good elmentary background in modern algebra including group theory, linear algebra and category theory.

The Chapter 1 begins with a number of preliminary definitions. These are given here for the convenience of later reference as well as setting up notations and conventions. Since we make extensive use of categories (small, concrete categories) in this work, we make a brief review of the standard concepts and results from category theory needed in this work. We also define the notion of categories with subobjects. Similarly we introduce the relavent definitions of grouoids and related concepts. The concept of ordered groupoids, used extensivel elsewhere, is introduced here. The chapter end with an investigation of the relations between ordered groupoids and categories with subobjects.

Chapter 2 discuss some topics from elementary theory of semigroups.

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set
relation
set!subset
Cartesian product
xRy: x and y are R-related
dom *R: domain of R*
Im *R: image of R*
$R \circ R'$: composite of R and R'
relation!composition of relations
relation!composable

CHAPTER 1

Preliminary Definitions

As we already have noted in Preface, prerequisite for the material in this book is a good elmentary background in modern algebra including group theory, linear algebra and category theory. An understanding of the contents of Herstein [1988] or Hungerford [1974] will be adequate for algebra and that of MacLane [1971] for category theory.

This chapter mainly consists of a number of preliminary definitions; these are given here for the convenience of later reference as well as setting up notations and conventions.We shall also introduce a few concepts that will be used through out the rest of this book.

1.1 SETS, RELATIONS AND FUNCTIONS

We shall *not* define sets or related concepts here; instead, we shall adopt the definitions and conventions of MacLane [1971] unless indicated otherwise.

Let X and Y be sets. A *relation* R of X with Y is a subset of the Cartesian product $X \times Y$. In this case the statement $(x, y) \in R$ will also be written as xRy . We also write:

$$\operatorname{dom} R = \{x : (x, y) \in R \ \text{ for some } \ y \in Y\} \tag{1.1a}$$

$$\operatorname{Im} R = \{y : (x, y) \in R \ \text{ for some } \ x \in X\}. \tag{1.1b}$$

If $R \subseteq X \times Y$ and $R' \subseteq Y \times Z$ are relations, the relation $R \circ R'$ defined by

$$R \circ R' = \{(x, z) \in X \times Z : \text{for some } y \in Y,\ xRy \text{ and } yR'z\} \tag{1.2}$$

is called the *composite* of the relations R and R'. Note that composite of R and R' are defined only when $\operatorname{Im} R$ and $\operatorname{dom} R'$ are subsets of the same set. If this is the case, we shall say that the pair (R, R') are *composable*. If the pairs $(R,\ R')$ and $(R',\ R'')$ of relations are composable, it is easy to see that

$$(R \circ R') \circ R'' = R \circ (R' \circ R''). \tag{1.3}$$

associative
relation!converse –
relation!single-valued
$|X|$*: cardinal number of X*
transformation!partial
function
map
transformation
function!domain
function!co-domain
function!value of –
dom f*:domain of f*
cod f*: codomain of f*
function!injective (one-to-one)
function!surjective (onto)
function!bijection
1_X*: identity function on X*

Thus the operation of forming the composite is associative when ever the relevant pairs of relations are composable.

Given a relation $R \subseteq X \times Y$, we can form a relation from Y to X, called the *converse* of R, as follows:

$$R^{-1} = \{(y, x) : (x, y) \in R\}. \tag{1.4}$$

Further, we shall find it convenient to use the following notations: For all $X' \subseteq X$ and $Y' \subseteq Y$,

$$RY' = R(Y') = \{x \in X : \text{for some } \ y \in Y',\ xRy\} \tag{1.5a}$$

and

$$X'R = R^{-1}(X') = \{y \in Y : \text{for some } \ x \in X',\ xRy\}. \tag{1.5b}$$

where $R \subseteq X \times Y$, $X' \subseteq X$ and $Y' \subseteq Y$. Especially, when $Y' = \{y\}$, a singleton, we shall use these later notations to be in conformity with the traditional notations for functions. Thus we write

$$\begin{aligned} Ry = R(y) &= \{x \in X : (x, y) \in R\} \quad \text{for all} \quad y \in \operatorname{Im} R; \\ xR = R^{-1}(x) &= \{y \in Y : (x, y) \in R\} \quad \text{for all} \quad y \in \operatorname{dom} R. \end{aligned}$$

A relation $R \subseteq X \times Y$ is said to be *single-valued* if for all $x \in X$, there at most one $y \in Y$ such that $(x, y) \in R$; that is, $|R(x)| \leq 1$, where for any set X, $|X|$ denote the cardinal number of X. If R is single-valued, for every $x \in \operatorname{dom} R$, by the above,$xR = R(x)$ to denote the unique element $y \in Y$ with $(x, y) \in R$. When $x \notin \operatorname{dom} R$, $R(x)$ is not defined. A single-valued relation on X is also called a *partial transformation*. R is called a *function* if R is single-valued and $\operatorname{dom} R = X$. Note that the relation R is a function if and only if

$$|R(x)| = 1 \quad \forall\, x \in X. \tag{1.6}$$

Functions are also called maps, transformations, etc. We denote a function $f \subseteq X \times Y$ by $f : X \to Y$; the set X [Y] is called the *domain* [*co-domain*] of f. For $x \in \operatorname{dom} f$, the unique element $f(x) \in \operatorname{cod} f$ is called the *value* of f at x. We shall use the notation dom f and cod f to indicate the domain and co-domain of the function f respectively. A function f is said to be *injective* (or *one-to-one*) if f^{-1} is single-valued and it is *surjective* (or *onto*) if $\operatorname{Im} f = \operatorname{cod} f$. f is a *bijection* if both f and f^{-1} are functions. In this case, we have

$$f \circ f^{-1} = 1_{\operatorname{dom} f} \quad \text{and} \quad f^{-1} \circ f = 1_{\operatorname{cod} f}$$

where, for any set X, $1_X = \{(x, x) : x \in X\}$.

relation
$\boldsymbol{B}_X$: set of relations on X

If $f : X \to Y$ and $g : Y \to Z$ are functions, it is easy to verify using Equation (1.2) that $f \circ g$ is also a function with $\operatorname{dom}(f \circ g) = \operatorname{dom} f$ and $\operatorname{cod}(f \circ g) = \operatorname{cod} g$ defined for each $x \in \operatorname{dom} f$ by:

$$x(f \circ g) = (xf)g \quad \text{or} \quad (f \circ g)(x) = g(f(x))$$

The composite function $f \circ g$ can be indicated by the following "commutative diagram":

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow^{f \circ g} & \downarrow g \\ & & Z \end{array} \tag{1.7}$$

By Equation (1.3), composition of functions is associative when ever the relevant functions are composable.

Remark* 1.1*: The rule for composition used by many authors is:

$$(g \circ f)(x) = g(f(x)) \quad \forall \; x \in X \tag{1.2*}$$

where $f : X \to Y$ and $g : Y \to Z$. Notice that this is different from the composition relations defined by Equation (1.2) above. In this book, we will have occation to use both these rules for composition of functions. However, unless otherwise made explicit otherwise, the rule for composition will be assumed to be the one given by Equation (1.2). This will also agree with commutative diagrams of functions (see also Section 1.2).

1.1.1 Equivalence relations

Let X be a set. By a *relation on the set* X we mean a subset of $X \times X$. We denote the set of all relations on X by $\boldsymbol{B}_X$. Note any two relations in X are composable and by Equation (1.3), composition of relations in X is associative. Also, if $\rho \in \boldsymbol{B}_X$, then so is its converse ρ^{-1}.

DEFINITION 1.1. Let $\rho \in \boldsymbol{B}_X$. We say that

(R1) ρ is *reflexive* if $1_X \subseteq \rho$;

(R2) *symmetric* if $\rho^{-1} \subseteq \rho$;

(R3) *transitive* if $\rho \circ \rho \subseteq \rho$; and

(R4) *antisymmetric* if $\rho^{-1} \cap \rho \subseteq 1_X$.

A relation $\rho \in \boldsymbol{B}_X$ is reflexive if and only if $(x, x) \in \rho$ for all $x \in X$ and it is symmetric if and only if

$$(x, y) \in \rho \;\Rightarrow\; (y, x) \in \rho.$$

ρ^n:composite of n copies of ρ
equivalence relation
partition
decomposition

Hence for any $\rho \in \boldsymbol{B}_X$, it is clear that $1_X \cup \rho$ is the smallest reflexive relation containing ρ and $\rho^{-1} \cup \rho$ is the smallest symmetric relation containing ρ. By the definition of composite of relations, the transitivity is equivalent to the property

$$(x, y), (y, z) \in \rho \Rightarrow (x, z) \in \rho.$$

If ρ^n denote the composite of n copies of the relation ρ, it follows by induction from the definition of composition that for all $n \geq 1$

$$\rho^n = \{(x, y) : \exists\, z_i \in X, i = 0, \ldots, n \quad \text{with} \\ z_0 = x,\ z_n = y,\ (z_{i-1}, z_i) \in \rho,\ i = 1, \ldots, n\}.$$

Further, by condition (R3) of the definition above, ρ is transitive if and only if

$$\rho^n \subseteq \rho \quad \text{for all} \quad n \geq 1.$$

If $\rho \in \boldsymbol{B}_X$, it easy to verify that

$$\rho^{(t)} = \bigcup_{n \in \mathbb{N}} \rho^n \quad \text{where} \quad \rho^0 = 1_X, \tag{1.8a}$$

is the smallest reflexive and transitive relation on X containing ρ. If ρ is symmetric, so is $\rho^{(t)}$.

A relation $\rho \in \boldsymbol{B}_X$ is called an *equivalence relation* if it satisfies the properties (Ri), $i = 1, 2, 3$. Given any relation ρ on X, it is easy to deduce from the discussion above that

$$\rho^{\epsilon} = (\rho \cup \rho^{-1})^{(t)}. \tag{1.8b}$$

is the smallest equivalence relation on X that contain ρ.

Let X be a set. A collection $\mathcal{P}$ of subsets of X is a *partition* or *decomposition* of X if

$$Y_1, Y_2 \in \mathcal{P},\ Y_1 \neq Y_2 \Rightarrow Y_1 \cap Y_2 = \emptyset \\ \bigcup_{Y \in \mathcal{P}} Y = X. \tag{1.9a}$$

If $\mathcal{P}$ is a partition, then the relation

$$\rho_{\mathcal{P}} = \{(x, y) \in X \times X : \exists\, Y \in \mathcal{P} \text{ such that } x, y \in Y\} \tag{1.9b}$$

is the unique equivalence relation such that

$$\rho_{\mathcal{P}}(x) \in \mathcal{P} \quad \text{for all} \quad x \in X.$$

Conversely, if ρ is any equivalence relation, then

$$X/\rho = \{\rho(x) : x \in X\} \quad \text{is a partition such that} \quad \rho_{X/\rho} = \rho. \tag{1.9c}$$

In view of this, we shall often use the terms *equivalence relation* and *partition* or *decomposition* as synonyms.

map!quotient – quasi-order quasi-order!quasi-ordered set interval ideal!order – ideal

Moreover, if ρ is an equivalence relation, there is a unique surjective map $\rho^\# : X \to X/\rho$ which maps x to the unique set $\rho(x)$ containing x; $\rho^\#$ is called the *quotient map* determined by the equivalence relation ρ (or the partition X/ρ).

Let $f : X \to Y$ be a function. Then the relation defined by

$$\pi_f = \{(x, y) : f(x) = f(y)\} \tag{1.10a}$$

is an equivalence relation and there is an injective map $\psi_f : X/\pi_f \to Y$ defined by

$$\psi_f(\pi_f(x)) = f(x) \quad \text{such that} \quad f = (\pi_f)^\# \circ \psi_f \tag{1.10b}$$

Hence the function f is injective if and only if $\pi_f = 1_X$ and it is surjective if and only if ψ_f is a bijection.

1.1.2 Partially ordered sets

A relation ρ on X is called a *quasi-order* if it is reflexive and transitive; that is, ρ is a quasi-order if it satisfies conditions (R1) and (R2). If ρ is a quasi-order on the set X, the pair (X, ρ) is called a *quasi-ordered set*.

Note that every equivalence relation is a quasi-order. On the other hand, if ρ is any quasi-order on X, then clearly,

$$\rho \cap \rho^{-1}$$

is an equivalence relation. Moreover, if ρ is any relation on X, $\rho^{(t)}$ is the smallest quasi-order that contain ρ. If $Y \subseteq X$, then

$$\rho|Y = \rho \cap (Y \times Y)$$

is a quasi-order on Y; $(Y, \rho|Y)$ is called a *quasi-ordered subset* of (X, ρ). If $x, y \in X$, then

$$[x, y] = (Y, \rho|Y) \quad \text{where} \quad Y = \{z \in X : x\rho z\rho y\} \tag{1.11a}$$

is called the *closed interval* with end points x and y; other type of intervals may be defined similarly. If $Y \subseteq X$ has the property that

$$x \in Y \;\Rightarrow\; \rho^{-1}(x) \subseteq Y \tag{1.11b}$$

(see Equation (1.5b)) then $(Y, \rho|Y)$ is called an *order ideal* (or simply, *ideal*) of (X, ρ). In particular, when no confusion is likely regarding the quasi-order under consideration, we write:

$$X(x) = (\rho^{-1}(x), \rho|\rho^{-1}(xq)). \tag{1.11c}$$

principal ideal
dual
filter
maximal
maximum
minimal
minimum
partial order
map!order preserving
order embedding

This is clearly an order ideal; it is called the *principal order ideal* (or principal ideal) generated by x. Note that in a quasi-ordered set, principal ideals may have more than one generator.

If ρ is a quasi-order, so is ρ^{-1}; it is called the quasi-order on X *dual* to ρ. If T is a statement about a quasi-ordered set (X, ρ), the statement T^* obtained by replacing every occurrence of ρ in T by the dual quasi-order ρ^{-1} is called the *dual* of T. We will have several occasion to use this *duality* (process of deriving T^* from T) in the sequel. An ideal [principal ideal] in the dual quasi-ordered set is called a *filter* [*principal filter*].

An element x in a quasi-ordered set X (with quasi-order ρ) is said to be *maximal* if $x\rho y$ with $y \in X$ implies $y\rho x$; x is *maximum* if for every $y \in X$, $y\rho x$. *Minimal* and *minimum* elements in a quasi-ordered set are defined dually. If $Y \subseteq X$, an element $y \in Y$ is maximal [minimal] in Y if y is maximal [minimal] in the quasi-ordered set $(Y, \rho|Y)$ (that is, the quasi-ordered subset Y of X). Maximum and minimum element of a subset Y is defined in the obvious way.

A relation $\rho \in \boldsymbol{B}_X$ is called a *partial order* if it is a quasi-order which is antisymmetric (so that ρ satisfies (Ri), $i = 1, 3, 4$). If ρ is a partial order, so is ρ^{-1}. Note that an equivalence relation σ is a partial order if and only if $\sigma = 1_X$. In the sequel, we shall use symbols $\leq, \geq, \preceq, \succeq$, etc., to denote partial orders. As above, if $\leq$ is a partial order on X, we shall say that $(X, \leq)$ is a *partially ordered set* or that X is a *partially ordered set* (*poset* for short) with respect to $\leq$.

In a partially ordered set X the maximum element or the largest element [the minimum element or smallest element], if it exists, is unique and is denoted by $\mathbf{1}$ [$\mathbf{0}$]. Note that $\mathbf{0}$ is the dual of $\mathbf{1}$; that is, the element $\mathbf{1}$ in the poset $(X, \leq^{-1})$. The element $\mathbf{1}$ is, often referred to as the *identity* of X and $\mathbf{0}$ is called the *zero* of X.

A mapping $f : X \to Y$ of partially ordered sets is said to be *order preserving* if for all $x, y \in X$

$$x \leq y \text{ in } X \Rightarrow f(x) \leq f(y) \text{ in } Y. \tag{1.12a}$$

f is called an *order embedding* if f satisfies the following:

$$x \leq y \text{ in } X \iff f(x) \leq f(y) \text{ in } Y. \tag{1.12b}$$

Note that every order embedding is injective.

***Remark* 1.2:** Again for simplicity, we shall often say that X is a partially ordered set; unless explicitly provided otherwise, in this case, the notation for the partial order on X under consideration will be $\leq$. We also denote partially ordered subsets, ideals, intervals, etc., by their underlying sets.

1.1.3 Semilattices and lattices

Birkhoff, G.
meet
$\wedge\Lambda'$:
semilattice
semilattice!complete – join
$\vee\Lambda'$:meet of Λ':
lattice
complete lattice

Here we list a few definitions and results needed later on. For more details, the reader may refer to Birkhoff [1967].

Let Λ be a poset. If $\Lambda' \subseteq \Lambda$, the *greatest lover bound* (or *meet*) of Λ' in Λ is the element σ such that

$$\begin{aligned} \sigma \leq \lambda \quad \forall\, \lambda \in \Lambda'; \\ \tau \leq \lambda \quad \forall\, \lambda \in \Lambda' \;\Rightarrow\; \tau \leq \sigma. \end{aligned} \tag{1.13}$$

The properties of the partial order (specifically, the antisymmetry) implies that the meet of a subset Λ' of Λ, if it exists, is unique; we denote the unique element by $\wedge\Lambda'$. If $\Lambda' = \{\lambda_\alpha : \alpha \in I\}$, then we write

$$\wedge\Lambda' = \bigwedge_{\alpha\in I} \lambda_\alpha. \tag{1.14a}$$

In particular, if $\Lambda' = \{\lambda_1, \ldots, \lambda_n\}$ for some $n \in \mathbb{N}$, then we write

$$\wedge\Lambda' = \lambda_1 \wedge \lambda_2 \wedge \cdots \wedge \lambda_n. \tag{1.14b}$$

A partially ordered set Λ is called a *meet-semilattice* (or lover semilattice) if every finite subset of Λ has meet. A *meet-subsemilattice* of Λ is a subset Λ' such that meet of every finite non-empty subset of Λ' is again belongs to Λ'. It is a *complete meet-semilattice* if every non-empty subset of Λ has meet. Note that in a complete meet-semilattice Λ, $\wedge\Lambda$ must exist. It denotes the least element in Λ so that $\wedge\Lambda = \mathbf{0}$. A *complete* meet-subsemilattice is defined in the obvious way.

The *least upper bound* (or *join*) of a subset Λ' of Λ, is defined dually (that is, by replacing $\leq$ by $\geq$ through out in 1.13 above). When it exists, it is unique and we denote it by $\vee\Lambda'$. Notations dual to those given in Equations (1.14a) and (1.14b) will also be used in this connection (with $\bigwedge$ and $\wedge$ replaced by $\bigvee$ and $\vee$ respectively). Similarly *complete join-semilattice* [*join-semilattice*] is a partially ordered set in which every non-empty [finite] subset has join. These concepts are dual to meet-semilattices and complete meet-semilattices respectively. We can define *join-subsemilattice*, etc., in the obvious manner. As above, a complete join-semilattice Λ must have the largest element $\vee\Lambda = \mathbf{1}$.

A poset Λ is a *lattice* if every finite subset of Λ has both join and meet; that is, Λ is both a join-semilattice and a meet-semilattice. Λ is a *complete lattice* if every non-empty subset Λ' of Λ has both join and meet. Note that in a complete lattice Λ, $\mathbf{1}$ and $\mathbf{0}$ always exists and we have

$$\vee\Lambda = \mathbf{1} \quad \text{and} \quad \wedge\,\Lambda = \mathbf{0}.$$

The following Proposition is useful in characterizing complete lattices:

$\mathcal{E}_X$:
semilattice!– homomorphism
lattice!sublattice

PROPOSITION 1.1. *Every complete meet-semilattice with identity is a complete lattice.*

Proof. Let Λ' be a non-empty subset of Λ and let

$$M = \{\sigma : \forall\ \rho \in \Lambda', \quad \rho \leq \sigma\}.$$

Since $\mathbf{1} \in M, M \neq \emptyset$. Let $\bar{\sigma} = \wedge M$. Since Λ is a complete meet-semilattice and M non-empty, $\bar{\sigma}$ exists. If $\rho \in \Lambda'$, then ρ is a lover bound of M and hence, by the definition of $\wedge$ (see Equation (1.13)) $\rho \leq \bar{\sigma}$. Hence $\bar{\sigma} \in M$ and so $\bar{\sigma} = \vee\Lambda'$. □

As an example, we have:

COROLLARY 1.2. *Let $\mathcal{E}_X$ be the set of all equivalence relations on the set X. Then $\mathcal{E}_X$ is a complete lattice with respect to inclusion.*

Proof. Clearly $\mathcal{E}_X$ is a poset with respect to the inclusion $\subseteq$. Given any non-empty set E of equivalence relations on X, it is easy to verify that their intersection is an equivalence relation on X which is clearly $\wedge E$. Hence $\mathcal{E}_X$ is a complete meet-semilattice. Also $X \times X$ is an equivalence relation on X and is clearly the identity of $\mathcal{E}_X$. Hence, by the above, $\mathcal{E}_X$ is a complete lattice. □

Let Λ and Λ' be meet-semilattices. Then f is a *$\wedge$-homomorphism* (or *semilattice homomorphism*) if f preserves meet of finite subsets of Λ; that is, for all $\lambda, \lambda' \in \Lambda$,

$$f(\lambda \wedge \lambda') = f(\lambda) \wedge f(\lambda');$$

it is a *complete $\wedge$-homomorphism* if for all non-empty $M \subseteq \Lambda$,

$$f(\wedge M) = \wedge f(M).$$

By Equation (1.12a), every meet-homomorphism is, in particular, an order preserving map and any one-to-one meet-homomorphism is an order embedding (see Equation (1.12b)). Notice that $\Lambda' \subseteq \Lambda$, then Λ' is a subsemilattice of Λ if and only if Λ' is a semilattice and the inclusion is a meet homomorphism. The corresponding $\vee$-concepts such as $\vee$-*homomorphisms*, etc., are defined dually.

If Λ and Λ' are lattices, then they are meet-semilattices as well; a $\wedge$-homomorphism of the associated meet-semilattices will be called a $\wedge$-homomorphism of the lattice Λ to Λ'. $\vee$-homomorphisms are defined dually. One can extend in the obviouse way these definitions to complete $\wedge$-homomorphisms and complete $\vee$-homomorphisms of lattices and complete lattices. A partially ordered subset Λ' of a lattice Λ is a *sublattice* of Λ if Λ' is a lattice and the inclusion $\Lambda' \subseteq \Lambda$ is a lattice homomorphism.

If Λ and Λ' are lattices [complete lattices] $f : \Lambda \to \Lambda'$ is a lattice homomorphism or [complete lattice homomorphism] if it preserve join and meet of finite non-empty subsets [arbitrary non-empty subsets]. Also, given any family non-empty of [complete] lattices $\{\Lambda_\alpha : \alpha \in \Omega\}$, the Cartesian product

lattice!modular – modular law
dual!self-dual
lattice!distributive
lattice!complement
lattice!complement

$$\Lambda = \prod_{\alpha \in \Omega} \Lambda_\alpha$$

becomes a [complete] lattice when we define join and meet in Λ by

$$\pi_\alpha(\vee\Lambda') = \vee\pi_\alpha(\Lambda') \quad \text{and} \quad \pi_\alpha(\wedge\Lambda') = \wedge\pi_\alpha(\Lambda') \tag{1.15}$$

for all non-empty finite subsets [arbitrary non-empty subsets] $\Lambda' \subseteq \Lambda$. Here $\pi_\alpha : \Lambda \to \Lambda_\alpha$ denote projections of the product to the co-ordinate lattices. Note that when join and meet are defined in Λ as above, $\pi_\alpha : \Lambda \to \Lambda_\alpha$ becomes a lattice homomorphism for each $\alpha \in \Omega$.

Complemented and modular lattices: A lattice Λ is said to be *modular* if

$$\lambda, \sigma, \tau \in \Lambda, \;\; \lambda \leq \tau \;\Rightarrow\; (\lambda \vee \sigma) \wedge \tau = \lambda \vee (\sigma \wedge \tau). \tag{1.16}$$

The statement above is called the *modular law*. Note that the dual of this statement is essentially the same and hence modular law is *self-dual*. The lattice Λ is said to be *distributive* if for all $\lambda, \sigma, \tau \in \Lambda$, we have

$$(\alpha \vee \sigma) \wedge \tau = (\alpha \wedge \tau) \vee (\sigma \wedge \tau) \tag{1.17a}$$

$$(\alpha \wedge \sigma) \vee \tau = (\alpha \vee \tau) \wedge (\sigma \vee \tau) \tag{1.17b}$$

Note that a distributive lattice is modular. Every sublattice of a modular [distributive] lattice is modular [distributive] and products of modular [distributive] lattices are modular [distributive].

Let Λ be a lattice with $\mathbf{0}$ and $\mathbf{1}$. A *complement* of an element $\lambda \in \Lambda$ is an element $\lambda' \in \Lambda$ satisfying the following:

$$\lambda \vee \lambda' = \mathbf{1} \quad \text{and} \quad \lambda \wedge \lambda' = \mathbf{0}. \tag{1.18}$$

A lattice Λ is said to be *complemented* if every element in Λ has a complement.

It is clear that products of complemented lattices are complemented. However, a sublattice of a complemented lattice need not be complemented. If Λ is complemented and modular, every interval $[\lambda, \sigma]$ in Λ is complemented and modular. In fact, if α' is a complement of $\alpha \in \Lambda$, then

$$\alpha^* = (\alpha' \wedge \sigma) \vee \lambda \tag{1.19}$$

lattice!complement!relative
Birkhoff, G.
MacLane, S.

can be shown to be a complement of α in $[\lambda, \sigma]$; α^* is called the *relative* complement of α in $[\lambda, \sigma]$. Also, it is easy to see that a complemented distributive lattice is *uniquely* complemented; that is, every element in a complemented distributive lattice has a unique complement. Complemented distributive lattices are called *Boolean algebras* (see Birkhoff [1967] for more details).

Example 1.1: For any set X, $\boldsymbol{B}_X$ is clearly a complete lattice with respect to inclusion. By Corollary 1.2, $\mathcal{E}_X$ is a complete lattice. By Equation (1.8b), the map $R \mapsto R^\epsilon$ is an order preserving map of $\boldsymbol{B}_X$ onto $\mathcal{E}_X$. It is easy to show that this map is a complete join homomorphism which is not a lattice homomorphism. Clearly, $\mathcal{E} \subseteq \boldsymbol{B}_X$ and the inclusion is a meet homomorphism but not a join homomorphism. Thus $\mathcal{E}$ is a meet subsemilattice of $\boldsymbol{B}_X$, but not a sublattice.

Example 1.2: Let G be a group and let $\mathcal{N} = \mathcal{N}_G$ be the partially ordered set of normal subgroups of G under inclusion. Then $\mathcal{N}$ is a lattice with

$$N_1 \vee N_2 = N_1N_2 \quad \text{and} \quad N_1 \wedge N_2 = N_1 \cap N_2$$

where $N_1N_2 = \{n_1n_2 : n_1 \in N_1,\ n_2 \in N_2\}$ denote the product of N_1 and N_2. It is easy toverify that N_1N_2 is the join of N_1 and N_2 in $\mathcal{N}$. Let $H, K, N \in \mathcal{N}$ and $H \subseteq N$. Then $x \in HK \cap N$ if and only if $x = hk$ with $h \in H$, $k \in K$ and $hk = x \in N$. This is true if and only if $h \in H$ and $k = h^{-1}x \in K \cap N$. It follows that

$$HK \cap N = H(K \cap N)$$

which is the modular law for $\mathcal{N}$. Thus $\mathcal{N}$ is a modular lattice.

Example 1.3: Let $\mathcal{P}_V = \mathcal{P}$ denote the partially ordered set of all subspaces of a vector space V over the field $\Bbbk$ under inclusion. Then $\mathcal{P}$ is a lattice with

$$V_1 \vee V_2 = V_1 + V_2 \quad \text{and} \quad V_1 \wedge V_2 = V_1 \cap V_2$$

for all $V_1, V_2 \in \mathcal{P}$. It is easy to see using elementary linear algebra that $\mathcal{P}$ is a complemented modular lattice which is not a Boolean algebra.

1.2 CATEGORIES

The aim of this section is to list some preliminary definitions and results about categories; this will enable us to set up notations and conventions to be followed in the sequel. In the first section we review some definitions from category theory for the convenience of later use. The remainder of the chapter is devoted to describing certain results and constructions of category theory needed later. Most of these results are quite standard and can be found in any standard work on categories. In our formulation of these results, we have followed MacLane [1971] as far as possible.

1.2.1 Definitions and notations

In the following we assume that the reader is familiar with the concepts of categories, functors and related concepts (see Hungerford [1974], MacLane

[1971], etc., for details). Here our aim is limited to introducing notations and terminology needed in the sequel.

We shall generally follow notations and terminology established in Nambooripad [1994] (except for some occasional modifications). However, for completeness, we shall reproduce most of them here. For those notation and / or terminology not explicitly defined here, the reader should refer MacLane [1971], Nambooripad [1979] or Nambooripad [1994].

DEFINITION 1.2. A *category* C consists of the following data:

1. A class called the class of *vertices* or *objects*.
2. a class of disjoint sets $C(a, b)$, one for each pair $(a, b) \in \mathfrak{v}C \times \mathfrak{v}C$. An element $f \in C(a, b)$ is called a *morphism* (or an arrow) from a to b, written $f : a \to b$; $a = \operatorname{dom} f$ is called the *domain* of f and $b = \operatorname{cod} f$ is called the *codomain* of f.
3. For $a, b, c \in \mathfrak{v}C$, a map
$$\circ : C(a, b) \times C(b, c) \to C(a, c), \quad (f, g) \mapsto f \circ g.$$
$\circ$ is called the *composition* of morphisms in C.
4. For each $a \in \mathfrak{v}C$, a unique $1_a \in C(a, a)$ called the identity morphism on a.

These must satisfy the following axioms:

(Cat 1) The composition is associative: for $f \in C(a, b)$, $g \in C(b, c)$ and $h \in C(c, d)$, we have
$$f \circ (g \circ h) = (f \circ g) \circ h).$$

(Cat 2) For each $a \in \mathfrak{v}C$, $f \in C(a, b)$ and $g \in C(c, a)$,
$$1_a \circ f = f \quad \text{and} \quad g \circ 1_a = g.$$

Observe that the order of the composition given by item (3) is from left to right and agree with the composition of function defined earlier (cf. (1.2)) as well as the usage in Nambooripad [1994].

Let C be a category. The symbol C will also denote its morphism class . As in MacLane [1971], the sets $C(a, b)$ will also be called *home-sets*. The home-set $C(a, a)$ is often abbreviated as $C(a)$. Morphisms in $C(a)$ are called *endomorphisms* of a. Since the morphism sets $C(a, b)$ are disjoint (by item (2) above), the correspondance $a \mapsto 1_a$ is an injection of the class $\mathfrak{v}C$ into C. It is convenient to identify $\mathfrak{v}C$ as a subclass of C by this injection so that we have $\mathfrak{v}C \subseteq C$. With this identification, it is possible to define categories in terms of morphisms (arrows) alone. Notice that the class $\mathfrak{v}C$ need not be a set whereas the morphism set $C(a, b)$ (by (2) above) is required to be a set (*small set* — see

Hungerford, W.
Nambooripad, K. S. S.
category
vertices
objects
$\mathfrak{v}C$:*vertex class of* C
morphism
$C(a, b)$:*set of morphisms from a to b*
dom f:*domain of f*
domain
codomain
cod f:*codomain of f*
composition
morphism!identity –
class!morphism –
home-sets
endomorphisms
small set

category!small –
partial binary operartion
domain!domain of a partial binary operartion
partial algebra
D_X:domain of the partial binary operartion on X
identity!categorical identity
identity!categorical left identity
identity!categorical left identity
e_g and f_g:unique left and right identity of $g \in C$

MacLane [1971],pp 21–24.) The category C is said to be *small* if the class C (that is, the class of all morphisms in C) is a set. In view of item (2) above, this true if and only if $\mathfrak{v}C$ is a set.

In this work, we will use categories not only as a language but also as a mathematical structure which is a generalization of partially ordered sets. In the later usage the categories considered will be small. For small categories, the arrows-only definition is more appropriate. To formulate this definition, we need some additional concepts. A *partial binary operartion* on a set X is a function from a subset $D \subseteq X \times X$ to X; the set D is called the *domain* of the partial binary operartion. A *partial algebra* X is a set (again denoted by X) on which a partial binary operartion is given. If no ambiguity is likely, we shall denote the partial binary operation on X by juxtaposition and its domain by $D = D_X$. Note that the statement $(g, h) \in D$ is equivalent to the statement that the product gh exists (or is defined) in X. An element $u \in X$ is a *categorical identity* or simply, an *identity*, if

$$ug = g \quad \text{whenever } (u, g) \in D \text{ and} \quad hu = h \quad \text{whenever } (h, u) \in D.$$

We are now ready for the arrow-only definition of small categories (see also MacLane [1971], pp 9).

DEFINITION 1.3. A small category C is a partial algebra satisfying the following axioms:

(Ar 1) The composite $(gh)k$ is defined if and only if the composite $g(hk)$ is defined. When either is defined they are equal. The common value of thess triple composites is denoted by ghk.

(Ar 2) If the composites gh and hk are defined, then the triple composite ghk is defined.

(Ar 3) For all $g \in C$, there exist identities $u, v \in C$ such that ug and gv are defined.

If $g \in C$ an identity $u \in C$ with $ug = g$ [$gu = g$] is called a *left identity* [*right identity*]. Axiom (Ar 3) shows that every $g \in C$ has a left [right] identity. The strong associtivity implied by axioms (Ar 1) and (Ar 2) will mean that these are unique. For if u, u' are left identities of g. Then products $ug = g$ and $u'g = u'(ug)$ exists in C. Hence by (Ar 1), $(u'u)g$ exists whcich implies that $u'u$ exists. Since these are identities, we have $u = u'u = u'$ by definition. Similarly right identities are also unique. We use the notation e_g and f_g to denote the unique left and the right identity of the morphism $g \in C$. Moreover, the composite gh is defined in C if and only if $f_g = e_h$. For from the fact that the composite $gh = (gf_g)h$ exists we conclude that the product $g(f_gh)$ exists and so

$f_g h$ exists. This gives $f_g = e_h$. Conversely, if $f_g = e_h = u$, form the fact that the products gu and uh exists, we conclude by axion (Ar 2) that $(gu)h = gh$ exists in C. It follows that, taking $\mathfrak{v}C$ as the set of identities in C, C becomes a category as per Definition 1.2. On the other hand, if C is a category according to Definition 1.3, then for any $a \in \mathfrak{v}C$, $u = 1_a$ is a categorical identity. For if ug exists, then by item (3) of the definition, $g \in C(a, b)$ for some $b \in \mathfrak{v}C$ and by axion (Cat 2), $ug = g$. Similarly, if hu exists, $h \in C(c, a)$ and $hu = h$. It now follows immediately that axioms (Ar i), $i = 1, 2, 3$ holds so that C is a small category according to the arrow-only definition.

C^{op}:*the opposite category of C*
the opposite category of C
dual
principle of duality
1_a:
category!trivial – on X
Set:*category of sets*
category! – of sets
Grp:*category of groups*
category!– of groups
Ab:*category of abelian groups*
category! – of abelian groups
functor!covariant –
vertex map
$\mathfrak{v}F$:*The vertex map of F*

Suppose that C is a category (not necessarily small). Then there exists a category C^{op} defined as follows:

$$\mathfrak{v}C^{op} = \mathfrak{v}C, \quad C^{op}(a, b) = C(b, a) \tag{1.20}$$

for all $a, b \in \mathfrak{v}C$ and the composition $*$ in C^{op} is given by

$$g * h = h \circ g$$

for all $g, h \in C^{op} = C$ for which $h \circ g$ is defined in C. Indeed, one can readily see from the definition above that, these data give a category C^{op} called the *the opposite category of C*. Any statement T regarding C corresponds to a suitable statement T^* regarding C^{op} obtaind by reversing arrows and composition. The statement T^* is called the *dual* of T. Clearly, if T is true for C, then T^* is true for C^{op}. This method of inforing the truth of a statement T^* for C^{op} from the truth of T for C is called the *principle of duality*. Also, $T^{**} = T$. Note that if T holds for arbitrary categories, it holds for C^{op} and so, both T and T^* holds for C.

Observe that with any *class* X, we can trivially associate a category C with $\mathfrak{v}C = X$ and for $a, b \in X$, $C(a, b)$ is empty if $a \neq b$ and $C(a) = \{1_a\}$ where 1_a denotes the identity morphism on a. Since no confusion is likely, we shall denote this *trivial category* on X by X itself.

Example 1.4: Some of the most frequently used examples of categories are the following:

1. **Set**: the category in which vertices are sets and morphisms are maps. It is called the *category of sets*.
2. **Grp**: the category with groups as vertices and morphisms as homomorphisms. **Grp** is called the *category of groups*.
3. **Ab**: the category in which vertices are abelian groups and morphisms are homomorphisms. The *category of abelian groups* is a subcategory of **Grp**.

The reader may verify that the above list are valid examples of categories.

DEFINITION 1.4. A *covariant functor* $F : C \to \mathcal{D}$ from a category C to a category $\mathcal{D}$ consists of a *vertex map* $\mathfrak{v}F : \mathfrak{v}C \to \mathfrak{v}\mathcal{D}$ which assigns to each $a \in \mathfrak{v}C$ a vertex

morphism map
functor!contravariant –
partial algebra! – homomorphism
partial algebra! – anti-homomorphism
Cat*:The category of small categories*
$\mathfrak{v}$*:functor from* **Cat** *to* **Set**
category! – of small categories

$\mathfrak{v}F(a) \in \mathfrak{v}\mathcal{D}$ and a *morphism map* $F : C \to \mathcal{D}$ which assigns to each morphism $f : a \to b \in C$, a morphism

$$F(f) : \mathfrak{v}F(a) \to \mathfrak{v}F(b) \in \mathcal{D}$$

such that

(Fn 1) $F(1_a) = 1_{\mathfrak{v}F(a)}$ for all $a \in \mathfrak{v}C$; and

(Fn 2) $F(f)F(g) = F(fg)$ for all morphisms $f, g \in C$ for which the composite fg exists.

F is a *contravariant functor* if $\mathfrak{v}F$ is as above and the morphism map assigns to each $f : a \to b \in C$, a morphism

$$F(f) : \mathfrak{v}F(b) \to \mathfrak{v}F(a) \in \mathcal{D}$$

such that they satisfy axiom (Fn 1) and the following:

(Fn* 2) $F(g)F(f) = F(fg)$ for all morphisms $f, g \in C$ for which the composite fg exists.

In the following, unless otherwise stated, a functor will mean a covariant functor. Observe that a functor $F : C \to \mathcal{D}$ is contravariant if and only if $F: C^{op} \to \mathcal{D}$ is a covariant functor.

If we identify $\mathfrak{v}C$ as a subset of C by identifying vertices with the corresponding identity, the condition (Fn 1) implies that

$$\mathfrak{v}F = F \mid \mathfrak{v}C$$

for any functor $F : C \to \mathcal{D}$. Therefore we may use the same notation for the morphism map as well as the vertex map of a functor. Thus the symbol $F(x)$ will mean a vertex in $\mathcal{D}$ if $x \in \mathfrak{v}C$ and a morphism in $\mathcal{D}$ if x is a morphism in C. We may now define a covariant functor $F : C \to \mathcal{D}$ as a mapping of the class C to the class $\mathcal{D}$ that preserves identities and composition. A contravariant functor is similarly a map that preserves identities but reverses composition.

A functor $F : C \to \mathcal{D}$ is said to *small* if C is a small category. In this case, it is easy to see that there is a small subcategory $\mathcal{D}'$ of $\mathcal{D}$ such that F is a functor of C to $\mathcal{D}'$. Thus a small functor is a partial algebra homomomorphism that preserve categorical identities. Similarly, a contravariant small functor is a partial algebra anti-homomorphism which preserves identities. It is clear that there is a category **Cat** in which vertices are (small) categories and morphisms are (small) functors. Moreover the assignments

$$C \mapsto \mathfrak{v}C \quad \text{and} \quad F \mapsto \mathfrak{v}F \tag{1.21}$$

is a functor $\mathfrak{v}$ from **Cat** (the category of small categories) to the category **Set**.

For any category C, there always exists a functor, denoted by 1_C, whose vertex map is the identity map on the vertex set of C and whose morphism map is the identity map on the morphism class of C. A category $\mathcal{D}$ is a subcategory of a category C if the class $\mathcal{D}$ is a subclass of C and the composition in $\mathcal{D}$ is the restriction of the composition in C to $\mathcal{D}$. In this case, the inclusion $\mathcal{D} \subseteq C$ preserves composition and identites and so, represents a functor of $\mathcal{D}$ to C which is called the *inclusion functor* of $\mathcal{D}$ into C. Observe that for any category C, the trivial category $\mathfrak{v}C$ is a subcategory of C. In particular the inclusion $\mathfrak{v}C \subseteq C$ can be regarded as a *category inclusion*.

1_C:
functor!incluysion functor inclusion
category!– inclusion
functor!$\mathfrak{v}$ injective
functor!$\mathfrak{v}$-surjective
functor!faithful
functor!injective embedding
functor!embedding
functor!full
functor!surjective
functor!strictly full
functor!full embedding
functor!fully-faithful
isomorphism! – of categories
isomorphism!inverse of isomorphisms
F^{-1}*:inverse of F*
$C(c, f)$*:function* $g \mapsto gf$
$C(c, -)$*:covariant hom-functor*
functor!covariant hom- –
$C(f, c)$*:function from* $C(c'', c)$ *to* $C(c', c)$

Let C and $\mathcal{D}$ be two categories. We shall say that a functor $F : C \to \mathcal{D}$ is $\mathfrak{v}$-*injective* if $\mathfrak{v}F$ is injective and F is $\mathfrak{v}$-*surjective* if $\mathfrak{v}F$ is surjective. F is said to be *faithful* if the morphism map is injective on each hom-set of C and F is *injective* or an *embedding* if it is faithful and $\mathfrak{v}$-injective. Note that this is equivalent to requiring that F is injective as a partial algebra homomorphism. We shall say that F is *full* if its morphism map is surjective on each hom-set of F. It is *surjective* if it is surjective as a partial algebra homomorphism (or, its morphism map is surjective). In this case, it is easy to see that F is $\mathfrak{v}$-surjective. F is *strictly full* if it is full and $\mathfrak{v}$-surjective. If F is strictly full then it is clearly surjective. We shall say that F is a *full embedding* if it is *fully-faithful* (that is, full and faithful) and $\mathfrak{v}$-injective. An *isomorphism* of categories is a full embedding in which $\mathfrak{v}F$ is a bijection. If F is an isomorphism, the *inverse* F^{-1} exists and is also an isomorphism of categories.

We now describe two classes of set-valued functors that will be of use later. Let C be a category. For fixed $c \in \mathfrak{v}C$ and $f : c' \to c''$ in C, let $C(c, f)$ denote the function from $C(c, c')$ to $C(c, c'')$ defined as follows:

$$C(c, f)(g) = gf \quad \text{for all } g \in C(c, c'). \tag{1.22}$$

Then the assignments

$$c' \mapsto C(c, c') \qquad f \mapsto C(c, f) \tag{1.23}$$

for all $c' \in \mathfrak{v}C$ and $f : c' \to c'' \in C$, defines a functor $C(c, -)$ from C to the category **Set**. $C(c, -)$ is called the *covariant hom-functor* determined by c.

Again, as above, for fixed $c \in \mathfrak{v}C$ and $f : c' \to c''$ in C, let $C(f, c)$ denote the function from $C(c'', c)$ to $C(c', c)$ defined as follows:

$$C(f, c)(g) = fg \tag{1.22*}$$

for all $g \in C(c'', c)$. The assignments

$$c' \mapsto C(c', c) \qquad f \mapsto C(f, c) \tag{1.23*}$$

C(−, c):contravariant hom-functor
functor!contravariant hom-functor
natural transformation
natural transformation!component of −
η(a) :component of the natural transformation η
natural isomorphism
naturally equivalent
F ≅ G:F is naturally equivalent to G
functor!morphism of functors

for all $c' \in \mathfrak{v}\mathcal{C}$ and $f : c' \to c'' \in \mathcal{C}$ defines a contravariant functor $\mathcal{C}(-, c) : \mathcal{C} \to$ **Set** which is called the *contravariant hom-functor*. Notice that the definition of contravariant hom-functor is obtained by dualising the definition of covariant hom-functor.

Natural transformations Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ be two functors (with the same domain and codomain). A *natural transformation* $\eta : F \xrightarrow{n} G$ is a map $a \mapsto \eta(a)$ from the vertex class $\mathfrak{v}\mathcal{C}$ of $\mathcal{C}$ to the morphism class of $\mathcal{D}$ (which by the convention introduced above is denoted by $\mathcal{D}$ itself) such that for each $a \in \mathfrak{v}\mathcal{C}$, component $\eta(a) : F(a) \to G(a)$ is a morphism in $\mathcal{D}$ such that the following diagram commutes for all $f : a \to b$ in $\mathcal{C}$:

$$\begin{array}{ccc} F(a) & \xrightarrow{\eta(a)} & G(a) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(b) & \xrightarrow[\eta(b)]{} & G(b) \end{array} \tag{1.24}$$

In the following we will denote the component of η at a either as $\eta(a)$(as above) or as η_a (as in MacLane [1971]). If every component of η is an isomorphism, then η is called a *natural isomorphism*. Functors F and G from $\mathcal{C}$ to $\mathcal{D}$ are *naturally equivalent* (written $F \cong G$) if there is a natural isomorphism $\eta : F \xrightarrow{n} G$. Notice that for any functor $F : \mathcal{C} \to \mathcal{D}$ the map $a \mapsto 1_{F(a)}$ is a natural isomorphism of F to itself which is denoted by 1_F.

1.2.2 Functor categories

Suppose that $\mathcal{F}$ is a class of functors. If $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{D} \to \mathcal{D}'$ are functors in $\in \mathcal{F}$, a *morphism* $\mu : F \xrightarrow{n} G$ is a triple $\mu = (\alpha, \eta, \alpha')$ where $\alpha : \mathcal{C} \to \mathcal{D}$, $\alpha' : \mathcal{C}' \to \mathcal{D}'$ are functors and $\eta : F \circ \alpha' \xrightarrow{n} \alpha \circ G$ is a natural transformation. If

$$\mu = (\alpha, \eta, \alpha') : F \to G \quad \text{and} \quad \tau = (\beta, \zeta, \beta') : G \to H$$

are morphisms of functors, the composite $\mu \circ \tau = \sigma$ is defined as follows:

$$\sigma = (\alpha \circ \beta, \xi, \alpha' \circ \beta'). \tag{1.25}$$

Here, ξ denote the map $: c \to \xi_c$ where for each $c \in \mathfrak{v}\mathcal{C}$

$$\xi_c = \beta'(\eta_c)\zeta_{\alpha(c)}. \tag{1.26}$$

It is easy to see that ξ is a natural transformation $\xi : F \circ \alpha' \circ \beta' \xrightarrow{n} \alpha \circ \beta \circ H$ and thus $\sigma : F \to H$ is a morphism of functors. With this morphism we can define a category $\mathcal{X}$ in which $\mathfrak{v}\mathcal{X} = \mathcal{F}$ provided that for all $F, G \in \mathcal{F}$, the class of morphisms from F to G is a set. The reader can verify that a sufficient condition for this to hold is that dom F and cod F of every $F \in \mathcal{F}$ is a small category. Any subcategory of $\mathcal{X}$ will be called a *functor category* (or a category of functors).

category!functor – $\mathcal{D}$-valued functors
transformation of functors
$[C, \mathcal{D}]$:*category of functors*
Nat(S, T):*natural transformations from S to T*
component-wise product

We proceed to discuss some particular instances of this construction that will be of use in the sequal. Suppose that $\mathcal{F}$ is a class of functors taking values in some fixed category $\mathcal{D}$. For example $\mathcal{D}$ may be the category **Set**, **Grp** (the category of groups) or the category **Ab** of abelian groups, etc. A category of *$\mathcal{D}$-valued functors* is a category $\mathcal{E}$ with

$$\mathfrak{v}\mathcal{E} = \mathcal{F} \tag{1.27a}$$

and for $F, G \in \mathcal{F}$ morphisms $\mu : F \to G$ are of the form

$$\mu = (\alpha, \eta, 1_{\mathcal{D}}). \tag{1.27b}$$

A sufficient condition that this will in fact define a category $\mathcal{E}$ is that $\mathcal{F}$ consists of small functors. Obviousely, $\mathcal{E}$ is a subcategory of $\mathcal{X}$ if the later exists. Morphisms in $\mathcal{E}$ are called *transformations* . If $\mu = (\alpha, \eta, 1_{\mathcal{D}}) : F \to G$ is a fransformation in $\mathcal{E}$, it will be convenient to write $\alpha = \mathfrak{v}\mu$ and use the symbol μ to denote the the natural transformation η also if there will be no ambiguity. It follows from Equations (1.27a) and (1.27b) that composition $\tau = \mu \circ \nu$ of transformations $\mu : F \to G$ and $\nu : G \to H$ is defined by

$$\mathfrak{v}\tau = \mathfrak{v}\mu \circ \mathfrak{v}\nu, \quad \text{and} \quad \tau_c = \mu_c \nu_{\mathfrak{v}\mu(c)} \tag{1.27c}$$

for all $c \in \mathfrak{v}C$.

Suppose that C and $\mathcal{D}$ are categories in which C is small. Then there is a category $[C, \mathcal{D}]$ whose objects are functors from C to $\mathcal{D}$ and whose morphisms are natural transformations (see MacLane [1971]). Notice that this construction can be obtained as a particular case of the construction of $\mathcal{X}$ (or $\mathcal{E}$) if we take $\mathcal{F}$ as the set of all functors from C to $\mathcal{D}$ and morhisms as transformations of the form

$$\mu = (1_C, \eta, 1_{\mathcal{D}}). \tag{1.27d}$$

If S and T are functors from C to $\mathcal{D}$, we shall also use the more usual notation Nat(S, T) to denote the set $[C, \mathcal{D}](S, T)$ of all morphisms (natural transformations) in $[C, \mathcal{D}]$ from S to T. Notice that composition in this category is defined as the *component-wise product* of natural transformations: if $\eta \in \mathrm{Nat}(S, T)$ and $\zeta \in \mathrm{Nat}(T, U)$, then $\eta\zeta \in \mathrm{Nat}(S, U)$ is the natural transformation defined by

$$(\eta\zeta)_c = \eta_c\zeta_c. \tag{1.27e}$$

for all $c \in \mathfrak{v}C$ (see MacLane [1971]). Clearly, $[C, \mathcal{D}]$ is a subcategory of the category $[-, \mathcal{D}]$ of all small $\mathcal{D}$-valued functors.

product category
category!product –
$\mathcal{C} \times \mathcal{D}$:
bifunctor
functor! – in n variables
bifunctor! – criterion

Bifunctors and bifunctor criterion Let $\mathcal{C}$, $\mathcal{D}$ be categories. Recall that the *product category* $\mathcal{C}\times\mathcal{D}$ is the category with vertex class $\mathfrak{v}\mathcal{C}\times\mathfrak{v}\mathcal{D}$, morphism class $\mathcal{C} \times \mathcal{D}$ and in which composition of morphisms are defined componentwise; that is, if $(f, g) : (c, d) \to (c', d')$ and $(f', g') : (c', d') \to (c'', d'')$ are morphisms in $\mathcal{C} \times \mathcal{D}$, then the composition in the category $\mathcal{C} \times \mathcal{D}$ is given by the equation

$$(f, g)(f', g') = (ff', gg').$$

A *bifunctor* or a functor in two variables is a (covariant) functor $B : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ (where $\mathcal{E}$ is another category). A bifunctor $B : \mathcal{C}^{op} \times \mathcal{D} \to \mathcal{E}$ is said to be contravariant in the first variable and covariant in the second. In an obvious manner, the definition above can be extended to functors in n variables which is contravariant in $r \le n$ variables, etc.

The following principle, called the *bifunctor criterion* is useful in checking whether a given assignments of functors and natural transformations constitute a bifunctor:

THEOREM 1.3 (BIFUNCTOR CRITERION). *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories. For each $c \in \mathfrak{v}\mathcal{C}$ and $d \in \mathfrak{v}\mathcal{D}$, let*

$$G_c : \mathcal{D} \to \mathcal{E} \quad \textit{and} \quad F_d : \mathcal{C} \to \mathcal{E}$$

be functors such that

$$F_d(c) = G_c(d) \quad \textit{for all} \quad (c, d) \in \mathfrak{v}\mathcal{C} \times \mathfrak{v}\mathcal{D}.$$

Then there exists a bifunctor $B : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ with $B(c, -) = G_c$ for all c and $B(-, d) = F_d$ for all d if and only if for every pair of morphisms $f : c \to c' \in \mathcal{C}$ and $g : d \to d' \in \mathcal{D}$ the following diagram commutes:

$$\begin{array}{ccc} F_d(c) & \xrightarrow{G_c(g)} & G_c(d') \\ {\scriptstyle F_d(f)}\downarrow & & \downarrow{\scriptstyle F_{d'}(f)} \\ F_d(c') & \xrightarrow[G_{c'}(g)]{} & G_{c'}(d') \end{array}$$

If this holds, then B is defined by the assignments: a

$$B(c, d) = F_d(c) = G_c(d) \tag{1.28a}$$

for all $(c, d) \in \mathfrak{v}\mathcal{C} \times \mathfrak{v}\mathcal{D}$ and

$$B(f, g) = F_d(f)G_{c'}(g) = G_c(g)F_{d'}(f). \tag{1.28b}$$

for all $(f, g) : (c, d) \to (c', d') \in \mathcal{C} \times \mathcal{D}$. □

$C(-,-)$:*the hom-functor*
hom-functor
category! – isomorphism
bifunctor! – criterion

We refer the reader to MacLane [1971], Proposition 1 on page 37 for further information about this principle.

Given any category C, it is easy to check that the contra-variant and covariant hom-functors

$$C(-,c) : C^{op} \to \mathsf{Set} \qquad C(c,-) : C \to \mathsf{Set} \tag{1.29}$$

(cf. Equation (1.23) and Equation (1.23*)) satisfy the bifunctor criterion above and hence determines a unique bifunctor $C(-,-) : C^{op} \times C \to \mathsf{Set}$. $C(-,-)$ is called the *hom-functor*. Notice that $C(-,-)$ sends each $(c,d) \in \mathfrak{v}C \times \mathfrak{v}C$ to the set $C(c,d)$ and $(f,g) \in C(c',c) \times C(d,d')$ to the function $C(f,g)$ defined by

$$C(f,g) : h \mapsto fhg. \tag{1.30}$$

Clearly the bifunctor $C(-,-)$ is contravariant in the first variable and covariant in the second.

An isomorphism of functor categories It is well-known that, if C, $\mathcal{D}$ are small categories, and $\mathcal{E}$ is any category, we have the following category isomorphisms:

$$[C,[\mathcal{D},\mathcal{E}]] \;\cong\; [C\times\mathcal{D},\mathcal{E}] \;\cong\; [\mathcal{D},[C,\mathcal{E}]]\,. \tag{1.31}$$

(see MacLane [1971]). In fact the first isomorphism is defined by the assignments:

$$F \mapsto F(-,-); \qquad \text{and} \qquad \eta \mapsto \eta_{-,-}. \tag{1.31*}$$

Here $F(-,-)$ is defined, for any functor $F \in \mathfrak{v}\,[C,[\mathcal{D},\mathcal{E}]]$, as follows. For each $c \in \mathfrak{v}C$, let $G_c = F(c)$. By hypothesis $G_c : \mathcal{D} \to \mathcal{E}$ is a functor. Also for each $d \in \mathfrak{v}\mathcal{D}$, let F_d be defined by the assignments

$$c \mapsto F(c)(d) \quad \text{and} \quad f \mapsto F(f)(d).$$

It is easy to see that $F_d = F(-)(d) : C \to \mathcal{E}$ is a functor. If $f : c \to c' \in C$, then $F(f) : F(c) \to F(c')$ is a natural transformation and hence the following diagram commutes for each $g : d \to d' \in \mathcal{D}$:

$$\begin{array}{ccc} F(c,d) & \xrightarrow{F(f)_d} & F(c',d) \\ {\scriptstyle F(c)(g)}\downarrow & & \downarrow{\scriptstyle F(c')(g)} \\ F(c,d') & \xrightarrow[F(f)_{d'}]{} & F(c',d') \end{array} \tag{1.32}$$

It follows from bifunctor criterion (see Subsection 1.2.2) that the functors F_d and G_c determines a unique bifunctor $F(-,-) : C \times \mathcal{D} \to \mathcal{E}$ defined as follows:

$$F(c,d) = F(c)(d) \tag{1.33a}$$

category!small – category!big – higher universe representation functor! – category $C^* = [C, \mathsf{Set}]$:

for each $(c,d) \in \mathfrak{v}C \times \mathfrak{v}\mathcal{D}$, and for $(f,g) : (c,d) \to (c',d')$, let

$$F(f,g) = (F(f)_d)(F(c')(g)) = (F(c)(g))(F(f)_{d'}). \tag{1.33b}$$

Then we clearly have $F(c,-) = F(c)$ and $F(-,d) = F(-)(d)$ for all c and d. Similarly if η is a natural transformation in $[C,[\mathcal{D},\mathcal{E}]]\,(F,G)$, and if we define

$$\eta_{c,d} = (\eta_c)_d \tag{1.33c}$$

then it is easily seen that $\eta_{-,-} : F(-,-) \xrightarrow{n} G(-,-)$ is a natural transformation of bifunctors.

Conversely, let $F(-,-) \in \mathfrak{v}[C \times \mathcal{D}, \mathcal{E}]$ and $\eta_{-,-} \in [C \times \mathcal{D}, \mathcal{E}]$. For each $c \in \mathfrak{v}C$, $F(c,-) : \mathcal{D} \to \mathcal{E}$ is a functor and for each $f : c \to c' \in C$, by the bifunctor criterion, $F(f,-) : F(c,-) \xrightarrow{n} F(c',-)$ is a natural transformation. Define $\tilde{F}$ and $\tilde{\eta}$ as follows:

$$\tilde{F}(c) = F(c,-); \quad \tilde{F}(f) = F(f,-); \qquad \tilde{\eta}_c = \eta_{c,-} \tag{1.34}$$

It can be shown that $\tilde{F} : C \to [\mathcal{D}, \mathcal{E}]$ is the unique functor such that the bifunctor $\tilde{F}(-,-)$ determined by $\tilde{F}$ as above (using Equations 1.33a and 1.33b) coincides with $F(-,-)$. Also it is easy to see that $\tilde{\eta} \colon \tilde{F} \to \tilde{G}$ is the unique natural transformation such that the natural transformation of bifunctors determined by $\tilde{\eta}$ (as in Equation 1.33c) is the same as $\eta_{-,-}$. It follows that the assignments given by Equation (1.31*) is a category isomorphism. Since categories $C \times \mathcal{D}$ and $\mathcal{D} \times C$ are isomorphic, the second isomorphism of Equation 1.31 can be obtained in the obvious way.

Remark* 1.3*: Notice that even if C and $\mathcal{D}$ are not small $[C,\mathcal{D}]$ can still be interpreted as a category though the hom-sets of this category is no longer small; also Equation (1.31) remains valid where the isomorphisms are isomorphisms of "large" categories (that is, categories whose hom-sets belongs to a *higher universe* so that they are not small sets—see MacLane [1971], pp 21–24). In any case, given any bifunctor F from $C \times \mathcal{D}$ to $\mathcal{E}$, Equation (1.34) gives a *representation* $\tilde{F}$ sending each object in C to a functor from $\mathcal{D}$ to $\mathcal{E}$ and morphisms to natural transformations between such functors and this assignment is functorial in the sense that it preserve identities and composition. When C and $\mathcal{D}$ are not small, $\tilde{F}$ will be a functor from a category with small hom-sets to a category whose hom-sets may not be small sets.

Yoneda lemma For any category C, we use the notation C^* to denote the functor category $[C, \mathsf{Set}]$. If C and $\mathcal{D}$ are any two categories, by Equation (1.31), we have the following isomorphisms:

$$[C, \mathcal{D}^*] \;\cong\; (C \times \mathcal{D})^* \;\cong\; [\mathcal{D}, C^*]. \tag{1.35}$$

In particular, setting $\mathcal{D} = C^{op}$ it follows from Equation (1.35) that there are unique functors (representations)

$$\mathsf{H}_C : C^{op} \to C^* \quad \text{and} \quad \mathsf{H}^C : C \to (C^{op})^* \tag{1.36}$$

that corresponds to the bifunctor $C(-,-)$ under the isomorphisms given in Equation 1.35 (see Equations 1.33a, 1.33b, and 1.34). It follows that $\mathsf{H}_C : C^{op} \to C^*$ is a unique contravariant representation of C by covariant set-valued functors on C. Similarly $\mathsf{H}^C : C \to (C^{op})^*$ is a unique covariant representation of C by contravariant set-valued functors on C.

representations
H_C, H^C*:contra, co-variant representations*
representation!contravariant –
representation!covariant –
natural transformation
Yoneda, N.
Yoneda lemma
natural
E_C*:evaluation functor*
functor!evaluation –
$\mathfrak{Y}_C$*:Yoneda functor*
Y:Yoneda equivalence

Let $F \in C^*$ and $u \in F(c)$ with $c \in \mathfrak{v}C$. It is easy to see that for each $c' \in \mathfrak{v}C$ and $f \in C(c, c')$,

$$\zeta^u_{c'}(f) = F(f)(u) \tag{1.37}$$

defines a map $\zeta^u_{c'} : C(c, c') \to F(c')$ such that the assignment $c' \mapsto \zeta^u_{c'}$ is a natural transformation ζ^u of $C(c, -)$ to F. Every element of $\mathrm{Nat}(C(c, -), F)$ is of this form. This leads to the following well-known result, due to N. Yoneda Yoneda [1954], which we shall need in the sequel (see also MacLane [1971], pp 59–62).

Theorem 1.4 (Yoneda Lemma). *Let C be a category, $c \in \mathfrak{v}C$ and $F \in \mathfrak{v}C^*$. Then the map*

$$Y_{c,F} : u \mapsto \zeta^u$$

is a bijection of $F(c)$ onto $\mathrm{Nat}(C(c, -), F)$ *which is* natural *in c and F.* □

The last statement that $Y_{c,F}$ is *natural* in c and F may be explained as follows. Let E_C be defined on objects and morphisms of the category $C \times C^*$ as follows:

$$\mathsf{E}_C(c, F) = F(c), \quad \mathsf{E}_C(f, \eta) = F(f)\eta_{c'} = \eta_c G(f) \tag{1.38}$$

where $f \in C(c, c')$ and $\eta \in \mathrm{Nat}(F, G)$. The equality $F(f)\eta_{c'} = \eta_c G(f)$ follows from the fact that η is a natural transformation. It is easy to see that E_C is a set-valued bifunctor on $C \times C^*$ and is called the *evaluation functor*. Similarly, $\mathfrak{Y}_C$ defined on objects and morphisms of $C \times C^*$ to **Set** by

$$\mathfrak{Y}_C(c, F) = \mathrm{Nat}(\mathsf{H}_C(c), F), \quad \mathfrak{Y}_C(f, \eta) = C^*(\mathsf{H}_C(f), \eta) \tag{1.39}$$

is a bifunctor. Here H_C denotes the functor from C^{op} to C^* satisfying Equation (1.36) and $C^*(\mathsf{H}_C C(f), \eta)$ is the function defined by Equation (1.30). Yoneda lemma is equivalent to the following:

Corollary 1.5. *The assignment*

$$Y : (c, F) \mapsto Y_{c,F}$$

is a natural isomorphism $Y : \mathsf{E}_C \to \mathfrak{Y}C$. □

representation
Yoneda representations
embedding!contravariant Yoneda –
embedding!covariant Yoneda –
universal! – arrow
universal
universal! – element
representable
representing object

Another consequence of Yoneda lemma is that it gives some useful representations, called *Yoneda representations*. In fact, the functors $\mathsf{H}_{\mathcal{C}}$ and $\mathsf{H}^{\mathcal{C}}$ are embedding of categories; $\mathsf{H}_{\mathcal{C}}$ is called the *contravariant Yoneda representation (or embedding)* and $\mathsf{H}^{\mathcal{C}}$ is called the *covariant Yoneda representation (or embedding)*.

1.2.3 Universal arrows, representable functors and limits

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Recall that a *universal arrow* from $d \in \mathfrak{v}\mathcal{D}$ to the functor F is a pair (c, g) where $c \in \mathfrak{v}\mathcal{C}$ and $g \in \mathcal{D}(d, F(c))$ such that given any pair (c', g') with $g' \in \mathcal{D}(d, F(c'))$, $c' \in \mathfrak{v}\mathcal{C}$, there is a unique $f \in \mathcal{C}(c, c')$ such that $g' = g \circ F(f)$ (cf. MacLane [1971], p 55). In this case, we say that the morphism g is *universal* from d to F. A universal arrow from F to d is defined dually.

The following are standard examples of universal arrows.

Example 1.5: Let $U : \mathbf{Grp} \to \mathbf{Set}$ be the forgetful functor from the category **Grp** of groups to **Set**. Let $F(X)$ be the free group on the set X [see Hungerford, 1974, page. 65] for definition of free groups). Let $j_X : X \to U(F(X))$ be the natural insertion of generators in $F(X)$. Then the pair $(F(X), j_X)$ is a universal arrow from X to U. Also there is a functor $F : \mathbf{Set} \to \mathbf{Grp}$ sending each set X to the free group $F(X)$ generated by it. If G is any group, then there is a unique quotient homomorphism $q_G : F(U(G)) \to G$ where $F(U(G))$ is the free group generated by the set $U(G)$ of G. The pair $(F(U(G)), q_G)$ is a universal arrow from the functor to G.

The remainder of this section deals with some applications of this concept which we shall find useful later.

Universal elements Let $F \in \mathcal{C}^*$ and let (c, g) be a universal arrow from a one point set $*$ to F. Then the map $g : * \to F(c)$ is uniquely determined by the element $x = g(*)$. In this case the pair (c, x) (or, the element x alone, if the object c is clear from the context) is called a *universal element* for F. Note that $x \in F(c)$ is a universal element for F if and only if for every $c' \in \mathfrak{v}\mathcal{C}$ and $y \in F(c')$, there is a unique $f : c \to c'$ such that $F(f)(x) = y$. It is easy to see that the natural transformation ζ^x defined by Equation (1.37) is a natural isomorphism if and only if the element $x \in F(c)$ is a universal element for F. By Yoneda lemma every natural isomorphism of F with a covariant hom-functor $\mathcal{C}(c, -)$ is obtained in this manner.

Representable functors A functor $F \in \mathcal{C}^*$ is said to be *representable* if F is naturally isomorphic to some $\mathcal{C}(c, -)$; in this case the object $c \in \mathfrak{v}\mathcal{C}$ is called a *representing object* for F. Remarks above imply that c is a representing object for F if and only if $F(c)$ contains a universal element for F. In particular, F is representable if and only if F has a universal element.

Limits Let C and $\mathcal{D}$ be two categories and let $d \in \mathfrak{v}\mathcal{D}$. In the following we denote by Δ_d the *constant functor* from C to $\mathcal{D}$ with value d; that is, the functor which sends every object of C to d and every morphism to 1_d. By a *cone* we mean a natural transformation σ belonging to either $\mathrm{Nat}[F, \Delta_d]$ or $\mathrm{Nat}[\Delta_d, F]$ where $F : C \to \mathcal{D}$ is a functor. If $\sigma \in \mathrm{Nat}[F, \Delta_d]$ then it is called a *cone from the base F to the vertex d*. Clearly $\sigma : c \mapsto \sigma_c$ is a function from $\mathfrak{v}C$ to $\mathcal{D}$ such that for any $f : c \to c' \in C$, the following diagram commutes:

functor!constant –
Δ_d:constant functor with value d
cone
$\sigma : F \xrightarrow{n} d$:cone from base F to vertex d
$\sigma : coneC \xrightarrow{n} d$
$\Delta : \mathcal{D} \to [C, \mathcal{D}]$
cone! – to F from d
$\eta : d \xrightarrow{n} F$:cone to the base F from vertex d
universal! – cone
limit!direct –
$\varinjlim F$:direct limit of F
cone!limiting –
limit
limit!inverse –
$\varprojlim F$:inverse limit of F

$$\begin{array}{ccc} F(c) & \xrightarrow{F(f)} & F(c') \\ & {\scriptstyle\sigma_c}\searrow \quad \swarrow{\scriptstyle\sigma_{c'}} & \\ & d & \end{array} \tag{1.40}$$

We shall write $\sigma : F \xrightarrow{n} d$ to mean that σ is a cone from the base F to d. In particular, if F is the inclusion functor of C in $\mathcal{D}$, we shall say that σ is a cone from the base C to d; in this case we write $\sigma : C \xrightarrow{n} d$. If $F = \Delta_{d'}$, another constant functor, them any $\sigma : F \xrightarrow{n} \Delta_d$ is a constant mapping of $\mathfrak{v}C$ to $\mathcal{D}(d', d)$ which may be represented as Δ_g where $g = \sigma(c)$ for any $c \in \mathfrak{v}C$. Moreover, the assignments

$$d \mapsto \Delta_d \quad \text{and} \quad g \mapsto \Delta_g \tag{1.41}$$

is a functor $\Delta : \mathcal{D} \to [C, \mathcal{D}]$

Dually if $\sigma \in \mathrm{Nat}[\Delta_d, F]$ then it is called a *cone to the base F from the vertex d* (see MacLane [1971], pp 62–71). In this case, we write $\eta : d \xrightarrow{n} F$ to indicate this natural transformation.

A cone $\sigma : F \xrightarrow{n} d$ is a *universal cone* if for each cone $\tau : F \xrightarrow{n} d'$ there is a unique $g : d \to d'$ such that the following diagram commutes for every $c \in \mathfrak{v}C$:

$$\begin{array}{ccc} F(c) & \xrightarrow{\sigma_c} & d \\ & {\scriptstyle\tau_c}\searrow & \downarrow{\scriptstyle g} \\ & & d' \end{array} \tag{1.42}$$

A cone $\sigma : F \xrightarrow{n} d$ is universal if and only if the natural transformation $\sigma : F \xrightarrow{n} \Delta_d$ is a universal arrow from F to the functor Δ in the sense defined earlier in this section. The *direct limit (or inductive limit or colimit)* of F is a pair (d, σ) where $d \in \mathfrak{v}\mathcal{D}$ and $\sigma : F \xrightarrow{n} d$ is a universal cone (see MacLane [1971] pp 67–68). In this case we write

$$d = \varinjlim F$$

and σ is called the *limiting cone*. Dually the *limit* (or *inverse limit or projective limit*) of F is a pair $(\varprojlim F, \tau)$ where $\varprojlim F \in \mathfrak{v}\mathcal{D}$ and $\tau : \varprojlim F \xrightarrow{n} F$ is a universal cone to F from $\varprojlim F$.

pushout square
pullback square
category!complete –
category!cocomplete –
relation!equivalence –

We end this section with some useful examples of limits.

Example 1.6 (Pushout square): A *pushout square* of a pair $\langle f, g\rangle$ of morphisms in a category C (with common domain) is a commutative square on the left below such that whenever a commutative square, such as the one on the right below is given, there is a unique isomorphism $t : b \coprod_a c \to s$ such that $w = ut$ and $z = vt$.

$$\begin{array}{ccc} a & \xrightarrow{f} & c \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle u} \\ b & \xrightarrow[v]{} & b \coprod_a c \end{array} \qquad \begin{array}{ccc} a & \xrightarrow{f} & c \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle w} \\ b & \xrightarrow[z]{} & s \end{array} \tag{1.43}$$

A push out square can be interpreted as a direct limit of a functor from the category $\cdot \leftarrow \cdot \rightarrow \cdot$ to C. Observe that it is a particular case of the *fibered sum* or *coproduct* over a, the common domain of f and g (see 2.23). ([see MacLane, 1971, Page 66]).

Example 1.7 (Pullbacks): A *pullback square* of a pair $\langle f, g\rangle$ of morphisms in a category C (wth common codomin) is a commutative square on the left below such that whenever a commutative square such as the one on the right below is given, there is a unique morphism $t : s \to a \times_c b$ such that $w = ut$ and $z = vt$.

$$\begin{array}{ccc} a & \xrightarrow{f} & c \\ {\scriptstyle u}\uparrow & & \uparrow{\scriptstyle g} \\ a \times_c b & \xrightarrow[v]{} & b \end{array} \qquad \begin{array}{ccc} a & \xrightarrow{f} & c \\ {\scriptstyle w}\uparrow & & \uparrow{\scriptstyle g} \\ s & \xrightarrow[z]{} & b \end{array} \tag{1.44}$$

Show that a push out square can be interpreted as a limit of a functor from the category $\cdot \rightarrow \cdot \leftarrow \cdot$ to C. Moreover it is a particular case of the *fibered product* or *product* over $c = \operatorname{cod} f = \operatorname{cod} g$. A pullback square is the dual of a pushout square. ([see MacLane, 1971, Page 71]).

Example 1.8: It is well-known that if C is a small category then for any functor $F : C \to$ **Set**, both $\varinjlim F$ and $\varprojlim F$ exists, since the category **Set** is *complete* and *cocomplete* (see MacLane [1971], pp 105–108). In fact, let X denote the disjoint union of sets $\{F(c) : c \in \mathfrak{v}C\}$ and let ρ denote the smallest equivalence relation containing the relation

$$\{(x, y) \in X \times X : F(f)(x) = y \quad \text{for some} \quad f \in C\}.$$

Also, let $\rho^\natural : X \to X/\rho$ denote the quotient map. Then it can be checked that

$$\varinjlim F = X/\rho$$

and the map $c \mapsto \rho^\natural|F(c)$ gives the limiting cone. The inverse limit of F can be constructed in a similar fashion (see MacLane [1971], Theorem1, p 106).

1.2.4 Adjoints and equivalence of categories

It is clear that given any functor $F : C \to \mathcal{D}$, the assignments

$$(c, d) \mapsto (F(c), d); \qquad (f, g) \mapsto (F(f), g)$$

is a bifunctor $F \times 1_{\mathcal{D}} : \mathcal{C} \times \mathcal{D} \to \mathcal{D} \times \mathcal{D}$. Hence the composite

$$\mathcal{D}(F(-), -) = (F \times 1_{\mathcal{D}}) \circ \mathcal{D}(-, -) : \mathcal{C} \times \mathcal{D} \to \mathsf{Set}$$

is a set-valued bifunctor which is contravariant in the first variable. Here $\mathcal{D}(-, -)$ denote the hom-functor of $\mathcal{D}$. Similarly,

$$\mathcal{D}(-, F(-)) = (1_{\mathcal{D}} \times F) \circ \mathcal{D}(-, -) : \mathcal{D} \times \mathcal{C} \to \mathsf{Set}$$

is a set-valued bifunctor which is also contravariant in the first variable.

Let $\eta \in \mathrm{Nat}[F, G]$ be a natural transformation where $F, G : \mathcal{C} \to \mathcal{D}$. If $H : \mathcal{D} \to \mathcal{X}$ and $K : \mathcal{A} \to \mathcal{C}$ are functors, it is easy to verify that the mappings

$$c \mapsto H(\eta_c) \quad \text{and} \quad a \mapsto \eta_{K(a)}$$

are natural transformations. We denote these by

$$\eta H : F \circ H \xrightarrow{n} G \circ H \quad \text{and} \quad K\eta : K \circ F \xrightarrow{n} K \circ G.$$

We use these notations in the statement below. See MacLane [1971]; page 81, Theorem 2 for a proof.

THEOREM 1.6. *The following statements are equivalent for functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$:*

(i) There exists a natural isomorphism

$$\phi : \mathcal{C}(-, G(-)) \xrightarrow{n} \mathcal{D}(F(-), -).$$

(ii) There exists a natural transformation $\eta : 1_{\mathcal{C}} \xrightarrow{n} F \circ G$ such that for each $c \in \mathfrak{v}\mathcal{C}$, η_c is a universal arrow from c to G.

(iii) There exists a natural transformation $\sigma : G \circ F \xrightarrow{n} 1_{\mathcal{D}}$ such that for each $d \in \mathfrak{v}\mathcal{D}$, σ_d is a universal arrow to d from F.

(iv) There exist natural transformations

$$\eta : 1_{\mathcal{C}} \xrightarrow{n} F \circ G \quad \textit{and} \quad \sigma : G \circ F \xrightarrow{n} 1_{\mathcal{D}}$$

such that

$$(\eta F)_c \circ (F\sigma)_c = 1_c \quad \textit{and} \quad (G\eta)_d \circ (\sigma G)_d = 1_d$$

for all $c \in \mathfrak{v}\mathcal{C}$ and $d \in \mathfrak{v}\mathcal{D}$.

adjoint
adjoint!left –
adjoint!right –
adjunction
unit
counit
$< F, G, \eta, \sigma >: \mathcal{C} \overset{\cdot}{\rightharpoonup} \mathcal{D}$:adjunction from $\mathcal{C}$ to $\mathcal{D}$
category! – equivalence
$< F, G; \eta, \nu >: \mathcal{C} \overset{\cdot}{\rightleftharpoons} \mathcal{D}$:A category equivalence
adjoint!– equivalence
adjoint! – inverse
category!reflective subcategory
reflector
monomorphism
cancelable
cancelable!right –
monomorphism!split –

Moreover, given $F : \mathcal{C} \to \mathcal{D}$ there exists $G : \mathcal{D} \to \mathcal{C}$ satisfying the equivalent conditions (i) – (iv) if and only if for each $d \in \mathfrak{v}\mathcal{D}$ there is a unique $G_0(d) \in \mathfrak{v}\mathcal{C}$ and $\sigma_d : F(G_0(d)) \to d$ which is universal from F to d. Dually, given $G : \mathcal{D} \to \mathcal{C}$ there exists $F : \mathcal{C} \to \mathcal{D}$ satisfying the equivalent conditions (i) – (iv) if and only if for each $c \in \mathfrak{v}\mathcal{C}$ there is a unique $F_0(c) \in \mathfrak{v}\mathcal{D}$ and $\eta_c : c \to G(F_0(c))$ which is universal to G from c. □

Given a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, we shall say that F is a *left adjoint* of G and G is the *right adjoint* of F if the pair (F, G) satisfies the equivalent conditions of the theorem above. The natural isomorphism ϕ of statement (i) above is often referred to as the *adjunction* between F and G. Also the natural transformation η of (ii) (or (iv)) is called the *unit* and the natural transformation σ of (iii) (or (iv)), is called the *counit* of the adjunction. By the statements (i) and (iv) above, the triple $< F, G, \phi >$ or the quadruple $< F, G, \eta, \sigma >$ completely determine the adjunction. We shall use the notation $< F, G, \eta, \sigma >: \mathcal{C} \overset{\cdot}{\rightharpoonup} \mathcal{D}$ for an adjunction from $\mathcal{C}$ to $\mathcal{D}$ where $F : \mathcal{C} \to \mathcal{D}$ is a left adjoint of $G : \mathcal{D} \to \mathcal{C}$, $\eta : 1_{\mathcal{C}} \overset{n}{\to} F \circ G$ is the unit and $\sigma : G \circ F \overset{n}{\to} 1_{\mathcal{D}}$ is the counit of the adjunction. Note that any two left adjoints [right adjoints] of G are naturally equivalent (see MacLane [1971], page 83).

We say that two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if if there exist functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ and such $\eta : 1_{\mathcal{C}} \to\overset{n}{\to} F \circ G$ and $\nu : 1_{\mathcal{D}} \overset{n}{\to} G \circ F$. We write $< F, G; \eta, \nu >: \mathcal{C} \rightleftharpoons \mathcal{D}$ for an equivalence between catwegories $\mathcal{C}$ and $\mathcal{D}$. In this case both $< F, G, \eta, \nu^{-1} >: \mathcal{C} \overset{\cdot}{\rightharpoonup} \mathcal{D}$ and $< G, F, \nu, \eta^{-1} >: \mathcal{D} \overset{\cdot}{\rightharpoonup} \mathcal{C}$ are adjunctions so that F is both left and right adjoint of G. An adjunction arising in this way from an equivalence is called an *adjoint equivalence*. If $< F, G; \eta, nu >: \mathcal{C} \overset{\cdot}{\rightleftharpoons} \mathcal{D}$ is an equivalence of categories $\mathcal{C}$ and $\mathcal{D}$, G is called the *adjoint inverse* of F (and F is the adjoint inverse of G). Note that if $F : \mathcal{C} \to \mathcal{D}$ is a category isomorphism with inverse G (so that $F \circ G = 1_{\mathcal{C}}$ and $G \circ F = 1_{\mathcal{D}}$), then $< F, G, 1_{\mathcal{C}}, 1_{\mathcal{D}} >: \mathcal{C} \overset{\cdot}{\rightharpoonup} \mathcal{D}$ is an adjoint equivalence. Therefore an inverse is, in particular, an adjoint inverse; but the converse is not true.

Let $\mathcal{D}$ be a subcategory of $\mathcal{C}$ and let $K : \mathcal{D} \to \mathcal{C}$ be the inclusion functor. If K has a left adjoint F, then $\mathcal{D}$ is called a *reflective* subcategory of $\mathcal{C}$ and F is called a *reflector* of $\mathcal{C}$ on $\mathcal{D}$.

1.2.5 Monomorphisms and epimorphisms

Monomorphisms Recall that a morphism f in a category $\mathcal{C}$ is a *monomorphism* if a

$$gf = hf \Rightarrow g = h \quad \text{for all} \quad g, h \in \mathcal{C}; \tag{1.45a}$$

that is, f is a monomorphism if it is *right cancelable*. A morphism $f \in \mathcal{C}(c, c')$ is called a *split monomorphism* if there exists a morphism $g \in \mathcal{C}(c', c)$ with $fg = 1_c$

inverse!right –
$\mathbb{M}C$:subcategory of monomorphisms
$f \le g$:
relation!quasi-order –
$f \sim g$:equivalent monomorphisms
monomorphism!equivalent monomorphisms

in which case g is called a *right inverse* of f. In this case, if $h, k \in C$ with $hf = kf$ then

$$h = h(fg) = (hf)g = (kf)g = k.$$

Thus a split monomorphism is a monomorphism; but not all monomorphisms are split.

Let $\mathbb{M}C$ denote the class of all monomorphisms in C. For any $c \in C$, $1_c \in \mathbb{M}C$ and $fg \in \mathbb{M}C$ for all $f, g \in \mathbb{M}C$. These imply that $\mathbb{M}C$ is a subcategory of C with $\mathfrak{v}\mathbb{M}C = \mathfrak{v}C$. It is useful to note that the subcategory $\mathbb{M}C$ has the following property:

$$f, g \in C \quad \text{and} \quad fg \in \mathbb{M}C \Rightarrow f \in \mathbb{M}C. \tag{1.45b}$$

On $\mathbb{M}C$ define the relation

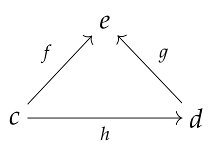
$$f \le g \quad \Longleftrightarrow \quad f = hg \quad \text{for some} \quad h \in C. \tag{1.45c}$$

Clearly if $f \le g$ then f and g have the same codomain and by Equation (1.45b), the morphism h such that $f = hg$ is also a monomorphism. Also the $\le$ is a *quasi-order* (that is satisfies (R1) and (R2) of Definition 1.1; see § Subsection 1.1.2) and so

$$\sim = \le \cap \le^{-1} \tag{1.45d}$$

is an equivalence relation on $\mathbb{M}C$. We have the following characterization of $\sim$:

Proposition 1.7. *For $f, g \in \mathbb{M}C$, $f \sim g$ if and only if there there is an isomorphism h such that the following diagram commutes:*



Proof. If an isomorphism h exists making the diagram commute, then $f = hg$ and so $f \le g$. Then $g = h^{-1}f$ and so, $g \le f$. Therefore $f \sim g$. Conversely if $f \sim g$, and if $h, k \in C$ with $f = hg$ and $g = kf$, then $1_c f = f = hkf$ where $c = \operatorname{dom} f$ and since f is a monomorphism, we have $hk = 1_c$. Similarly, $kh = 1_d$ where $d = \operatorname{dom} g$. Hence h is an isomorphism making the diagram above commute and $k = h^{-1}$. □

Two monomorphisms f and g are said to be *equivalent* if $f \sim g$ (see MacLane [1971], p 122).

epimorphism
cancelable!left –
epimorphism!split –
inverse!left –
$\mathbb{E}C$:
epimorphism!equivalent epimorphisms
morphism!balanced –

Epimorphisms Dually, $f \in C(c, c')$ is called an *epimorphism* if f satisfies the following: a

$$fg = fh \Rightarrow g = h \quad \text{for all} \quad g, h \in C; \tag{1.45a*}$$

so that f is *left cancelable*. f is called a *split epimorphism* if there is $g \in C(c', c)$ such that $gf = 1_{c'}$. As before a split epimorphism is an epimorphism and f is a split epimorphism if and only if its left inverse is a split monomorphism.

Definitions dual to that of $\mathbb{M}C$ give a subcategory $\mathbb{E}C$ of all epimorphisms in C satisfying the property:

$$f, g \in C \quad \text{and} \quad fg \in \mathbb{E}C \Rightarrow g \in \mathbb{E}C. \tag{1.45b*}$$

Moreover, dual of Equations 1.45c and 1.45d gives a quasi-order and an equivalence relation on $\mathbb{E}C$; since there is no possibility of confusion we shall use the same notations $\leq$ and $\sim$ to denote these relations on $\mathbb{E}C$ as well. Dual of Proposition 1.7 also hold for this relation on $\mathbb{E}C$. Two epimorphisms related by $\sim$ are said to be equivalent.

Balanced morphisms A morphism f is a *balanced* if it is both a monomorphism and an epimorphism. Clearly, an isomorphism is a balanced morphism; but there exist balanced morphisms that are not isomorphisms. The following observation will be of use later:

PROPOSITION 1.8. *A balanced morphism which is a split monomorphism or a split epimorphism is an isomorphism.*

Proof. . Suppose that $f : c \to d$ is a balanced morphism with right inverse $g : d \to c$. Then $fg = 1_c$ and

$$f(gf) = (fg)f = f = f1_d.$$

Since f is an epimorphism, we have $gf = 1_d$ which implies that f is an isomorphism with $f^{-1} = g$. If f is a split epimorphism, we can similarly (dually) see that f is an isomorphism. □

Example 1.9: Let $D \subseteq X$ be a proper dense subspace of a topological space X. Then the inclusion mapping $j : D \subseteq X$ is a balanced morphism in the category Top of topological spaces which is clearly not an isomorphism.

1.3 SMALL CATEGORIES

Recall that a category C is small if its morphism class C (or equivalently $\mathfrak{v}C$) is a set (see Subsection 1.2.1). We have noted that small categories can be considered as partial algebras and functors between small categories as partial algebra homomorphisms which preserve identities. We therefore have a category **Cat** in which objects are small categories and morphisms are functors. Recall from § Subsection 1.2.1 also that, for any morphism u in small category C, we use the notations e_u and f_u for identities corresponding to $\operatorname{dom} u$ and $\operatorname{cod} u$ respectively. *category!concrete –*

In this section we give some definitions and results, mainly relevant for small categories needed in the sequel. Note that some of these definitions are valid for arbitrary categories also.

1.3.1 Concrete categories and preorders

We shall say that a category C (not necessarily small) is *concrete* if there exists a faithful functor $U : C \to \mathsf{Set}$. If C is concrete, we may assume that there is a faithful functor $V : C \to \mathsf{Set}$ which is injective on objects. For, if $U : C \to \mathsf{Set}$ is faithful, define $V : C \to \mathsf{Set}$ by

$$V(c) = \{(x, c) : x \in U(c)\}$$

and for $f \in C(c, d)$,$x \in U(c)$, let

$$V(f)(x, c) = (U(f)(x), d).$$

Then V is faithful Set-valued functor on C which is injective on $\mathfrak{v}C$. In this case, the image $\operatorname{Im} V$ of C in Set is a subcategory of Set and V is an isomorphism of C onto $\operatorname{Im} V$. Therefore, with out loss of generality, any (small) concrete category C can be regarded as a category of sets; that is, objects in C are sets and morphisms are functions. However, such representation of C is not unique.

PROPOSITION 1.9. *Let C be a small category. Then there exists a faithful functor $\mathbf{U}_C : C \to \mathsf{Set}$ which is injective on $\mathfrak{v}C$. Hence C is isomorphic to a category of sets; in particular, C is concrete.*

Proof. We construct a functor $U = \mathbf{U}_C : C \to \mathsf{Set}$ as follows. For each $c \in \mathfrak{v}C$ define

$$U(c) = \{g \in C : \operatorname{cod} g = c\} \tag{1.47}$$

and for $f : c \to c' \in C$, define $U(f) : U(c) \to U(c')$ by

$$U(f)(g) = gf. \tag{1.48}$$

preorder

Since C is small, $U(c)$ is a set for all $c \in \mathfrak{v}C$. Moreover, since $1_c \in U(c)$, $U(c) \neq \emptyset$. For $f : c \to c'$, $U(f)$ is clearly a map of $U(c)$ to $U(c')$ and it is easy to verify that $U; C \to \mathsf{Set}$ is a functor. Now for $c \neq c'$,

$$U(c) \cap U(c') = \emptyset$$

and so U is injective on objects of C. Also, if $U(f) = U(h)$ for $f, h \in C(c, c')$, then $gf = gh$ for all $g \in U(c)$ and so

$$f = 1_c f = 1_c h = h.$$

Thus $U : C \to \mathsf{Set}$ is faithful. Consequently, $U(C) = \operatorname{Im} U$ is a subcategory of Set and $\mathbf{U}_C : C \to U(C)$ is an isomorphism. □

***Remark* 1.4:** Proposition 1.9 has the following consequence. Let Scat denote the category of all small subcategories of Set. Then Scat is a full subcategory of Cat. For $C \in \mathfrak{v}\mathsf{Cat}$, let $\mathbf{U}_C : C \to U(C)$ be the isomorphism constructed in the Proposition 1.9 above. It is easy to verify that $\mathbf{U}_C$ is a universal arrow from C to the inclusion functor $J : \mathsf{Scat} \to \mathsf{Cat}$ (see § Subsection 1.2.3). Hence by Theorem 1.6, J has a left adjoint. Therefore Scat is a reflective subcategory of Cat (see § Subsection 1.2.4). In fact, the construction $C \mapsto \operatorname{Im} \mathbf{U}_C$ can be naturally extended to a functor $\mathbf{U} : \mathsf{Cat} \to \mathsf{Scat}$ which is the reflector of Cat in Scat.

Preorders A category $\mathbf{P}$ is called a *preorder* if the hom-set $\mathbf{P}(p, q)$ contains at most one morphism for all $p, q \in \mathfrak{v}\mathbf{P}$. If $\mathbf{P}$ is a preorder and if $P = \mathfrak{v}\mathbf{P}$, the relation

$$\rho(P) = \{(p, q) \in P \times P : \mathbf{P}(p, q) \neq \emptyset\} \tag{1.49}$$

is a quasiorder (reflexive and transitive relation) on the class P. In particular, a small preorder is a quasiordered set. Conversely, if ρ is any quasiorder relation on a class X, then ρ may be considered as the morphism set of a preorder with vertex class X; composition in ρ is defined as follows: for all $(p, q), (r, s) \in \rho$,

$$(p, q)(r, s) = \begin{cases} (p, s) & \text{if } q = r; \\ \text{undefined} & \text{if } q \neq r. \end{cases} \tag{1.50}$$

Thus the quasiordered class (X, ρ) becomes a preorder with morphism class ρ and vertex class X. Note that if $\mathbf{P} = (X, \rho)$, then the relation $\rho(P)$ defined by Equation (1.49) coincides with ρ. We may therefore use the same notation to denote a preorder and the associated quasiordered class. Also, a mapping f of the vertex class of the preorder $\mathbf{P}$ to the vertex class of $\mathbf{Q}$ determines a unique functor of $\mathbf{P}$ to $\mathbf{Q}$ if and only if f is an order preserving mapping of the associated quasiordered classes; as above we shall use the same notation to denote functors of preorders as well as order preserving maps of the associated quasiordered classes.

A preorder **P** is said to be *strict* if the associated relation $\rho(P)$ defined by Equation (1.49) is antisymmetric; this is equivalent to the fact that quasiordered class is a partially ordered class.

preorder!strict – subobject
Krishnan, E.
Nambooripad, K. S. S.
subobject!choice of subobjects
category!– with subobjects
subobject!– relation

1.3.2 Categories with subobjects

Here we introduce the important preliminary notion of *subobject relations* in categories. Most of the familiar categories such as **Set**, **Grp**, **Top**, etc., are naturally endowed with subobject relations (the relation induced by the usual set inclusion). Moreover, morphisms in these categories satisfy a factorization property which enables us to identify image of a morphism with a universal subobject of its codomain. Here we shall be concerned mostly with small categories even though most of the definitions may apply for arbitrary categories.

Subobject relations According to the usual definition, *subobjects* in a category, are certain equivalence classes of monomorphisms (see MacLane [1971], page 122). While this is quite adequate in algebraic categories such as **Set**, **Grp**, $\mathbf{Vct}_{\Bbbk}$, etc., the natural subobject relation in categories such as **Top** (category of topological spaces), **Tvs** (category of topological vector spaces), etc., indicate embeddings rather than monomorphisms. We shall therefore give a new definition of subobject relation to take this distinction into account (see also Krishnan [1990], Krishnan and Nambooripad [1993]).

Definition 1.5. Let $\mathcal{C}$ be a category. A *choice of subobjects* in $\mathcal{C}$ is a subcategory $\mathbf{P} \subseteq \mathcal{C}$ satisfying the following:

(a) **P** is a strict preorder with $\mathfrak{v}\mathbf{P} = \mathfrak{v}\mathcal{C}$.

(b) Every $f \in \mathbf{P}$ is a monomorphism in $\mathcal{C}$.

(c) If $f, g \in \mathbf{P}$ and if $f = hg$ for some $h \in \mathcal{C}$, then $h \in \mathbf{P}$. par

If **P** is a choice of subobjects in $\mathcal{C}$, the pair $(\mathcal{C}, \mathbf{P})$ is called a *category with subobjects*.

In the following, to simplify the notation, we shall denote by $\mathcal{C}$, $\mathcal{D}$, etc., categories with subobjects. If **P** is the choice of subobjects in $\mathcal{C}$, then by axiom (a), **P** induces a partial order $\rho(P)$ on $\mathfrak{v}\mathcal{C}$ (see Equation (1.49)) and this partially order completely determine the preorder **P**. When $\mathcal{C}$ has subobjects, unless explicitly stated otherwise, $\mathfrak{v}\mathcal{C}$ will denote the choice of subobjects in $\mathcal{C}$. Also, in this case, the partial order defined by Equation (1.49) will be called the preorder of *inclusions* or *subobject relation* in $\mathcal{C}$ and will be denoted

$c \subseteq d$:c is a subobject of d
subobject
$\jmath_c^d$:inclusion of c in d
monomorphism!embedding
inclusion:split –
retraction
functor!inclusion preserving –
isomorphism!– of categories with subobjects

by $\subseteq$; as usual the statement $(c, d) \in \subseteq$ is written as $\jmath_c^d : c \subseteq d$ (or $c \subseteq d$ for short) where $\jmath_c^d$ denotes the unique morphism in $\mathfrak{v}C$ from c to d. When $c \subseteq d$, we say that c is a *subobject* of d; the morphism $\jmath_c^d$ is called the inclusion of c in d. Since we often identify vertices with identities, we shall continue to use these notations for identities also. Thus if e and f are identities in C the relation $e \subseteq f$ is synonymous with $\operatorname{dom} e \subseteq \operatorname{dom} f$ and the inclusion $\jmath_{\operatorname{dom} e}^{\operatorname{dom} f}$ is written also as $\jmath_e^f$. Any monomorphism f equivalent to an inclusion (with respect to the equivalence relation $\sim$ defined by Equation (1.45d)) is called an *embedding*. We say that an inclusion splits if it is split as a monomorphism (see § Subsection 1.2.5); thus $\jmath_c^d$ splits if there is a morphism $\epsilon : d \to c$ such that $\jmath_c^d \epsilon = 1_c$; in this case ϵ is called a *retraction*; a retraction is clearly a split epimorphism.

LEMMA 1.10. *Let C be a category with subobjects. Then*

1) *No two inclusions can be equivalent as monomorphisms.*
2) *If a split inclusion $\jmath_a^b$ is an epimorphism, then $a = b$ and $\jmath_a^b = 1_b$.*
3) *If a retraction $\epsilon : b \to a$ is a monomorphism, then $a = b$ and $\epsilon = 1_b$.*

Proof. 1) If inclusions $j = \jmath_a^c$ and $j' = \jmath_b^c$ are equivalent as monomorphisms, then by Equation (1.45d), there exist $p : a \to b$ and $q : b \to a$ such that $j = pj'$ and $j' = qj$. By axion (c), $p = \jmath_a^b$ and $q = \jmath_b^a$. Thus $a \subseteq b$ and $b \subseteq a$. Hence $a = b$ since the preorder of subobjects is strict. Therefore $p = q = 1_a$ and so $j = j'$.

2) Let $j = \jmath_a^b$ be a split inclusion with $j\epsilon = 1_a$. Then we have $j(\epsilon j) = j\epsilon j = j = j1_b$. If j is an epimorphism, we have $\epsilon j = 1_b$ and so, $j : a \to b$ is an isomorphism. Since $j = j1_b$, it follows that inclusions j and $1_b = \jmath_b^b$ are equivalent as monomorphisms and so, $a = b$ and $j = 1_b$ by 1).

3) Assume that the retraction $\epsilon : b \to a$ is a nonomorphism. If $j = \jmath_a^b$, then $(\epsilon j)\epsilon = \epsilon 1_a = \epsilon = 1_b\epsilon$ and so $\epsilon j = 1_b$. Hence ϵ is an isomorphism and j is its inverse. In particular, $j : a \to b$ is an isomorphism and by 2), $a = b$ and $j = 1_b$. Since ϵ is the inverse of j, we have $\epsilon = 1_b$. □

Let C and $\mathcal{D}$ be categories with subobjects. A functor $F : C \to \mathcal{D}$ is said to be *inclusion preserving* if $\mathfrak{v}F = F|\mathfrak{v}C$ is a functor of the preorder $\mathfrak{v}C$ to $\mathfrak{v}\mathcal{D}$; that is, for all $c, d \in \mathfrak{v}C$ with $c \subseteq d$, we have $F(c) \subseteq F(d)$. $F : C \to \mathcal{D}$ is an *isomorphism of categories with subobjects* if F is a category isomorphism such that $\mathfrak{v}F : \mathfrak{v}C \to \mathfrak{v}\mathcal{D}$ is an isomorphism of preorders. It is clear that, in this case the assignment

$$\mathfrak{v} : C \mapsto \mathfrak{v}C, \quad \text{and} \quad F \mapsto \mathfrak{v}F$$

is a functor of the category of small categories with subobjects to the category of small preorders.

Note that, if $\mathcal{D}$ is a subcategory of a category C with subobjects, then the class of all inclusions of C that belongs to $\mathcal{D}$ is a choice of subobjects for $\mathcal{D}$ and the inclusion functor $\mathcal{D} \subseteq C$ preserves subobject relation. However, for $d, d' \in \mathfrak{v}\mathcal{D}$, it is possible that $d \subseteq d'$ in C, but $\jmath_d^{d'} \notin \mathcal{D}$. We shall say that $\mathcal{D}$ is a *subcategory with subobject* if for $d, d' \in \mathfrak{v}\mathcal{D}$,

category!subcategory with subobject
functor!subfunctor

$$d \subseteq d' \text{ in } \mathcal{D} \iff d \subseteq d' \text{ in } C. \tag{1.51}$$

This is equivalent to requiring that $\mathfrak{v}\mathcal{D} \subseteq \mathfrak{v}C$ is a full embedding of preorders (that is, fully-faithful and injective on vertices).

A subobject relation on a category $\mathcal{D}$ may be extended to the functor category $[-, \mathcal{D}]$ of all $\mathcal{D}$-valued small functors (see § Subsection 1.2.2). If $F, G \in \mathfrak{v}[-, \mathcal{D}]$, we say F is a *subfunctor* of G, written $F \subseteq G$, if

$$\operatorname{dom} F = \operatorname{dom} G, \quad F(c) \subseteq G(c) \quad \text{for all } c \in \mathfrak{v} \operatorname{dom} F \text{ and the map } \jmath_F^G : c \mapsto \jmath_{F(c)}^{G(c)} \tag{1.52}$$

of $\mathfrak{v} \operatorname{dom} F$ to $\mathcal{D}$ is a natural transformation from F to G. It is easy to verify from the definition that

$$\mathbf{P} = \{\jmath_F^G : F, G \in \mathfrak{v}[-, \mathcal{D}], \quad F \subseteq G\}$$

is a strict preorder. Also, for each $F \subseteq G$, $\jmath_F^G$ is a monomorphism in $[-, \mathcal{D}]$. For let $s, t : H \to F$ are transformations in $[-, \mathcal{D}]$ such that $s \circ j = t \circ j$ where $j = \jmath_F^G$. Suppose that

$$\mathfrak{v}s = \alpha : \mathcal{A} \to C \quad \text{and} \quad \mathfrak{v}t = \beta : \mathcal{A} \to C.$$

Since $\mathfrak{v}j = 1_C$, we have

$$\alpha = \alpha \circ 1_C = \mathfrak{v}(s \circ j) = \mathfrak{v}(t \circ j) = \beta$$

and for any $a \in \mathfrak{v}\mathcal{A}$,

$$(s \circ j)_a = s_a \jmath_{F(\alpha(a))}^{G(\alpha(a))} = t_a \jmath_{F(\alpha(a))}^{G(\alpha(a))}.$$

Since $\jmath_{F(\alpha(a))}^{G(\alpha(a))}$ is a monomorphism in $\mathcal{D}$, $s_a = t_a$ for all $a \in \mathfrak{v}\mathcal{A}$. Hence $s = t$ which implies that $j = \jmath_F^G$ is a monomorphism in $[-, \mathcal{D}]$.

We have thus shown that $\mathbf{P}$ satisfies axioms (a) and (b) of definition Definition 1.5. To verify (c), let $f : F \subseteq G$ and $g : F \subseteq H$. If $h : G \xrightarrow{n} H$ is a transformation such that $f \circ h = g$, then $\operatorname{dom} G = \operatorname{dom} F = \operatorname{dom} H = C$ (say). Hence $h : G \xrightarrow{n} H$ is a natural transformation and

$$(f \circ h)_c = f_c h_c = g_c.$$

Since $f_c = \jmath_{F(c)}^{G(c)}$ and $g_c = \jmath_{F(c)}^{H_c}$ are inclusions in $\mathcal{D}$, by axion (c) of Definition 1.5, h_c is an inclusion in $\mathcal{D}$ for all $c \in \mathfrak{v}C$. Hence $h : G \subseteq H$. We have thus proved the following:

morphism!factorization
morphism!canonical factorization
category!– with factorization
category!– with unique factorization

Proposition 1.11. *Let $\mathcal{D}$ be a category with subobjects and let $[-,\mathcal{D}]$ denote the category of small functors. Define the relation $\subseteq$ on $\mathfrak{v}[-,\mathcal{D}]$ by Equation* (1.52)*. Then*

$$\mathbf{P} = \{\jmath_F^G : F, G \in \mathfrak{v}[-,\mathcal{D}], \quad F \subseteq G\}$$

is a choice of subobjects for $[-,\mathcal{D}]$. Moreover, if C is any small category, then the category $[C,\mathcal{D}]$ of all $\mathcal{D}$-valued functors on C is a subcategory of $[-,\mathcal{D}]$ with subobjects.

Example 1.10: In categories **Set**, **Grp**, $\mathbf{Vct}_{\Bbbk}$, etc., the relation on objects induced by the usual set-inclusion is a subobject relation in the sense of the definition above. Notice that in these categories, all monomorphisms are embeddings.

Example 1.11: Let C be a concrete category so that there is a faithful functor $U : C \to \mathbf{Set}$ which is injective on objects (see § Subsection 1.3.1). Let

$$\mathbf{P} = \{f \in C(c,d) : U(f) = U(c) \subseteq U(d)\}. \qquad (\bullet)$$

It is easy to verify that $\mathbf{P}$ is a choice of subobjects in C according to Definition 1.5 and with this subobject relation, $U : C \to \mathbf{Set}$ becomes an inclusion preserving functor. In this case, a morphism $f \in C$ is a monomorphism if $U(f)$ is injective; but the converse may not hold. In view of Proposition 1.9, this example also shows that every small category has at least one choice of subobjects.

Example 1.12: The category **Top** of all topological spaces and continuous maps is clearly a concrete category and so, the construction in the last example gives a choice of subobjects in **Top** consisting of all continuous inclusions. So, with this choice, subobjects of topological spaces will include spaces other than those with relative topology. For this category, the natural choice of subobjects is the collection of all inclusions that are homeomorphisms onto the range. This also shows that a category can have more than one choice of subobjects.

Example 1.13: Let $F : \mathbf{Set} \to \mathbf{Grp}$ be the functor given by the construction of free groups. It is clear that F is naturally inclusion preserving. Similarly many other familiar functors are inclusion preserving. On the other hand, functors that arises in the construction of fundamental groups or homology groups of topological spaces are not inclusion preserving.

Categories with factorization A morphism f in a category C with subobjects is said to have *factorization* if f can be expressed as $f = pm$ where p is an epimorphism and m is an embedding. The factorization of a morphism need not be unique. For if $f = pm$ is a factorization of f, then m is an embedding and so, $f \sim j$ for some inclusion j. Then by Equation 1.45d $m = uj$ where u is an isomorphism in C. But then $p' = pu$ is an epimorphism and $f = p'j$ is a factorization of f. This also shows that every morphism f with factorization has at least one factorization of the form $f = qj$ where q is epimorphism and j is an inclusion. Such factorizations are called *canonical* factorizations.

We shall say that C is a *category with factorization* if C has subobjects and if every morphism in C has factorization; the category has *unique factorization* property if every morphism in C has unique canonical factorization. If C

and $\mathcal{D}$ are categories with factorization, a functor $F : C \to \mathcal{D}$ is *factorization preserving* if whenever $f = xj$ is a canonical factorization of f in C, then $F(x)F(j)$ is a canonical of $F(f)$ in $\mathcal{D}$. Clearly if F is factorization preserving then F preserve inclusions and epimorphisms. The uniquiness of factorization is an important property. A sufficient condition for its existance is given in the following.

functor!factorization preserving
morphism!image of –

PROPOSITION 1.12. *Let C be a category with factorization property such that every inclusion in C splits. Then every morphism in C has unique canonical factorization.*

Proof. Let $f = xj = yj'$ be two canonical factorizations of $f \in C$. Since inclusions split, there exist $u, v \in C$ with $ju = 1_a$ and $j'v = 1_b$ where $a = \operatorname{dom} j$ and $b = \operatorname{dom} j'$. Then

$$yj'uj = xjuj = xj = yj'$$

and since y is an epimorphism, we have $(j'u)j = j'$. Similarly, $(jv)j' = j$. Hence j and j' are equivalent monomorphisms. Hence $j = j'$. Since $xj = yj$ and j is a monomorphism, $x = y$. □

Example 1.14: If $f : X \to Y$ is a mapping of sets and if $f(X) = \operatorname{Im} f$ then $f(X) \subseteq Y$ and we can write $f = f^\circ j^Y_{f(X)}$. Here f° denote the mapping of X onto $f(X)$ determined by f. Since surjective mappings are epimorphisms in **Set**, this gives a canonical factorization of f in **Set** which is clearly unique. Thus **Set** is a category with unique factorization. In a similar way it can be shown that categories such as **Grp**, $\mathbf{Vct}_{\Bbbk}$, etc., are also categories with unique factorization.

Example 1.15: Since surjective continuous mappings are epimorphisms in **Top**, it follows as in the last example that the category has factorization property. However, if Y is dense in X, $h = j^X_Y$ is an epimorphism in **Top** and $h = 1_Y j^X_Y = j^X_Y 1_X$. Then both $1_Y j^X_Y$ and $j^X_Y 1_X$ are canonical factorizations of h in **Top**. Thus **Top** does not have unique factorization property.

Images Here we introduce the concept of the *image* of a morphism in a category with factorization.

PROPOSITION 1.13. *Let C be a category with factorization. Suppose that the morphism $f \in C$ has the following property:*

(Im) *f has a canonical factorization $f = xj$ such that for any canonical factorization $f = yj'$ of f, there is an inclusion j'' with $y = xj''$.*

Then the factorization $f = xj$ is unique.

Proof. Suppose that $f \in C$ satisfies the given condition and that the factorization $f = xj$ has the property stated above. Hence if $f = yj'$ is any canonical factorization, then $y = xj''$ where $j'' : a = \operatorname{cod} x \subseteq \operatorname{cod} y = b$. Therefore if $f = yj'$ also has this property, then we have $a = b$ and so, $j'' \in C(a)$. Since the

f°:epimorphic component of f
j_f:inclusion of f
category!with images

inclusions form a preorder and $1_a \in C(a)$, we must have $j'' = 1_a$. Therefore $y = x$ so that $xj = xj'$. Since x is an epimorphism, we have $j = j'$. □

A morphism f in a category with factorization is said to have *image* if f satisfies the condition (Im) of the Proposition above. In this case the unique canonical factorization $f = xj$ with the property stated in (Im) is denoted by $f = f^\circ j_f$ where f° is called the *epimorphic component* of f and j_f is called the *inclusion* of f. The unique vertex

$$\operatorname{Im} f = \operatorname{cod} f^\circ = \operatorname{dom} j_f \tag{1.53}$$

is called the *image* of f.

Example 1.16: Since categories **Set**, **Grp**, etc., has unique factorization, morphisms in these categories have images by the observation above. Though the category **Top** does not have unique factorization, it can be seen that every morphism in **Top** also has image.

If $f \in C$ has image, we define the *direct image* of a subobject $a \subseteq \operatorname{dom} f$ by:

$$f(a) = \operatorname{Im}(f|a) \quad \text{where} \quad f|a = j_a^{\operatorname{Im} f} f. \tag{1.54}$$

Here $f|a = j_a^{\operatorname{Im} f} f$ is called the *restriction* of f to a. Clearly, $f|a$ is a morphism with domain a and codomain $\operatorname{cod} f$.

We say that C is a category *with images* if every morphism in C has image in the sense defined above. Note that if C has unique factorization, then C has images. Also, by Proposition 1.12 C has images if every inclusion in C splits.

Categories with unique factorization We have noted above that a category C with unique factorization has images. Hence for any $f \in C$ and $a \subseteq \operatorname{dom} f$, the direct image $f(a)$ is defined.

PROPOSITION 1.14. *Let C be a category with unique factorization. Then we have*

$$(fg)^\circ = f^\circ(g|\operatorname{Im} f)^\circ \quad \text{and} \quad \operatorname{Im} fg = g(\operatorname{Im} f).$$

for all $f, g \in C$ for which fg exists.

Proof. Since fg exists, $\operatorname{cod} f = \operatorname{dom} g$ and so $\operatorname{Im} f \subseteq \operatorname{dom} g$. Let $h = j_f g = g|\operatorname{Im} f$. Then

$$(fg)^\circ j_{fg} = fg = f^\circ j_f g = f^\circ h = f^\circ h^\circ j_h.$$

Now $f^\circ h^\circ$ is an epimorphism and so the $f^\circ h^\circ j_h$ is a canonical factorization of fg. Since C has unique factorization, we have $(fg)^\circ = f^\circ h^\circ$. This proves the first equality. Further, $j_{fg} = j_h$ and so,

$$\operatorname{Im} fg = \operatorname{dom} j_{fg} = \operatorname{dom} j_h = \operatorname{Im} h = g(\operatorname{Im} f)$$

by Equation (1.54). □

We have seen that every small category C is isomorphic to a subcategory of Set (see Proposition 1.9). However, if C has subobjects, this isomorphism may not preserve subobjects. We now use the Proposition above to obtain an inclusion preserving isomorphism of a small category C with unique factorization onto a subcategory of Set.

THEOREM 1.15. *Let C be a small category with unique factorization. Then there exists a faithful, inclusion preserving functor $U : C \to$ Set which is injective on objects with the following properties:*

(a) $c \subseteq d$ in $\mathfrak{v}C \iff U(c) \subseteq U(d)$ in Set.

(b) $f \in C$ is monomorphism in C if and only if $U(f)$ is injective.

(c) $f \in C$ is split epimorphism in C if and only if $U(f)$ is surjective.

Hence for $f \in C$, f° is a split epimorphism if and only if

$$U(\operatorname{Im} f) = \operatorname{Im} U(f). \quad \textit{In particular,} \quad U(f) = U(f^\circ)U(j_f)$$

is the canonical factorization of $U(f)$ in Set. Consequently, every epimorphism in C splits, if and only if $U : C \to$ Set is factorization preserving.

Proof. Define $U : C \to$ Set as follows: for $c \in \mathfrak{v}C$, let

$$U(c) = \{f^\circ : f \in C, \quad \operatorname{cod} f = c\}; \tag{1.55}$$

and for $f : c \to d \in C$, let

$$U(f) : g^\circ \mapsto (gf)^\circ, \quad g^\circ \in U(c). \tag{1.56}$$

We first observe that, since C is small, $U(c)$ is a set for all $c \in \mathfrak{v}C$ and that the maps $c \to U(c)$ and $f \to U(f)$satisfies (a). If $c \subseteq d$ and $f^\circ \in U(c)$, then $\operatorname{cod} f = c$ and so $g \in U(d)$ if $g = fj_c^d$. Then

$$g = f^\circ j_f j_c^d = f^\circ j_{\operatorname{Im} f}^d.$$

Since C has unique factorization, we have $f^\circ = g^\circ$ and so $f^\circ \in U(d)$. Thus $U(c) \subseteq U(d)$. By the definition of $U(j_c^d)$, we have

$$U(j_c^d) : f^o \in U(c) \mapsto (fj_c^d)^o \in U(d).$$

Since $fj_c^d = f^o j_f j_c^d$ is a canonical factorization of fj_c^d, we have $(fj_c^d)^o = f^o$. Therefore

$$U(j_c^d)(f^o) = f^o = j_{U(c)}^{U(d)}(f^o) \quad \text{for all} \quad f^o \in U(c).$$

Conversely, if $U(c) \subseteq U(d)$, then $1_c \in U(d)$. Hence $1_c = g^\circ$ for some g with $\operatorname{cod} g = d$. Then $\operatorname{Im} g = c$ and so, we have $c \subseteq d$. Since $\mathfrak{v}C$ is a strict preorder, this also shows that the map $c \mapsto U(c)$ is injective.

To show that U is a functor, let $f : c \to d, g : d \to e \in C$ and let $h^\circ \in U(c)$. Then by Equation (1.56),

$$\begin{aligned}(h^\circ)U(fg) &= (hfg)^\circ = h^\circ(j_h fg)^\circ && \text{by Proposition 1.14,}\\ &= h^\circ(f|\operatorname{Im} h)^\circ\,(g|f(\operatorname{Im} h))^\circ && \text{by Equation (1.54).}\end{aligned}$$

Similarly, using Proposition 1.14 and Equation (1.54), we get

$$\begin{aligned}(h^\circ)U(f)U(g) &= ((h^\circ(f|\operatorname{Im} h)^\circ)\,U(g)\\ &= h^\circ(f|\operatorname{Im} h)^\circ\,(g|f(\operatorname{Im} h))^\circ\,.\end{aligned}$$

Hence $U(fg) = U(f)U(g)$. If $c \subseteq d$ and $g^\circ \in U(c)$, then by Proposition 1.14, $(gj_c^d)^\circ = g^\circ(j_{\operatorname{Im} g}^d)^\circ = g^\circ$ by the unique factorization property of C. Hence, by Equation (1.54), $U(j_c^d) = j_{U(c)}^{U(d)}$. In particular, $U(1_c) = 1_{U(c)}$ and so, U is an inclusion preserving functor.

Let $f, g \in C(c, d)$. If $U(f) = U(g)$, then by Proposition 1.14, $f^\circ = (1_c f)^\circ = (1_c g)^\circ = g^\circ$. Hence by the definition of image, $\operatorname{Im} f = \operatorname{Im} g$ and so, $f = g$. Thus U is faithful.

To prove (b), assume that $f : c \to d$ is a monomorphism. If $(h^\circ)U(f) = (hf)^\circ = (g^\circ)U(f) = (gf)^\circ$, then, as above $hf = gf$ and so $h = g$ which imply that $U(f)$ is injective. Conversely, if $U(f)$ is injective and if $hf = gf$ for $h, g \in C(a, c)$, then $(h^\circ)U(f) = (hf)^\circ = (gf)^\circ = (g^\circ)U(f)$ and since $U(f)$ is one-to-one, $h^\circ = g^\circ$ which implies $h = g$. Therefore f is a monomorphism. This proves (b).

If $f : c \to d$ is a split epimorphism then there is $g : d \to c$ with $gf = 1_d$. Let $h^\circ \in U(d)$. If $k = hg$, then $\operatorname{cod} k = \operatorname{cod} g = c$ and $(k^\circ)U(f) = (kf)^\circ = (hgf)^\circ = h^\circ$. Hence $U(f)$ is surjective.

Conversely, if $U(f)$ is surjective, then there is $g^\circ \in U(c)$ such that $(g^\circ)U(f) = 1_d$. Then we have $(gf)^\circ = 1_d$ and so $\operatorname{Im}(gf) = d = \operatorname{cod} gf$. Hence $gf = 1_d$; thus f is a split epimorphism and (c) follows.

Let $f : c \to d$ be such that f° is a split epimorphism. Then $U(f) = U(f^\circ j_f) = U(f^\circ)U(j_f)$. By the above, $U(f^\circ)$ is surjective and hence an epimorphism in **Set**. Since U is inclusion preserving, $U(j_f) = U(j_{\operatorname{Im} f}^d) = j_{U(\operatorname{Im} f)}^{U(d)}$. Hence $U(f^\circ)U(j_f)$ is the canonical factorization of $U(f)$ in **Set**. Therefore

$$U(f^\circ) = U(f)^\circ, \quad j_{U(\operatorname{Im} f)}^{U(d)} = j_{\operatorname{Im} U(f)}^{U(d)} \quad \text{and so,} \quad U(\operatorname{Im} f) = \operatorname{Im} U(f).$$

Let $f : c \to d$ satisfy $U(\operatorname{Im} f) = \operatorname{Im} U(f)$. If $a = \operatorname{Im} f$, we have

$$U(f) = U(f^o j_a^d) = U(f^o)U(j_a^d) = U(f^o)j_{\operatorname{Im} f}^{U(d)} = U(f)^o j_{\operatorname{Im} f}^{U(d)}.$$

This implies that $U(f^o) = U(f)^o$ and so, $U(f^o)$ is surjective. Hence by (c), f^o is a split epimorphism. The last statement now follows from this. □

category!– of sets
groupoid
Higgins, P.
groupoid!ordered –
semigroup!inverse –
semigroup!regular –
groupoid
groupoid!connected –

***Remark* 1.5:** We have noted that every conctete category C is a category of sets so that objects in C can be identified as sets and morphisms as maps. C is said to be a *category of sets with subobjects* if, in addition, $U : C \to \mathbf{Set}$ is a $\mathfrak{v}$-isomorphism (so that U satisfies condition (a) of the theorem above). The theorem above shows that every small category C with unique factorization is isomorphic to and hence can be identified with a category of sets with subobjects. To an extent, such identification enables us to replace categorical arguments in C by elementary setheoretic arguments. However, the theorem above also shows the limitations in this: the factorization of a morphism f in C may be different from its factorization in **Set**. When epimorphisms in C splits, the factorization in C coincides with those in **Set**; in this case one can more-or-less replace completely replace categorical arguments in C by setheoretic arguments.

1.4 GROUPOIDS

In this section we shall briefly discuss a class of small categories, called *groupoids*, which we need in the sequel. Groupoids occur naturally in several branches of mathematics. Here, in § Subsection 1.4.1 we content ourself by giving necessary definitions, a few elementary properties and some examples. We refer the reader to Higgins [1971] for a more detailed discussion. In § Subsection 1.4.2 we discuss a class of groupoid, called *ordered groupoids* endowed with an additional structure in the form of a partial order. As we shall see later, ordered groupoids are important structural components of the class of *inverse semigroups* and *regular semigroups*. Finally, in § Subsection 1.4.3, we discuss the relation between ordered groupoids and categories with subobjects.

1.4.1 Definition and examples

A *groupoid* is a small category in which every morphism is an isomorphism. This means that when $\mathcal{G}$ is a groupoid and $a, b \in \mathfrak{v}\mathcal{G}$, then for any $u \in \mathcal{G}(a, b)$, there exists $u^{-1} \in \mathcal{G}(b, a)$ such that $uu^{-1} = e_u = 1_a$ and $u^{-1}u = f_u = 1_b$ (see § Subsection 1.2.1). Recall that, by the convention introduced in § Subsection 1.2.1, since groupoids are small categories, we regard them as partial algebras and identify vertices with identities.

If $\mathcal{G}$ is a groupoid, then it is easy to see that for all $a \in \mathfrak{v}\mathcal{G}, H_a = \mathcal{G}(a, a)$ is a group under the composition in $\mathcal{G}$. It is easy to see that maximal subgroups of the groupoid $\mathcal{G}$ are precisely the groups H_a for $a \in \mathfrak{v}\mathcal{G}$.

A groupoid $\mathcal{G}$ is said to be *connected* if for all $a, b \in \mathfrak{v}\mathcal{G}$, $\mathcal{G}(a, b) \neq \emptyset$. Given groupoid!component of –

any groupoid $\mathcal{G}$ the relation

$$a \sim b \iff \mathcal{G}(a,b) \neq \emptyset \quad \forall\ a, b \in X, \tag{1.57}$$

where $X = \mathfrak{v}\mathcal{G}$, is an equivalence relation on X. For $a \in X$, let X_a denote the $\sim$-class of X containing a. If

$$\mathcal{G}_a = \bigcup_{b,c \in X_a} \mathcal{G}(b.c) \tag{1.58}$$

then it can be seen that $\mathcal{G}_a$ is the maximal connected subgroupoid of $\mathcal{G}$ with $\mathfrak{v}\mathcal{G}_a = X_a$. The subgroupoid $\mathcal{G}_a$ is called a *component* of $\mathcal{G}$. If $a, b \in X$, then it is easy to see that either

$$\mathcal{G}_a = \mathcal{G}_b \quad \text{or} \quad \mathcal{G}_a \cap \mathcal{G}_b = \emptyset.$$

Thus we have:

PROPOSITION 1.16. *Let $\mathcal{G}$ be a groupoid with $\mathfrak{v}\mathcal{G} = X$. Then Equation* (1.57) *defines an equivalence relation $\sim$ on X and $\mathcal{G}_a$ defined by Equation* (1.58) *is the component (maximal connected subgroupoid) whose vertex set is the $\sim$-class containing a. Hence $\mathcal{G}$ is the disjoint union of its components.* □

We now give some examples of groupoids some of which will be of use later.

Example 1.17: Every group G is a groupoid with exactly one vertex.

Example 1.18: Let G be a group and X be a set. Let

$$\mathcal{G} = \{(x, g, y) : x, y \in X,\ g \in G\} = X \times G \times X.$$

Define composition in $\mathcal{G}$ by:

$$(x, g, y)(u, h, v) = \begin{cases} (x, gh, v) & \text{if } y = u; \\ \text{undefined} & \text{if } y \neq u. \end{cases}$$

$\mathcal{G}$, with this composition, is a connected groupoid such that we can identify $\mathfrak{v}\mathcal{G}$ with X. In the following we will denote by $X \times G \times X$, the connected groupoid with vertex set X and in which morphisms and composition is defined as above.

Example 1.19: Let H be a subgroup of a group G and let

$$\mathcal{G}(G/H) = \{xHy : x, y \in G\}.$$

Define composition in $\mathcal{G}(G/H)$ as follows:

$$(xHy)(uHv) = \begin{cases} xHyuv & \text{if } (yu)H = H(yu); \\ \text{undefined} & \text{if } (yu)H \neq H(yu). \end{cases}$$

With this composition $\mathcal{G}(G/H)$ is a connected groupoid whose vertex set (set of identities) is the set of all conjugates of H in G.

Higgins, P.
groupoid!simplecial
Munkers, J. R.
Spanier, E. H.
Singer, I. M.
Thorpe, J. A.

Example 1.20: Let σ be an equivalence relation on a set X. For $(x, y), (u, v) \in \sigma$, define a partial composition in σ by:

$$(x, y)(u, v) = \begin{cases} (x, v) & \text{if } y = u; \\ \text{undefined} & \text{if } y \neq u. \end{cases}$$

With this composition, σ becomes a groupoid whose vertex set can be identified with X. Note that in the groupoid σ, the hom-set $\sigma(x, y)$ contain utmost one element; in particular, maximal subgroups of σ are trivial. Conversely, any groupoid $\mathcal{G}$ with the property that any hom-set of $\mathcal{G}$ contain atmost one element can be represented as a groupoid determined by an equivalence relation on $\mathfrak{v}\mathcal{G}$ as above (see Higgins [1971]). A groupoigd with this property is called a *simplecial groupoid* . We shall use the same notation for the equivalence relation and the correspoinding simplecial groupoid.

Example 1.21: Let X be a set and let $\boldsymbol{I}_X$ be the set of all bijections between subsets of X. For $\alpha, \beta \in \boldsymbol{I}_X$, let

$$\alpha \cdot \beta = \begin{cases} \alpha\beta & \text{the usual composition, if } \operatorname{cod} \alpha = \operatorname{dom} \beta; \\ \text{undefined} & \text{if } \operatorname{cod} \alpha \neq \operatorname{dom} \beta. \end{cases} \tag{1.59}$$

With this product, $\boldsymbol{I}_X$ is a groupoid. We shall refer to the composition defined above as *groupoid composition*

Example 1.22: The example above can be generalized further by replacing X by any specified mathematical system and $\boldsymbol{I}_X$ by the class of all isomorphisms of suitable subsystems provided that these isomorphisms are closed with respect to groupoid composition (see Equation (1.59)). Thus if M is a (finite dimensional) manifold and if $\mathcal{M}$ denote the set of all homeomorphisms co-ordinate neighborhoods (of suitable type such as differentiable, smooth, analytic, etc), then $\mathcal{M}$ is a groupoid when composition is defined as in the last example. Similarly, if $\mathcal{A}$ denote the set of all analytic isomorphisms of regions in the complex plane, then $\mathcal{A}$ is a groupoid. Note that, by Riemann mapping theorem, the set of all analytic isomorphisms of simply connected regions different from the whole complex plane $\mathbb{C}$, is a component in $\mathcal{A}$.

Example 1.23: An important classical example of groupoid is the following: Let $[\alpha]$ denote the path-homotopy class of a path α in the topological space X. For paths α, β in X, let $\alpha \cdot \beta$ denote the usual product of paths in X which is defined if $\alpha(1) = \beta(0)$. Consider the set

$$\mathbf{H}(X) = \{[\alpha] : \alpha \quad \text{is a path in} \quad X\}.$$

Define composition in $\mathsf{H}(X)$ by:

$$[\alpha][\beta] = \begin{cases} [\alpha \cdot \beta] & \text{if } \alpha(1) = \beta(0); \\ \text{undefined} & \text{if } \alpha(1) \neq \beta(0). \end{cases}$$

$\mathbf{H}(X)$, with this composition, is a groupoid, called the homotopy group of paths in X. The vertex set of $\mathbf{H}(X)$ can be identified with X. The maximal subgroup H_x of $\mathbf{H}(X)$ at $x \in X$ is the fundamental group of X based at x. Also the groupoid $\mathbf{H}(X)$ is connected if and only if X is path connected. See Munkers [1984], Spanier [1971] or Singer and Thorpe [1967] for details and proofs.

We proceed to show that structure of connected groupoids is quite simple; all of them are isomorphic to a groupoid constructed as in Example 1.18 above.

PROPOSITION 1.17. *Let $\mathcal{G}$ be a connected groupoid and let $X = \mathfrak{v}\mathcal{G}$. Let 1 denote a fixed element in X and suppose that $H = \mathcal{G}(1, 1)$. Then we have the following:*

- *(a) $\mathcal{G}$ is equivalent (as categories) to the group H.*
- *(b) $\mathcal{G}$ is isomorphic to the groupoid $X \times H \times X$ (see Example 1.18).*

In particular, all maximal subgroups (hom-sets $\mathcal{G}(a, a)$ for $a \in X$) of $\mathcal{G}$ are isomorphic to H.

Proof. For each $a \in X$, choose $\eta_a \in \mathcal{G}(a, 1)$ such that η_1 is the identity on the vertex 1. Since $\mathcal{G}$ is connected $\mathcal{G}(a, 1) \neq \emptyset$ for all $a \in X$ and so it is possible to choose η_a as above.

(a) Define $F : \mathcal{G} \to H$ by

$$F(u) = \eta_a^{-1} u \eta_b \quad \forall \;\; u \in \mathcal{G}(a, b).$$

Then clearly, $F(u) \in H$ for all $u \in \mathcal{G}(a, b)$. Also, F is the morphism map of a functor of $\mathcal{G}$ to H whose vertex map is the constant map on X with value 1. Let $J : H \subseteq \mathcal{G}$ be the inclusion functor. It is easy to verify that for each $a \in X$, η_a is universal from a to J and that the map $\eta : a \mapsto \eta_a$ is a natural isomorphism of $1_{\mathcal{G}} \xrightarrow{n} FJ$. Hence by Theorem 1.6, F is a left adjoint to J and $\eta : 1_{\mathcal{G}} \xrightarrow{n} FJ$ is the unit of adjunction. Now $JF = 1_H$ and so, the counit of the adjunction is the identity on JF. It follows that

$$< F, J, \eta, 1 >: \mathcal{G} \overset{\cdot}{\rightharpoonup} H$$

is an adjoint equivalence of $\mathcal{G}$ to H.

(b) Let $K : \mathcal{G} \to X \times H \times X$ and $K' : X \times H \times X \to \mathcal{G}$ in the reverse direction be defined by

$$\mathfrak{v}K = 1_X = \mathfrak{v}K'$$

and for $a, b \in X, u \in \mathcal{G}(a, b)$ and $v \in H$, let

$$K(u) = (a, F(u), b) \quad \text{and} \quad K'(a, v, b) = \eta_a v \eta_b^{-1}.$$

Then K and K' are mutually inverse functors from $\mathcal{G}$ to $X \times H \times X$ and back respectively. Hence K and K' are isomorphisms.

Clearly, the functor $F : \mathcal{G} \to H$ is fully faithful and so, its restriction to maximal subgroups H_a are isomorphisms of H_a onto H. □

The statement (a) above implies that any groupoid is equivalent (as categories) to a groupoid whose components are all groups (that is, a disjoint union of groups). Also, by statement (b), every groupoid is isomorphic to a disjoint union of groupoids of the form $X \times H \times X$.

groupoid!ordered – restriction restriction!domain – $e \centerdot x$ or $x|e$:restriction of x to e partial order!restriction – functor!order preserving –

1.4.2 Ordered groupoid

Many groupoids that occur naturally carries additional structures. For example, consider the groupoid $\boldsymbol{I}_X$ of all partial bijections of the set X (see Example 1.21). Clearly, $\mathfrak{v}\boldsymbol{I}_X = \mathcal{P}(X)$, the set of all subsets of X. The inclusion relation is a natural partial order on $\mathcal{P}(X)$. If $\alpha \in \boldsymbol{I}_X$ and $D \subseteq \operatorname{dom}\alpha$, then the restriction $\alpha|D$ of α to D is an injective map of D into $\operatorname{cod}\alpha$. We denote by $(\alpha|D)^\circ$, the unique bijection of D onto $(D)\alpha$ determined by $\alpha|D$. We can be extended the partial order on $\mathcal{P}(X)$ to a partial order on $\boldsymbol{I}_X$ by setting

$$\alpha \leq \beta \iff \operatorname{dom}\alpha \subseteq \operatorname{dom}\beta \quad \text{and} \quad \alpha = (\beta|\operatorname{dom}\alpha)^\circ. \tag{1.60}$$

Thus, the usual restriction of functions induces a partial order on $\boldsymbol{I}_X$. We formalize this as follows:

DEFINITION 1.6. Let $\mathcal{G}$ be a groupoid and $\leq$ be a partial order on $\mathcal{G}$ satisfying the following:

(OG1) Suppose $u \leq x$ and $v \leq y$ in $\mathcal{G}$. If products uv and xy exists in $\mathcal{G}$, then $uv \leq xy$.

(OG2) If $u \leq x$, then $u^{-1} \leq x^{-1}$.

(OG3) If $x \in \mathcal{G}$ and $e \leq e_x$ with $e \in \mathfrak{v}\mathcal{G}$, then there exists a unique $e \centerdot x \in \mathcal{G}$ such that $e \centerdot x \leq x$ and $e_{e \centerdot x} = e$.

Then $\mathcal{G}$ is called an *ordered groupoid* with respect to $\leq$.

The unique element $e \centerdot x$ of axiom (3) is called the *restriction* or *domain restriction* of x to e. Often, we shall also use the usual notation $x|e$ to denote $e \centerdot x$. If $u \leq x$, it follows from axioms (1) and (2) that

$$e_u = uu^{-1} \leq xx^{-1} = e_x \quad \text{and similarly,} \quad f_u \leq f_x.$$

Therefore, in view of the uniqueness in axiom (3), $u \leq x$ implies $u = e_u \centerdot x$. The relation $\leq$ is called the *restriction* (or restriction order) on $\mathcal{G}$.

If $\mathcal{G}$ and $\mathcal{G}'$ are ordered groupoids, a functor $f : \mathcal{G} \to \mathcal{G}'$ is said to be *order preserving* if whenever $x \leq y$ in $\mathcal{G}$, $f(x) \leq f(y)$ in $\mathcal{G}'$. In the following, unless otherwise stated explicitly, by a *functor* of ordered groupoids, we shall mean

isomorphism!$\mathfrak{v}$-isomorphism
embedding!– of ordered groupoids
isomorphism!– of ordered groupoids
$x \cdot f$:co-restriction of x to f
co-restriction
restriction!range –

an order preserving functor. The collection of all ordered groupoids forms a category $\mathfrak{OG}$ with morphisms as order preserving functors.

An order preserving functor $f : \mathcal{G} \to \mathcal{H}$ of ordered groupoids is said to be a $\mathfrak{v}$-*isomorphism* if $\mathfrak{v}f$ is an order-isomorphism. f is an *embedding* of ordered groupoids if f preserves and reflects partial orders; that is,

$$x \leq y \quad \text{in } \mathcal{G} \quad \iff f(x) \leq f(y) \quad \text{in } \mathcal{H}. \tag{1.61}$$

The functor f is an isomorphism of ordered groupoids if f is an isomorphism of groupoids as well as an order isomorphism.

A subgroupoid $\mathcal{G}'$ of an ordered groupoid $\mathcal{G}$ is an ordered subgroupoid if and only if $e \cdot x \in \mathcal{G}'$ for all $x \in \mathcal{G}'$ and $e \in \mathfrak{v}\mathcal{G}'$ with $e \leq e_x$. Note that, in this case, $\mathcal{G}' \subseteq \mathcal{G}$ an embedding of ordered groupoids. Note that if f is an embedding of $\mathcal{G}$ to $\mathcal{H}$, then $f(\mathcal{G})$ is an ordered subgroupoid of $\mathcal{H}$ and f is an isomorphism of $\mathcal{G}$ onto $f(\mathcal{G})$.

Observe that axioms (1) and (2) above are (left-right) self-dual. We show below that the dual statement of (3) is equivalent to (3).

PROPOSITION 1.18. *Let $\mathcal{G}$ be a groupoid and $\leq$ denote a partial order on $\mathcal{G}$ such that axioms (1) and (2) of Definition 1.6 hold. Then $\mathcal{G}$ satisfies axiom (3) if and only if it satisfies the following:*

(3) For every $x \in \mathcal{G}$ and $f \leq f_x$ with $f \in \mathfrak{v}\mathcal{G}$, there exists a unique $x \cdot f \in \mathcal{G}$ such that $x \cdot f \leq x$ and $f_{x \cdot f} = f$.*

Proof. Assume that $\mathcal{G}$ satisfies axiom (3). Given $x \in \mathcal{G}$ and $f \leq f_x$, define

$$x \cdot f = (f \cdot x^{-1})^{-1}. \tag{1.62}$$

Since $f \leq f_x = e_{x^{-1}}$, by axiom (3), $f \cdot x^{-1} \leq x^{-1}$ and so, by axiom (2), $x \cdot f = (f \cdot x^{-1})^{-1} \leq x$. Also, $f_{x \cdot f} = e_{f \cdot x^{-1}} = f$ by axion (3). If $y \in \mathcal{G}$ also satisfies the conditions $y \leq x$ and $f_y = f$ then by axiom (2), $y^{-1} \leq x^{-1}$ and $e_{y^{-1}} = f_y = f$. Therefore by (3), we must have $y^{-1} = f \cdot x^{-1}$ and so $y = x \cdot f$. This proves the uniqueness of $x \cdot f$. Thus $x \cdot f$ satisfies the conditions in (3)*.

Conversely, if (3)* holds, defining

$$e \cdot x = (x^{-1} \cdot e)^{-1}. \tag{1.62*}$$

we can show, as above, that axiom (3) holds. □

For $x \in \mathcal{G}$ and $f \leq f_x$, the morphism $x \cdot f$ defined by Equation (1.62) is called the *co-restriction* (or *range restriction*) of x to f.

Example 1.24: By the remarks at the beginning of this section, for any set X, $\boldsymbol{I}_X$ is an ordered groupoid with order relation induced by restriction (see Equation (1.60)).

Similarly, if X is any partially ordered set, the set of all order isomorphisms of order ideals (see § Subsection 1.1.2 for definition of order ideals) is an ordered groupoid with respect to the usual restriction of maps (again defined as in Equation (1.60)). We denote this by $\boldsymbol{OI}_X$. Note that $\mathfrak{v}\boldsymbol{OI}_X$ is the partially ordered set of all order ideals under inclusion. Similarly, isomorphisms of pricipal order ideals of X gives an ordered subgroupoid $\mathsf{T}^*(X)$ of $\boldsymbol{OI}_X$. Since there exists an order-isomorphism of X onto the set of principal orderideals of X, we may identify X with $\mathfrak{v}\mathsf{T}^*(X)$ and regard $\mathsf{T}^*(X)$ as an ordered groupoid with $\mathfrak{v}\mathsf{T}^*(X) = X$. Also, the groupoids $\mathcal{M}$ and $\mathcal{A}$ of Example 1.22 are also ordered groupoids under inclusion.

The next proposition lists a few useful properties of ordered groupoids

PROPOSITION 1.19. *For an ordered groupoid $\mathcal{G}$, we have the following:*

(1) Let $x \in \mathcal{G}$, and $e, f \in \mathfrak{v}\mathcal{G}$ with $e \leq e_x$, $f \leq f_x$. Then $f = f_{e.x}$ if and only if $e = e_{x.f}$. When e and f satisfies this, we have $e \,.\, x = x \,.\, f$.

(2) Assume that xy exists in $\mathcal{G}$. If $e \leq e_x$, then

$$e \,.\, (xy) = (e \,.\, x)(f_{e.x} \,.\, y).$$

Dually, if $f \leq f_y$, we have

$$(xy) \,.\, f = (x \,.\, e_{y.f})(y \,.\, f)$$

Proof. **(1)**. Let $f = f_{e.x}$. Then $e_{(e.x)^{-1}} = f$ and by axiom (2), $(e \,.\, x)^{-1} \leq x^{-1}$. Hence by the uniqueness in axiom (3), $(e \,.\, x)^{-1} = f \,.\, x^{-1}$ and so, by Equation (1.62), $e \,.\, x = x \,.\, f$. But then $e = e_{e.x} = e_{x.f}$. The converse can be proved similarly.

(2) Suppose that xy exists in $\mathcal{G}$ and $e \leq e_x$. If $h = f_{e.x}$, then using axioms (1) and (2), we have

$$h = (e \,.\, x)^{-1}(e \,.\, x) \leq x^{-1}x = f_x = e_y.$$

Hence the product $z = (e \,.\, x)(h \,.\, y)$ exists in $\mathcal{G}$ and $z \leq xy$ by (1). Further,

$$\begin{aligned} e_z = zz^{-1} &= (e \,.\, x)(h \,.\, y)(h \,.\, y)^{-1}(e \,.\, x)^{-1} \\ &= (e \,.\, x)h(e \,.\, x)^{-1} = (e \,.\, x)(e \,.\, x)^{-1} \\ &= e_{e.x} = e. \end{aligned}$$

Therefore, by axiom (3), $z = e.xy$. The remaining assertion is proved dually. □

Next Proposition give a representation of an ordered groupoid (not necessarily faithful) as an ordered subgroupoid of $\mathsf{T}^*(X) \subseteq \boldsymbol{OI}_X$ for a suitable partially ordered set X. This will be useful later on.

PROPOSITION 1.20. *Let $\mathcal{G}$ be an ordered groupoid and $V = \mathfrak{v}\mathcal{G}$. Then we have:*

(1) *V is an order ideal in $\mathcal{G}$.*

(2) *If $x \in \mathcal{G}$, the map $\mathfrak{a}(x) : e \mapsto f_{e\centerdot x}$ is an order isomorphism of $V(e_x)$ onto $V(f_x)$.*

(3) *The map $\mathfrak{a} : x \mapsto \mathfrak{a}(x)$ is a $\mathfrak{v}$-isomorphism of $\mathcal{G}$ onto $\mathsf{T}^*(V)$.*

Proof. **(1)** To show that V is an order ideal in $\mathcal{G}$, it is sufficient to show that for $e \in V$ and $x \leq e$ in $\mathcal{G}$ implies that $x \in V$. Since $x \leq e$, $x^{-1} \leq e^{-1} = e$ and so, $e_x = xx^{-1} \leq ee = e$ by axiom (1). Then $x = e_x \centerdot e = e_x$ by axiom (3).

(2) If $h \leq e \leq e_x$, then $h \centerdot x \leq e \centerdot x \leq x$. Hence $f_{h\centerdot x} \leq f_{e\centerdot x} \leq f_x$; that is $h\mathfrak{a}(x) \leq e\mathfrak{a}(x) \leq f_x$. Hence $\mathfrak{a} : V(e_x) \to V(f_x)$ is order preserving. Dually, it can be seen that the map $\mathfrak{a}'(x) : f \mapsto e_{x\centerdot f}$ is an order preserving map of $V(f_x)$ into $V(e_x)$. Now

$$(e\mathfrak{a}(x))\,\mathfrak{a}'(x) = (f_{e\centerdot x})\mathfrak{a}'(x) = e$$

by Proposition 1.19(1). Hence $\mathfrak{a}(x)\mathfrak{a}'(x) = 1_{V(e_x)}$. Similarly $\mathfrak{a}'(x)\mathfrak{a}(x) = 1_{V(f_x)}$ and so $\mathfrak{a}(x)$ is an order isomorphism.

(3) By the above, $\mathfrak{a}(x) \in \boldsymbol{OI}_V$ for all $x \in \mathcal{G}$. Hence $\mathfrak{a} : x \mapsto \mathfrak{a}(x)$ is a mapping of $\mathcal{G}$ into $\boldsymbol{OI}_V$. Assume that xy exists in $\mathcal{G}$ and $e \leq e_x$. Then

$$\begin{aligned} e\mathfrak{a}(xy) &= f_{e\centerdot xy} \\ &= f_{f_{e\centerdot x}\centerdot y} && \text{by Proposition 1.19(2)} \\ &= (f_{e\centerdot x})\mathfrak{a}(y) \\ &= (e\mathfrak{a}(x))\,\mathfrak{a}(y) \\ &= e\,(\mathfrak{a}(x)\mathfrak{a}(y)). \end{aligned}$$

Hence $\mathfrak{a}(xy) = \mathfrak{a}(x)\mathfrak{a}(y)$. If $e \in V$, then it is easy to see that $\mathfrak{a}(e) = 1_{V(e)}$ and so $\mathfrak{a} : \mathcal{G} \to \boldsymbol{OI}_V$ is a functor. If $e \leq e_x$ then for any $h \in V(e)$,

$$h\mathfrak{a}(e \centerdot x) = f_{h\centerdot(e\centerdot x)} = f_{h\centerdot x} = h\mathfrak{a}(x)$$

which shows that $\mathfrak{a}(e \centerdot x) = (\mathfrak{a}(x)|V(e))^{\circ}$. Therefore $\mathfrak{a}$ is order preserving. Since, for $e, h \in V$,

$$\mathfrak{a}(h) \leq \mathfrak{a}(e) \iff \mathfrak{a}(h) = (\mathfrak{a}(e)|V(h))^{\circ} \iff V(h) \subseteq V(e) \iff h \leq e,$$

$\mathfrak{v}\mathfrak{a} : \mathcal{G} \to \boldsymbol{OI}_V$ is an embedding. If $\tilde{e} \in \mathfrak{v}\mathsf{T}^*(V)$ then, by definition of $\mathsf{T}^*(V)$, $\tilde{e}$ must be an identity map on some principal ideal $V(e)$ of $e \in V$. Clearly, $\mathfrak{a}(e) = \tilde{e}$. Hence $\mathfrak{v}\mathfrak{a} : \mathfrak{v}\mathcal{G} \to \mathfrak{v}\mathsf{T}^*(V)$ is an order isomorphism. ☐

The construction given in the Proposition above represents any ordered groupoid $\mathcal{G}$ as a groupoid of order isomorphisms of principal order ideals of $\mathfrak{v}\mathcal{G}$. Though this representation has important applications, it may not be faithful. The next theorem shows that any ordered groupoid $\mathcal{G}$ can be embedded (represented faithfully) as an ordered subgroupoid of $\boldsymbol{OI}_X$ of a suitable partially ordered set X. In the construction below, we take $X = \mathcal{G}$ and consider X as a partially ordered set.

THEOREM 1.21. *Every ordered groupoid is isomorphic to an ordered subgroupoid of $\boldsymbol{OI}_X$ for a suitable partially ordered set X.*

Proof. Let $\mathcal{G}$ be an ordered groupoid with $\mathfrak{v}\mathcal{G} = V$. For each $e \in V$ let

$$\Lambda(e) = \{x \in \mathcal{G} : f_x \leq e\} \tag{1}$$

If $x \in \Lambda(e)$ and $y \leq x$, then $f_y \leq f_x \leq e$ and so $y \in \Lambda(e)$. Hence $\Lambda(e)$ is an order ideal of $\mathcal{G}$. Also, it is easy to see that

$$e \leq f \iff \Lambda(e) \subseteq \Lambda(f); \tag{2}$$

in particular, the mapping $e \mapsto \Lambda(e)$ is injective from V to the partially ordered set of order ideals of $\mathcal{G}$ under inclusion.

Now, for each $x \in \mathcal{G}(e, f)$, let

$$y\theta(x) = y(f_y \centerdot x) \quad \text{for all} \quad y \in \Lambda(e). \tag{3}$$

Since $y \in \Lambda(e)$, $f_y \leq e = e_x$ and so $f_y \centerdot x \leq x$. Then $f_{y(f_y \centerdot x)} = f_{(f_y \centerdot x)} \leq f_x = f$. Hence $\theta(x)$ is a well defined map of $\Lambda(e)$ into $\Lambda(f)$. Let $z \leq y \in \Lambda(e)$. Then $f_z \leq f_y \leq e$ and so, $f_z \centerdot x \leq f_y \centerdot x$. Hence by axiom (1),

$$z\theta(x) = z(f_z \centerdot x) \leq y(f_y \centerdot x) = y\theta(x)$$

and so $\theta(x) : \Lambda(e) \to \Lambda(f)$ is order preserving. Suppose that the product xy exists in $\mathcal{G}$ so that $f_x = e_y = f$ (say). Also, let $e = e_x$ and $g = f_y$. Then for any $u \in \Lambda(e)$,

$$\begin{aligned} u\theta(x)\theta(y) &= u(f_u \centerdot x)(h \centerdot y) && \text{where } h = f_{f_u \centerdot x} \\ &= u(f_u \centerdot xy) && \text{by Proposition 1.19} \\ &= u\theta(xy) && \text{by Equation (3).} \end{aligned}$$

Hence

$$\theta(x)\theta(y) = \theta(xy).$$

If $e \in V$, then clearly, $\theta(e) = 1_{\Lambda(e)}$. Hence, for any $x \in \mathcal{G}$,

$$\theta(x)\theta(x^{-1}) = \theta(e_x) = 1_{\Lambda(e_x)}$$

and similarly $\theta(x^{-1})\theta(x) = 1_{\Lambda(f_x)}$. Therefore $\theta(x) : \Lambda(e_x) \to \Lambda(f_x)$ is a bijection. It follows that $\theta : \mathcal{G} \to \boldsymbol{OI}_{\mathcal{G}}$ is a functor. If $x \leq y$, then $x = e_x \centerdot y$ and so, for any $u \in \Lambda(e_x)$, $f_u \leq e_x \leq e_y$ and we have

$$\begin{aligned} u\theta(x) &= u(f_u \centerdot x) = u(f_u \centerdot (e_x \centerdot y)) \\ &= u(f_u \centerdot y) = u\theta(y) \qquad \text{by axion (3).} \end{aligned}$$

Therefore $\theta(x) = (\theta(y)|\Lambda(e_x))^\circ$ which implies that $\theta(x) \leq \theta(y)$ in $\boldsymbol{OI}_{\mathcal{G}}$. On the other hand, if $\theta(x) = (\theta(y)|\Lambda(e_x))^\circ$, then $e_x \in \Lambda(e_x) \subseteq \Lambda(e_y)$ so that $e_x \leq e_y$ and by the definition of θ (Equation (3)), $x = e_x\theta(x) = e_x\theta(y) = e_x \centerdot y$. Thus $x \leq y$. It follows that

$$\theta(x) = (\theta(y)|\Lambda(e_x))^\circ \iff x \leq y.$$

Therefore $\theta : \mathcal{G} \to \boldsymbol{OI}_{\mathcal{G}}$ is an embedding of ordered groupoids. □

1.4.3 Categories generated by ordered groupoids

We now discuss the relation between ordered groupoids and categories with subobjects.

Recall that, if C is a category with subobjects, then $\mathfrak{v}C$, which is the same as the set of identities of C, is a partially ordered set. Here we shall also use the notations and conventions of § Subsection 1.3.2. Furthermore, it is clear that, the set of isomorphisms of C is a subgroupoid $\mathcal{G}(C)$ of C with $\mathfrak{v}C = \mathfrak{v}\mathcal{G}(C)$.

Definition 1.7. We shall say that a small category C with subobjects is *generated* by an ordered groupoid $\mathcal{G}$ if

(CG1) there is an injection of groupoids $\theta : \mathcal{G} \to C$ which induces an isomorphism of $\mathcal{G}$ on to the groupoid of isomorphisms of C. the set of isomorphisms of C which induces an order isomorphism of partially ordered set of identities of $\mathcal{G}$ onto $\mathfrak{v}C$;

(CG2) given a morphism σ in C there exists $x \in \mathcal{G}$ with $f_{\theta(x)} \leq f_\sigma$ such that σ has the factorization
$$\sigma = \theta(x) j^{f_\sigma}_{f_{\theta(x)}}.$$

We now characterizes small categories generated by ordered groupoids:

Proposition 1.22. *A category C with subobjects is generated by an ordered groupoid if and only if C has the following property:*

(CG*) *Every morphism $f \in C$ has a factorization $f = uj$ in C where u is an isomorphism and j is an inclusion.*

In particular, if C is generated by an ordered groupoid, then C has images.

Proof. If C is generated by the ordered groupoid $\mathcal{G}$, then by axioms (CG1) and (CG2) of Definition 1.7, C satisfies (CG*).

Conversely assume that C satisfies the condition (CG*) and that $\mathcal{G}$ denote the set of all isomorphisms in C. Then $\mathcal{G}$ is a subgroupoid of C containing all identities of C. Let θ denote the inclusion of $\mathcal{G}$ in C. Now the set of identities of $\mathcal{G}$ (which is the same as those of C) is a partially ordered set and the inclusion θ is clearly an order isomorphism. Thus axiom (CG1) holds and (CG2) follows from (CG*). It remains to show that we can define a partial order on $\mathcal{G}$ with respect to which $\mathcal{G}$ is an ordered groupoid.

We first observe that for any $f \in C$, the factorization $f = uj$ given by (CG*) is unique and that $f^\circ = u$. For if $f = qj'$ is any canonical factorization of f, we have $j = u^{-1}qj'$ and hence by axiom (c) of Definition 1.5, there is an inclusion j'' such that $u^{-1}q = j''$; that is, $q = uj''$. Hence the factorization $f = uj$ satisfies condition (Im) of Proposition 1.13 which implies that it is unique and that $f^\circ = u \in \mathcal{G}$. For $x \in \mathcal{G}$ and $e \le e_x$, define

$$e \centerdot x = (\jmath_e^{e_x} x)^\circ; \tag{1.}$$

and

$$u \le x \iff u = e \centerdot x. \tag{2.}$$

Since the factorization given by axiom (CG*) is unique, the morphism $e \centerdot x$ given by (1.) is uniquely determined by x and e. It follows, again from the uniqueness of the factorization, that the relation defined by (2.) is a partial order on $\mathcal{G}$. Clearly, axiom (3) of Definition 1.6 holds. Now if for $x, y \in \mathcal{G}$, the product xy exists in $\mathcal{G}$, then for $e \le e_x$,

$$\begin{aligned} e \centerdot xy\jmath_{f_{e \centerdot xy}}^{f_{xy}} &= \jmath_e^{e_x} xy \\ &= (e \centerdot x)\jmath_h^{f_x} y, && \text{where } h = f_{e \centerdot x} \\ &= (e \centerdot x)(h \centerdot y)\jmath_{f_{h \centerdot y}}^{f_y} \end{aligned}$$

since $f_x = e_y$. Hence by the uniqueness of factorization, we have

$$e \centerdot xy = (e \centerdot x)(f_{e \centerdot x} \centerdot y)$$

from which axiom (1) of Definition 1.6 follows. From

$$\jmath_e^{e_x} x = (e \centerdot x)\jmath_h^{f_x}$$

where $h = f_{e \centerdot x}$, we obtain

$$\begin{aligned} (e \centerdot x)^{-1}\jmath_e^{e_x} &= \jmath_h^{f_x} x^{-1} \\ &= (h \centerdot x^{-1})\jmath_{f_{h \centerdot x^{-1}}}^{e_x}. \end{aligned}$$

It follows, again by uniqueness of factorization, that

$$(e \centerdot x)^{-1} = f \centerdot x^{-1}$$

which shows that axiom (2) also holds. Therefore $\mathcal{G}$ is an ordered groupoid with respect to the partial order defined by (2.). This completes the proof. □

Suppose that $\mathcal{G}$ is an ordered groupoid. Then the embedding θ constructed in Theorem 1.21 embeds the groupoid $\mathcal{G}$ into the category of sets. Let $\mathfrak{C}(E)$ denote the smallest subcategory of sets containing $\theta(\mathcal{G})$ and the set of all inclusions

$$\mathbf{P} = \{ j_{\Lambda(e)}^{\Lambda(f)} : e, f \in \mathfrak{v}\mathcal{G}, \quad e \leq f \}$$

Clearly, $\mathbf{P}$ is a preorder isomorphic to $\mathfrak{v}\mathcal{G}$. Also $\mathfrak{C}(E)$ is a category with subobjects whose preorder of inclusions is $\mathbf{P}$. Since $\mathfrak{C}(E)$ is generated by $\theta(\mathcal{G})$ and $\mathbf{P}$, any morphism in $\mathfrak{C}(E)$ is a finite product of the form

$$j_i\sigma_1 \dots j_r\sigma_r, \quad j_i \in \mathbf{P}, \quad \sigma_i \in \theta(\mathcal{G})$$

for all $i = 1, \dots, r$, where all indicated compositions exists in **Set**; that is $\operatorname{cod} j_i = \operatorname{dom} \sigma_i$ for $i = 1, \dots r$ and $\operatorname{cod} \sigma_i = \operatorname{dom} j_{i+1}$ for $i = 1, \dots r-1$. Now let $j = j_{\Lambda(e)}^{\Lambda(f)}$ with and $\sigma : \Lambda(f) \to \Lambda(g) = \theta(x)$. Then, in **Set**, we have

$$j\sigma = \sigma|\Lambda(e) = (\sigma|\Lambda(e))^{\circ} j_{\sigma(\Lambda(e))}^{\Lambda(g)}.$$

As in the proof of the Theorem 1.21, $(\sigma|\Lambda(e))^{\circ} = \theta(x \centerdot x)$. Hence

$$j_{\Lambda(e)}^{\Lambda(f)}\theta(x) = \theta(e \centerdot x) j_{\Lambda(h)}^{\Lambda(g)} \tag{imfc}$$

where $h = f_{e \centerdot x}$. It follows inductively that every morphism has a unique factorization of the form σj where $\sigma = \theta(x)$ for some $x \in \mathcal{G}$ and j is an inclusion in $\mathbf{P}$. Thus $\mathfrak{C}(E)$ satisfies conditions (CG1) and (CG2) above and hence $\mathfrak{C}(E)$ is generated by $\mathcal{G}$.

Theorem 1.23. *Let $\mathcal{G}$ be an ordered groupoid. Then $\mathcal{G}$ generates a category $\mathfrak{C}(E)$ which is unique up to isomorphism. Further, if $\phi : \mathcal{G} \to \mathcal{H}$ is a morphism of ordered groupoids, there is a unique inclusion preserving functor $\mathfrak{C}(\phi) : \mathfrak{C}(E) \to \mathfrak{C}(H)$. Every inclusion preserving functor of $\mathfrak{C}(E)$ to $\mathfrak{C}(H)$ arises in this way.*

Proof. The discussion preceding the statement of the theorem shows that every ordered groupoid generates a category $\mathfrak{C}(E)$. Let C be another category generated by $\mathcal{G}$. Let

$$\psi = \theta^{-1}\theta'$$

where θ and θ' are embeddings of $\mathcal{G}$ in $\mathfrak{C}(E)$ and C respectively. By axiom (CG1) of Definition 1.7, ψ also induces an order isomorphism of the preorder of inclusions of $\mathfrak{C}(E)$ to $\mathcal{C}$. In view of axiom (CG2), ψ has a unique extension to an isomorphism of $\mathfrak{C}(E)$ to $\mathcal{C}$, defined by

$$\psi(j_1\sigma_1 \dots j_k\sigma_k) = \psi(j_1)\psi(\sigma_1)\dots\psi(j_k)\psi(\sigma_k). \tag{$\star$}$$

Thus $\mathfrak{C}(E)$ is unique up to isomorphism.

Let $\phi : \mathcal{G} \to \mathcal{H}$ be a morphism of ordered groupoids. The embeddings $\theta : \mathcal{G} \to \mathfrak{C}(E)$ and $\theta' : \mathcal{H} \to \mathfrak{C}(H)$ are isomorphisms of $\mathcal{G}$ onto the ordered groupoid $\theta(\mathcal{G})$ of isomorphisms of $\mathfrak{C}(E)$ (see proof of Proposition 1.22) and $\mathcal{H}$ onto $\theta'(\mathcal{H}) \subseteq \mathfrak{C}(H)$ respectively. Then

$$\phi^* = \theta^{-1} \circ \phi \circ \theta' : \theta(\mathcal{G}) \to \theta'(\mathcal{H})$$

is a morphism of ordered groupoids and so ϕ^* is an inclusion preserving functor of $\theta(\mathcal{G})$ to $\theta'(\mathcal{H})$. Then as in ($\star$), we can define a unique extension of ϕ^* to an inclusion preserving functor $\mathfrak{C}(\phi) : \mathfrak{C}(E) \to \mathfrak{C}(H)$. Conversely if $F : \mathfrak{C}(E) \to \mathfrak{C}(H)$ is an inclusion preserving functor, then $F|\theta(\mathcal{G})$ is a morphism of ordered groupoids and so

$$\phi = \theta \circ (F|\theta(\mathcal{G})) \circ (\theta')^{-1}$$

is a morphism of ordered groupoids $\mathcal{G}$ to $\mathcal{H}$. It follows from axiom (CG2) that $\mathfrak{C}(\phi) = F$. □

Remark* 1.6*: The construction of the theorem above can be routinely extended to a category equivalence $\mathfrak{C}$ of the category of ordered groupoid with the category of small categories satisfying conditions of Proposition 1.22. This means that in any discussion, we can always replace ordered groupoids and morphisms of ordered groupoids by categories generated by those groupoids and inclusion preserving functors respectively.

$\mathbb{N}$: *natural numbers*
$\mathbb{Q}$: *rational numbers*
$\mathbb{R}$: *real numbers*
$\mathbb{C}$: *complex numbers*
$\mathbb{N}^*$, $\mathbb{Q}^*$, $\mathbb{R}^*$, $\mathbb{C}^*$: *set of non-zero numbers*
X^n: *Cartesian product of n copies of X*
n-airy operation
null-airy operation
unary operation
binary operation
universal algebras
binary operation!associative –

CHAPTER 2

Semigroups

In this chapter we introduce some of the basic concepts of semigroup theory. The aim of this discussion is limited to setting up notations and to presenting those results of semigroup theory needed in the sequel. For details of topics and results covered here, we refer the reader to the standard books on semigroup theory such as Clifford and Preston [1961], Howie [1976], Lallement [1979], etc.

2.1 ELEMENTARY DEFINITIONS

In this section we give basic definitions of semigroups, homomorphisms, etc., and provide a list of standard examples.

Notation: In this book, we use the symbol $\mathbb{N}$ for natural numbers $\{0, 1, 2, \ldots\}$, $\mathbb{Q}$ for rational numbers, $\mathbb{R}$ for real numbers and $\mathbb{C}$ for complex numbers. For $X = \mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, X^* denote the set of non-zero numbers in X.

2.1.1 Monoids and semigroups

Let X be a set. For $n \in \mathbb{N}$, X^n denotes Cartesian product of n copies of X if $n \geq 1$ and a fixed singleton set $*$ if $n = 0$. A function $\odot : X^n \to X$ is called an *n-airy operation* on X. For $n = 0$, this is a mapping from $*$ to X and hence represent a choice of an element in X; it is called a *null-airy operation*. For $n = 1$, $\odot$ is a mapping of X to X and is called a *unary operation*. For $n = 2$, $\odot$ is called a *binary operation* on X. In This book we are mainly interested in binary operations. For a more general discussion of operations and the algebraic structures (called *universal algebras*) determined by them we refer the reader to Cohn [1965].

Thus a binary operation on X is a mapping $\cdot : X \times X \to X$; the value of the function $\cdot$ at $(x, y) \in X \times X$ is usually denoted by $x \cdot y$. The binary operation $\cdot$ is *associative* if

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \quad \forall\, x, y, z \in X;$$

semigroup
·, +, ∗, ∘: symbols for binary operations
binary operation!commutative –
semigroup!subsemigroup
semigroup!extension
dual
dual!left-right –
S^{op}: Left-right dual of S
T^{op}: dual of statement T
duality
self-dual
identity!left –

The binary operation $\cdot$ is commutative if

$$x \cdot y = y \cdot x \quad \forall\, x, y \in X.$$

A *semigroup* is a pair $(S, \cdot)$ consisting of a set S and an associative binary operation $\cdot$ on S. The binary operation $\cdot$ will be called the "product" of the semigroup. Other symbols such as $+, *, \circ$ etc., may also be used to represent product in a semigroup. However, often we shall not use any particular symbol to represent the product in a semigroup if it does not lead to any ambiguity. In this case, the product of $x, y \in S$ is simply indicated as xy. Again, for convenience, the set S itself will be used to denote the corresponding semigroup. A semigroup S is *commutative* if the product in S is commutative.

A subset T of a semigroup S is a *subsemigroup* of S if T is a semigroup with respect to the restriction of the binary operation of S to T; equivalently, if the subset T has the property that

$$T^2 = \{xy : x, y \in T\} \subseteq T$$

where xy denote the product of x and y in S. If T is a subsemigroup of S, then S is called an *extension* of T.

Left-right duality: If S is any semigroup, we can form the semigroup, denoted by S^{op}, as follows: the set underlying S^{op} is the same as the set underlying S and the binary operation of S^{op} (denoted by $\circ$ here) is defined by

$$x \circ y = yx \quad \forall\, x, y \in S. \tag{2.1}$$

It is clear that the binary operation $\circ$, called the *left-right dual* of the binary operation of S, is associative and hence S^{op} is a semigroup. We call the semigroup S^{op} also as the *left-right dual* of S. If T any statement about a semigroup, then we denote by T^{op} the statement obtained by replacing every occurrence of the binary operation in T by its left-right dual. The statement T^{op} is called the *left-right dual* of T. If T is true for S, it is clear that T^{op} must be true for S^{op}. Consequently, if T is a statement which is true for arbitrary semigroups, then T^{op} must also be true for arbitrary semigroups. The relation between statements T and T^{op} is called the *left-right duality* in semigroups. A statement T is *left-right self-dual* if T is the same as T^{op}. Note that statements about commutative semigroups are always left-right self-dual.

Identities and zeros: If S is a semigroup and $A \subseteq S$, an element $x \in S$ is a *left identity* of A if

$$xa = a \qquad \forall\, a \in A.$$

An element $x \in S$ is a *right identity* of A in S if it is a left identity of A in S^{op}. If x is both a left as well as a right identity of A, then it is called a [two-sided] *identity* of A. The element x is a left [right, two-sided] identity of S if the equation above and its left-right dual holds with $A = S$. A subset of semigroup may have more than one left [right] identities. However, while a proper subset $A \subseteq S$ may have more than one identity in S, identity of S, if it exists, is unique. The unique identity of S, if it exists, will usually be denoted by 1.

identity!right –
identity
monoid
monoid!sub–
S^1: monoid obtained by adjoining 1 to S
zero
zero!left, right –

Given any semigroup S we can always *adjoin a new* left [right] identity as follows: Let $T = S \cup \{e\}$ where e does not represent an element of S. Extend the multiplication in S to T by setting $ex = x$ [$xe = x$] for all $x \in S$ and $ee = e$. Clearly, this makes T a semigroup and e, a left [right] identity of T having S as a subsemigroup. Note that this construction works even if S already have left [right] identities. However, the old left [right] identities of S will no longer left [right] identities in T. Similarly, a *new* identity can be adjoined to S by extending the multiplication in S to T by

$$ex = x = xe \quad \forall\, x \in S \quad \text{and} \quad ee = e. \tag{2.2}$$

Again, as before, S is a subsemigroup of T and if S has identity, it will cease to be identity in T.

A semigroup S with identity is called a *monoid*. A *submonoid* M' of a monoid M is a subsemigroup such that the identity of M belongs to M' (which implies that M and M' have the same identity). Note that, it is possible that a subsemigroup S' of a monoid M may itself be a monoid with out being a submonoid of M. The remarks above implies that any semigroup can be extended to a monoid by adjoining a new identity to S. Given any semigroup S, we denote by S^1 the monoid defined as follows:

$$S^1 = \begin{cases} S & \text{if } S \text{ is a monoid,} \\ T & \text{if } S \text{ has no identity} \end{cases} \tag{2.3}$$

where T is the monoid obtained by adjoining an identity 1 to S.

An element z in a semigroup S is called a [*respectively left, right, two-sided*] *zero* of a subset $A \subseteq S$ if $za = z$ [$az = z$, $az = z = za$] for all $a \in A$. When $A = S$, we say that z is a [respectively left, right, two-sided] zero of S. Left and right zeros of S need not be unique. But a two-sided zero (or just *zero* for short) of S, when it exists, is unique and will be denoted by 0. As in the case of identities it is possible to adjoin a new left, right or two-sided zero to S. Thus if 0 does not represent an element of S, then $T = S \cup \{0\}$ becomes a semigroup with zero 0 having S as a subsemigroup if we extend the multiplication in S to T by:

$$0x = 0 = x0 \quad \forall\, x \in S \quad \text{and} \quad 00 = 0.$$

S^0: semigroup obtained by adjoining 0
idempotent
***E**(S): the biordered set of S*
ideal!left –
ideal!right –
ideal
ideal!proper –
$\mathfrak{I}_S$: lattice of ideals
$\mathfrak{LI}_S$: lattice of left ideals
$\mathfrak{RI}_S$: lattice of right ideals
ideal!maximal –
ideal!minimal –
ideal!0-minimal –

Again, as in Equation (2.2), we define S^0 by

$$S^0 = \begin{cases} S & \text{if } S \text{ has zero,} \\ T & \text{if } S \text{ has no zero} \end{cases} \tag{2.4}$$

where T is the semigroup obtained by adjoining a zero 0 to S. Note that an element e [z] in a semigroup S is the identity [zero] of S if and only if every element of S is a zero [identity] of the set $\{e\}$ [$\{z\}$]. Also given a semigroup S, for brevity, we shall often write $S = S^0$ to mean that the semigroup S has zero 0.

An element e in a semigroup is called an *idempotent* if $ee = e^2 = e$. Left identities, right identities, identity, left zero, right zero and zero of a semigroup S are all idempotents in S. Also if e is an idempotent, then the set of elements of S for which e is a left identity [respectively right identity, identity, left zero, right zero or zero] is non-empty (since each of this set contain e). In the following, we shall denote the set of all idempotents of S by $\boldsymbol{E}(S)$.

Ideals A subset I of a semigroup S is called a *left ideal* [*right ideal*] if for all $x \in I$ and $a \in S$, $ax \in I$ [$xa \in I$]. I is said to be a *two-sided ideal* (or simply an *ideal*) if I is both a left as well as a right ideal. Clearly S is an ideal. An ideal respectively, left or right ideal is said to be *proper* if it is different from S (so that it is a proper subset of S). If S has 0, then $\{0\}$ is clearly an ideal of S. In the following, if no confusion is likely, we shall denote this ideal also by 0. It is easy to see that the set of all ideals [respectively left ideals, right ideals] is a complete lattice under union and intersection; consequently, these are distributive lattices. We shall denote these lattices by $\mathfrak{I}_S$, (or $\mathfrak{I}$ if S is clear from the context) $\mathfrak{LI}_S$ (or $\mathfrak{LI}$) and $\mathfrak{RI}_S$ (or $\mathfrak{RI}$) respectively. Note that the empty subset $\emptyset$ of S is clearly an ideal in S and is the smallest ideal in S. We will follow that convention that $\mathbf{0}$ of $\mathfrak{I}_S$ (respectively $llat[S]$ and $\mathfrak{RI}_S$) is $\emptyset$ if S has no 0 and $\mathbf{0} = 0$ if S has 0. Thus in a semigroup with 0, an ideal is always non-empty. An ideal I is said to be maximal if I is maximal in the partially ordered set of all proper ideals and it is minimal if it is minimal in the partially ordered set of all non-empty ideals. If S has 0, then an ideal I is said to be 0-*minimal* if I is minimal in the partially ordered set of all non-zero ideals. Maximal, minimal and 0-minimal left or right ideals are defined in the obvious way.

If $\{I_\alpha : \alpha \in \Lambda\}$ is any set of left, right or two-sided ideals, then $\cap_{\alpha\in\Lambda} I_\alpha$ is an ideal of the same type. Hence, given any subset $A \subseteq S$, the set of ideal that contain A is not empty since S itself is a member of this set. Hence intersection $L(A)$ of all left ideals of S containing A is the smallest left ideal of S containing A; $L(A)$ is called the left ideal *generated* by A. Similarly the intersection $R(A)$ [$J(A)$] of all right [two-sided] ideals of S containing A is the right [two-sided] ideal generated by A. Given subsets A and B of a semigroup S, we use the notation $AB = \{ab : a \in A,\ b \in B\}$; AB is called the *set-product* (or, simply, the product) of A and B in S. It is easy to show, using the notation introduced in Equation (2.3) that

ideal!principal left –
$\mathbf{J}_S$, Λ_S, $\mathbf{I}_S$*: partially ordered set of principal ideals*
semigroup!simple –
semigroup!0-simple –
homomorphism
isomorphism
f°*: homomorphism onto the image of* f
embedding

$$L(A) = SA \cup A = S^1A; \tag{2.5}$$
$$R(A) = A \cup SA = AS^1; \tag{2.6}$$
$$J(A) = S^1AS^1. \tag{2.7}$$

When $A = \{a\}$, as usual, we write $L(a)$ for $L(\{a\})$; $L(a) = S^1a$ is called the *principal left ideal* generated by a. Similarly, $R(a) = aS^1$ denote the principal right ideal and $J(a) = S^1aS^1$ denote the principal ideal generated by a. The set $\mathbf{J}_S$ [Λ_S, $\mathbf{I}_S$] of all principal ideals, [principal left ideals, principal right ideals] is clearly a partially ordered subset of $\mathfrak{I}$ [respectively $\mathfrak{L}\mathfrak{I}$, $\mathfrak{R}\mathfrak{I}$]. Again the suffix S will be omitted if the semigroup S is clear from the context.

A semigroup S is said to be *simple* if S has no proper ideal; it is said to be 0-simple if S has 0 and 0 is the only proper ideal in S. Obviously similar definitions can be given for semigroups that are left [right] simple, left [right] 0-simple, etc.

2.1.2 Homomorphisms

Let S and T be two semigroups. A mapping $f : S \to T$ is called a *homomorphism* of S into T if

$$f(xy) = f(x)f(y) \quad \forall\ x, y \in S.$$

If $f : S \to T$ and $g : T \to U$ are homomorphisms of semigroup, it is easy to verify that $fg : S \to U$ is a homomorphism. A homomorphism f is *injective or surjective* if the map f is injective or surjective. A homomorphism $f : S \to T$ is said to be an *isomorphism* if the map f is a bijection. Clearly f is an isomorphism if and only if $f^{-1} : T \to S$ is a homomorphism. In particular, the identity map $1_S : S \to S$ is an isomorphism. Notice that for any homomorphism $f : S \to T$, $f(S) = \{f(s) : s \in S\}$ is a subsemigroup of T. Also, f considered as a homomorphism of S onto $f(S)$ (that is, the range restriction of f to $f(S)$) is a surjective homomorphism $f^\circ : S \to f(S)$ and we can factorize f as:

$$f = f^\circ j^T_{f(S)}. \tag{2.8}$$

Consequently, if $f : S \to T$ is an injective homomorphism $f^\circ : S \to T$ is an isomorphism of S onto $f(S)$. Thus an injective homomorphism is also called an *embedding* of S in T. If S' is a subsemigroup of S then the inclusion map $j^S_{S'}$ is clearly an injective homomorphism or an embedding of S' into S.

homomorphism!monoid – homomorphism!anti–isomorphism!anti–involution
$x \mapsto x^*$: involution
$\mathfrak{S}$: category of semigroups
homomorphism!image
products

If S and T are monoids, a homomorphism $f : S \to T$ is a monoid homomorphism if $f(1) = 1'$ where 1 [1′] is the identity of S [T]. Note that a monoid homomorphism is, in particular, a semigroup homomorphism; but there are semigroup homomorphisms of monoids that are not monoid homomorphisms. A monoid homomorphism is said to be injective, surjective or is an isomorphism if the corresponding semigroup homomorphism has the respective property. It is clear that $f : S \to T$ is a monoid isomorphism if and only if it is a semigroup isomorphism. Also, if M' is a submonoid of the monoid M, then $j^M_{M'} : M' \to M$ is a monoid homomorphism.

A homomorphism [isomorphism] $f : S \to T^{\text{op}}$ is called an *anti-homomorphism* [anti-isomorphism]. An anti-homomorphism $\theta : S \to S$ such that

$$\theta^2 = \theta \circ \theta = 1_S \tag{2.9}$$

is called an *involution* on S. An involution θ is therefore a unary operation and is denoted by notations like $\theta(x) = x^*$ x° or x' etc. Hence the assignment $x \mapsto x^*$ is an involution on S if and only if for all $x, y \in S$, we have

$$(xy)^* = y^* x^* \quad \text{and} \quad x^{**} = (x^*)^* = x. \tag{2.10}$$

Note that the second condition above implies that the assignment $x \mapsto x^*$ is, in fact, an anti-isomorphism.

The category $\mathfrak{S}$ The discussion above implies that we have a category $\mathfrak{S}$ in which objects are semigroups and morphisms are homomorphisms. $\mathfrak{S}$ has a natural subobject relation (see § Subsection 1.3.2). It is easy to verify that those inclusions that are morphisms in $\mathfrak{S}$ gives a choice of subobjects in $\mathfrak{S}$ according to Definition 1.5. Further, for any homomorphism $f : S \to T$ in $\mathfrak{S}$, in the factorization given by Equation (2.8), f° is a surjective homomorphism onto $f(S)$. Hence it is an epimorphism in $\mathfrak{S}$(see Remark 2.1 below). Clearly, the inclusion $f(S) \subseteq T$ is a morphism in $\mathfrak{S}$. Hence Equation (2.8) gives a canonical factorization of f in $\mathfrak{S}$. It is easy to see that this factorization satisfies condition (Im) of Proposition 1.13. Hence f° denote the epimorphic component of f and by Equation (1.53),

$$\operatorname{Im} f = f(S) = \{y \in T : y = f(x) \quad \text{for some} \quad x \in S\}. \tag{2.8*}$$

is the image of f. Thus the category $\mathfrak{S}$ has images (see § Subsection 1.3.2). Since f° is an isomorphism if f is injective, it follows that every injective homomorphism is an embedding (see § Subsection 1.3.2). Therefore, by Remark 2.1 below, every monomorphism is an embedding in $\mathfrak{S}$. The discussion in § Section 2.3 shows that $\mathfrak{S}$ has *products* in the usual categorical sense (see MacLane [1971], page 69,70).

Similarly, there exists a category $\mathfrak{M}$ whose objects are monoids and morphisms are monoid homomorphisms. Thus $\mathfrak{M}$ is a subcategory of $\mathfrak{S}$ with subobjects, factorizations and images.

Remark* 2.1*: In the category $\mathfrak{S}$, a homomorphism is a monomorphism if it is injective and an epimorphism if it is surjective. This follows from the fact that in the category **Set**, a map (morphism in **Set**) is a monomorphism if and only if it is injective and an epimorphism if and only if it is surjective.

Conversely, every monomorphism in $\mathfrak{S}$ is injective. To see this, we first observe that $N = \{1, 2, \ldots\}$ is a semigroup under addition and if x is any element of a semigroup S, then there is a unique homomorphism $\theta_x : N \to S$ sending 1 to x (set $\theta_x(n) = x^n$ for all $n \in \mathbb{N}$). Now if $f : S \to T$ is a homomorphism which is not injective, then there is $x, y \in S$ with $x \neq y$ such that $f(x) = f(y)$. Then $\theta_x \circ f = \theta_y \circ f$ and $\theta_x \neq \theta_y$. Hence f is not a monomorphism in $\mathfrak{S}$.

However, not all epimorphisms in $\mathfrak{S}$ are surjective. We can construct an counter example (which is an adaptation of the example given in Remark 1.6 of Lallement [1979]) as follows. Let R^* be the set of non-zero elements of an integral domain (commutative and with identity) R. Then R^* is a subsemigroup of the multiplicative semigroup of R. Let D^* denote the group of non-zero elements of the field of fractions of R. Then $D^* = \{\frac{a}{b} : a, b \in R^*\}$. Let $f : a \mapsto \frac{a}{1}$ be the embedding of R^* in D^*. Then, if $\theta : D^* \to T$ is any homomorphism of D^* to a semigroup T, then $\operatorname{Im}\theta$ is a subgroup of T and

$$\theta\left(\frac{a}{b}\right) = (\theta(f(a)))\,(\theta(f(b)))^{-1}$$

for all $\frac{a}{b} \in D^*$. Hence if $\theta_i : D^* \to T$, $i = 1, 2$ are homomorphisms such that $f \circ \theta_1 = f \circ \theta_2$, then $\theta_1 = \theta_2$. Therefore f is an epimorphism. If R is not a field, then f is not surjective.

The arguments above can be easily adopted for the category $\mathfrak{M}$. Thus a monoid homomorphism f is a monomorphism in $\mathfrak{M}$ if and only if f is injective; f is an epimorphism if it is surjective; but the converse is not true.

2.1.3 Examples

Here we give a list of examples of semigroups. These are standard examples and we shall have occasion later to refer back to some of these.

The semigroup of relations on the set* X*: From the discussion in Subsection 1.1.1 it follows that $\boldsymbol{B}_X$ is a semigroup in which product is the composition of relations defined by Equation (1.2). It has identity 1_X and zero $\emptyset$. This semigroup has additional structures. Since every $R \in \boldsymbol{B}_X$ has the unique converse R^{-1} defined by Equation (1.4) (see also § Subsection 1.1.1) the assignment $R \mapsto R^{-1}$ is a mapping and hence a unary operation on $\boldsymbol{B}_X$. It is easy to see that it is an involution (that is, satisfies conditions given in Equation (2.10)). Moreover, $\boldsymbol{B}_X$ is an *ordered semigroup* in the sense that *inclusion* $\subseteq$ is a partial order on $\boldsymbol{B}_X$ compatible with the binary operation:

$$R_1 \subseteq R_2 \Rightarrow \begin{cases} R \circ R_1 \subseteq R \circ R_2 & \text{and} \\ R_1 \circ R \subseteq R_2 \circ R. \end{cases}$$

The involution $R \mapsto R^{-1}$ is also admits the order on $\boldsymbol{B}_X$:

$$R_1 \subseteq R_2 \;\Rightarrow\; R_1^{-1} \subseteq R_2^{-1}.$$

Note that $\emptyset$, the zero of $\boldsymbol{B}_X$, is the smallest element with respect to this order and $X \times X$ is the largest.

This semigroup has several important subsemigroups; we list some of them below.

The semigroup of partial transformations: Let $\mathscr{PT}_X$ denote the set of all partial transformations (single-valued relations) on the set X. Since composite of single-valued relations are single-valued, $\mathscr{PT}_X$ is a subsemigroup of the semigroup $\boldsymbol{B}_X$. Each $\alpha \in \mathscr{PT}_X$ is a surjective function $\alpha : \operatorname{dom}\alpha \to \operatorname{Im}\alpha$ and hence determines an equivalence relation π_α on (partition of) $\operatorname{dom}\alpha$ and a bijection of $\operatorname{dom}\alpha/\pi_\alpha$ onto $\operatorname{Im}\alpha$ (see Equations (1.10a) and (1.10b)). Hence each $\alpha \in \mathscr{PT}_X$ determines a symmetric and transitive relation π_α and a subset $Y \subseteq X$ such that

$$|\operatorname{dom}\alpha/\pi_\alpha| = |Y|.$$

α is an idempotent in $\mathscr{PT}_X$ if and only if Y is a cross-section of π_α (that is, a subset such that it intersect every partition class in exactly one element). Also every pair (π, Y) where π is a symmetric and transitive relation and Y is a cross-section determines a unique idempotent in $\mathscr{PT}_X$. Moreover, any $\alpha \in \mathscr{PT}_X$ can be factorized relative to a cross-section Y of π_α as

$$\alpha = e \circ \tilde{\alpha}$$

where e is the idempotent determined by (π_α, Y) and $\tilde{\alpha} = \alpha|Y$ is a bijection.

The semigroup I_X: We denote by $\boldsymbol{I}_X$ the set of all injective elements of $\mathscr{PT}_X$. If $\alpha, \beta \in \boldsymbol{I}_X$, it is easy to verify that $\alpha \circ \beta \in \boldsymbol{I}_X$. Hence $\boldsymbol{I}_X$ is a subsemigroup of $\mathscr{PT}_X$. Since every $\alpha \in \boldsymbol{I}_X$ is injective, the equivalence relation π_α induced by α on $\operatorname{dom}\alpha$ is the identity relation on $\operatorname{dom}\alpha$. Therefore $\boldsymbol{I}_X$ consists of all bijections of subsets of X. In particular, idempotents in $\boldsymbol{I}_X$ are identity maps on subsets of X. In fact $\boldsymbol{I}_X$ is a subsemigroup of $\boldsymbol{B}_X$ which inherits the structure of $\boldsymbol{B}_X$. Thus, since $\alpha^{-1} \in \boldsymbol{I}_X$ for all $\alpha \in \boldsymbol{I}_X$, the involution $R \mapsto R^{-1}$ restricts to an involution on $\boldsymbol{I}_X$. $\boldsymbol{I}_X$ is also an ordered semigroup with respect to inclusion and the set of idempotents of $\boldsymbol{I}_X$ has the structure of a lattice (Boolean algebra) with respect to inclusion (see also Section Subsection 1.4.2).

***Remark* 2.2:** Clearly, the inclusion gives $\mathscr{P}\mathscr{T}_X$ the structure of an ordered semigroup. However, this is not of much significance for $\mathscr{P}\mathscr{T}_X$. On the other hand, the inclusion is an important structural part of the semigroup $\boldsymbol{I}_X$ and is called the *natural partial order* on $\boldsymbol{I}_X$. It may also be noted that the groupoid $\boldsymbol{I}_X$ considered in Example 1.21 is obtained from the semigroup $\boldsymbol{I}_X$ by restricting the product in the semigroup to pairs α, β with $\operatorname{cod}\alpha = \operatorname{dom}\beta$ in the semigroup $\boldsymbol{I}_X$; in other words, the product in the semigroup $\boldsymbol{I}_X$ is an extension of the composition in the groupoid $\boldsymbol{I}_X$.

The semigroup $\mathbf{T}_X$**:** The set $\mathscr{T}_X$ of all transformations on X (maps of X into X) is clearly a subsemigroup of $\mathscr{P}\mathscr{T}_X$. Hence most of the remarks for the semigroup $\mathscr{P}\mathscr{T}_X$ can be adopted for $\mathscr{T}_X$. Thus idempotents in $\mathscr{T}_X$ are uniquely determined by pairs (π, Y) where π is an equivalence relation on X and Y is a cross-section of π. Further every $f \in \mathscr{T}_X$ can be factorized as

$$f = e \circ \alpha$$

where e is an idempotent in $\mathscr{T}_X$ and α is a bijection of the cross-section $\operatorname{Im} e$ of $\pi_f = \pi_e$ onto $\operatorname{Im} f$. Also, composition in $\mathscr{T}_X$ (same as relational composition) is written in the order in which they appear in commutative diagrams and elements of $\mathscr{T}_X$ (transformations of X) are regarded as operating on the right. Often it will also be necessary to consider the left-right dual $\mathscr{T}_X^{\mathrm{op}}$ of $\mathscr{T}_X$ or subsemigroups of $\mathscr{T}_X^{\mathrm{op}}$. In this case transformations of X, considered as elements of $\mathscr{T}_X^{\mathrm{op}}$, are written as left operators.

This (that is $\mathscr{T}_X$) gives an important class of examples; we shall discuss other properties of these semigroups later. All examples of semigroups given so far are all monoids.

Semilattices: A *semilattice* is a *commutative semigroup of idempotents* (that is, a semigroup in which every element is an idempotent). If E is a lover semilattice (meet-semilattice) as defined in Subsection 1.1.2, then clearly, the map

$$(e, e') \in E \times E \mapsto e \wedge e'$$

which assign to each (e, e'), the meet $e \wedge e'$ is a binary operation on E. It follows from the definition of $\wedge$ (see Equation (1.13)) that this binary operation is associative, commutative and idempotent. Hence E is a semilattice with respect to the binary operation $\wedge$. In the partially ordered set does not have $\mathbf{1}$, then E is a semigroup which is not a monoid. Similarly, if E is an upper semilattice, E is a semilattice in the sense above with respect to $\vee$.

Conversely, any semilattice E (as defined above) can be considered as a lover semilattice as follows: for $e, e' \in E$, define

$$e \leq e' \iff ee' = e.$$

semigroup!cyclic – semigroup!cyclic semigroup!generator

Then $\leq$ is a partial order on E such that

$$ee' = e \wedge e' \quad \forall\, e, e' \in E.$$

Hence E becomes the lover semilattice with respect to the partial order defined above. On the other hand if we set

$$e \leq e' \iff ee' = e',$$

then $\leq$ is a partial order on E and E becomes the upper semilattice with respect to $\leq$. Thus a semilattice is a semigroup; it can be regarded as an order structure in two ways: as a lover semilattice or an upper semilattice as above. In the following, unless otherwise stated, a semilattice will be regarded as a lover semilattice.

Cyclic semigroups: A semigroup S is said to be *cyclic* if every element of S is a positive integral power of an element in S; that is, $S = \{a^n : n \in \mathbb{N}^*\}$ for some $a \in S$, where $\mathbb{N}^* = \{1, 2, \dots\}$. The element a is called the *generator* of S and S is denoted by $\langle a \rangle$. There are two possibilities:

1. Powers of a are distinct. In this case $\langle a \rangle$ is clearly infinite and is isomorphic to the additive semigroup $(\mathbb{N}^*, +)$.
2. Not all powers of a are distinct; that is, $a^n = a^m$ for some $n, m \in \mathbb{N}^*$, $n \neq m$.

In the second case, there exists the smallest integer $s > 1$ such that $a^r = a^s$ for some $r < s$ with $r \geq 1$. The choice of s implies that

$$a, a^2, \dots, a^r, \dots, a^{s-1}$$

are distinct powers of a in $\langle a \rangle$. We show that these are precisely the set of all distinct elements of $\langle a \rangle$ so that, in this case, the order of $\langle a \rangle$ is $s - 1$.

PROPOSITION 2.1. *Let $S = \langle a \rangle$ be the cyclic semigroup generated by a. Then either S is isomorphic to $(\mathbb{N}^*, +)$ or there exists positive integers r and m such that*

$$S = \{a, a^2, \dots, a^r, a^{r+1}, \dots, a^{r+m-1}\};$$

the order of S being $r + m - 1$. The set

$$K_a = \{a^r, a^{r+1}, \dots, a^{r+m-1}\}$$

is a cyclic subgroup of S of order m with identity a^t where t is the unique integer satisfying

$$t \equiv 0 \pmod{m}, \quad r \leq t < r + m.$$

The integer r is called the index of S and m is called the period of S.

order!– of an element
semigroup!periodic

Proof. In view of the discussion preceding the statement, it is sufficient to consider the case in which there is the smallest positive integer s and $0 < r < s$ such that $a^r = a^s$. In this case, powers a^i, $1 \leq i < s$ are distinct elements of S. Let $m = s - r$. Then we have $a^r = a^{r+m} = a^{r+km}$ for all $k \in \mathbb{N}$. If $n \geq r$, we can find $k \in \mathbb{N}$ and $0 \leq j < m$ such that $n - r = km + j$ and so

$$a^n = a^{r+km+j} = a^{r+j} \quad \text{where} \quad n \equiv r + j \pmod{m}, \quad r \leq r + j < r + m. \qquad (\star)$$

It follows that $S = \{a^i : 1 \leq i < r + m\}$. Since these powers are all distinct, the order of S is $r + m - 1$. We now show that for any $n, n' \in \mathbb{N}^*$, we have

$$a^n = a^{n'} \iff \begin{cases} n = n' & \text{if } \min\{n, n'\} < r, \\ n \equiv n' \pmod{m} & \text{if } \min\{n, n'\} \geq r. \end{cases} \qquad (\bullet)$$

Assume that $n < n'$. If $n < r$, then by the definition of $s = r + m$, $n' \geq s$. By $(\star)$, there is a unique n'' with $r \leq n'' < s$ such that $a^{n''} = a^{n'} = a^n$ which contradicts the definition of s. Hence we must have $n = n'$. Let $n \geq r$. Choose $p, q \in \mathbb{N}^*$ with $r \leq p, q < s$, $n \equiv p \pmod{m}$ and $n' \equiv q \pmod{m}$. Then $a^n = a^{n'}$ implies by Equation $(\star)$ that $a^p = a^q$. Since, for $i < s$, powers a^i are distinct, it follows that $p = q$ and so, $n = n' \pmod{m}$. On the other hand, if $n = n' \pmod{m}$, then it is immediate from $(\star)$ that $a^n = a^{n'}$.

It follows from $(\bullet)$ that the map $\phi : a^n \mapsto n \pmod{m}$, $n \geq r$ is a bijection of K_a onto $\mathbb{Z}_m$, the cyclic group of integers $\pmod{m}$. Also, for $p, q \geq r$,

$$\begin{aligned} \phi(a^p a^q) &= \phi(a^{p+q}) = (p + q) \pmod{m} \\ &= p \pmod{m} + q \pmod{m} = \phi(a^p) + \phi(a^q). \end{aligned}$$

Therefore $\phi : K_a \to \mathbb{Z}_m$ is an isomorphism. If $a^t \in K_a$, $r \leq t < r+m$, is the identity in K_a, then we have $a^t a^t = a^t$ and so, by Equation $(\bullet)$, $t = 0 \pmod{m}$. □

The *order* of an element a in a semigroup S is the order of the cyclic subsemigroup $\langle a \rangle$ of distinct powers of a in S. The order of a is finite if the order of $\langle a \rangle$ is finite; otherwise, the order of a is infinite. The semigroup S is said to be *periodic* if the order of every $a \in S$ is finite.

Example 2.1: Given two positive integers r and m there is a finite cyclic semigroup with index r and period m. Consider the transformation α of the set $M = \{0, 1, \ldots, r, \ldots, r + m - 1\}$ defined as follows:

$$i\alpha = \begin{cases} i + 1 & \text{if } i < r + m - 1, \\ r & \text{if } i = r + m - 1. \end{cases}$$

It is easy to show that $\alpha^r = \alpha^{r+m}$ which implies that the cyclic semigroup $\langle \alpha \rangle$ of all powers of α is a finite cyclic semigroup of index r and period m.

monoid!cyclic –
group with 0
matrix!Rees –
matrix!monomial –
$\mathcal{M}^0(G; I, \Lambda; P)$: *Rees* $I \times \Lambda$*-matrix semigroup over* G^0
semigroup!Rees matrix –
matrix!sandwich –

The *cyclic monoid* M generated by a is the set $\{a^n : n \in \mathbb{N}\}$ (including a^0 which is defined as 1). Note that $M = \langle a \rangle^1$, the monoid obtained by adjoining identity 1 to $\langle a \rangle$. It follows from the above that the cyclic monoid $\langle a \rangle^1$ is either infinite in which case all powers of a (including a^0) are distinct and is isomorphic to $(\mathbb{N}, +)$, or there exist integers r and m such that $a^r = a^{r+m}$, in which case the monoid $\langle a \rangle^1$ is of order $r + m$.

Rees-matrix semigroups: Let G be a group and let G^0 be the semigroup obtained by adjoining 0 to G (see Equation (2.4)). G^0 is called a *group with* 0. Let Λ and I be sets. A mapping $P : \Lambda \times I \to G^0$ is a $\Lambda \times I$-matrix over G^0; we denote the value of P at (λ, i) by $p_{\lambda i}$. Let a

$$\mathcal{M}^0(G; I, \Lambda; P) = (G \times I \times \Lambda) \cup \{0\}. \tag{2.11a}$$

Define product in $\mathcal{M}^0(G; I, \Lambda; P)$ by

$$(a, i, \lambda) \cdot (b, j, \mu) = \begin{cases} (ap_{\lambda j}b, i, \mu) & \text{if } p_{\lambda j} \neq 0; \\ 0 & \text{otherwise.} \end{cases} \tag{2.11b}$$

Non-zero elements (a, i, λ) of $\mathcal{M}^0(G; I, \Lambda; P)$ can be interpreted as $I \times \Lambda$-matrices as:

$$(a, i, \lambda) = (a_{i'\lambda'})_{I \times \Lambda}$$

in which

$$a_{i'\lambda'} = \begin{cases} a & \text{if } (i', \lambda') = (i, \lambda); \\ 0 & \text{if } (i', \lambda') \neq (i, \lambda). \end{cases}$$

Such matrices are called *monomial matrices* or *Rees matrices*. The element 0 is treated as $I \times \Lambda$ 0-matrix $(0)_{I \times \Lambda}$. If we do this, the product defined above reduces to the row-column product

$$(a, i, \lambda) \cdot (b, j, \mu) = (a, i, \lambda) P (b, j, \mu).$$

If $x = (a, i, \lambda)$, $y = (b, j, \mu)$ and $z = (c, k, \nu)$ are elements of $\mathcal{M}^0(G; I, \Lambda; P)$, then using Equation (2.11b), we have

$$(xy)z = (ap_{\lambda j}bp_{\mu k}c, i, \nu) = x(yz).$$

Hence $\mathcal{M}^0(G; I, \Lambda; P)$ is a semigroup. $\mathcal{M}^0(G; I, \Lambda; P)$ is called the *Rees* $I \times \Lambda$*-matrix semigroup over* G^0 with *sandwich matrix* P. The sandwich matrix P is said to be *regular* if every row and every column contain at least one non-zero semigroup!Rees matrix – with out zero

entry; that is, for each $\lambda \in \Lambda$ there is some $i \in I$ such that $p_{\lambda i} \neq 0$ and for each $i \in I$ there is $\lambda \in \Lambda$ with $p_{\lambda i} \neq 0$.

Let $x = (a, i, \lambda) \in \mathcal{M}^0(G; I, \Lambda, P)$. If P is regular, we can find $\mu \in \Lambda$ such that $p_{\mu i} \neq 0$ and $j \in I$ such that $p_{\lambda j} \neq 0$. Let

$$x' = (b, j, \mu) \quad \text{where} \quad p_{\lambda j} \neq 0,\ p_{\lambda j} \neq 0,\ b = (p_{\mu i} a p_{\lambda j})^{-1}.$$

An easy computation with the product defined above shows that

$$xx'x = x.$$

Hence, to each $x \in \mathcal{M}^0(G; I, \Lambda; P)$ there is some $x' \in \mathcal{M}^0(G; I, \Lambda; P)$ satisfying the equation above if P is regular. Conversely, if the semigroup $\mathcal{M}^0(G; I, \Lambda; P)$ has this property, it can be shown easily that the matrix P must be regular as defined. For if $x = (a, i, \lambda) \neq 0$ (that is, $a \neq 0$) the condition implies that for some $x' = (b, j, \mu)$, $x = xx'x$ and so we have, in particular, $xx' \neq 0$. This implies that $p_{\lambda j} \neq 0$ and so for $\lambda \in \Lambda$, there is $j \in I$ with $p_{\lambda j} \neq 0$. Similarly from $x'x \neq 0$ we infer that for $i \in I$, there is $\mu \in \Lambda$ with $p_{\mu i} \neq 0$. Semigroups satisfying this condition is said to be *regular* (see Subsection 2.6.2).

By Equation (2.11b), the set of non-zero elements of $\mathcal{M}^0(G; I, \Lambda; P)$ is a subsemigroup of $\mathcal{M}^0(G; I, \Lambda; P)$ if and only if $p_{\lambda i} \neq 0$ for all $(\lambda, i) \in \Lambda \times I$. When P satisfies this condition, we denote the subsemigroup of non-zero elements by $\mathcal{M}(G; I, \Lambda; P)$; it is called the *Rees matrix semigroup over* the group G or a Rees matrix semigroup *with out zero*. Note that $\mathcal{M}(G; I, \Lambda; P)$ is always regular. In particular, if we choose $G = \{1\}$, the one element group and P as the constant mapping with value 1, then the Rees matrix semigroup over G can be identified with a semigroup on the set $I \times \Lambda$ with product defined by

$$(i, \lambda) \cdot (j, \mu) = (i, \mu) \quad \forall\ (i, \lambda), (j, \mu) \in I \times \Lambda.$$

This semigroup is called the $I \times \Lambda$-*rectangular band*. Note that every element in the $I \times \Lambda$-rectangular band is idempotent.

Semigroup of matrices and linear transformations: Let V be a vector space over the field $\Bbbk$. It is well known that the set $\mathscr{L}\mathscr{T}(V)$ of all linear endomorphisms of V is a semigroup under composition and so it is a subsemigroup of $\mathscr{T}_V$. In this case $\epsilon \in \mathscr{L}\mathscr{T}(V)$ is an idempotent if and only if

$$\boldsymbol{N}(\epsilon) \oplus \operatorname{Im} \epsilon = V$$

where $\boldsymbol{N}(\epsilon)$ denote the subspace of V given by:

$$\boldsymbol{N}(\epsilon) = \{v \in V : \epsilon(v) = 0\}.$$

congruences
congruence
congruence!right
congruence!left

Conversely, given any direct-sum decomposition $N \oplus U = V$, there is a unique idempotent $\epsilon \in \mathscr{L}\mathscr{T}(V)$ with $\boldsymbol{N}(\epsilon) = N$ and $\operatorname{Im}\epsilon = U$ (as well as an idempotent ϵ' with $\boldsymbol{N} = U$ and $\operatorname{Im}\epsilon' = N$; we have $\epsilon' = 1 - \epsilon$). As in Subsection 2.1.3, every $f \in \mathscr{L}\mathscr{T}(V)$ can be factorized as $f = \epsilon \circ \alpha$ where ϵ is an idempotent in $\mathscr{L}\mathscr{T}(V)$ and $\alpha : \operatorname{Im}\epsilon \to \operatorname{Im} f$ ia a linear isomorphism.

Further properties of this semigroup will be considered later.

2.2 CONGRUENCES

Let $\phi : G \to H$ be a surjective homomorphism of groups. The basic homomorphism theorem for groups states that the quotient group $G/\ker\phi$ is isomorphic to H. This implies that, up to isomorphism, ϕ is completely determined by the normal subgroup $\ker\phi$ (see for example, Hungerford [1974]). Moreover, $\ker\phi$ is an object of the same type as G and is the kernel of the morphism in the category **Grp** of groups. On the other hand, for homomorphisms of semigroups there exist no sub-semigroup, or an object in the category $\mathfrak{S}$ which determines homomorphisms in this way. In particular the category $\mathfrak{S}$ does not have kernels. This is an important point of difference between group theory and semigroup theory. If $\psi : S \to T$ is a homomorphism of semigroups, it is necessary to replace the kernel in the theory of group homomorphisms with the equivalence relation π_ψ determined by the function ψ as in Equation (1.10a). Equivalence relations arising in this way are called *congruences*.

In this section, we give preliminary definitions of congruences and derive some of the basic properties of homomorphisms. We also give a brief discussion of the lattice of congruences on a semigroup.

2.2.1 Congruences and homomorphisms

Let S be a semigroup. A relation ρ on S is *right compatible* if ρ satisfies the following: a

$$(x, y) \in \rho \;\Rightarrow\; (xa, ya) \in \rho \quad \forall a \in S^1. \tag{2.12a}$$

A relation ρ is *left compatible* if it is right compatible as a relation on S^{op}. ρ is *compatible* if it is both left and right compatible. For any $\rho \in \boldsymbol{B}_S$

$$R^S = \{(axb, ayb) : a, b \in S^1 \quad \text{and} \quad (x, y) \in R\}. \tag{2.12b}$$

can be shown to be the smallest compatible relation that contain R.

A *right* [*left*] *congruence* on a semigroup S is an equivalence relation ρ on S which is right [respectively, left] compatible. ρ is a *congruence* on S if it is compatible so that it is both a right and a left congruence on S. This is equivalent to the fact that ρ satisfies the following:

$$(x, y), (x', y') \in \rho \;\Rightarrow\; (xx', yy') \in \rho. \tag{2.13}$$

S/ρ: quotient semigroup
homomorphism!quotient –

We have the following:

PROPOSITION 2.2. *Let ρ be a congruence on the semigroup S. For each $x \in S$, let $\rho(x)$ denote the ρ-class containing x (the equivalence with respect to ρ that contain x). Then*

$$\rho(x) \circ \rho(y) = \rho(xy) \quad \forall\ x, y \in S \tag{2.13*}$$

defines a single-valued binary operation $\circ$ on

$$S/\rho = \{\rho(x) : x \in S\} \tag{2.14}$$

which is associative. Hence S/ρ is a semigroup with respect to $\circ$. Moreover, the quotient map $\rho^\# : x \mapsto \rho(x)$ is a surjective homomorphism of S onto S/ρ.

Proof. The fact that $\circ$ is single-valued is equivalent to Equation (2.13). The remaining statements are immediate consequence of the definitions. □

We denote the semigroup constructed above also by S/ρ and is called the *quotient* of S with respect to the congruence ρ.

Example 2.2: If G is a group, an equivalence relation ρ on G is a left congruence if and only if the equivalence class $\rho(e) = K$ containing the identity e of G is a subgroup of G and $\rho(g) = gK$ for all $g \in G$. Thus ρ is a left congruence on G if and only if the partition of ρ is a left-coset decomposition of G with respect to the subgroup $\rho(e)$ of G. Similarly a right congruence ρ on G is a right-coset decomposition with respect to the subgroup $\rho(e)$. Thus ρ is a congruence on G if and only if for all $g \in G$, $\rho(g)$ is a left as well as a right coset of G. This is true if and only if $\rho(e)$ is a normal subgroup of G.

Isomorphism theorems of group theory can be extended to semigroups. The following is the analogue of the first homomorphism for groups. The routine verification is omitted.

THEOREM 2.3 (FIRST ISOMORPHISM THEOREM). *Let $\phi : S \to T$ be a homomorphism of the semigroup S =into T. Then*

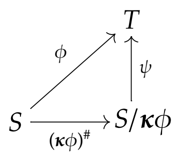
$$\boldsymbol{\kappa}\phi = \{(x, y) : x\phi = y\phi\}$$

is a congruence on S. Further, the map $\psi : S/\boldsymbol{\kappa}\phi \to T$ defined by

$$(\boldsymbol{\kappa}\phi(x))\psi = x\phi \quad \forall\ x \in S$$

is an injective homomorphism such that the following diagram commutes:

(D2)



The homomorphism ϕ is injective if and only if $\boldsymbol{\kappa}\phi = 1_S$ and surjective if and only if $\psi : S/\boldsymbol{\kappa}\phi \to T$ is an isomorphism. □

Other isomorphism theorems for groups can also be extended to semigroups by replacing *subgroups* by *subsemigroups* and *normal subgroups* by *congruences* in the corresponding statements for groups. Thus the second isomorphism theorem can be stated as follows.

THEOREM 2.4 (SECOND ISOMORPHISM THEOREM). *Let σ be a congruence on the semigroup S and let T be a subsemigroup of S. Then the restriction $\sigma_T = \sigma \cap (T \times T)$ of σ to T is a congruence on T and there is an isomorphism*

$$\phi : T/\sigma_T \to \sigma(T)/\sigma$$

where

$$\sigma(T) = \bigcup_{t \in T} \sigma(t)$$

denote the union of all congruence classes of σ that intersect T.

Proof. Let $\Phi = \sigma^{\#}|T$ denote the restriction of the quotient homomorphism $\sigma^{\#} : S \to S/\sigma$ to T. Then Φ is a homomorphism of T into S/σ. It is easy to see that $\boldsymbol{\kappa}\Phi = \sigma_T$ and $\operatorname{Im}\Phi = \sigma(T)/\sigma$. By Theorem 2.3, there is an isomorphism of T/σ_T onto $\sigma(T)/\sigma$. □

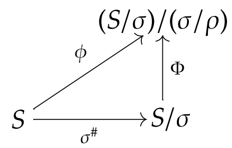
THEOREM 2.5 (THIRD ISOMORPHISM THEOREM). *suppose that $\rho, \sigma \in \mathfrak{L}$ are congruences on S such that $\rho \subseteq \sigma$. Then*

$$\sigma/\rho = \{(\rho(x), \rho(y)) : (x, y) \in \sigma\}$$

is a congruence on S/ρ such that there is an isomorphism

$$\Phi : S/\sigma \to (S/\rho)/(\sigma/\rho)$$

making the following diagram commute.



(2.15)

Moreover, $\sigma \mapsto \sigma/\rho$ is an inclusion preserving bijection of the set of all congruences on S containing ρ and the set of all congruences on S/ρ.

Proof. It is easy to verify that σ/ρ is a congruence of S/ρ and that the map $\phi : x \mapsto \sigma/\rho(\rho(x))$ is a homomorphism such that $\boldsymbol{\kappa}\phi = \sigma$. By Theorem 2.3, there is an isomorphism $\Phi : S/\sigma \to (S/\rho)/(\sigma/\rho)$ The last statement is also easy to verify (see Proposition 2.8 and Remark 2.5). □

congruence!Rees –
semigroup!Rees quotient –
S/I: Rees quotient semigroup of S by ideal I
extension!ideal –

Rees congruences Let I be an ideal in a semigroup S and let

$$\rho_I = \{(x, y) : \text{either } x = y \text{ or } x, y \in I\}. \tag{2.16}$$

It is easy to verify that ρ_I is a congruence on S such that

$$\rho_I(x) = \begin{cases} I & \text{if } x \in I; \\ \{x\} & \text{if } x \notin I. \end{cases}$$

Note that the congruence ρ_I is completely determined by the ideal I. ρ_I has atmost one non-trivial congruence class I in S. Congruences determined by ideals in this way are called *Rees congruences*. The quotient (or factor) semigroup S/ρ_I is called the *Rees quotient semigroup* or *Rees factor semigroup* and is denoted (for brevity) as S/I. Note that S/I is the semigroup obtained by identifying all elements in I as a single element I in S/I which is the 0 of S/I and leaving every other element (not in I) unaltered. We have noted that $\emptyset$ is an ideal in S. We follow the convention that the Rees quotient $S/\emptyset = S$. Similarly, if S has 0, then for $I = 0$, the Rees quotient S/I is isomorphic to S; in this case also, we will assume that $S/I = S/0 = S$.

Let T and N be semigroups. A semigroup S is called an *ideal extension* of the semigroup N by the semigroup T if N is isomorphic an ideal N' of S and the Ree's factor semigroup S/N' is isomorphic to T.

We observe that, if ρ is any congruence on S the S/ρ is a semigroup with zero if and only if there is a congruence class $\rho(x)$ which is an ideal in S; in this case, the zero in S/ρ is the ideal $I = \rho(x)$ and $\rho_I \subseteq \sigma$. Thus every such congruence contains a Rees congruence.

Remark 2.3: Isomorphism theorems for Rees congruences can be stated in much more simpler way. Thus the second and third isomorphism theorems can be stated as follows:

(a) Let I be an ideal in the semigroup S and let T be a subsemigroup. Then

$$(I \cup T)/I \cong T/(I \cap T).$$

(b) Let I and J be ideals in the semigroup S such that $I \subseteq J$. Then

$$S/J \cong (S/I)/(J/I).$$

Moreover, $J \mapsto J/I$ is an inclusion preserving bijection of the set of all ideals of S containing I and the set of all ideals of S/I.

These statements follows immediately from Theorems 2.4 and 2.5 respectively.

Remark 2.4: For a more detailed discussion of ideal theory of semigroups, including the theory of ideal series such as composition series, principal series, etc. and the Jordan-Hö lder-Schreier refinement theorem, we refer the reader to Clifford and Preston [1961], Rees [1940]. Since we have no occasion to use these results in this book, we shall not discuss them here.

2.2.2 The lattice of congruences

Let $\mathfrak{L}$ denote the set of all congruences on the semigroup S. Then $\mathfrak{L}$ is nonempty since the identity relation 1_S and the universal relation $S \times S$ belongs to it. Clearly, $\mathfrak{L}$ is a partially ordered set with respect to inclusion in which $S \times S$ is the largest element, **1** and 1_S is the smallest, **0** (see § Subsection 1.1.2.

Recall that a complete lattice is a partially ordered set in which every subset has both join and meet (see § Subsection 1.1.3). The Proposition below describes the join and meet in the partially ordered set $\mathfrak{L}$ and shows that it is a complete lattice.

PROPOSITION 2.6. *Let S be a semigroup. Then $\mathfrak{L}$ is a complete sublattice of the lattice $\mathcal{E}_S$ of all equivalence relations on S with join and meet defined as follows:*

$$\wedge\Lambda = \bigcap_{i\in I} \rho_i; \tag{2.17}$$

$$\vee\Lambda = \left(\bigcup_{i\in I} \rho_i\right)^{(t)} \tag{2.18}$$

for any subset $\Lambda = \{\rho_i : i \in I\}$ of $\mathfrak{L}$.

Proof. It is easy to verify that $\wedge\Lambda \in \mathfrak{L}$; clearly it is the largest congruence contained in each ρ_i. Hence $\wedge\Lambda$ is the meet of Λ in $\mathfrak{L}$.

Let $\rho = \bigcup_{i\in I} \rho_i$ and $\sigma = \rho^{(t)}$. Since ρ is reflexive and symmetric, by definition, σ is the join of Λ in $\mathcal{E}_S$. Hence it is sufficient to show that σ is compatible. Let $a \in S^1$ and $(x, y) \in \sigma$. Then by Equation (1.8a), $(x, y)\rho^n$ for some $n \in \mathbb{N}$; that is, there exist $u_t \in S$, $t = 0, 1, \ldots, n$ with $u_0 = x$ and $u_n = y$ such that $(u_{t-1}, u_t) \in \rho$. Then for each t, there is $i_t \in I$ with $(u_{t-1}, u_t) \in \rho_{i_t}$. Since ρ_{i_t} is a congruence for every t, $(u_{t-1}a, u_ta), (au_{t-1}, au_t) \in \rho_{i_t}$. It follows that $(xa, ya), (ax, ay) \in \sigma$ and so $\sigma \in \mathfrak{L}$.

For each $\Lambda \subseteq \mathfrak{L}$, it is clear from the definitions above that $\wedge\Lambda$ and $\vee\Lambda$ are respectively meet and join of Λ in $\mathcal{E}_S$. Hence $\mathfrak{L}$ is a sublattice of $\mathcal{E}_S$. □

Note that, if G is a group, $\mathfrak{L}_G$ can be identified with the lattice of normal subgroups of G (see Example 2.2). The following result gives some useful consequences of the Proposition above:

PROPOSITION 2.7. *Let S be a semigroup. We have:*

(a) For any $R \in \boldsymbol{B}_S$, there is a congruence $R^{(c)}$ such that $R^{(c)}$ is the smallest congruence containing R. The map $R \mapsto R^{(c)}$ is a complete $\vee$-homomorphism of the lattice $\boldsymbol{B}_S$ of all relations on S onto $\mathfrak{L}$.

(b) *Let E be an equivalence relation on S and let*

$$E_{(c)} = \{(x, y) : (axb, ayb) \in E \ \forall\, a, b \in S^1\}.$$

Then $E_{(c)}$ is the largest congruence contained in E. The mapping $E \mapsto E_{(c)}$ is a complete $\wedge$-homomorphism of the lattice $\mathcal{E}_S$ of all equivalences on S onto $\mathfrak{L}$.

Proof. To prove (a), let

$$R^c = \{(axb, ayb) : a, b \in S^1 \quad \text{and} \quad (x, y) \in R \cup R^-\}.$$

Then by Equation (2.12b), R^c is the smallest relation containing $R \cup R^{-1}$. Hence R^c is the smallest symmetric and compatible relation of S containing R. It follows from the construction of the transitive closure (see Equation (1.8a)) that the transitive closure of a symmetric and compatible relation is again symmetric and compatible. Hence

$$R^{(c)} = (R^c)^{(t)}$$

is the smallest congruence containing R^c and hence containing R. The map $R \mapsto R^{(c)}$ is clearly inclusion preserving. Let $M \subseteq \boldsymbol{B}_S$. Then $(\vee M)^{(c)} \supseteq R^{(c)}$ for all $R \in M$ and so

$$\bigvee_{R \in M} R^{(c)} \subseteq (\vee M)^{(c)}.$$

Now, since for each $R \in M$,

$$\bigvee_{R \in M} R^{(c)} \supseteq R^{(c)} \supset R,$$
$$\bigvee_{R \in M} R^{(c)} \supseteq (\vee M)^{(c)}.$$

Hence $\bigvee_{R \in M} R^{(c)} = (\vee M)^{(c)}$. Thus the map $R \mapsto R^{(c)}$ is a complete $\vee$-homomorphism.

To prove (b), we observe that, since E an equivalence relation, so is $E_{(c)}$. If $(x, y) \in E_{(c)}$, then from the definition of $E_{(c)}$, we see that $(xa, ya), (ax, ay) \in E_{(c)}$ for all $a \in S^1$. Hence $E_{(c)}$ is both a left as well as a right congruence. Thus $E_{(c)}$ is a congruence which is clearly contained in E. Now let ρ be any congruence contained in E. If $(x, y) \in \rho$, then for all $a, b \in S^1$, $(axb, ayb) \in \rho \subseteq E$. By the definition of $E_{(c)}$, we conclude that $(x, y) \in E_{(c)}$; thus $\rho \subseteq E_{(c)}$.

Again, the map $E \mapsto E_{(c)}$ is inclusion preserving. Let $M \subseteq \mathcal{E}_S$. Since, for all $E \in M$, $\wedge M \subseteq E$ and so,

$$(\wedge M)_{(c)} \subseteq E_{(c)} \qquad \text{and so,}$$
$$(\wedge M)_{(c)} \subseteq \bigwedge_{E \in M} E_{(c)}.$$

Since $\bigwedge_{E\in M} E_{(\mathfrak{c})} \subseteq \wedge M$, we have

$$\bigwedge_{E\in M} E_{(\mathfrak{c})} \subseteq (\wedge M)_{(\mathfrak{c})}.$$

Hence

$$\bigwedge_{E\in M} E_{(\mathfrak{c})} = (\wedge M)_{(\mathfrak{c})}$$

which proves that the map $E \mapsto E_{(\mathfrak{c})}$ is a complete $\wedge$-homomorphism. □

Recall (see Equation (1.11a) and Remark 1.2) that in a partially ordered set Λ, we use the notation

$$[\alpha, \beta] = \{\gamma : \alpha \leq \gamma \leq \beta\} \subseteq \Lambda.$$

for $\alpha, \beta \in \Lambda$. If Λ is a complete lattice, so is $[\alpha, \beta]$. Recall also that an order preserving map of a lattice is a $\vee$-homomorphism [$\wedge$-homomorphism] if it preserve join [meet](see Subsection 1.1.3).

Proposition 2.8. *Let $f : S \to T$ be a surjective homomorphism of the semigroup S onto T. For each $\rho \in \mathfrak{L}$ and $\rho' \in \mathfrak{L}_T$, define*

$$\begin{aligned} f^*(\rho) &= \{(xf, yf) \in T \times T : (x, y) \in \rho\}; \\ f_*(\rho') &= \{(x, y) \in S \times S : (xf, yf) \in \rho'\}. \end{aligned} \tag{2.19}$$

Then we have the following:

(a) $f^* : \mathfrak{L} \to \mathfrak{L}_T$ *is a surjective complete $\vee$-homomorphism.*

(b) $f_* : \mathfrak{L}_T \to [\delta, \mathbf{1}]$ *is a lattice isomorphism such that*

$$f_* \circ f^* = 1_{\mathfrak{L}_T}$$

where $\boldsymbol{\kappa} f = \delta$ and $\mathbf{1} = S \times S$.

(c) For each $\rho \in [\delta, \mathbf{1}]$, there is a unique isomorphism $f_\rho : S/\rho \to T/f^*(\rho)$ such that the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\ f\ } & T \\ {\scriptstyle \rho^\#}\downarrow & & \downarrow{\scriptstyle f^*(\rho)^\#} \\ S/\rho & \xrightarrow[f_\rho]{} & T/f^*(\rho) \end{array} \tag{D3}$$

Here $\rho^\#$ and $f^*(\rho)^\#$ denote the quotient homomorphisms (see Proposition 2.2).

Proof. It is clear that $f^*(\rho)$ is a congruence on T for any congruence $\rho \in \mathfrak{L}$; also the mapping $\rho \mapsto f^*(\rho)$ is order preserving from $\mathfrak{L}$ to $\mathfrak{L}_T$. For $\rho' \in \mathfrak{L}_T$, it is clear that $f_*(\rho')$ defined in the statement, is a congruence in $[\delta, \mathbf{1}]$. Moreover, it is easy to see that f_* is also an order preserving map of $\boldsymbol{\kappa}T$ into $[\delta, \mathbf{1}]$ and for all $\rho' \in \mathfrak{L}_T$,

$$f^*\left(f_*(\rho')\right) = \rho'. \quad \text{Hence} \quad f_* \circ f^* = 1_{\mathfrak{L}_T}.$$

It follows that f_* is a one-to-one order preserving map of $\mathfrak{L}_T$ into $[\delta, \mathbf{1}]$ and that f^* is surjective.

We now show that $f_* : \mathfrak{L}_T \to [\delta, \mathbf{1}]$ is surjective. Let $\rho \in [\delta, \mathbf{1}]$ so that $\delta \subseteq \rho$. Let $\bar{\rho} = f_*(f^*(\rho))$. Then $\rho \subseteq \bar{\rho}$. If $(x, y) \in \bar{\rho}$, then $(xf, yf) \in f^*(\rho)$ and so, by the definition of f^*, there is $(u, v) \in \rho$ such that $(xf, yf) = (uf, vf)$. Hence $(x, u), (v, y) \in \delta$. Therefore

$$(x, y) \in \delta \circ \rho \circ \delta \subseteq \rho^3 = \rho$$

since ρ is transitive. Hence

$$(x, y) \in \rho \iff (xf, yf) \in f^*(\rho). \tag{$\star$}$$

This shows that $\bar{\rho} = \rho$. Therefore f_* is surjective and

$$(f^*|[\delta, \mathbf{1}]) \circ f_* = 1_{[\delta, \mathbf{1}]}.$$

Thus $f_* : \mathfrak{L}_T \to [\delta, \mathbf{1}]$ is an order isomorphism.

To prove that f^* is a complete $\vee$-homomorphism, assume that $\Lambda \subseteq \mathfrak{L}$, $\sigma = \vee\Lambda$ and $\sigma' = \vee f^*(\Lambda)$ where $f^*(\Lambda) = \{f^*(\rho) : \rho \in \Lambda\}$. Since f^* is order preserving, $\sigma' \subseteq f^*(\sigma)$. Since $f^*(\rho) \subseteq \sigma'$ for all $\rho \in \Lambda$,

$$\rho \subseteq f_*\left(f^*(\rho)\right) \subseteq f_*(\sigma')$$

so that

$$\sigma \subseteq f_*(\sigma') \quad \text{which implies} \quad f^*(\sigma) \subseteq f^*\left(f_*(\sigma')\right) = \sigma'.$$

Thus $f(\sigma) = \sigma'$ and this proves (a) and (b).

To prove (c), define f_ρ by:

$$(\rho(x))f_\rho = f^*(\rho)(xf) \quad \text{for all} \quad x \in S.$$

It follows from ($\star$) that $f_\rho : S/\rho \to T/f^*(\rho)$ is a bijection. Using Equation (2.13) and Proposition 2.2 we can easily show that f_ρ is a homomorphism. Hence f_ρ is an isomorphism. Using quotient maps the definition of f_ρ may be rewritten as

$$x\rho^\# f_\rho = xf\left(f^*(\rho)\right)^\# \quad \text{for all} \quad x \in S$$

which shows that the Diagram (D3) is commutative. □

product
sets!Cartesian product of –
$\prod_{i\in I} S_i$:Direct product of the family $\{S_i\}$ of semigroups
product!direct product
senigroups!direct product of –

***Remark* 2.5:** The statement Proposition 2.8(c), in particular, implies the third isomorphism theorem Theorem 2.5. For if $\rho, \sigma \in \mathfrak{L}$ are such that $\rho \subseteq \sigma$, then by Equation (2.19), $\sigma/rho = (\rho^{\#})^{*}(\rho)$ is a unique congruence on S/ρ induced by the quotient homomorphism $\rho^{\#}$ and by Proposition 2.8(c), there is an isomorphism

$$\rho^{\#}_{\sigma} : S/\sigma \to (S/\rho)/(\sigma/\rho).$$

which is the the third isomorphism theorem. Applied to Rees congruences, the statement Proposition 2.8(b) implies that given any ideal I in S, the map $A \mapsto A/I$ is an inclusion preserving bijection of the set of all ideals A in S containing I and all ideals in S/I such that $S/A \cong (S/I)(A/I)$ (by Theorem 2.5).

2.3 PRODUCTS

Various types of products are basic methods of constructing new semigroups. The reader can find a good discussion of direct products and coproducts of sets, groups, etc., in any good book on set theory / algebra (for example Hungerford [1974] gives a good account of these). In fact we can define these concepts categorically (see MacLane [1971]).

2.3.1 Direct product of semigroups

Recall that the *Cartesian product* of a family of sets $\mathcal{A} = \{A_i : i \in I\}$ is the set of all functions $f : I \to \cup_i A_i$ where $f(i) \in A_i$ for all $i \in I$. The function on the indexset Isatisfying the condition above will also be denoted as $f = (f_i)_{i\in I}$ (as I-tuples). When I is a finite set having cardinality $n \in \mathbb{N}$, this definition coincides with the definition of n-tuples. We use these notations below.

PROPOSITION 2.9 (DIRECT PRODUCTS). *Let $\mathcal{F} = \{S_i : i \in I\}$ be a family of semigroups. Assume that*

$$S = \prod_{i\in I} S_i = \prod \mathcal{F} \tag{2.20a}$$

denote the cartesian product of sets S_i. Define a binary operation on S pointwise:

$$xy = (x_i y_i) \quad \text{for all} \quad x = (x_i), y = (y_i) \in S. \tag{2.20b}$$

Then S with the binary operation above, is a semigroup such that for each $i \in I$

$$\pi_i(x) = x_i \quad \text{for all} \quad x \in S \tag{2.20c}$$

is a homomorphism $\pi_i : S \to S_i$. □

The semigroup $S = \prod \mathcal{F}$ constructed above is called the *Direct product* of the family $\mathcal{F} = \{S_i : i \in I\}$. When $\mathcal{F}$ is finite, say $\mathcal{F} = \{S_1, S_2, \ldots, S_n\}$ we use the usual notation

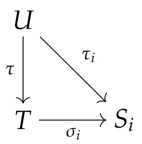
$$S = S_1 \times S_2 \times \cdots \times S_n$$

to denote the product. The above equations are valid for an arbitrary family $\mathcal{F}$ of semigroups (where I is a set). Hence product of any family of semigroups exists. Moreover, products can be characterized upto isomorphisms abstractly (categorically) in terms of homomorphisms. The proof of the following is routine exercise.

product!subdirect product

THEOREM 2.10. *A semigroup T is isomorphic to the product $\prod_{i\in I} S_i$ if and only if T satisfies the following universal property:*

(Π) *For each $i \in I$, there is a homomorphism $\sigma_i : T \to S_i$ such that, to each semigroup U and each family $\{\tau_i : U \to S_i, i \in I\}$ of homomorphisms, there corresponds a unique homomorphism $\tau : U \to T$ making the following diagram commute:*



(2.21)

□

There are several constructions related to direct products that are useful in structure theory of semigroups. We discuss two such constructions below that are of interest to us in the sequel.

Remark* 2.6*: The proposition above proves the existance and gives a construction of direct products in the category of semigroups and the theorm gives the unvesal property of direct products. If, in these results one replaces semigroups by monoids or semigroups with zero and homomorphisms with monoid homomorphisms or homomorphisms that preserve zero, then it can be shown easily that the resulting product will also be of the same type; that is, the category of monoids and the category of semigroups with zero have products and is the same as the product in the category of semigroups.. In particular, it is useful to note that, the category of all groups with zero also has this property. However, other products discussed below does not have this property.

2.3.2 Subdirect products

A subsemigroup T of the direct product S of the family $\mathcal{F} = \{S_i\}_{i\in I}$ is called a *subdirect product* provided that, for each $i \in I$, $\sigma_i = \pi_i|T : T \to S_i$ is a surjective homomorphism.

Subdirect products is a concept from universal algebra that has been useful in semigroup theory. In many constructions of semigroups, we need a subdirect product rather than direct product. However, it may be noted that a subdirect product is not uniqiuly specified by the family $\mathcal{F}$; we need additional conditions to fix it uniquely. The following results are consequences of G. Birkhoff's basic work on universal algebras **?** (see also Grillet Grillet).

semigroup!subdirectly irreducible –

If $\theta : S \to T$ is an isomorphism of S to a subdirect product T of $\mathcal{F}$, then for each $i \in I$, $\phi_i = \theta \circ \sigma_i$ where $\sigma_i = \pi_i | T$ is a surjective homomorphism. Now, for each $t \in T$,

$$\theta(t) = (\sigma_i(\theta(t))) = (\phi_i(t))$$

Since θ is an isomorphism, for any $t, t' \in T$, $t \neq t'$, $\theta(t) \neq \theta(t')$. Hence for any $t, t' \in T$, $t \neq t'$ there exists $i \in I$ with $\phi_i(t) \neq \phi_i(t')$. Therefore the family $\{\phi_i\}_{i \in I}$ of surjective homomorphisms separate points of S. Conversely if S is a semigroup and if $\{\phi_i : S \to S_i\}_{i \in I}$ is family of surjective homomorphisms of S to semigroups in $\mathcal{F}$, then by Theorem 2.10, there is a unique homomorphism θ of S to the product $\prod \mathcal{F}$ such that the Diagram 2.21 commutes for all $i \in I$. If $T = \operatorname{Im} \theta$, this implies in particular that $\pi_i(T) = S_i$ for all $i \in I$. Hence T is a subdirect product of $\mathcal{F}$. Moreover, θ is injective if and only if the family $\{\phi_i\}$ separates points of S. Thus we have

Proposition 2.11. *A semigroup S is isomorphic to a subdirect product of a family $\mathcal{F} = \{S_i\}_{i \in I}$ if and only if there is a family $\{\phi_i : S \to S_i\}_{i \in I}$ of surjective homomorphisms that seperate points of S.* □

When S satisfies the conditions of the proposition above, we will refer to S as a subdirect product of $\mathcal{F}$ with projections ϕ_i. We can formulate the result above in terms of congruences as follows.

Corollary 2.12. *Let $\{\rho_i : i \in I\}$ be a set of congruences on the semigroup S and let $\rho = \cap_i \rho_i$. Then $\bar{S} = S/\rho$ is a subdirect product of semigroups $S_i = S/\rho_i$.*

Proof. By Theorem 2.5, for each $i \in I$, $\sigma_i = \rho_i/\rho$ is a congruence on $\bar{S} = S/\rho$ such that $\bar{S}/\sigma_i$ is isomorphic to $S_i = S/\rho_i$. Hence there exists surjective homomorphisms $\phi_i : \bar{S} \to S_i$, $i \in I$. Suppose that $\bar{a}, \bar{b} \in \bar{S}$ are such that $\phi_i(\bar{a}) = \phi_i(\bar{b})$ for all $i \in I$. Since the map $\theta : x \mapsto \rho(x)$ is a surjective homomorphism of S onto $\bar{S}$ we can find $a, b \in S$ with $\bar{a} = \rho(a)$, $\bar{b} = \rho(b)$. Then

$$(\rho(a), \rho(b)) \in \sigma \Rightarrow (a, b) \in \rho_i \quad \text{for all} \quad i \in I.$$

This gives $\bar{a} = \rho(a) = \rho(b) = \bar{b}$. Hence the family of homomorphisms $\{\phi_i\}_{i \in I}$ separates points of $\bar{S}$. □

A semigroup S is said to be *subdirectly irreducible* if S has more than one element and has the following property: if S is isomorphic to a subdirect product of semigroups S_i, $i \in I$, there is atleat one $i \in I$ such that the corresponding projection $S \to S_i$ is an isomorphism. By the proposition above, this is equivalent to the statement that intersection of any set of proper (non-trivial) congruences on S is proper.

The following result is due to G. Birkhoff **?**.

THEOREM 2.13. *Every semigroup is a subdirect product of subdirectly irreducible semigroups.*

Proof. Let S be a semigroup. Consider $(a, b) \in S^2$ with $a \neq b$. Let $\mathcal{R}_{a,b}$ denote the set of all congruences ρ on S for which $(a, b) \notin \rho$. Clearly, union of any chain (under inclusion) of congruences in $\mathcal{R}_{a,b}$ again belongs to $\mathcal{R}_{a,b}$. Hence by Zorn's lemma $\mathcal{R}_{a,b}$ contains maximal congruences. For each $(a, b) \in S^2$ with $a \neq b$ choose a maximal congruence $\rho_{a,b} \in \mathcal{R}_{a,b}$. The maximality of $\rho_{a,b}$ implies that for every congruence $\rho \supseteq \rho_{a,b}$ with $\rho \neq \rho_{a,b}$, $(a, b) \in \rho$. Hence intersection of any set of congruences on S properly containing $\rho_{a,b}$ properly contains $\rho_{a,b}$. This implies that intersection of any set of proper congruences of $S_{a,b} = S/\rho_{a,b}$ is proper. Therefore the semigroup $S_{a,b}$ is subdirectly irreducible for all $(a, b) \in S^2$, $a \neq b$. Also since

$$\bigcap \left\{\rho_{a,b} : (a, b) \in S^2,\ a \neq b\right\} = 1_S,$$

by Corollary 2.12, S is a subdirect product of semigroups $S_{a,b}$. □

2.3.3 Fibered products

Let $\phi : S \to U$ and $\theta : T \to U$ be homomorphisms of semigrroups. Then

$$S \times_U T = \{(s, t) \in S \times T : s\phi = t\theta\} \tag{2.22}$$

is easily seen to be a subsemigroup of the direct product $S \times T$. We use the notations introduced above in the statement:

PROPOSITION 2.14. *Let S, T, ϕ and θ be as above. Assume that*

$$\psi_1 = \pi_1|F, \quad \psi_2 = \pi_2|F$$

where

$$\pi_1 : S \times T \to S, \quad \pi_2 : S \times T \to T, \quad \text{are projections and} \quad F = S \times_U T.$$

Then the first diagram below is commutative:

$$\begin{array}{ccc} S \times_U T & \xrightarrow{\psi_1} & S \\ {\scriptstyle \psi_2}\downarrow & & \downarrow{\scriptstyle \phi} \\ T & \xrightarrow[\theta]{} & U \end{array} \qquad \begin{array}{ccc} W & \xrightarrow{\eta_1} & S \\ {\scriptstyle \eta_2}\downarrow & & \downarrow{\scriptstyle \phi} \\ T & \xrightarrow[\theta]{} & U \end{array} \tag{2.23}$$

Moreover, if $\eta_1 : W \to S$ and $\eta_2 : W \to T$ are homomorphisms such that the second diagram above is commutative, then there exist a unique homomorphism $\xi : W \to S \times_U T$ such that

$$\eta_1 = \xi \circ \psi_1, \quad \text{and} \quad \eta_2 = \xi \circ \psi_2. \tag{$*$}$$

$S \times_U T$:fibered product of S and T over U
product!fibered product
homomorphism!fiber homomorphism
homomorphism!induced fiber –
diagram!pullback
word!normalized –

Proof. The definition of $S \times_U T$ implies that the first diagram in 2.23 is commutative. If η_i, $i = 1, 2$ are homomorphisms making the second diagram commute, then

$$u\xi = (u\eta_1, u\eta_2) \quad \text{for all} \quad u \in W$$

defines a homomorphism $\xi : W \to S \times_U T$ satisfying the conditions $(*)$. These conditions imply that, for all $u \in W$,

$$u\eta_1 = (u\xi)\psi_1, \quad \text{and} \quad u\eta_2 = (u\xi)\psi_2.$$

Since ψ_i, $i = 1, 2$ are projections the equations above shows that $u\xi = (u\eta_1, u\eta_2)$ for all $u \in W$ and so, ξ is unique. □

The semigroup $S \times_U T$ is called the *fibered product* of semigroups S and T over U. Homomorphisms ϕ and θ are refered to as *fiber homomorphisms* while ψ_1 and ψ_2 are *induced fiber homomorphisms*. Here ψ_1 is induced by ϕ and ψ_2 is induced by θ. The result above says that the first diagram in (2.23) defining the fibered product is a pullback diagram (see MacLane [1971], page 71).

We can generalize the construction for an arbitrary family of semigroups in the obvious manner.

2.3.4 Coproducts

To define coproducts of $\mathcal{F}$, we may assume with out loss of generality that

$$S_i^1 \cap S_j^1 = \{1\} \quad \text{for all} \quad i \neq j.$$

Consider the set

$$X = \bigcup_{i \in I} S_i.$$

A word in X is the concatenation

$$w = x_{i_1} x_{i_2} \dots x_{i_n}, \qquad x_{i_t} \in S_{i_t}$$

of a finite sequence

$$(x_{i_i}, \dots, x_{i_n}) = (x_{i_t})_{1 \le t \le n}$$

in of elements in X. The word w is said to be *normalized* if no two adjacent terms of the sequence $(x_{i_t})_{1 \le t \le n}$ belongs to the same semigroup; equivalently, $i_s \neq i_{s+1}$ for any s, $1 \le s < n$. Given an arbitrary finite sequence $(x_{i_t})_{1 \le t \le n}$, if x_{i_s} and $x_{i_{s+1}}$ belongs to the same semigroup, we may multiply these and obtain a modified sewquence in which the term x_{i_s} is replaced by the product $x_{i_s}x_{i_{s+1}}$ and $x_{i_{s+t}}$ is replaced by $x_{i_{s+t+1}}$, $t \geq 1$. Repeating this process a finite number of time, we will obtain a unique normalized word. For convenience we denote the normalizes word obtained from $(x_{i_t})_{1\leq t\leq n}$ by $w(x_{i_i}, \ldots, x_{i_n})$. Notice that for $a \in S_i$, $i \in I$, the normalized word $w(a)$ given by the sequence with the only term a is a itself. It is not difficult to verify the following.

$\coprod_{i\in I} S_i$:free product of $\{S_i\}_{i\in I}$
semigroups!coproduct of –
semigroups!free product of –

PROPOSITION 2.15. *Let $\mathcal{F} = \{S_i : i \in I\}$ be a family of semigrpoups and let $X = \bigcup_{i\in I} S_i$. Suppose that*

$$P = \coprod_{i\in I} S_i = \{w : w \text{ is a normalized word in } X\}. \tag{2.24a}$$

For any words $w = x_{i_1}x_{i_2}\ldots x_{i_n}$ and $w' = y_{j_1}\ldots y_{j_m}$ in P, define the product ww' by

$$ww' = w(x_{i_i}, \ldots, x_{i_n}, y_{j_1}, \ldots, y_{j_m}). \tag{2.24b}$$

This defines a product in P and with this product P is a semigroup. Further, for each $i \in I$, the map

$$\jmath_i : a \mapsto w(a)$$

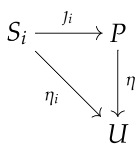
is an injective homomorphism of S_i into P. □

The semigroup P constructed above is called the *Coproduct* or *free product* of the family $\mathcal{F}$. Free products are (categorical) duals of products.

The following is the dual of Theorem 2.10; its proof is left as an exercise.

THEOREM 2.16. *A semigroup P is isomorphic to the coproduct of a family $\{S_i\}_{i\in I}$ of semigroups if and only if P satisfies the following universal property:*

($\amalg$) *For each $i \in I$, there is a homomorphism $\jmath_i : S_i \to P$ such that, to each semigroup U and each family $\{\eta_i : S_i \to U, i \in I\}$ of homomorphisms, there corresponds a unique homomorphism $\eta : P \to U$ making the following diagram commute for each $i \in I$:*



(2.25)

□

We discuss an important particular case of free products in the next section.

2.4 FREE SEMIGROUPS AND PRESENTATIONS OF SEMIGROUPS

Free semigroups form one of the most important and naturally occuring class of semigroups. In this section we provide the elementary definitions and discuss presentations of semigroups by generators and relations.

word
alphabet
X^+: Free semigroup on X
semigroup!free –
word!empty
X^*: Free monoid on X
monoid!free –

2.4.1 Free semigroups and monoids

Let X be a set. A *word* over [the *alphabet*] X is a finite sequence $(x_1, x_2, \dots, x_r)$ of symbols representing elements of X in which repetitions are allowed. We denote words by juxtaposition as $w = x_1x_2 \dots x_r$. We can define a binary operation in the set of all words by juxtaposition: if $w_1 = x_1 \dots x_r$ and $w_2 = y_1 \dots y_s$ are words, we set

$$w_1w_2 = (x_1 \dots x_r)(y_1 \dots y_s) = x_1 \dots x_r y_1 \dots y_s.$$

This binary operation is clearly associative. Thus the collection X^+ of all non-empty words over X is a semigroup; X^+ is called the *free semigroup* on X. A semigroup S is said to be *free* if it is isomorphic to X^+ for some $X \neq \emptyset$.

If w_1 and w_2 are non-empty words over X, then clearly it is not possible to have

$$w_1w_2 = w_1 \quad \text{or} \quad w_1w_2 = w_2.$$

Hence X^+ is a semigroup and not a monoid. If we include, in the set of words over X, a word e which does not have any symbol, called the *empty word*, then for any word w over X, we must have

$$we = ew = w$$

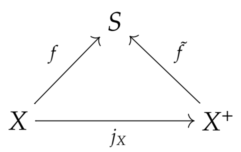
Hence the set X^* of all words over X (including empty word) is a monoid and is called the *free monoid* over X.

Example 2.3: Let $X = \{x\}$ be a singleton set. Then words over X are precisely powers of x and so $x^+ = \{x^n : n \in \mathbb{N}^*\}$. Hence $x^+ = \langle x \rangle$. Since no two distinct words in x^+ can be equal, by Proposition 2.1, it is an infinite cyclic semigroup which is isomorphic to $(\mathbb{N}^*, +)$. Similarly the free monoid x^* is the infinite cyclic monoid isomorphic to $(\mathbb{N}, +)$.

The free semigroup is characterized by the following universal property:

PROPOSITION 2.17. *Let $j_X : X \to X^+$ denote the map which identifies each $x \in X$ with the word containing the only symbol x. Then the pair (X^+, j_X) has the following property:*

- *Let S be a semigroup and $f : X \to S$ be a map. Then there exists a* unique *homomorphism $\tilde{f} : X^+ \to S$ such that the following diagram commute:*



(D1)

The free semigroup X^+ is characterized up to isomorphism by the property above.

generating set

Proof. For $w = x_1 \dots x_r \in X^+$, define

$$\tilde{f}(w) = f(x_1) \dots f(x_r) \tag{2.26}$$

where, on the right-hand side, the product is taken in the semigroup S. Clearly, $\tilde{f} : X^+ \to S$ is a homomorphism. If $x \in X$, then $j_X(x)$ is the word in X^+ having the only symbol x and so, by the definition above, $f(j_X(x)) = f(x)$; this proves that the diagram (D1) commute. The uniqueness of $\tilde{f}$ is clear from the definition.

Suppose that (S, f) is a pair consisting of a semigroup S and a mapping $f : X \to S$ that satisfies the property above. Since X^+ is a semigroup and $j_X : X \to X^+$ is a mapping, by the above, there exists a homomorphism $h : S \to X^+$ such that $f \circ h = j_X$. By the diagram above $j_X \circ \tilde{f} = f$. Hence

$$\begin{aligned} j_X = j_X \circ 1_{X^+} &= f \circ h = (j_X \circ \tilde{f}) \circ h \\ &= j_X \circ (\tilde{f} \circ h) = j_X \circ g \end{aligned}$$

where $\tilde{f} \circ h = g : X^+ \to X^+$ is a homomorphism. By the uniqueness, we have $g = 1_X$. Similarly $h \circ \tilde{f} = 1_S$. Thus $\tilde{f} : X^+ \to S$ is an isomorphism. □

Example 2.4: Let X be a set. Show that there is an isomorphism $\theta : X^+ \to \coprod_{x \in X} x^+$.

The proposition above holds for monoid also; in fact, we have:

PROPOSITION 2.18. *Let $j_X : X \to X^*$ denote the map which identifies each $x \in X$ with the word containing the only symbol x. Then the pair (X^*, j_X) has the following property:*

- *Let M be a monoid and $f : X \to M$ be a map. Then ere exists a* unique *monoid homomorphism $\hat{f} : X^* \to M$ such that*

$$j_X \circ \hat{f} = f.$$

The free monoid X^* is characterized up to isomorphism by the property above.

Proof. The proof for Proposition 2.17 goes through in this case if we replace X^+ by X^*, the semigroup S by the monoid M, homomorphism h by a monoid homomorphism, and define $\hat{f}$ by Equation (2.26) and the condition that $\hat{f}(e) = 1$, the identity of M. □

A subset A of a semigroup [monoid] S [M] is called a *generating set* (or a set of generators) for S [M] if every element of S [M] can be written as a finite product of elements in A. Note that X (identified as a subset of X^+ by the function j_X) is a generating set for X^+, where as $X \cup \{e\}$ is a generating set for X^*. It is easy to see that every semigroup S [monoid M] has at least one generating set (the trivial generating set S [M] of all elements of S [M]). We have:

semigroup!presentation
$\langle A; \{w_i = w'_i, i \in I\}\rangle$:semigroup presented with generators A and relations R
word!directly derivable
word!derivable
consequence

COROLLARY 2.19. *Every semigroup [monoid] is a homomorphic image of a free semigroup [monoid].*

Proof. Let A be a generating set for the semigroup [monoid] S [M]. By the Proposition 2.17 [Proposition 2.18], $\tilde{f} : A^+ \to S$ [$\hat{f} : A^* \to M$] is a homomorphism [monoid homomorphism], where f denote the inclusion of A in S [M]. Clearly $A \subseteq \operatorname{Im} \tilde{f}$ [$A \subseteq \operatorname{Im} \hat{f}$] and hence $\tilde{f}$ [$\hat{f}$] is surjective. □

Remark* 2.7*: The universal property of the construction of X^+ given in Proposition 2.17 implies that the construction X^+ gives a left-adjoint of forgetful functor $U : \mathfrak{S} \to \mathbf{Set}$ (see Theorem 1.6). Similar remark holds for the free monoid construction also.

2.4.2 Presentations

We have seen that every semigroup S is a quotient (homomorphic image) of a free semigroup A^+ where A is a generating set for S (see Corollary 2.19). Hence by Theorem 2.3, there is a congruence ρ on A^+ such that A^+/ρ is isomorphic to S. If R is any relation that generate the congruence ρ (that is, $R^{(c)} = \rho$, see Proposition 2.7) then S is determined, up to an isomorphism by the set A and the relation R. We say that $\langle A; R\rangle$ is a *presentation* of S. If R and R' are both relations generating ρ they give two equivalent presentations of S. Since

$$R^{(c)} = (R \cup R^{-1})^{(c)},$$

we may assume that R is symmetric. If $R = \{(w_i, w'_i) : i \in I\}$, is symmetric, we indicate the presentation $\langle A; R\rangle$ as

$$\langle A; \{w_i = w'_i, i \in I\}\rangle.$$

Note that, if $f : A \subseteq S$ denotes the inclusion, then the homomorphism $\tilde{f} : A^+ \to S$ defined by Equation (2.26) maps both w_i and w'_i to the same element in S. Now, since R is symmetric, from the proof of Proposition 2.7, we have

$$\rho = (R^S)^{(t)}$$

By Equation (2.12b), $(u, v) \in R^S$ if and only if $u = rw_is$, $v = rw'_is$ or $u = rw'_is$, $v = rw_is$ for some $i \in I$. We say that the word v is *directly derivable* from the word u; if $(u, v) \in R^{(c)}$ then v is said to be *derivable* from u. In this case the relation $u = v$ is said to be a *consequence* of relations $\{w_i = w'_i, i \in I\}$.

Note that every semigroup $(S, \cdot)$ admits at least the trivial presentation

$$\langle S; \{xy = x \cdot y \ \forall\ x, y \in S\}\rangle$$

semigroup!finitely generated
semigroup!finitely related
semigroup!finitely presented

If S admits a presentation $\langle A; \{w_i = w_i', i \in I\}\rangle$ in which A [I] is finite, then S is said to be *finitely generated* [*finitely related*]. If both A and I are finite, then S is said to be *finitely presented* .

Presentations of monoids can be defined as above by replacing A^+ by A^* and $\tilde{f}$ by $\hat{f}$ (see Proposition 2.8) in the discussion above. Note that a monoid can have a semigroup presentation if there is a word $v \in A^+$ such that $v = 1$ is a consequence of the the relations $R = \{w_i = w_i',\ i \in I\}$.

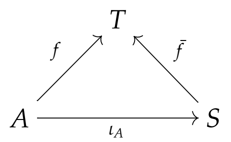
We have the following universal property for semigroups [monoids] with a given presentation. We formulate the result for semigroups. The same result holds for presentations of monoids also with appropriate modification.

PROPOSITION 2.20. *Let $S = \langle A; \{w_i = w_i', i \in I\}\rangle$ be a semigroup presented with generators A and relations $w_i = w_i'$, $i \in I$. Let $f : A \to T$ be a mapping of A into a semigroup T. If for every $i \in I$,*

$$x_1 f \dots x_n f = x_1' f \dots x_m' f \quad \textit{where} \quad w_i = x_1 \dots x_n, \quad w_i' = x_1' \dots x_m' \qquad (1^*)$$

then there exists a unique homomorphism $\bar{f} : S \to T$ such that the following diagram commutes.

(D1*)



where ι_A denote the insertion of the generators A in S.

Proof. Since $f : A \to T$ is a mapping, by Proposition 2.17 there is $\tilde{f} : A^+ \to T$ such that $j_A \circ \tilde{f} = f$. Let $\sigma = \boldsymbol{\kappa}\tilde{f}$. Since equations (1*) holds for each i, $w_i\tilde{f} = w_i'\tilde{f}$ for all i. Hence if $R = \{w_i = w_i' : i \in I\}$ denote the relation determined by the presentation of S, then $R \subseteq \sigma$. Hence $\rho = R^{(c)} \subseteq \sigma$. Therefore if we set

$$(\rho(w))\bar{f} = w\tilde{f} \quad \text{for all} \quad w \in A^+$$

then $\bar{f} : S \to T$ is clearly a homomorphism. Since $\iota_A : A \to S$ is a mapping, by Proposition 2.17, there is a unique homomorphism $\tilde{\iota}_A = \phi : A^+ \to S$ such that $j_A \circ \phi = \iota_A$. Also, by the definition of ρ, ι_A sends each $a \in A$ to the ρ-class containing the word a; that is,

$$a\iota_A = \rho(aj_A) = aj_A \circ \rho^{\#}$$

for all $a \in A$. Hence by the uniqueness, $\phi = \rho^{\#}$. Therefore by the definition of $\bar{f}$, $\phi \circ \bar{f} = \tilde{f}$. Hence,

$$\begin{aligned} \iota_A \circ \bar{f} &= j_A \circ \phi \circ \bar{f} \\ &= j_A \circ \tilde{f} = f. \end{aligned}$$

Thus the given diagram commutes. The uniqueness of $\bar{f}$ is clear from its definition. □

Example 2.5: A semigroup S is free if and only if it has a presentation of the form $\langle X; \emptyset \rangle$.

Example 2.6: $\langle x \rangle$ is a finite cyclic semigroup if and only if it has a presentation of the form $\langle x; x^r = x^{r+m} \rangle$ with $r \in \mathbb{N}$ and $m \in \mathbb{N}^*$. This shows in particular that, given any two positive integers r and m, there is finite cyclic semigroup with index r and period m which is clearly unique up to isomorphism (see also the Example in § Subsection 2.1.3). Moreover, any presentation of a semigroup with one generator is a consequence of a presentation of the form $\langle x; x^r = x^{r+m} \rangle$ if it is not free. For any presentation of a semigroup S with one generator x, relations must be a set of equations of the form $\{x^{r_i} = x^{s_i} : i \in I\}$ with $r_i, s_i \in \mathbb{N}^*$ and $r_i \neq s_i$. If $I \neq \emptyset$, by Proposition 2.1, S must be a finite cyclic semigroup and hence these relations must be consequence of a single relation of the form $x^r = x^{r+m}$.

Example 2.7: Let $S = \langle p, q; pq = 1 \rangle$ be the monoid generated by elements p and q with relation $pq = 1$. Then in S, we have

$$q^n p^n \neq 1 \quad \text{for} \quad n \in \mathbb{N}^*. \tag{a}$$

To see this holds for $n = 1$, by Proposition 2.20, it is sufficient to find a monoid T and $a, b \in T$ with $ab = 1$ and $ba \neq 1$. For example consider $T = \mathscr{T}_{\mathbb{N}}$, and $a, b \in \mathscr{T}_{\mathbb{N}}$ be the maps defined by $a : n \mapsto n + 1$, $0b = 0$, $nb = n - 1$ for $n > 0$. Then it is readily seen that $ab = 1$ and $ba \neq 1$. For $n > 1$, assume that the result holds for $1 \leq r < n$. If $q^n p^n = 1$, then, using the relation $pq = 1$, we obtain

$$q^{n-1} p^{n-1} = p(q^n p^n) q = pq = 1$$

which is a contradiction. Further, for any $m, n \in \mathbb{N}$, again using the relation $pq = 1$, we deduce that

$$p^m q^n = \begin{cases} p^{m-n} & \text{if } m \geq n; \\ q^{n-m} & \text{if } m < n. \end{cases} \tag{b}$$

Here, we write $p^0 = 1 = q^0$. Moreover,

$$q^m p^n = q^r p^s \iff m = r, \quad n = s. \tag{c}$$

To prove (c), we first observe that p and q are of infinite order in S. For, if p is of finite order, by Proposition 2.1, there exist $r, m \in \mathbb{N}^*$ such that $p^r = p^{r+m}$. Then $p^m = p^{r+m} q^r = p^r p^r = 1$ and so, $qp = p^m qp = p^{m-1} p = p^m = 1$ which contradicts (a). In Equation (c) we note that, if $m = r$ then $p^n = p^s$ which implies the p is of finite order if $n \neq s$. Assume that $m > r$. If $n < s$ then we get $q^{m-r+s-n} = 1$ which implies that q is of finite order. If $n > s$, then $p^{m-r} = p^{n-s}$ which implies that p is of finite order if $m - r \neq n - s$. If $m - r = n - s$, this gives $q^{m-r} p^{m-r} = 1$ which contradicts (a). The case $m < r$ can be treated similarly. It follows that (c) holds. Therefore the monoid S can be described as:

$$S = \{q^m p^n : m, n \in \mathbb{N}\} \tag{d}$$

with product defined by

semigroup!bicyclic semigroup
representation
representation!– by transformations
representation!linear –
representation!faithful –
action

$$(q^m p^n)(q^r p^s) = \begin{cases} q^m p^{n-r+s} & \text{if } n \geq r; \\ q^{m+r-n} p^s & \text{if } n < r. \end{cases} \tag{e}$$

The monoid S can also be given a semigroup presentation:

$$S = \langle p, q; pqp = p^2 q = p, qpq = pq^2 = q \rangle.$$

Suppose that T is the semigroup presented as above. If $pq = \epsilon$, it is easy to verify that ϵ is the identity in T and so the relation $pq = 1$ is a consequence of the relations of T. It is clear that the relations of T is a consequence of the relation $pq = 1$. Hence $S = T$. The semigroup (or monoid) S is called the *bicyclic* semigroup (or monoid).

2.5 REPRESENTATIONS

By a *representation* of a semigroup S, we shall mean a homomorphism $\phi : S \to T$ of S into a semigroup T of some specific type. If S is a monoid, then ϕ is a representation of monoids if T is also a monoid and ϕ is a monoid homomorphism. Thus if $T = \mathscr{T}_X$, ϕ is a representation by transformations on X, if $T = \mathscr{P}\mathscr{T}_X$, it is called a representation by partial transformations, etc. Linear representations, that is, representations by linear transformations on vector spaces are also important. This is particularly true if V is finite dimensional. Note that such a representation is equivalent to a representation by $n \times n$ matrices over a field. A representation ϕ is said to be *faithful* if ϕ is a one-to-one homomorphism.

In the first subsection below, we consider representations of semigroups by functions on sets. In Subsection 2.5.2 we examine a specific representation and in Subsection 2.9.3 we discuss a representation by row-monomial matrices over a group with 0.

2.5.1 Representation by functions

We begin by showing that every representation of a semigroup S by functions on a set X determines an *action* of S on X.

PROPOSITION 2.21. *Let S be a semigroup and X be a set. Suppose that $\phi : S \to \mathscr{T}_X$ is a representation of S by functions on X. Define*

$$\bar{\phi}(x, s) = x\phi(s) \quad \textit{for all} \quad (x, s) \in X \times S. \tag{2.27}$$

Then $\bar{\phi} : X \times S \to X$ is a map such that

$$\bar{\phi}(x, st) = \bar{\phi}\left(\bar{\phi}(x, s), t\right) \quad \textit{for all} \quad x \in X, \ \ s, t \in S. \tag{2.28a}$$

Conversely, if $\bar{\phi} : X \times S \to X$ is a map satisfying the condition (2.28a) *above, then for each $s \in S$ Equation* (2.27) *defines a map $\phi(s) : X \to X$ such that $\phi : s \mapsto \phi(s)$ is*

action
action!right –
S-set:right –
S-set
S-set!faithful –
action!left –

a representation of S by functions on X. Moreover, if S is a monoid (with identity 1*), then* $\bar\phi$ *satisfies the following:*

$$\bar\phi(x,1) = x \quad \textit{for all} \quad x \in X. \tag{2.28b}$$

if and only if ϕ *is a representation of monoids.*

Proof. Since ϕ is a representation, for $s, t \in S$

$$\begin{aligned}
\bar\phi(x, st) &= x\phi(st) \\
&= \big(x\phi(s)\big)\,\phi(t), && \text{since } \phi \text{ is a homomorphism;} \\
&= \big(\bar\phi(x,s)\big)\,\phi(t) && \text{by Equation (2.27)} \\
&= \bar\phi\,\big(\bar\phi(x,s), t\big)
\end{aligned}$$

which proves Equation (2.28a). If ϕ is a monoid homomorphism, then $\bar\phi$ clearly satisfies Equation (2.28b).

Conversely assume that the function $\bar\phi$ satisfies Equation (2.28a). It is clear that for fixed $s \in S$, Equation (2.27) defines a function $\phi(s) : X \to X$. Moreover, if $s, t \in S$, then for all $x \in X$,

$$\begin{aligned}
x\phi(st) &= \bar\phi\,\big(\bar\phi(x,s), t\big) && \text{by Equation (2.28a)} \\
&= \big(x\phi(x)\big)\,\phi(t) && \text{by Equation (2.27).}
\end{aligned}$$

It follows that ϕ is a representation and clearly ϕ is a monoid homomorphism if Equation (2.28b) holds. □

The Proposition above shows that there is a bijection between representations of a semigroup S by functions on X and functions $\bar\phi : X \times S \to X$ satisfying Equation (2.28a). A function $\bar\phi : X \times S \to X$ is called an *right action* of the semigroup S on the set X if it satisfies Equation (2.28a). $\bar\phi$ is the right action of a monoid S on X if and only if it also satisfies Equation (2.28b). When the action of S on X is clear from the context, we may simplify the notation by writing $\bar\phi(x, s)$ as xs. With this simplification, Equations 2.28a and 2.28b becomes $x(st) = (xs)t$ and $x1 = x$ respectively for all $x \in X$ and $s, t \in S$. If $\bar\phi$ is a right action of the semigroup [monoid] S on the set X, then the pair $(X, \bar\phi)$ is called a *right S-set*. Again we abbreviate the notation to X and say that X is a right S-set if $\bar\phi$ is clear from the context. A right S-set X is said to be *faithful* if the associated representation is faithful.

Dually a *left action* of S on X is defined as a function $\bar\psi : S \times X \to X$ satisfying:

$$\bar\psi(st, x) = \bar\phi\,(s, \bar\psi(t, x)) \quad \text{for all} \quad x \in X, \quad s, t \in S. \tag{2.28a*}$$

If S is a monoid, $\bar{\psi}$ is the left action of the monoid if, in addition, we have

$$\bar{\psi}(1, x) = x \quad \text{for all} \quad x \in X. \tag{2.28b*}$$

representation!dual or left
S-morphism
S-set:S-subset
S-set!orbit
xS^1 :cyclic right S-set generated by x
S-set!cyclic –
S^1x :cyclic left S-set generated by x
Set_S:category of right S-sets

In this case, the dual of Proposition 2.21 also holds. However, it may be noted that, the function $\psi : S \to \mathscr{T}_X$ defined by (the dual of Equation (2.27))

$$\bar{\psi}(s, x) = x\psi(s) \quad \text{for all} \quad (s, x) \in S \times X$$

gives a homomorphism $\psi : S \to \mathscr{T}_X^{\text{op}}$ (that is an anti-homomorphism of S to $\mathscr{T}_X$). We shall refer to this as a *dual or left representation*. Again if $\bar{\psi}$ is clear from the context, we may write sx for $\bar{\psi}(s, x)$. A left S-set is a pair $(X, \bar{\psi})$ where X is a set and $\bar{\psi}$ is a left action of S on X; again we abbreviate this to X if the left action is clear from the context. A left S-set is faithful if the associated representation is faithful.

We observe that the concept of left S-sets is the left-right dual of right S-sets and so we may dualise every definition right S-sets to left S-sets and vice-versa and to every result that holds for right S-sets, the dual result holds for left S-sets. Consequently, in the following we shall not repeat the dual statements explicitly.

If X and Y are right S-sets, a mapping $\lambda : X \to Y$ is called a *morphism of right S-sets* or an S-morphism if (using simplified notations)

$$\lambda(xs) = (\lambda(x))\, s \quad \text{for all} \quad x \in X;\ s \in S. \tag{2.30}$$

We shall follow the convention that morphism of right [left] S-sets are written as left [right] operators. As a consequence, the endomorphism semigroup of a right [left] S-set X is naturally identified as a subsemigroup of $\mathscr{T}_X^{\text{op}}$ [$\mathscr{T}_X$] (see § Subsection 2.1.3).

A subset X' of a right S-set X is called an *S-subset* if for all $x' \in X'$, $x'S^1 \subseteq X'$ where $x'S^1 = \{x's : s \in S^1\}$. Thus the subset $X' \subseteq X$ is an S-subset if and only if X' is a right S-set and the inclusion $j_{X'}^X$ is a morphism of right S-sets. Note that, for any $x \in X$, xS^1 itself is an S-subset of X and is called the *orbit* of x in X. A right S-set X for which $X = xS^1$ for some $x \in X$ is said to be a *cyclic* (or *monogenic*) S-set generated by x. Dually, a left S-set X is cyclic if $X = S^1x$ for some $x \in X$.

Given a semigroup S, the collection of all right S-sets with morphisms defined as above is clearly a category Set_S. Isomorphisms, endomorphisms, automorphisms, etc. of right S-sets are isomorphisms, endomorphisms, etc. in the category Set_S. The discussion of S-subsets above implies that Set_S has subobjects in the sense of § Subsection 1.3.2. Also it is easy to see that if $f : X \to Y$ is a morphism of right S-sets, then the factorization of f as a

*$_S$**Set**:category of left S-sets*
congruence!– on S-set
S-set!right regular –
S-set!left regular –
S_r : right regular S-set
S_l : left regular S-set
representation!right regular –
$\boldsymbol{\rho}_S$:right regular representation of S
translation
translation!inner right –
translation!right –

mapping (that is, factorization of f in **Set**) gives a factorization of f in $\mathbf{Set}_S$ also. Therefore the category $\mathbf{Set}_S$ has images (see § Subsection 1.3.2). We also have a category $_S\mathbf{Set}$ of all left S-sets which has images.

Let X be a right S-set. A congruence on X is an equivalence relation μ satisfying the following condition:

$$(x, y) \in \mu \iff (xs, ys) \in \mu \quad \text{for all} \quad s \in S^1. \tag{2.31}$$

A congruence on a left S-set is defined dually. The routine proof of the following statement is left as an exercise.

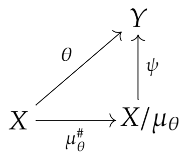
Proposition 2.22. *Let μ be a congruence on the right S-set X. Then X/μ is a right S-set with respect to the action defined by*

$$(\mu(x), s) \mapsto \mu(xs) \quad \textit{for all} \quad (\mu(x), s) \in X/\mu \times S \tag{2.32}$$

such that the quotient map $\mu^{\#} : X \to X/\mu$ is a morphism of S-sets. Moreover, if $\theta : X \to Y$ is a morphism of S-sets then

$$\mu_\theta = \{(x, x') \in X \times X : \theta(x) = \theta(x')\}$$

is a congruence on X and there exists an injective S-morphism $\psi : X/\mu_\theta : Y$ such that the following diagram commutes:



(D4)

In particular, θ is surjective, if and only if ψ is an isomorphism. □

2.5.2 Regular representations

If S is a semigroup, with respect to the product in S, S can be regarded as a right S-set as well as a left S-set. Here we shall refer to these as *right regular* and *left regular* S-set and use the notations S_r and S_l respectively to denote these. A subset X of S is an S-subset of S_r [S_l] if and only if X is a right [left] ideal of S and X is cyclic if and only if X is a principal right [left] ideal. The representation $\boldsymbol{\rho}_S = \boldsymbol{\rho}$ associated with the right regular S-set S_r is called the *right regular* representation of S. In this case, for any $a \in S$, we write ρ_a for $\boldsymbol{\rho}_S(a)$. The map $\rho_a : s \mapsto sa$ is called the *inner right translation* of S by a. Note $\rho_a : S \to S$ is an endomorphism of the left regular S-set S_l. More generally, a *right translation* ρ is an endomorphism of S_l. It follows from the discussion that

$$\boldsymbol{\rho} : S \to \mathrm{End}(S_l)$$

is a homomorphism of S to the endomorphism semigroup of S_l.

λ_S : *left regular representation*
translation!inner left –
translation!left –
semigroup!right reductive
semigroup!left reductive

These definitions can be dualized for the left regular S-set S_l. In particular, the left representation determined by S_l is denoted by $\boldsymbol{\lambda}_S$ and for $a \in S$, the map $\boldsymbol{\lambda}_S(a) = \lambda_a : s \mapsto as$ is called the *inner left translation* by a. As above, a *left translation* is an endomorphism of the right regular S-set. Note that, according to the convention adopted for morphisms of left and right S-sets (see Subsection 2.5.1), left translations are considered as maps in $\mathscr{T}_S^{\mathrm{op}}$ and written as left operators. Consequently $\mathrm{End}(S_r)$ is naturally a subsemigroup of $\mathscr{T}_S^{\mathrm{op}}$ and

$$\boldsymbol{\lambda}_S = \boldsymbol{\lambda} : S \to \mathrm{End}(S_r)$$

is a homomorphism of S into the semigroup of endomorphisms of the right regular S-set S_r.

Also, every inner right translations commute with every inner left translations; that is for all $s, t, a \in S$,

$$(\lambda_s a)\rho_t = \lambda_s(a\rho_t). \tag{2.33}$$

We say that a semigroup S is *right[left] reductive* if right regular [left regular] representation is faithful.

It is clear from Equation (2.31) that an equivalence relation μ on S is congruence on the right regular S-set if and only if μ is a right congruence on the semigroup S (see § Subsection 2.2.1). Dually μ is a congruence on the left regular S-set if and only if μ is a left congruence on the semigroup S.

Let M be a monoid. Cyclic right [left] M-sets and right [left] congruences on the monoid M are related as follows.

PROPOSITION 2.23. *Let μ be a right congruence on the monoid M. Then M/μ is the cyclic right M-set xM, generated by $x = \mu(1)$. Conversely, if $X = xM$ is a cyclic right M-set, then*

$$\mu(X) = \{(s, t) \in M \times M : xs = xt\}$$

is a right congruence on M such that X is isomorphic to $M/\mu(X)$. Moreover, if $X = xM$ and $Y = yM$ are cyclic M-sets, there is surjective morphism $\theta : X \to Y$ with $\theta(x) = y$ if and only if $\mu(X) \subseteq \mu(Y)$.

Proof. Since a right congruence on M is a congruence on the right regular M-set, it follows that M/μ is a right M-set by Proposition 2.22. Since $\mu^{\#} : M \to M/\mu$ is a surjective morphism, we have

$$\mu^{\#}(s) = \mu^{\#}(1s) = \mu^{\#}(1)s = xs$$

and so $M/\mu = xM$. Conversely, given $X = xM$, $(s, t) \in \mu(X)$ implies $xs = xt$ and so $x(su) = x(tu)$ for all $u \in M$. This implies that $(su, tu) \in \mu(X)$. Thus $\mu(X)$ is a

automaton
set!– of states
function!transition –
automaton!transition monoid
automaton!finite state –

right congruence. Also, the map $\theta_X : s \mapsto xs$ is a surjective morphism of the right M-set onto X such that $\theta_X(s) = \theta_X(t)$ if and only if $(s,t) \in \mu(X)$. Therefore, by Proposition 2.22 X is isomorphic to $M/\mu(X)$.

Suppose that $f : s \mapsto xs$ and $g : s \mapsto ys$ where $X = xM$ and $Y = yM$. Then, by the above, f and g are morphisms of the right regular M-set to X and Y such that the right congruences on M induced by f and g are $\mu(X)$ and $\mu(Y)$ respectively. Also we have $f(1) = x$ and $g(1) = y$. If $\mu(X) \subseteq \mu(Y)$ then it is easy to see that

$$\theta(f(s)) = g(s) \quad \text{for all} \quad s \in M$$

defines a surjective morphism $\theta : X \to Y$ with $\theta(x) = y$. Conversely, assume that a surjective morphism $\theta : X \to Y$ exists with $\theta(x) = y$. If $(s,t) \in \mu(X)$ then by the definition of $\mu(X)$ we have $xs = xt$. Then

$$ys = \theta(x)s = \theta(xs) = \theta(xt) = \theta(x)t = yt$$

which shows that $(s,t) \in \mu(Y)$. Thus $\mu(X) \subseteq \mu(Y)$. □

Remark* 2.8*: The Proposition above shows that cyclic right [left] actions of monoids are characterized, up to isomorphisms, as quotients of monoids by right [left] congruences. A similar characterization of semigroup actions is not possible. However, we can always associate a cyclic S-set with every right [left] congruence μ on a semigroup S. For if $X = S/\mu$, let u denote some symbol not representing any element in X. Then $X^1 = \{u\} \cup X$ becomes a cyclic S-set by defining action of S on u by $us = \mu(s)$. But the cyclic S-set X^1 may not a quotient of the right [left] regular S-set even though it is a quotient of S^1. Notice that S^1, the semigroup obtained by adjoining identity to S (see Equation (2.3)) is always a faithful, cyclic [right, left] S-set.

Remark* 2.9*: Let M is a monoid with identity 1. A right M-set $\mathcal{A} = (X, \bar{\phi})$ is also called an *M-automaton* (see Eilenberg [1974], Lallement [1979]). In this case the set X is called the *set of states* of $\mathcal{A}$ and $\bar{\phi} : X \times M \to X$ is called its *transition function*. If $\phi : M \to \mathscr{T}_M$ is the representation determined by $\mathcal{A}$, then $\operatorname{Im}\phi = \phi(M)$ is a sub-semigroup of $\mathscr{T}_X$; $\phi(M)$ is called the *transition monoid* of $\mathcal{A}$. If X is a finite set, then $\mathcal{A}$ is called a *finite state automaton*. Concepts such as sub-automaton, morphism of automata, etc., can be defined in the obvious way.

2.6 IDEALS GREEN'S RELATIONS

Study of the structure the set ideals (both one-sided and two-sided) has been an important technique for analyzing the structure of various types of algebraic systems. For semigroups this technique has proved to be of great importance. Of particular importance are the classes of principal left and right ideals. These are usually studied via certain equivalence relations induced by them on the semigroup. These relations were first introduced and studied by Green Green [1951] and has shed considerable light on the local structure of semigroups in general and the class of regular semigroups in particular. Here we shall study these relations in terms of certain categories of principal left and right ideals.

$\mathbb{L}(S)$:*l-category of S*
category!l-category of S
category!r-category of S
$\mathbb{R}(S)$:*r-category of S*

2.6.1 Green's relations

Let S be a semigroup. Recall from Subsection 2.5.2 (see also § Subsection 2.1.1) that $I \subseteq S$ is a left ideal of S if and only if I is an S-subset of the left-regular S-set S_l. In particular, for any $a \in S$, $L(a)$ is a cyclic S-subset of S_l and is a quoitent of the left S-set S_l^1. Dual remarks hold for right ideals. I is a two-sided ideal of S if and only if I is an S-subset of both S_l and S_r.

Definition 2.1. Let $\mathbb{L}(S)$ denote the subcategory of the category ${}_S\mathbf{Set}$ of left S-sets (cf.Subsection 2.5.1) for which vertices are:

$$\mathfrak{v}\mathbb{L}(S) = \{L(a) : s \in S\}; \tag{2.34}$$

and $\rho \in {}_S\mathbf{Set}(L(a), L(b))$ is a morphism in $\mathbb{L}(S)$ if and only if there is $t \in S^1$ such that

$$x\rho = xt \quad \text{for all} \quad x \in L(a). \tag{2.35}$$

$\mathbb{L}(S)$ is called, for brevity, the *l-category* of S. The *r-category* $\mathbb{R}(S)$ of S is defined dually.

Thus $\mathbb{L}(S)$ is a clearly a subcategory of ${}_S\mathbf{Set}$. Therefore if $\rho : L(a) \to L(b)$ is a morphism in $\mathbb{L}(S)$, it is a morphism in ${}_S\mathbf{Set}$. The converse may not be true (see Example 2.8). Also if $L(a) \subseteq L(b)$, $j_{L(a)}^{L(b)}$ satisfies Equation (2.35) with $t = 1$. Hence inclusions are moriphisms in $\mathbb{L}(S)$. It is easy to verify that this provides a choice of suboblects in $\mathbb{L}(S)$ (see Subsection 1.3.2). Furtermore, every morphism in $\mathbb{L}(S)$ has unique factorization so that $\mathbb{L}(S)$ has images. Dual remarks hold for $\mathbb{R}(S)$.

Remark* 2.10*: If I and J are left [right] ideals, any S-morphism (in the category ${}_S\mathbf{Set}$) is a morphism of ideals. In particular, if $a \in J$, then $\rho_a|I$ [$\lambda_a|I$] is a morphism of I to J. However, if I and J are principal ideals, by a morphism $\theta : I \to J$, we shall mean a morphism in the category $\mathbb{L}(S)$ [$\mathbb{R}(S)$].

Since $\mathbb{L}(S)$ has subobjects, $\mathfrak{v}\mathbb{L}(S)$ is a partially ordered set which we denote by $\Lambda_S = \Lambda$. Thus Λ is the partially ordered set of principal left ideals under inclusion. Dually we denote by $I_S = I$ the paritally ordered set of principal right ideals and we have $\mathfrak{v}\mathbb{R}(S) = I$.

$\leq_l$, $\leq_r$, $\leq_j$: quasi-orders induced by principal ideals
Green's relations
$\mathscr{L}, \mathscr{R}, \mathscr{H}, \mathscr{D}, \mathscr{J}$: Green's relations
$\mathscr{L}[\mathscr{R}, \mathscr{H}, \mathscr{D}, \mathscr{J}]$-class
L_a, R_a, H_a, D_a, J_a: equivalence class of Green's relations

Definition 2.2 (Fundamental quasi-orders). The set of principal left and right ideals of a semigroup S (§ Subsection 2.1.1) induce some fundamental relations on S:

$$a \leq_l b \iff L(a) \subseteq L(b); \tag{2.36a}$$
$$a \leq_r b \iff R(a) \subseteq R(b); \tag{2.36b}$$
$$a \leq_j b \iff J(a) \subseteq J(b). \tag{2.36c}$$

These are quasi-orders (that is reflexive and transitive relations) on S (see § Subsection 1.1.2) such that order ideals with respect to these are respectively left, right and two-sided ideals.

We shall write $\leq_l (S)$, $\leq_r (S)$, etc., if it is necessary to indicate the semigroup on which the relations are defined.

Recall (see § Subsection 1.1.2) that if ρ is a quasi-order on a set X, then $\rho \cap \rho^{-1}$ is an equivalence relation on X. The quasi-orders above generate certain equivalence relations on the semigroup S which are also of fundamental importance.

Definition 2.3 (Green's relations). The following equivalence relations on a semigroup S are called *Green's relations*:

$$\mathscr{L} = \leq_l \cap (\leq_l)^{-1} = \{(a,b) \in S \times S : L(a) = S^1a = L(b)\}; \tag{2.37a}$$
$$\mathscr{R} = \leq_r \cap (\leq_r)^{-1} = \{(a,b) \in S \times S : R(a) = aS^1 = R(b)\}; \tag{2.37b}$$
$$\mathscr{H} = \mathscr{L} \cap \mathscr{R}; \tag{2.37c}$$
$$\mathscr{D} = \mathscr{L} \vee \mathscr{R}; \tag{2.37d}$$
$$\mathscr{J} = \leq_j \cap (\leq_j)^{-1} = \{(a,b) \in S \times S : J(a) = S^1aS^1 = J(b)\}. \tag{2.37e}$$

Again we shall use the notations $\mathscr{L}(S)$, $\mathscr{R}(S)$, etc., to denote these relations in case it is necessary to specify the semigroup to which these relations corresponds.

If $a \in S$, then the $\mathscr{L}[\mathscr{R}, \mathscr{H}, \mathscr{D}, \mathscr{J}]$-class of a is denoted by L_a [respectively, R_a, H_a, D_a, J_a]. It may be noted that L_a, R_a and J_a are generating sets of the principal ideal $L(a)$, $R(a)$ and $J(a)$ respectively. If $(a,b) \in \mathscr{L}$, then for any $s \in S^1$, $L(as) = L(bs)$ and so, $\mathscr{L}$ is a right congruence. Similarly $\mathscr{R}$ is a left congruence. However, in general, $\mathscr{H}$ and $\mathscr{J}$ are neither left nor right congruences. Recall that $\vee$, in Equation (2.37d), denote the join of $\mathscr{L}$ and $\mathscr{R}$ in the lattice $\mathcal{E}_S$ of all equivalence relations on S (Corollary 1.2). Since $\mathscr{L} \subseteq \mathscr{J}$ and $\mathscr{R} \subseteq \mathscr{J}$, we have $\mathscr{L} \vee \mathscr{R} = \mathscr{D} \subseteq \mathscr{J}$. It is easy to see that Green's relations satisfy the following inclusions:

$$\mathscr{H} \subseteq \mathscr{L} \subseteq \mathscr{D} \subseteq \mathscr{J}; \quad \mathscr{H} \subseteq \mathscr{R} \subseteq \mathscr{D} \subseteq \mathscr{J}.$$

In general all these inclusions are proper (see examples at the end of this section).

semigroup!bisimple –
semigroup!0-bisimple –

The partially ordered sets of Green's classes The quasiorder $\leq_l$ induces a partial order on the quotient set $S/\mathscr{L}$ defined by

$$L_a \leq L_b \iff L(a) \subseteq L(b) \quad \text{for all} \quad L_a, L_b \in S/\mathscr{L} \tag{2.38}$$

so that $S/\mathscr{L}$ is order isomorphic with the partially ordered set $\Lambda_S = \Lambda = \mathfrak{v}\mathbb{L}(S)$ of all principal left ideals under inclusion (see § Subsection 2.1.1) and Definition 2.1. In the following, we shall identify the partially ordered set $S/\mathscr{L}$ with Λ_S (or Λ). Similar remarks are valid for quasi-order $\leq_r$; we will identify partially ordered sets $S/\mathscr{R}$ with $\mathbf{I}_S = \mathfrak{v}\mathbb{R}(S)$. $\mathbf{J}_S$ denotes the partially ordered set of j-classes (see Subsection 2.1.1).

It is clear that a semigroup S is simple [0-simple] if and only if it has only one [non-zero] $\mathscr{J}$-class. Similar remarks hold for left simple semigroups, right simple semigroups, etc. A semigroup having only one [non-zero] $\mathscr{D}$-class is said to be *[0-]bisimple*. Since $\mathscr{D} \subseteq \mathscr{J}$, a bisimple [0-bisimple] semigroup is simple [0-simple]; but the converse is not true (see Example 2.13).

The following results are statements regarding categories $\mathbb{L}(S)$ and $\mathbb{R}(S)$. Therefore morphisms and/or isomorphisms considered are morphisms and/or isomorphisms in $\mathbb{L}(S)$ or $\mathbb{R}(S)$. Snce these are left-right duals, the dual of any result proved for one of them holds for the other. In particular $\mathscr{R}$ is the left-right dual of $\mathscr{L}$.

Since principal ideals are cyclic S-sets, the following uniqueness property holds for morphisms of principal ideals.

LEMMA 2.24. *Let S be a semigroup and let $a, b \in$. If $\rho, \sigma : L(a) \to L(b)$ are morphisms in $\mathbb{L}(S)$ such that $a\rho = a\sigma$, then $\rho = \sigma$.*

Proof. If $a\rho = a\sigma$, for any $u = sa \in L(a)$ ($s \in S^1$), we have $u\rho = s(a\rho) = s(a\sigma) = (sa)\sigma = u\sigma$. Therefore $\rho = \sigma$. □

The following result exhibit certain connections between isomorphisms of principal ideals (in $\mathbb{L}(S)$ or $\mathbb{R}(S)$) and Green's relations $\mathscr{L}$ and $\mathscr{R}$.

THEOREM 2.25. *Let S be a semigroup and $\rho : L(a) \to L(b)$ be an isomorphism of left ideals. Then we have the following:*

(a) For any $x \in L(a)$, $x \mathscr{R} x\rho$.

(b) For any $x \in L(a)$, $\rho|L_x$ is a bijection of L_x onto $L_{x\rho}$.

(c) If $c = a\rho$, then $a \mathscr{R} c \mathscr{L} b$.

In particular, $\rho|H_x$ is a bijection of H_x onto $H_{x\rho}$.

Proof. By the definition of isomorphisms there exists $t, t' \in S^1$ such that $x\rho = xt$ for all $x \in L(a)$ and $y\rho^{-1} = yt'$ for all $y \in L(b)$. If $y = x\rho$, then this shows that $y \in R(x)$ and so $R(y) \subseteq R(x)$. Similarly, $R(x) \subseteq R(y)$. Hence $x \mathscr{R} y = x\rho$. This proves (a) and the relation $a \mathscr{R} c$ in (c). Since ρ maps $L(a)$ onto $L(b)$, and since $\operatorname{Im} \rho = L(c)$, we have $L(c) = L(b)$ which implies $c \mathscr{L} b$ proving (c). If $x \in L(a)$, then $L_x \subseteq L(a)$ and since $\mathscr{L}$ is a right congruence, $L_x\rho = L_xt \subseteq L_{xt} = L_{x\rho}$. Thus ρ maps L_x into $L_{x\rho}$. Similarly ρ^{-1} maps $L_{x\rho}$ into L_x. It follows that ρ is a bijection of L_x onto $L_{x\rho}$. This proves (b). Finally, if $u \mathscr{H} x$, then $u \mathscr{L} x$ which implies, by (b), that $u\rho \mathscr{L} x\rho$. Also, by (a), $u\rho \mathscr{R} u \mathscr{R} x \mathscr{R} x\rho$ and so $u\rho \mathscr{H} x\rho$. As in the proof of (b), it can be shown that ρ is a bijection of H_x onto $H_{x\rho}$. □

THEOREM 2.26 (GREEN). *Let a and b be elements of a semigroup S. Then $a \mathscr{R} b$ if and only if there is a unique isomorphism $\rho : L(a) \to L(b)$ such that $a\rho = b$.*

Proof. Assume that $a \mathscr{R} b$. Then $R(a) = R(b)$ and so $b \in R(a)$. Hence $b = at$ for some $t \in S^1$. Similarly there is $t' \in S^1$ with $a = bt'$. Let $\rho = \rho_t|L(a)$ and $\rho' = \rho_{t'}|L(b)$. Then $\rho : L(a) \to L(b)$ and $\rho' : L(b) \to L(a)$ are morphisms in $\mathbb{L}(S)$ such that $a\rho = b$ and $b\rho' = a$. Then $\rho\rho' : L(a) \to L(a)$ is a morphism with $a\rho\rho' = a$ and so by Lemma 2.24, $\rho\rho' = 1_{L(a)}$. Similarly $\rho'\rho = 1_{L(b)}$. Thus ρ is an isomorphism. Uniqueness of ρ also follows from Lemma 2.24. Conversely, if $\rho : L(a) \to L(b)$ is an isomorphism such that $a\rho = b$, then by Theorem 2.25(c), $a \mathscr{L} b$. □

Let $a, x \in S$. Then, by the above, $a \mathscr{R} ax$ if and only if there is an isomorphism $\sigma : L(a) \to L(ax)$ with $a\sigma = ax$ and $\rho_x|L(a)$ is a morphism of left ideals $L(a)$ to $L(ax)$. Hence, by Lemma 2.24, $\sigma = \rho_x|L(a)$. This remark is often useful and so we state it as:

COROLLARY 2.27. *Let $a, x \in S$. Then $a \mathscr{R} ax$ if and only if $\sigma = \rho_x|L(a)$ is the unique isomorphism of $L(a)$ onto $L(ax)$ such that $a\sigma = ax$.* □

If α, β are equivalence relations on a set X and if they commute, it is easy to see that $\gamma = \alpha \circ \beta = \beta \circ \alpha$ is an equivalence relation. Since $\alpha \subseteq \gamma$ and $\beta \subseteq \gamma$, $\alpha \vee \gamma \subseteq \gamma$. On the other hand, if ρ is any other equivalence relation with $\alpha, \beta \subseteq \rho$, then by the transitivity of ρ, $\gamma = \alpha \circ \beta \subseteq \rho$. Hence

$$\alpha \circ \beta = \beta \circ \alpha = \alpha \vee \beta.$$

We use these remarks in the following characterization of the Green's relation $\mathscr{D}$.

PROPOSITION 2.28. *Let $\mathscr{D}$ denote the relation defined by Equation* (2.37d) *on a semigroup S. Then*

$$\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L} .$$

Proof. Let $(a, b) \in \mathscr{L} \circ \mathscr{R}$. Then, by the definition of composition (Equation (1.2)), for some $c \in S$, $a \mathscr{L} c \mathscr{R} b$ and by Theorem 2.25, there is a unique isomorphism $\sigma : L(a) = L(c) \to L(b)$ such that $c\sigma = b$. Let $d = a\sigma$. Then by Theorem 2.25(b), $a \mathscr{R} d$ and by (c), $d \mathscr{L} b$. Hence $a \mathscr{R} d \mathscr{L} b$ and so $(a, b) \in \mathscr{R} \circ \mathscr{L}$. Thus

$$\mathscr{L} \circ \mathscr{R} \subseteq \mathscr{R} \circ \mathscr{L} . \quad \text{Similarly,} \quad \mathscr{R} \circ \mathscr{L} \subseteq \mathscr{L} \circ \mathscr{R}$$

and so $\mathscr{L}$ and $\mathscr{R}$ are commuting equivalence relations on S. Hence

$$\mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L} = \mathscr{L} \vee \mathscr{R} = \mathscr{D} .$$

by the remarks preceding the statement of the proposition and the definition of $\mathscr{D}$ (Equation (2.37d)). The last statement is now clear from Theorem 2.25. □

The following are some of the consequences of the Proposition above.

Corollary 2.29. *For $a, b \in S$, the principal ideals $L(a)$ and $L(b)$ are isomorphic in $\mathbb{L}(S)$ if and only if $a \mathscr{D} b$.*

Proof. If $\sigma : L(a) \to L(b)$ is an isomorphism, then by Theorem 2.25(a) and the Proposition above, $a \mathscr{D} b$. Conversely, if $a \mathscr{D} b$, then by the above, there is $c \in S$ with $a \mathscr{R} c \mathscr{L} b$ and by Theorem 2.26, there is an isomorphism $\sigma : L(a) \to L(c) = L(b)$ such that $a\sigma = c$. □

Corollary 2.30. *Let L denote an $\mathscr{L}$-class and R, an $\mathscr{R}$-class of a semigroup S. Then $H = L \cap R \neq \emptyset$ if and only if there is a $\mathscr{D}$-class D with $L \cup R \subseteq D$. Moreover, if $H \neq \emptyset$, then H is a $\mathscr{H}$-class of S.*

Proof. If $L \cap R \neq \emptyset$, then for any $a \in L$ and $b \in R$, $a \mathscr{L} c \mathscr{R} d$ for any $c \in L \cap R$. Hence $a \mathscr{D} b$. Therefore $L \cup R \subseteq D_a$. Conversely if $L \cup R \subseteq D$, then $a \mathscr{D} b$ for any $a \in L$ and $b \in R$. Hence by Proposition 2.28, there is c with $a \mathscr{L} c \mathscr{R} b$ so that $c \in L \cap R$. The last statement is a consequence of the definition of the relation $\mathscr{H}$ (Equation (2.37c)). □

Corollary 2.31. *Let H_1 and H_2 be two $\mathscr{H}$-classes contained in the same $\mathscr{D}$-class of S. Then there is a bijection of H_1 onto H_2.*

Proof. Let D be the $\mathscr{D}$-class such that $H_1 \cup H_2 \subseteq D$ and let $a \in H_1$ and $b \in H_2$. Then $a \mathscr{D} b$ and so, by Proposition 2.28, $a \mathscr{R} c \mathscr{L} b$ for some $c \in D$. By Theorem 2.25 there exist an isomorphism $\sigma : L(a) \to L(c)$ which is a bijection of $H_a = H_1$ onto H_c. Dually, there is an isomorphism $\lambda : R(c) \to R(b)$ which is a bijection of H_c onto H_2. Hence $\sigma \circ \lambda | H_1$ is a bijection of H_1 onto H_2. □

Corollary 2.32. *Let L denote an $\mathscr{L}$-class and R, an $\mathscr{R}$-class of a semigroup S. Then the set product LR of L and R is contained in some $\mathscr{D}$-class of S.*

egg-box picture

Proof. Let $a, a' \in L$ and $b, b' \in R$. Since $\mathscr{L}$ [$\mathscr{R}$] is a right [left] congruence we have $ab \mathrel{\mathscr{L}} a'b \mathrel{\mathscr{R}} a'b'$. Hence by Proposition 2.28, $ab \mathrel{\mathscr{D}} a'b'$. □

Remark* 2.11 *(The "egg-box" picture of $\mathcal{D}$-classes): Let D be a $\mathscr{D}$-class in S. Since $\mathscr{L} \subseteq \mathscr{D}$, D is the union of all $\mathscr{L}$-classes intersecting D. Similarly, D is the union of all $\mathscr{R}$-classes intersecting D. Let $\{R_i : i \in I\}$ and $\{L_\lambda : \lambda \in \Lambda\}$ be the sets of $\mathscr{R}$ and $\mathscr{L}$-classes contained in D. Then by Corollary 2.30, $H_{i,\lambda} = R_i \cap L_\lambda$ is not empty and so, is an $\mathscr{H}$-class for any $(i, \lambda) \in I \times \Lambda$. By Corollary 2.31 there is a bijection between any two of these $\mathscr{H}$-classes. Therefore D is a rectangular grid of $\mathscr{H}$-classes $H_{i,\lambda}$ having I rows, the $\mathscr{R}$-classes contained in D, and Λ columns, the $\mathscr{L}$-classes in D, and such that each cell contains the same number of elements. We may thus visualize $\mathscr{D}$-classes as "egg boxes" in which each cell contain the same number of "eggs". In the following, whenever we refer to the egg-box picture of $\mathscr{D}$-classes, we shall assume that *columns of the egg-box represent $\mathscr{L}$-classes and rows represent $\mathscr{R}$-classes*. The semigroup S itself may be viewed as a stalk of egg boxes placed one over the other with the box containing 1 (if it exists) at the top and the a single box containing 0 alone at the bottom.

Note that the fundamental quasiorders $\leq_l$, $\leq_r$ and $\leq_j$ (see Equations (2.36a)–(2.36c)) and Green's relations (see Equations (2.37a)–(2.37d)) on a semigroup S are defined in terms of the product in S. Therefore they are preserved under homomorphisms. For example, if $a \leq_l b$ in S then $s = sb$ for some $s \in S^1$. Hence if $\phi : S \to T$ is a homomorphism, then $a\phi = (s\phi)(b\phi)$ and so $a\phi \leq_l b\phi$ in T. In a similar way it can be shown that ϕ preserves other relations also. This fact is of constant use in the sequel and so we state it as:

LEMMA 2.33. *Let $\phi : S \to T$ be a homomorphism of a semigroup S into T. If $a, b \in S$ are related by any one of the fundamental quasi-orders defined by Equations* (2.36a)–(2.36c) *or Green's relations defined by Equations* (2.37a)–(2.37d)*, then $a\phi$ and $b\phi$ are related by the same relation in T.* □

Even though homomorphisms preserve these relations, they do not reflect them; that is, if $a\phi$ and $b\phi$ are related by, say $\mathscr{L}$ in T, it is clear that a and b may not be so related in S. It is not even true that if $a\phi \mathrel{\mathscr{L}} b\phi$ then they are so related in the subsemigroup $\phi(S) = \operatorname{Im}\phi$ of T (see Example 2.9).

We end this section with some counter examples and examples which illustrate the computation of Green's relations on some important classes of semigroups.

Example 2.8: Let $S = X^*$ over the set X and let u be a non-empty word. Then $S = L(e)$ where e is the empty word and $L(u) = Su$ are objects in the category $\mathbb{L}(S)$ amd the mapping $g : w \mapsto wu$ is a morphism of $\mathbb{L}(S)$ from S to Sw which is clearly injective. Infact, it can be seen that the map $g^{-1} : Sw \to S$ is a morohism od S-sets in the catehory ${}_S\mathbf{Set}$; but there exists no $t \in S = S^1$ such that $g^{-1} = \rho_t|S$. For, if it did, we must have $ut = 1$ which is imposible in a free monoid if $u \neq 1$. Thus g^{-1} is not a morphism in $\mathbb{L}(S)$.

equivalence relation!cross-section of

regular

Example 2.9: Let $S = (\mathbb{N}, +)$. It is clear that the quasi-orders $\leq_l$, $\leq_r$ and $\leq_j$ coincides with the natural order on $\mathbb{N}$ and so all Greeen's relations on S coincides with the identity relation on $\mathbb{N}$. It is clear that on any group G, all the fundamental quasiorders and Green's relations coincides with the universal relation $G \times G$. Therefore, even though $S = (\mathbb{N}, +)$ is a subsemigroup of the additive group $(\mathbb{Z}, +)$ of integers, the fundamental quasi-orders or the Green's relations on S is not the restriction of the corresponding relation on $(\mathbb{Z}, +)$ to S.

Example 2.10: Let $S = \mathscr{T}_X$ (cf. § Subsection 2.1.3 be the semigroup of transformations of a set X. Recall Corollary 1.2 that $\mathcal{E}_X$ denote the lattice of all equivalence relations on X and let $\mathbb{P}(X)$ denote the partially ordered set (under inclusion) of all non-empty subsets of X. We say that $Y \in \mathbb{P}(X)$ is a *cross-section* of $\pi \in \mathcal{E}_X$ if each π-class contains exactly one element of Y. If y_x denote the unique element in $Y \cap \pi(x)$, this implies that the map $\pi(x) \mapsto y_x$ is a bijection of X/π onto Y. By Zorn's lemma, every $\pi \in \mathcal{E}_X$ has at least one cross-section; it is also easy to see that for any $Y \in \mathbb{P}(X)$, there is at least one $\pi \in \mathcal{E}_X$ such that Y is a cross-section of π.

For each $f \in S$ we can associate a subset $\operatorname{Im} f$ Equation (1.1b) of X an equivalence relation π_f Equation (1.10a) such that $|X/\pi_f| = |\operatorname{Im} f|$ (see Equation (1.10b)). Conversely given $\pi \in \mathcal{E}_X$ and $Y \in \mathbb{P}(X)$ with $|X/\pi| = |Y|$, for any bijection $\psi : X/\pi \to Y$, the map $f = \pi^\# \circ \psi$ is a transformation of X such that $\pi_f = \pi$ and $\operatorname{Im} f = Y$. Given $f \in S$, choose $\pi \in \mathcal{E}_X$ such that $\operatorname{Im} f$ is a cross-section of π and let Y be a cross-section of π_f. Then $f|Y$ is a bijection of Y onto $\operatorname{Im} f$ and so $\psi : \pi(x) \mapsto (f|Y)^{-1}(y_x)$, where y_x denote the unique element in $\operatorname{Im} f \cap \pi(x)$, is a bijection of $|X/\pi|$ onto Y. Hence by the remarks above, $f' = \pi^\# \circ \psi$ is a transformation of X with $\pi_{f'} = \pi$ and $\operatorname{Im} f' = Y$. Moreover, for this f', we have

$$ff'f = f \qquad \text{and} \qquad f'ff' = f'.$$

This shows that the semigroup $S = \mathscr{T}_X$ is *regular* (see § Subsection 2.6.2 for definition of regular semigroups). These Equations shows in particular that $ff' : X \to Y$ and $f'f : X \to \operatorname{Im} f$ are idempotents (so that $ff'|Y = 1_Y$ and $f'f|\operatorname{Im} f = 1_{\operatorname{Im} f}$).

If $g = sf$ for $s \in S^1 = S$, then it is clear that $\operatorname{Im} g \subseteq\in f$. Suppose conversely that $\operatorname{Im} g \subseteq \operatorname{Im} f$. If $f' : X \to Y \in \mathscr{T}_X$ is constructed as in the paragraph above, then $gf'f = g$. So $g = sf$ where $s = gf'$. Hence

$$g \leq_l f \iff \operatorname{Im} g \subseteq \operatorname{Im} f \quad \text{and by (2.37a), we have} \quad g \mathscr{L} f \iff \operatorname{Im} g = \operatorname{Im} f. \tag{1}$$

Again, if $g = ft$ with $t \in S$, then $\pi_g \supseteq \pi_f$. Conversely, if $\pi_g \supseteq \pi_f$, then any cross-section Y of π_f contains a cross-section of g and so $ff'g = g$ where f' is constructed as above. Then $g = ft$ where $t = f'g$. Thus we have

$$g \leq_r f \iff \pi_f \subseteq \pi_g \quad \text{hence by (2.37b),} \quad g \mathscr{R} f \iff \pi_f = \pi_g. \tag{2}$$

By Equation (2.37c) and (1) and (2) above, we obtain

$$g \mathscr{H} f \iff \operatorname{Im} g = \operatorname{Im} f \quad \text{and} \quad \pi_f = \pi_g. \tag{3}$$

If $f \mathscr{D} g$, then by Proposition 2.28 $f \mathscr{L} h \mathscr{R} g$ for some $h \in S$. Hence by (1) and (2) above, $\operatorname{Im} f = \operatorname{Im} h$ and $\pi_h = \pi_g$ and so,

$$|\operatorname{Im} f| = |\operatorname{Im} h| = |X/\pi_h| = |X/\pi_g| = |\operatorname{Im} g|.$$

On the other hand, if $|\operatorname{Im} f| = |\operatorname{Im} g|$, we can find $t \in S$ such that $\alpha = t|\operatorname{Im} f$ is a bijection of $\operatorname{Im} f$ onto $\operatorname{Im} g$. Then by (1), $ft \mathscr{L} g$ and since $ft = f\alpha$, $\pi_f = \pi_{ft}$ so that $f \mathscr{R} ft$. Hence by Proposition 2.28, we have

$$g \mathscr{D} f \iff |\operatorname{Im} g| = |\operatorname{Im} f|. \tag{4}$$

If $g \leq_j f$, then $g = sft$ for $s, t \in S$. Then t maps $\operatorname{Im} f$ onto $\operatorname{Im} g$ and so $|\operatorname{Im} g| \leq |\operatorname{Im} f|$. If f and g satisfies this condition, it is easy to see that for some $t \in S$, t maps $\operatorname{Im} f$ onto $|\operatorname{Im} g$ and so, by (1), $ft \mathscr{L} g$; in particular, $g \leq_j ft$. Also $ft \leq_r f$ and so $g \leq_j f$. Thus

$$g \leq_j f \iff |\operatorname{Im} g| \leq |\operatorname{Im} f|. \tag{5}$$

Hence by (2.37e) and Equation (4), we have

$$\mathscr{J} = \mathscr{D}\ . \tag{6}$$

We have noted above that the quasiorder $\leq_l$ induces a partial order on the quotient set $S/\mathscr{L}$ which is order-isomorphic with the partially ordered set of principal left ideals under inclusion. The Equation (1) above shows that $S/\mathscr{L}$ is order isomorphic with $\mathbb{P}(X)$. Similarly, it follows from Equation (2) that $S/\mathscr{R}$ is order isomorphic with $\mathcal{E}_X$. If L_Y denote the unique $\mathscr{L}$-class corresponding to $Y \in \mathbb{P}(X)$ and R_π denote the the unique $\mathscr{R}$-class corresponding to $\pi \in \mathcal{E}_X$, then $L_Y \cap R_\pi \neq \emptyset$ and hence an $\mathscr{H}$-class if and only if there is $f \in S$ with $\operatorname{Im} f = Y$ and $\pi_f = \pi$; this is true if and only if $|X/\pi| = |Y|$. If we set

$$H_{\pi,Y} = \begin{cases} \{f : \pi_f = \pi,\ \operatorname{Im} f = Y\} & \text{if } |X/\pi| = |Y|; \\ \emptyset & \text{otherwise.} \end{cases}$$

then we have

$$S/\mathscr{H} = \{H_{\pi,Y} : |X/\pi| = |Y|\}.$$

By Equation (5), $S/\mathscr{J}$ is order isomorphic with the linearly ordered set of all cardinal numbers $\alpha \leq |X|$.

Example 2.11: Let $S = \mathscr{L}\mathscr{T}(V)$, (cf. § Subsection 2.1.3 the semigroup of all linear transformations on a vector space V over some field $\Bbbk$. Most of the arguments in the last example carries over to this situation if we replace maps by appropriate linear transformations. In this case, if $f \in S$, $\operatorname{Im} f$ is a subspace of V and π_f is the coset decomposition of V with respect to the *null-space* $\boldsymbol{N}(f) = \{v \in V : vf = 0\}$. Recall that, for $f \in \mathscr{L}\mathscr{T}(V)$, $\operatorname{Rank} f = \dim(\operatorname{Im} f)$. We have the following description of Green's relations on $S = \mathscr{L}\mathscr{T}(V)$.

$$g \leq_l f \iff \operatorname{Im} g \subseteq \operatorname{Im} f, \qquad g \mathscr{L} f \iff \operatorname{Im} g = \operatorname{Im} f; \tag{1}$$

$$g \leq_r f \iff \boldsymbol{N}(f) \subseteq \boldsymbol{N}(g), \qquad g \mathscr{R} f \iff \boldsymbol{N}(g) = \boldsymbol{N}(f); \tag{2}$$

$$g \leq_j f \iff \operatorname{Rank} g \leq \operatorname{Rank} f, \qquad g \mathscr{J} f \iff g \mathscr{D} f. \tag{3}$$

It follows from (1) above that, for the semigroup $S = \mathscr{L}\mathscr{T}(V)$, the partially ordered set $S/\mathscr{L}$ is order isomorphic with the lattice $\mathfrak{P}(V)$ of all subspaces of V and by (2), $S/\mathscr{R}$ is dually isomorphic with $\mathfrak{P}(V)$ (or isomorphic to $\mathfrak{P}(V)$ with dual order— see § Subsection 1.1.2). If for $N, U \in \mathfrak{P}(V)$, R_N the unique $\mathscr{R}$-class corresponding to N and L_U denote the unique $\mathscr{L}$-class corresponding to U, then $R_N \cap L_U \neq \emptyset$ if and only if $\dim N + \dim U = \dim V$; when N and U satisfy this condition, $R_N \cap L_U = H_{N,U}$ is an $\mathscr{H}$-class of S consisting of all linear transformations with null space N and image U. Again, using Equation (3), we can see that $S/\mathscr{J}$ is order isomorphic with the linearly ordered set of all cardinal numbers $\alpha \leq \dim V$.

Example 2.12: Let $S = \langle p, q : pq = 1\rangle$ denote the bicyclic semigroup (see Example 2.7). For any $x = q^n p^m, y = q^r p^s \in S$, using Equations (c) and (e) in 2.7, we find that there is $t = q^u p^v \in S$ with $y = xt$ if and only if $n \leq r$; that is

$$q^r p^s \leq_r q^n p^m \iff n \leq r \quad \text{and} \quad q^r p^s \,\mathscr{R}\, q^n p^m \iff r = n. \tag{1}$$

Similarly

$$q^r p^s \leq_l q^n p^m \iff m \leq s \quad \text{and} \quad q^r p^s \,\mathscr{L}\, q^n p^m \iff s = m. \tag{2}$$

It follows that every $\mathscr{R}$-class of S is of the form

$$R_n = R_{q^n} = \{q^n p^s : s \in \mathbb{N}\}, \tag{3r}$$

for $n \in \mathbb{N}$ and every $\mathscr{L}$-class of S has the form

$$L_m = L_{p^m} = \{q^r p^m : r \in \mathbb{N}\}, \tag{3l}$$

for $m \in \mathbb{N}$. It follows from (1) and (2) that $S/\mathscr{R} = \{R_n : n \in \mathbb{N}\}$ and $S/\mathscr{L} = \{L_m : m \in \mathbb{N}\}$ are both order isomorphic with $\mathbb{N}$. Now for any $n, m \in \mathbb{N}$, by (1) and (2), we have $R_n \cap L_m = \{q^n p^m\}$. Hence every $\mathscr{H}$-class of S contain exactly one element. Also, this shows that any $\mathscr{R}$-class of S intersect any $\mathscr{L}$-class. Therefore, by Corollary 2.30, any two elements of S are $\mathscr{D}$-related. Hence S has exactly one $\mathscr{D}$-class (and so, one $\mathscr{J}$-class); that is, S is bisimple.

Example 2.13: Let $\mathbb{R}^+$ denote the set of all positive real numbers and $A = \mathbb{R}^+ \times \mathbb{R}^+$. Define product in A by:

$$(a, b)(c, d) = (ac, bc + d). \tag{1}$$

If $\alpha = (a, b), \beta = (c, d), \gamma = (e, f) \in A$, then, using (1), we compute

$$\begin{aligned}(\alpha\beta)\gamma &= (ac, bc + d)(e, f) = (ace, (bc + d)e + f) \\ &= (ace, bce + de + f) = (a, b)(ce, de + f) \\ &= \alpha(\beta\gamma).\end{aligned}$$

Hence A is a semigroup. Given $\alpha = (a, b), \beta = (c, d) \in A$, choose positive real numbers u and v satisfying $bu + v < d$. Let x and y be solutions of Equations

$$c = xau, \quad \text{and} \quad d = yau + bu + v.$$

Then $\sigma = (x, y), \tau = (u, v) \in A$ and $\beta = \sigma\alpha\tau$. This implies that $\beta \in J(\alpha)$ and since β is arbitrary, we have $J(\alpha) = A$. It follows that A does not contain any proper ideal. Therefore A is simple. Now suppose that $\alpha \,\mathscr{R}\, \beta$. If $\alpha \neq \beta$, there exists $\tau, \tau' \in A$ such that $\beta = \alpha\tau$ and $\alpha = \beta\tau'$. Then $\alpha = \alpha\tau\tau'$. If $\alpha = (a, b)$ and $\tau\tau' = (u, v)$, then we have $(a, b) = (a, b)(u, v) = (au, bu + v)$ and so $u = 1$ and $v = 0$. This implies that $(u, v) \notin A$ which is a contradiction. Hence $\alpha = \beta$; that is, $\mathscr{R} = 1_A$. Similarly, $\mathscr{L} = 1_A$ and so, $\mathscr{D} = \mathscr{L} \vee \mathscr{R} = 1_A$. Therefore A is simple, but not bisimple; in particular, $\mathscr{D} \neq \mathscr{J}$.

2.6.2 Regular $\mathscr{D}$-classes

Here we introduce the concept of regular elements and investigate the structure of $\mathscr{D}$-classes containing regular elements. Most of the results are reformulation of results due to A. H. Clifford and D. D. Miller (from Miller and Clifford [1956]). The first result is of basic importance in the discussion of regularity Clifford and Preston [1961], Miller and Clifford [1956].

THEOREM 2.34. *Let a and b be elements of a semigroup S. Then* $ab \in R_a \cap L_b$ *if and only if* $L_a \cap R_b$ *contains an idempotent.*

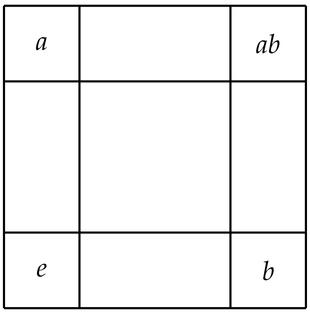


Fig. 1

Proof. The result stated can be illustrated, using the "egg-box picture" (see Remark 2.11), given on the left.

Suppose that $ab \in R_a \cap L_b$. Then by Corollary 2.27 the map $\sigma = \rho_b|L(a)$ is an isomorphism of $L(a)$ onto $L(ab) = L(b)$. Since, by Theorem 2.25, σ preserves $\mathscr{L}$-classes, there is a unique $e \in L_a \cap R_b$ such that $e\sigma = eb = b$. Now $e^2 \in L(a)$ and $e^2\sigma = e^2b = eb = b$. Hence $e\sigma = e^2\sigma$ and since σ is an isomorphism, $e = e^2$. Conversely, if there is an idempotent $e \in L_a \cap R_b$, then $b = et$ for some $t \in S^1$ and so $eb = e^2t = et = b$. Hence, as above $\sigma = \rho_b|L(a)$ is an isomorphism of $L(a)$ onto $L(b)$ and by Theorem 2.25, $ab = a\sigma \in R_a \cap L_b$. □

Remark* 2.12*: Theorem 2.34 is one of the few theorems in semigroup theory that assert the existence of an idempotent. Since idempotents have strong relation to the structure of important classes of semigroups such as regular semigroups, finite semigroups, etc., we will find this an indispensable tool in the sequel.

As a consequence of Theorem 2.34, we have the following characterization of $\mathscr{H}$-classes that contain idempotents.

COROLLARY 2.35. *Let H be an* $\mathscr{H}$*-class of a semigroup S. Then there exists* $a, b \in H$ *such that* $ab \in H$ *if and only if H contains an idempotent.*

Proof. If there exist $a, b \in H$ such that $ab \in H$, then by Theorem 2.34, H contains an idempotent. Conversely, if $e = e^2 \in H$, then $e, e, ee = e^2 \in H$. □

The following properties of idempotents are useful frequently and so, for convenience of reference, we state them as:

LEMMA 2.36. *For an idempotent e in a semigroup S, we have the following:*

group!subgroup group!automorphism – regular inverse!generalized – inverse

(a) e is a right identity of every element in $L(e)$ *and hence right identity of every element in* L_e*. Further,* $L(e) = Se$*.*

(b) e is a left identity of every element in $R(e)$ *and hence of every element in* R_e*. Further,* $R(e) = eS$*.*

(c) e is a two-sided identity of every element in eSe and hence of every element in H_e*.*

Proof. If $a \in L(e) = S^1e$, then $a = se$ for some $s \in S^1$ and so, $ae = se^2 = se = a$. Since $e = e^2 \in Se$, it follows that

$$Se \subseteq S^1e = Se \cup \{e\} \subseteq Se.$$

This proves (a). Proof of (b) is dual and (c) is an immediate consequence of (a) and (b). □

PROPOSITION 2.37. *Let e be an idempotent in a semigroup S. Then* H_e *is a subgroup of S and there are isomorphisms* $\sigma : H_e \to \mathrm{Aut}[L(e)]$ *and* $\tau : H_e \to \mathrm{Aut}[R(e)]$ *of* H_e *onto the group of automorphisms of* $L(e)$ *and* $R(e)$ *respectively. Moreover, maximal subgroups of S are precisely those* $\mathscr{H}$*-classes that contain idempotents.*

Proof. Since e is an idempotent in S, by Theorem 2.34, $ab \in H_e$ for all $a, b \in H_e$ and so H_e is a subsemigroup of S. By Lemma 2.36, $ea = a$ for all $a \in H_e$. Hence for any $a \in H_e$, $\sigma_a = \rho_a|L(e)$ is the unique automorphism of $L(e)$ such that $e\sigma_a = a$ by Corollary 2.27. If $a, b \in H_e$,

$$e\sigma_a\sigma_b = eab = e\sigma_{ab}$$

and by the uniqueness (Lemma 2.24), we have $\sigma_a\sigma_b = \sigma_{ab}$. Thus $\sigma : a \mapsto \sigma_a$ is a homomorphism of H_e into the group $\mathrm{Aut}[L(e)]$ of automorphisms of $L(e)$ which is injective by Lemma 2.24. If $\alpha \in \mathrm{Aut}[L(e)]$, then $a = e\alpha \in H_e$ by Theorem 2.25 and, again by Lemma 2.24, $\alpha = \sigma_a$. Thus $\sigma : H_e \to \mathrm{Aut}[L(e)]$ is an isomorphism. In particular H_e is a group with identity e. Dually the map $\tau : a \mapsto \tau_a = \lambda_a|R(e)$ is an isomorphism of H_e onto the group $\mathrm{Aut}[R(e)]$ of automorphisms of $R(e)$.

Now suppose that G is a subgroup of S with identity e. Then clearly, e is an idempotent in S. If $a \in G$, equations $ae = ea = a$ and $aa^{-1} = a^{-1}a = e$ in G implies that $e \mathscr{H} a$ and so $G \subseteq H_e$. This proves that a subgroup G of S is a maximal subgroup if and only if $G = H_e$ for some idempotent e of S. □

An element $a \in S$ is said to be *regular* if there is $t \in S^1$ such that $ata = a$; in this case t is called a *generalized inverse* of a. An element $a' \in S$ is called a *semigroup inverse* or simply an *inverse* if there is no ambiguity if a' is a generalized inverse of a and vice versa; that is a and a' satisfies the following:

$$aa'a = a \quad \text{and} \quad a'aa' = a'; \tag{2.39}$$

$\mathscr{V}(a)$: The set of inverses of a semigroup!regular –

a' is called a *group-inverse* of a if

$$aa'a = a, \qquad a'aa' = a' \quad \text{and} \quad a'a = aa'. \tag{2.40}$$

The set of all inverses of a is denoted by $\mathscr{V}(a)$. A semigroup S is said to be *regular* if every element of S is regular.

If a is regular with generalized inverse t, and if $a' = tat$ then

$$a'aa' = t(ata)tat = t(ata)t = tat = a';$$
$$aa'a = (ata)ta = ata = a.$$

Hence $a' = tat$ is an inverse of a. Moreover $e = aa'$ is an idempotent such that $ea = a$ and so $e \mathscr{R} a$. Similarly $e \mathscr{L} a'$ and if $f = a'a$, then $a \mathscr{L} f \mathscr{R} a'$. These relations are shown in the figure on the right. These imply that $a \mathscr{D} a'$ and so, $\mathscr{V}(a) \subseteq D_a$. Further, if a' is a group inverse of a (so that $e = f$), then by the above, $a \in H_e$ and a' is the inverse of a in the group H_e. Conversely, if a is an element of a maximal subgroup H_e of S, the inverse of a in the group H_e is clearly a group inverse as defined above.

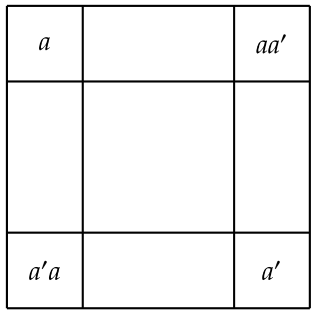


Fig. 2

For convenience of later reference, we summarize the discussion as:

Lemma 2.38. *An element a in a semigroup S is regular if and only if $\mathscr{V}(a) \neq \emptyset$. Further, if $a' \in \mathscr{V}(a)$, then $e = aa'$ and $f = a'a$ are idempotents such that*

$$a \mathscr{R} e \mathscr{L} a' \mathscr{R} f \mathscr{L} a.$$

In particular, $\mathscr{V}(a) \subseteq D_a$. Moreover, $a' \in \mathscr{V}(a)$ is a group inverse of a if and only if a belongs to a maximal subgroup H_e of S and a' is the inverse of a in the group H_e. □

We now characterize regular elements in terms of Green's relations.

Proposition 2.39. *For an element a in a semigroup S, the following statements are equivalent:*

(a) a is regular;

(b) L_a contains an idempotent;

(c) R_a contains an idempotent.

Further, if a is regular, every element in D_a is also regular.

Proof. The statement (a) implies (b) by Lemma 2.38. Conversely, if e is an idempotent in L_a, then $e = sa$ for some $s \in S$ and $asa = ae = a$ by Lemma 2.36.

Thus statements (a) and (b) are equivalent. Dually, statements (a) and (c) are equivalent. Also, the equivalence of (a) and (b) implies that, if one element of a $\mathscr{L}$-class L is regular, then every element of L is regular. Dually if one element of an $\mathscr{R}$-class R is regular, then every element of R is regular by the equivalence of (a) and (c). Now if if a is regular and if $b \in D_a$, then by Corollary 2.30, $R_a \cap L_b \neq \emptyset$. Hence there is c such that $a \mathrel{\mathscr{R}} c \mathrel{\mathscr{L}} b$. Since $R_c = R_a$, by the remarks above, c is regular. Again, this implies that every element of $L_c = L_b$ is regular and so, b is regular. □

regular!– $\mathscr{D}$-class

If D is a $\mathscr{D}$-class of a semigroup S, the result above shows that either every element of D is regular or none of them are regular. We say that D is a *regular $\mathscr{D}$-class* of S if every element of D is regular.

The next result locates all inverses of a regular element. Recall from Lemma 2.38 that every inverse a' of a regular element a of S belongs to D_a. In the following, for any $X \subseteq S$, we write $E(X)$ for the set of idempotents in X.

PROPOSITION 2.40. *Let a be a regular element of a semigroup S and let a' be an inverse of a.*

(a) *For every $e \in E(R_a)$ and $f \in E(L_a)$, $fa'e$ is an inverse of a in $L_e \cap R_f$ (see the figure on the right).*

(b) *If $a' \mathrel{\mathscr{H}} a''$, $a', a'' \in \mathscr{V}(a)$ then $a' = a''$.*

(c) *$\mathscr{V}(a) = \{fa'e : e \in E(R_a), f \in E(L_a)\}$.*

In particular, $a \in S$ has unique inverse if and only if L_a and R_a contains exactly one idempotent each.

a		aa'	e
$a'a$		a'	a'e
f		fa'	fa'e

Fig. 3

Proof. Since a' is an inverse of a, by Lemma 2.38, aa' and $a'a$ are idempotents such that $a \in R_{aa'} \cap L_{a'a}$ and $a' \in L_{aa'} \cap R_{a'a}$. Hence $a'a \in R_{a'} \cap L_f$ and so $fa' \in R_f \cap L_{a'}$ by Theorem 2.34. Similarly, $aa' \in L_{fa'} \cap R_e$ and so, again by Theorem 2.34, $fa'e \in R_{fa'} \cap L_e = R_f \cap L_e$. Further,

$$a(fa'e)a = (af)a'(ea) = aa'a = a;$$
$$(fa'e)a(fa'e) = fa'(eaf)a'e = f(a'aa')e = fa'e.$$

This proves (a) (see the egg-box diagram on the right).

To prove (b), suppose that a' and a'' are $\mathscr{H}$-equivalent inverses of a. Then aa', aa'', $a'a$ and $a''a$ are idempotents such that $aa' \mathrel{\mathscr{H}} aa''$ and $a'a \mathrel{\mathscr{H}} a''a$ (by Lemma 2.38). By Proposition 2.37, $aa' = aa''$ and $a'a = a''a$. Hence we have

$$a' = a'aa' = a''aa' = a''aa'' = a''.$$

E(*S*)*: the biordered set of idempotents of S*

Finally, Let X denote the set on the right of the equation in item (c). By (a), $X \subseteq \mathscr{V}(a)$. If $a'' \in \mathscr{V}(a)$, then by Lemma 2.38,

$$e = aa'' \in R_a \cap L_{a''}; \quad f = a''a \in L_a \cap R_{a''}.$$

Then by (a), $fa'e$ is an inverse of a in $L_e \cap R_f = H_{a''}$. Hence by (b), $fa'e = a''$ and so $a'' \in X$. The last statement is am immediate consequence of (c). □

Remark* 2.13*: The result above throws considerable light on the structure of the set of inverses of a regular elements. An $\mathscr{H}$-class H_b contains a an inverse of an element a if and only if $\mathscr{H}$-classes $R_a \cap L_b$ and $L_a \cap R_b$ contains idempotents. If e and f are respectively these idempotents, one can calculate the unique inverse in H_b in terms of any other inverse a' of a as $fa'e$. It may be noted that, in this case, a', fa', $a'e$ and $fa'e$ are all inverses of a (see the figure above). It follows that there is a bijection $(e, f) \mapsto fa'e$ of $E(L_a) \times E(R_a)$ onto $\mathscr{V}(a)$. We may therefore visualize the set $\mathscr{V}(a)$ as a rectangular $E(L_a) \times E(R_a)$-array of elements in D_a.

Recall that for any $a \in S$, L_a [R_a, J_a] is the set of generators of the principal ideal $L(a)$ [respectively $R(a)$, $J(a)$]. Therefore $L(a)$ has an idempotent generator if and only if L_a contains an idempotent. We have the following important characterization of regular semigroups and their homomorphisms in terms of Green's relations and idempotents. Note that an idempotent is always a regular element.

In the following, we denote by $\boldsymbol{E}(S)$ the set of all idempotents of the semigroup S.

THEOREM 2.41. *The following statements are equivalent for a semigroup S:*

(a) S is regular.

(b) Every principal left ideal has an idempotent generator.

(c) Every principal right ideal has an idempotent generator.

(d) For every $\mathscr{D}$*-class D,* $E(D) \neq \emptyset$*.*

Moreover, if $\phi : S \to T$ *is a homomorphism of a regular semigroup S onto a semigroup T, then T is regular and*

$$\boldsymbol{E}(S)\phi = \boldsymbol{E}(T).$$

Proof. Equivalence of (a), (b) and (c) are immediate consequences of statements (a), (b) and (c) of Proposition 2.39. If (a) holds, then every $\mathscr{D}$-class D is regular and hence by Lemma 2.38, $E(D) \neq \emptyset$. Therefore (a) implies (d). If (d) holds, then every $\mathscr{D}$-class contain idempotents and hence regular elements. Therefore, by Proposition 2.39, (a) holds.

To prove the last statement, let $y = x\phi \in T$. If $x' \in \mathscr{V}(x)$, clearly, $x'\phi \in \mathscr{V}(x\phi)$ and so y is a regular element of T. Hence T is regular. Also it is clear that $\boldsymbol{E}(S) \subseteq \boldsymbol{E}(T)$ and so, to complete the proof, it is sufficient to show that given any $e' \in \boldsymbol{E}(T)$, there is an idempotent $e \in S$ with $e\phi = e'$. So, assume that $e' \in \boldsymbol{E}(T)$. Since ϕ is onto, there is $x \in S$ with $x\phi = e'$. Choose $f \in E(L_x)$, $g \in E(R_x)$, $a \in \mathscr{V}(fg)$ and let $e = gaf$. Then

$$e^2 = ga(fg)af = gaf = e.$$

Now, by the choice of f and g, we have

$$f' = f\phi \mathscr{L} x\phi = e' \mathscr{R} g\phi = g'$$

and so, by Theorem 2.34, $f'g' \in L_{g'} \cap R_{f'}$. It is easy to verify that e' is the inverse of $f'g'$ in $L_{f'} \cap R_{g'}$. Also, $a\phi$ is an inverse of $(fg)\phi = f'g'$ and by statement (a) of Proposition 2.40, $g'(a\phi)f'$ is the inverse of $f'g'$ in $L_{f'} \cap R_{g'}$. Hence by Proposition 2.40(b),

$$e' = g'(a\phi)f' = (gaf)\phi = e\phi.$$

Whence the proof is complete. □

The foregoing theorem is essentially due to Clifford and Miller Miller and Clifford [1956]. The last assertion of the theorem implies that idempotents in the homomorphic image of a regular semigroup are images of idempotents. As far as we know, this important property of homomorphisms of regular semigroups was first noted by Lallement Lallement [1967]

If T is a subsemigroup of a semigroup S, then it is clear that, if ρ denote any one of the Green's relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$, $\mathscr{D}$ or $\mathscr{J}$, then

$$\rho(T) \subseteq \rho(S) \cap (T \times T)$$

where $\rho(S)$ denote the relation ρ on the semigroup S. In general these inclusions are proper (see Example 2.14). However, we have following result due to Hall Hall [1972].

COROLLARY 2.42. *Let T be a regular subsemigroup of a semigroup S. Then*

$$\rho(T) = \rho(S) \cap (T \times T)$$

for $\rho = \mathscr{L}, \mathscr{R}$ or $\mathscr{H}$.

Proof. We shall varify the assertion for $\rho = \mathscr{L}$. Let $a \mathscr{L}(S) b$ where $a, b \in T$. Since T is regular, by Theorem 2.41, there are idempotents $e, f \in T$ with $e \mathscr{L}(T) a$ and $b \mathscr{L}(T) f$. Then

$$e \mathscr{L}(S)\, a \mathscr{L}(S)\, b \mathscr{L}(S)\, f$$

semigroup!orthodox – band

and so, $e \mathscr{L}(S) f$. Hence by Lemma 2.36, $ef = e$ and $fe = f$. This implies that $e \mathscr{L}(T) f$. Hence $a \mathscr{L}(T) b$. □

Note that the statement of the Corollary above is not true for relations $\mathscr{D}$ and $\mathscr{J}$ (see Example 2.16).

Before ending this section, we give preliminary classification of regular semigroups that illustrate the use of concepts and results developed in this section. We consider two subclasses of the class of regular semigroups:*orthodox* semigroups and *inverse* semigroups. These classes appread early in the development of the theory of regular semigroups; many structure theorem that exist to day (especially for regular semigroups) were modeled on theorems developed for these semigroups. Moreover, they will provide a rich source of examples in the sequel.

Orthodox semigroups A regular semigroup S is said to be *orthodox* if $\boldsymbol{E}(S)$ is a subsemigroup of S. A semigroup in which every element is an idempotent (or briefly, an idempotent semigroup) is called a *band*. Thus S is orthodox if and only if $\boldsymbol{E}(S)$ is a band. As far as we know, the following characterization of orthodox semigroups is due to Schein Schein [1966].

THEOREM 2.43. *A regular semigroup S is orthodox if and only if it satisfies the following condition: for all $x, y \in S$*

$$x' \in \mathscr{V}(x), \quad y' \in \mathscr{V}(y) \Rightarrow y'x' \in \mathscr{V}(xy). \tag{2.41}$$

Further, if $\phi : S \to T$ is a homomorphism of an orthodox semigroup onto a semigroup T, then T is orthodox.

Proof. Assume that S is orthodox. Let $x, y \in S$, $x' \in \mathscr{V}(x)$ and $y' \in \mathscr{V}(y)$. Since S is orthodox, $(x'x)(yy')$ and $(yy')(x'x)$ are idempotents. Using this we deduce

$$\begin{aligned}(xy)(y'x')(xy) &= x(yy')(x'x)y\\ &= x(x'x)(yy')(x'x)(yy')y\\ &= x(x'x)(yy')y = (xx'x)(yy'y)\\ &= xy\end{aligned}$$

and similarly,

$$\begin{aligned}(y'x')(xy)(y'x') &= y'(yy')(x'x)(yy')(x'x)x'\\ &= (y'yy')(x'xx')\\ &= y'x'.\end{aligned}$$

Hence $y'x' \in \mathscr{V}(xy)$. Conversely suppose that S satisfies the given condiyion and let $f, g \in \boldsymbol{E}(S)$. Then $f \in \mathscr{V}(f)$ and $g \in \mathscr{V}(g)$ and so, by the given condition, $gf \in \mathscr{V}(fg)$. Hence

semigroup!inverse – a^{-1}: unique inverse of a

$$fg = (fg)(gf)(fg) = (fg)(fg)$$

which shows that $fg \in \boldsymbol{E}(S)$ so that S is orthodox.

Finally if $\phi : S \to T$ is a homomorphism of the orthodox semigroup S onto T, then by Theorem 2.41, T is regular and $\boldsymbol{E}(S)\phi = \boldsymbol{E}(T)$. Since $\boldsymbol{E}(S)$ is a subsemigroup of S,it follows that $\boldsymbol{E}(T)$ is a subsemigroup of T and hence T is orthodox. □

***Remark* 2.14:** Note that the condition stated in Equation (2.41) is the analogue of the group-theoretic fact that

$$(uv)^{-1} = v^{-1}u^{-1}$$

for any two elements u and v in a group G. The theorem above therefore implies that this property does not hold in an arbitrary regular semigroup which is not orthodox. However, we will show later in the next chapter (Chapter 3,) that there is a suitable interpretation of this condition which is valid for arbitrary regular semigroups. ref ?

Note also that in a group the mapping $u \mapsto u^{-1}$ is an involution of groups (see Equation (2.9)). For arbitrary orthodox semigroups the relation

$$\mathscr{V} = \{(x, x') : x \in S,\ x' \in \mathscr{V}(x)\}$$

is not single valued. However, in the class of semigroups defdined below, (inverse semigroup) this property also holds. Furthermore, the theorem above implies that when S is othodox the set $\mathscr{V}$ is closed with respect to the product

$$(x, x')(y, y') = (xy, y'x')$$

and $\mathscr{V}$ is a semigroup.

Inverse semigroups A semigroup S is called an *inverse semigroup* if every element in S has a unique inverse. In this case, a^{-1} will usually denote the unique inverse of $a \in S$. Note that an inverse semigroup is regular; in fact, by the theorem below, it is orthodox. The theorem below also gives some useful characterizations of inverse semigroups using Green's relations. Recall that a semilattice is a commutative semigroup of idempotents (see Subsection 1.1.3).

THEOREM 2.44. *The following conditions on a semigroup S are equivalent.*

(1) S is regular and $\boldsymbol{E}(S)$ *is a subsemilattice of S;*

(2) Every principal left ideal and every principal right ideal has a unique idempotent generator;

(3) S is an inverse semigroup.

When S is an inverse semigroup the map $x \mapsto x^{-1}$ is an involution S.

Moreover, if $\phi : S \to T$ is a homomorphism of an inverse semigroup S onto a semigroup T, then T is an inverse semigroup.

Proof. **(1) ⇒ (2).** It is sufficient to show that every $\mathscr{L}$-class and every $\mathscr{R}$-class contain exactly one idempotent. By Theorem 2.41, every $\mathscr{L}$ [$\mathscr{R}$] class contains idempotents. Let $e \mathscr{L} f$. Then by Lemma 2.36 and (1), we have $e = ef = fe = f$. Similarly, every $\mathscr{R}$-class also contains exactly one idempotent.

(2) ⇒ (3). By Proposition 2.40, (2) implies that each element in S has exactly one inverse (see Proposition 2.40). Hence S is an inverse semigroup.

(3) ⇒ (1). Since, by definition, inverse semigroups are regular, it is sufficient to show that the set of idempotents is a commutative subsemigroup of S. Let e and f be idempotents in S and $a \in \mathscr{V}(ef)$. If $h = fae$, then

$$(ef)h(ef) = (ef^2)a(e^2f) = (ef)a(ef) = ef; \quad h(ef)h = f(a(ef)a)e = fae = h \quad \text{and}$$
$$h^2 = (fae)(fae) = f(a(ef)a)e = fae = h.$$

Hence h is an idempotent inverse of ef and so, $h, ef \in \mathscr{V}(h)$. Therefore by (3), $h = ef$. Hence ef is an idempotent. Similarly fe is also an idempotent. Consequently,

$$(ef)(fe)(ef) = (ef)(ef) = ef \quad \text{and similarly,} \quad (fe)(ef)(fe) = fe.$$

Therefore $fe, ef \in \mathscr{V}(ef)$ and so, $fe = ef$ by (3).

To show that the map $\theta : x \mapsto x^{-1}$ is an involution, consider $x, y \in S$. Then, using statement (1) above, we get

$$\begin{aligned} y^{-1}x^{-1}xyy^{-1}x^{-1} &= y^{-1}(x^{-1}x)(yy^{-1})x^{-1} \\ &= y^{-1}(yy^{-1})(x^{-1}x)x^{-1} = y^{-1}x^{-1}; \end{aligned}$$

and

$$\begin{aligned} xyy^{-1}x^{-1}xy &= x(yy^{-1})(x^{-1}x)y \\ &= x(x^{-1}x)(yy^{-1})y = xy. \end{aligned}$$

Therefore, the uniquiness of inverse implies that $(xy)^{-1} = y^{-1}x^{-1}$ and so θ is an involution.

Finally, if $\phi : S \to T$ is a surjective homomorphism of the inverse semigroup S, then by Theorem 2.41, T is regular and $\boldsymbol{E}(S)\phi = \boldsymbol{E}(T)$. This implies that, since $\boldsymbol{E}(S)$ is a commutative subsemigroup of S, $\boldsymbol{E}(T)$ is a commutative subsemigroup of T and so, T is inverse. □

The following statement intutively mean that every $\mathscr{D}$-class has the same length breadth; that is the number of $\mathscr{R}$-class contained in a $\mathscr{D}$-class D is the same as the number of $\mathscr{L}$-class contained in D.

COROLLARY 2.45. *If D is a $\mathscr{D}$-class of an inverse semigroup, then there is a bijection θ of the set $D/\mathscr{L}$ of all $\mathscr{L}$-classes contained in D onto the set $D/\mathscr{R}$ such that the $\mathscr{H}$-class $L \cap \phi(L)$ contains an idempotent.*

Proof. For each $\mathscr{L}$-class L, let e_L denote the unique idempotent in L. Clearly $L \mapsto e_L$ is a bijection of $D/\mathscr{L}$ onto $E(D)$. Since each $\mathscr{R}$-class contain a unique idempotent the map θ defined by $\theta(L) = R_{e_L}$ is a bijection such that $L \cap \theta(L)$ contains e_L. □

Example 2.14: In the semigroup $(\mathbb{N}, +)$, the $\mathscr{D}$-relation is the identity relation and the only regular $\mathscr{D}$-class is $\{0\}$. $(N, +)$ is a subsemigroup of the group $(\mathbb{Z}, +)$ and all Green's relations on $(\mathbb{Z}, +)$ is the universal relation (see Corollary 2.42). Similarly, for the semigroup A of Example 2.13 each $\mathscr{D}$-class is singleton and has no regular $\mathscr{D}$-class. However, if S is a semilattice, $\mathscr{D}$-classes are singletons, but every $\mathscr{D}$-class is regular.

Example 2.15: Let $S = \mathscr{T}_X$. Let $f \in S$, $U = \operatorname{Im} f$ and $\pi = \pi_f$. It follows from Example 2.10 that an inverse of f can be uniquely constructed from a cross-section Y of π and an equivalence relation π' having U as a cross-section. If I_U denote the set of equivalence relations having U as a partition and Λ_π denote the set of all cross-sections of π, then there is a bijection between $\Lambda_\pi \times I_U$ and $\mathscr{V}(f)$. Note that there are bijections between I_U and $E(L_f)$ and Λ_π and $E(R_f)$. It is clear that if $|X| > 1$ and if f is not a bijection, then f has more than one inverse. Similarly if $S = \mathscr{L}\mathscr{T}(V)$ and $\dim V > 1$, then any $f \in S$ which is not invertible has more than one inverse; in fact there is a bijection between $\mathscr{V}(f)$ and $C(N) \times C(U)$ where $C(N)$ $[C(U)]$ denote the set of all complements of the subspace $N = \boldsymbol{N}(f)$ $[U = \operatorname{Im} f]$.

Example 2.16: Let $S = \langle p, q : pq = 1 \rangle$ be the bicyclic semigroup (see Examples 2.7 and 2.12). Each $\mathscr{L}$-class L_{p^n} contain exactly one idempotent $q^n p^n$ and each $\mathscr{R}$-class R_{q^m} contain exactly one idempotent $q^m p^m$. Hence S is an inverse semigroup (which is bisimple—see Example 2.12). It can be seen that the unique inverse of $q^m p^n$ is $q^n p^m$.

Now, the set of idempotents E of S is a regular subsemigroup of S and the $\mathscr{D}$-relation on E is the identity relation where as the $\mathscr{D}$-relation on S is the universal relation.

2.6.3 The Schützenberger group of an $\mathscr{H}$-class

If H_e is any $\mathscr{H}$-class of a semigroup S containing the idempotent e, then by Proposition 2.37 H_e is a group isomorphic with both the automorphism groups $\operatorname{Aut}[L(e)]$ and $\operatorname{Aut}[R(e)]$ of $L(e)$ and $R(e)$ respectively. P. M. Schützenberger Schützenberger [1957] gave an appropriate extension of this result to an arbitrary $\mathscr{H}$-class.

Here we give a different formulation of his result which exhibits its relation with the ideal structure of the semigroup as well as its left-right symmetry.

Note that since $H \subseteq L_a$ $[H \subseteq R_a]$ $L(H) = L(a)$ $[R(H) = R(a)]$ for any $a \in H$. Now elements $\sigma \in \mathrm{Aut}[L(H)]$ $[\tau \in \mathrm{Aut}[R(H)]]$ are induced by inner right [left] translations of S. Hence by Equation (2.33)

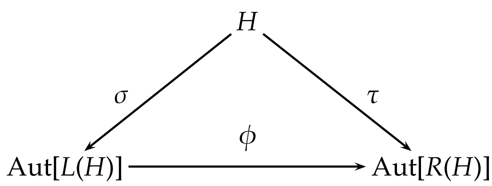
$$(\tau u)\sigma = \tau(u\sigma)$$

for all $u \in H$. We use this remark in the proof below.

Proposition 2.46. *For any $\mathscr{H}$-class H in the semigroup S, there is a isomorphism*

$$\phi : \mathrm{Aut}[L(H)] \to \mathrm{Aut}[R(H)].$$

In particular, if H contains an idempotent e, then we can choose the isomorphism ϕ so that the following diagram commute:



Here σ and τ denote isomorphisms defined in Proposition 2.37.

Proof. Fix $a \in H$. For each $\sigma \in \mathrm{Aut}[L(H)]$, by Theorem 2.25, $a\sigma \in H_a$ and by the dual of Lemma 2.24 and Theorem 2.26, there is a unique $\tau \in \mathrm{Aut}[R(H)]$ such that $\tau a = a\sigma$. For each $\sigma \in \mathrm{Aut}[L(H)]$, let

$$\phi(\sigma) = \tau, \quad \text{where } \tau \in \mathrm{Aut}[R(H)] \text{ with } \quad \tau a = a\sigma.$$

This defines a mapping ϕ of $\mathrm{Aut}[L(H)]$ to $\mathrm{Aut}[R(H)]$. By Lemma 2.24, Theorem 2.26 and their duals, ϕ is a bijection. If $\sigma_1, \sigma_2 \in \mathrm{Aut}[L(H)]$, by the definition of ϕ, we have

$$\begin{aligned}\phi(\sigma_1\sigma_2)a = a\sigma_1\sigma_2 &= (a\sigma_1)\sigma_2 \\ &= (\phi(\sigma)a)\sigma_2 = \phi(\sigma_1)(a\sigma_2) \\ &= \phi(\sigma_1)\left(\phi(\sigma_2)a\right) \\ &= \phi(\sigma_1)\phi(\sigma_2)a.\end{aligned}$$

Since $\phi(\sigma_1\sigma_2)$ and $\phi(\sigma_1)\phi(\sigma_2)$ are morphisms of principal right ideals, this implies by the dual of Lemma 2.24 that $\phi(\sigma_1\sigma_2) = \phi(\sigma_1)\phi(\sigma_2)$. Hence ϕ is an isomorphism.

To prove the last statement, let ϕ be the isomorphism determined by the condition $\phi(\sigma)e = e\sigma$ where $e \in H$ is the idempotent. Then using the definition of isomorphisms $\sigma : a \mapsto \sigma_a = \rho_a|L(H)$ and $\tau : a \mapsto \tau_a = \lambda_a|R(H)$ in Proposition 2.37, we have

$$\left((a\sigma)\phi\right)e = \phi(\sigma_a)e = e\sigma_a = a = (\tau_a)e$$

$\mathfrak{g}(H)$: *Schützenberger group of H*
Schützenberger group
group!Schützenberger –
action!simply transitive –
group!Schützenberger –!right
group!Schützenberger –!left

Hence

$$a\sigma\phi = \tau_a = a\tau \quad \text{for all} \quad a \in H.$$

This proves that the given diagram commutes. □

Note that the isomorphism $\phi : \text{Aut}[L(H)] \to \text{Aut}[R(H)]$ constructed above depends on a and so is not "natural" as isomorphism of the corresponding automorphism groups. However we can associate an abstract group $\mathfrak{g}(H)$ which is isomorphic to both $\text{Aut}[L(H)]$ and $\text{Aut}[R(H)]$. It is clear that the group $\mathfrak{g}(H)$ does not depend on the element a used in the definition of ϕ and only on the $\mathscr{H}$-class H. The group $\mathfrak{g}(H)$ is called the *Schützenberger group* of the $\mathscr{H}$-class H.

Let G be a group acting on the set X. We say that the action (or the G-set) is [*simply*] *transitive* if given $(a, b) \in X \times X$ there is [a unique] $g \in G$ such that $ag = b$. Note that the G-set X is transitive if and only if it is cyclic (this does not hold if G is not a group). Now, by Theorem 2.25, $\sigma|H$ is a permutation of H for any $\sigma \in \text{Aut}[L(H)]$. Thus $\text{Aut}[L(H)]$ acts on H on the right. By Lemma 2.24 and Theorem 2.26, the action is faithful and simply transitive. It follows that the corresponding representation of $\text{Aut}[L(H)]$ is an injective homomorphism of $\text{Aut}[L(H)]$ into the symmetric group $S(H)$ of all permutations of H and hence, an isomorphism of $\text{Aut}[L(H)]$ onto a simply transitive permutation group $\Gamma(H)$ which clearly depend depend only on the $\mathscr{H}$-class H. Consequently, by the definition above, $\mathfrak{g}(H)$ is isomorphic to the permutation group $\Gamma(H)$ of H. When H contains an idempotent, by Proposition 2.37, there is the isomorphism $\sigma : H \to \text{Aut}[L(H)]$ which induces an isomorphism $a \mapsto \sigma_a|H$ of H onto $\Gamma(H)$. It is easy to see that this isomorphism is in fact the right regular (or Keyley's) representation of H. Thus, when H is a group, $\Gamma(H)$ is the image of the right regular representation of H. Similarly $\text{Aut}[R(H)]$ acts simply transitively on the left on H and so it is isomorphic to a subgroup of $S(H)^{\text{op}}$. Since $\text{Aut}[R(H)]$ is a group of automorphisms of the right S-set $R(H)$, it acts on the left of $R(H)$ and hence on H. By lemma 2.24 the map $\alpha \mapsto \alpha \mid H$ is an embedding of the group $\text{Aut}[R(H)]$ into $S(H)^{\text{op}}$. Thus $\text{Aut}[R(H)]$ is isomorphic to a permutation group acting on the left of H. This group is then anti-isomorphic to a permutation group $\Gamma^*(H) \subseteq S(H)$ which is faithful and simply transitive. The groups $\Gamma(H)$ and $\Gamma^*(H)$ are callef the *right Schützenberger group* and *left Schützenberger group* of the $\mathscr{H}$-class H respectively. As above, when H is a group, $\Gamma^*(H)$ is the image of the left regular representation of H. Consequently the group $\mathfrak{g}(H)$ acts on H both on the right and the left and these actions are faithful and simply transitive. We summarize the discussion as:

S_λ: symmetric group of degree α
$\mathsf{Gl}_\alpha(\Bbbk)$: general linear group of degree α

Theorem 2.47. *Let H be an $\mathscr{H}$-class of a semigroup S. Then there exists a group $\mathfrak{g}(H)$, called the Schützenberger group of H, satisfying the following:*

(a) $\mathfrak{g}(H)$ is isomorphic to both $\operatorname{Aut}[L(H)]$ and $\operatorname{Aut}[R(H)]$.

(b) There exist simply transitive permutation groups $\Gamma(H)$ and $\Gamma^*(H)$ acting on H such that $\mathfrak{g}(H)$ is isomorphic to $\Gamma(H)$ and anti-isomorphic to $\Gamma^*(H)$.

If H contains an idempotent, then H is isomorphic to $\mathfrak{g}(H)$ and $\Gamma(H)$ [$\Gamma^*(H)$] is the image of the right [left] regular representation of H. □

The result above shows that we can associate a group, the Schützenberger group $\mathfrak{g}(H)$ with every $\mathscr{H}$-class H in such a way that it is isomorphic to H when H is a group. Since $\mathfrak{g}(H)$ is isomorphic to $\operatorname{Aut}[L(a)]$ for any $a \in H$, it is clear that $\mathfrak{g}(H)$ is isomorphic to $\mathfrak{g}(H')$ if $L(H)$ and $L(H')$ are isomorphic (or dually, if $R(H)$ and $R(H')$ are isomorphic). By Corollary 2.29 $L(H)$ and $L(H')$ are isomorphic if H and H' are $\mathscr{H}$-classes contained in the same $\mathscr{D}$-class. Hence we have:

Corollary 2.48. *If H and H' are $\mathscr{H}$-classes contained in the same $\mathscr{D}$-class of S, then $\mathfrak{g}(H)$ and $\mathfrak{g}(H')$ are isomorphic.* □

Example 2.17: Let α be a cardinal number. By the *symmetric group of degree α*, denoted by S_α, we shall mean the group isomorphic to the group $S(U)$ of all permutations of a set U with $|U| = \alpha$. By Example 2.10, there is a bijection $\alpha \mapsto D_\alpha$ of the set of all cardinal numbers $\alpha \le |X|$ and the set $S/\mathscr{D}$ of all $\mathscr{D}$-classes $\mathscr{T}_X$. It follows from Example 2.10 that for $\alpha \le |X|$, an $\mathscr{H}$-class H of the $\mathscr{D}$-class D_α has the form

$$H = H_{\pi,Y} = \{f : \pi_f = \pi,\ \operatorname{Im} f = Y\} \quad \text{where} \quad |X/\pi| = |Y| = \alpha\}.$$

If e is any idempotent with $\pi_e = \pi$, then $e \mathscr{R} f$ for any $f \in H$. Also, if $U = \operatorname{Im} e$, then it is easy to verify that the map $f \mapsto f|U$ is an isomorphism of H_e onto $S(U)$. It follows from Theorem 2.47 and Corollary 2.48 that the Schützenberger group $\mathfrak{g}(H)$ of H is isomorphic to S_α.

Example 2.18: Let V be a vector space over the field $\Bbbk$ and let α be a cardinal with $\alpha \le \dim V$. We denote by $\mathsf{Gl}_\alpha(\Bbbk)$, the group $\mathsf{Gl}(U)$ of all linear isomorphisms of a subspace U of V with $\dim U = \alpha$; $\mathsf{Gl}_\alpha(\Bbbk)$ is called the *general linear group* of degree α. Let $S = \mathscr{L}\mathscr{T}(V)$ be the semigroup of all linear endomorphisms of V (see Example 2.11). Any $\mathscr{H}$-class in S has the form (Example 2.15)

$$H(N, U) = \{f \in S : \boldsymbol{N}(f) = N, \operatorname{Im} f = U\} \quad \text{with} \quad \dim N + \dim U = \dim V.$$

If U' is any complement of N, then it is easy to see that the map $f \mapsto f|U'$ is an isomorphism of $H(N, U')$ onto $\mathsf{Gl}(U')$. It follows from Corollary 2.48 that $\mathfrak{g}\,(H(N, U))$ is isomorphic to $\mathsf{Gl}_\alpha(\Bbbk)$. Thus if $f \in S$ with $\operatorname{Rank} f = \alpha$, then $\mathfrak{g}(H_f)$ is isomorphic to $\mathsf{Gl}_\alpha(\Bbbk)$.

Example 2.19: If $S = \langle p, q; pq = 1\rangle$ is the bicyclic semigroup (see Examples 2.7 and 2.12) then every $\mathscr{H}$-class in S contain only one element and so the Schützenberger of every $\mathscr{H}$-class in S is trivial. Similarly, Schützenberger group of every $\mathscr{H}$-class of the semigroup A of Example 2.13 is also trivial.

ideal!minimal
ideal!0-minimal

Example 2.20: Let H be an $\mathscr{H}$-class of a semigroup S and $a \in H$. For each $b \in H$, Lemma 2.24 and Theorem 2.26, there is a unique automorphism $\sigma_b \in \mathrm{Aut}[L(H)]$ such that $a\sigma_b = b$ and by Theorem 2.25, $\sigma_b^* = \sigma_b|H$ is a permutation of H belonging to $\Gamma(H)$. Since the action of $\Gamma(H)$ on H is simply transitive, the map $\sigma^* : b \mapsto \sigma_b^*$ is a bijection of H onto $\Gamma(H)$. It is therefore clear that

$$bc = a\sigma_b^*\sigma_c^*$$

defines a binary operation on H with respect to which H becomes a group with identity a. Further, the map $\sigma^* : b \mapsto \sigma_b^*$ is the right regular representation of the group H. In particular, if $a = e$ is an idempotent, the binary operation of H defined above, coincides with the binary operation of the maximal subgroup $H_e = H$ and σ^* coincides with the isomorphism induced by the isomorphism of Proposition 2.37.

2.7 SIMPLE AND 0-SIMPLE SEMIGROUPS

2.7.1 Minimal and 0-minimal ideals

Recall § Subsection 2.1.1 that a left [right,two-sided] ideal L in a semigroup S is a *minimal* if L is minimal in the lattice $\mathfrak{L}\mathfrak{I}$ [respectively $\mathfrak{R}\mathfrak{I}$, $\mathfrak{I}_S$]. If S has 0, the left [right or two-sided] ideal L is 0-*minimal* (§ Subsection 2.1.1) if L satisfies the following:

(i) $L \neq 0$; and

(ii) if $L' \neq 0$ is an ideal of the same type as L in S with $L' \subseteq L$, then $L' = L$.

***Remark* 2.15:** If S is a semigroup with out 0, then any ideal (of any type) I in S corresponds to the non-zero ideal $I^0 = I \cup \{0\}$ of the same t ype in the semigroup S^0 and the correspondence $I \mapsto I^0$ is an inclusion preserving bijection of the set of all left [right, two-sided] ideals of S onto to the set of all non-zero ideals of the same type in S^0. Therefore from any result about non-zero ideals (of some type) in a semigroup with 0, one can derive a result about ideals (of the same type) in a semigroup with out 0. In the following such results will not be stated explicitly unless there is some strengthening in case of ideal in a semigroup with out 0 or emphasis is desired. It should be noted that the reverse derivation of results about non-zero ideals in semigroups with 0 from results about ideals in semigroups with out 0 may not always possible.

Note that if I is 0-minimal ideal of any type, then either $I^2 = 0$ or $I^2 = I$. For if $I^2 \neq 0$, then I^2 is a non-zero ideal of the same type as I, contained in I and so $I^2 = I$ by the 0-minimality of I. Thus we have

LEMMA 2.49. *Let I be a [left,right or two-sided] 0-minimal ideal of a semigroup with 0. Then either $I^2 = 0$ or $I^2 = I$.* □

The following result gives the structure of left [right, two-sided] 0-minimal ideals in terms of the corresponding Green's relations.

LEMMA 2.50. *Let L $[R, J]$ be a left [right, two-sided] ideal in a semigroup S. Then L $[R, J]$ is 0-minimal if and only if $L = L_a \cup \{0\}$ $[R = R_a \cup \{0\}, J = J_a \cup \{0\}]$ for all $a \in L - \{0\}$ $[a \in R - \{0\}, a \in J - \{0\}]$. Further L $[R, J]$ is minimal if and only if $L = L_a$ $[R = R_a, J = J_a]$ for all $a \in L$ $[a \in R, a \in J]$.*

Proof. We prove the result for left ideals. Proofs for the other cases are obtained by appropriate modification of this.

Suppose that $L = L_a \cup \{0\}$ where L_a is the $\mathscr{L}$-class of S of a non-zero element $a \in L$. If $L' \subseteq L$ is any non-zero left ideal and if is $b \neq 0$ in L' then $a \mathscr{L} b$ and so, $L_a = Lb \subseteq L'$. Since $0 \in L'$, we have $L \subseteq L'$. Thus L is 0-minimal.

Conversely assume that L is 0-minimal. If for some $a \in L - \{0\}$ $Sa = 0$ then $L' = \{0, a\}$ is a non-zero left ideal in S contained in L and by the 0-minimality of L, $L = L'$. Hence $L_a = \{a\}$ and so, $L = L_a \cup \{0\}$. Suppose now that $Sa \neq 0$ for any non-zero $a \in L$. Since Sa is a left ideal contained in L, we have $Sa = L$ for all $a \in L - \{0\}$. Hence $a \in Sa$ and so, $L = Sa = S^1a = L(a)$ for all $a \in L - \{0\}$. By Equation (2.37a) (the definition of $\mathscr{L}$ relation) it follows that the set of all non-zero elements of L is a $\mathscr{L}$-class in S.

In view of Remark 2.15, the statement about minimal ideals follow from that of 0-minimal ideals. □

LEMMA 2.51. *Let S be a semigroup with 0 and let I be an ideal in S such that $I^2 \neq 0$. Then I is 0-minimal if and only if $IaI = I$ for all $a \in I$ with $a \neq 0$.*

An ideal I is minimal if and only if $IaI = I$ for all $a \in I$. Moreover if S has a minimal ideal I, then it is the minimum ideal in the partially ordered set of all non-empty ideals and hence unique.

Proof. Let I be a 0-minimal ideal in S with $I^2 \neq 0$. By Lemma 2.49, $I^2 = I$. Let $J = \{x \in I : IxI = 0\}$. Then J is an ideal in S and so, either $J = 0$ or $J = I$ by the 0-minimality of I. If $J = I$, then $IxI = 0$ for all $x \in I$ and so $I^3 = 0$. But $I^3 = I^2I = II = I^2 = I$. Since I is 0-minimal, $I \neq 0$ and so $I^3 \neq 0$. Hence $J = I$ is not possible. Thus $J = 0$ which implies that $IxI \neq 0$ for any $x \in I$ with $x \neq 0$. Since IxI is an ideal contained in I, by 0-minimality of I, we have $I = IxI$ for all $x \in I$ with $x \neq 0$. Conversely, assume that I is an ideal in S satisfying the given conditions. Since $I^2 \neq 0$, we have $I \neq 0$. Let $J \neq 0$ be an ideal contained in I and $0 \neq x \in J$. Since J is an ideal, we have

$$I = IxI \subseteq S^1xS^1 \subseteq J$$

which implies that $J = I$. Thus I is 0-minimal.

If I is minimal and $x \in I$, then $IxI = I$ as above. Conversely if I satisfies this condition and if J is any ideal in S, then for any $x \in I \cap J$, we have $I = IxI \subseteq S^1xS^1 \subseteq J$. Thus I is minimum. □

semigroup!kernel
$K(S)$: kernel of the semigroup S

Examples can be constructed to show that there may exist 0-minimal ideals I with $I^2 = 0$; however, this condition is not sufficient to ensure 0-minimality. For if S is any null semigroup (that is, if S is a non-empty semigroup with 0 such that $S^2 = 0$). If Y is any subset of S containing 0, then is an ideal in S such that $Y^2 = 0$. If Y contains more than one non-zero element, then Y is clearly not 0-minimal.

Note also that a minimal ideal is minimum and hence unique. The unique minimal ideal of a semigroup S is called the *kernel* of S. We shall denote by $K(S)$ the kernel of S when it exists.

It should also be noted that there is no uniqueness for minimal left or right ideals (see Example 2.21 below) and for 0-minimal ideals of any type (see Example 2.22 below).

Notice the difference in the corresponding characterization of one-sided 0-minimal ideals below. The analogue of the condition $IaI = I$ for one sided ideals should have been $Ia = I$ for left ideal and $aI = I$ for right ideals. In fact the corresponding statements for left and right ideals are not true for 0-minimal ideals(see Example 2.23). In fact the condition for left [right] ideal is sufficient but not necessary (see also Corollary 2.58). However, these are both necessary and sufficient for minimality.

Most of the results that follows (about one-sided ideals) are stated for left ideals; the corresponding results for right ideals follow by duality.

LEMMA 2.52. *Let L be a left ideal in a semigroup S such that $L^2 \neq 0$. Then L is 0-minimal if and only if $Sa = L$ for all $a \in L$ with $a \neq 0$.*

A left ideal L in S is minimal if and only if $La = L$ for all $a \in L$.

Proof. Suppose that L is 0-minimal and that $a \in L - \{0\}$. Then Sa is a left ideal contained in L and so, by 0-minimality of L, either $Sa = L$ or $Sa = 0$. If $Sa = 0$, then $L' = \{0, a\}$ is a non-zero left ideal contained in L and so, $L' = L$. But then $L^2 = 0$, a contradiction. Hence $Sa = L$ for all $a \neq 0 \in L$. Conversely, assume that L satisfies the condition $Sa = L$ for all $a \neq 0 \in L$. Let $\bar{L}$ be any non-zero left ideal contained in L. Choose $a \in \bar{L}$ with $a \neq 0$. Then $L = Sa \subseteq S^1 a \subseteq \bar{L}$ since $\bar{L}$ is a left ideal. Hence $L = \bar{L}$.

If L is any left ideal and $a \in L$, then aL is clearly a left ideal contained in L and so $aL = L$ if L is minimal. Conversely if L satisfies the given condition and if $L' \subseteq L$ is any left ideal, then $L = La \subseteq S^a \subseteq L'$ for some $a \in L'$. Hence L is minimal. □

Recall from § Subsection 2.1.1 that for any $X \subseteq S$, X^2 denote the set product of X with itself.

semigroup![left,right, two-sided] simple –
semigroup![left,right, two-sided]0-simple –

A semigroup S is said to be [*left, right, two-sided*]*simple* if S is the only [left, right, two-sided] ideal of S. If S has 0, then S is said to be [*left,right, two-sided*] 0-*simple* if

(1) $S^2 \neq 0$; and

(2) if J is any ideal [left ideal, right ideal] of S then either $J = 0$ or $J = S$.

Note that a semigroup S is simple if and only if the semigroup S^0 obtained by adjoining 0 to S is 0-simple. Note that condition (2) 'nearly' implies condition (1).For, we have

Lemma 2.53. *Let S be a semigroup with* 0 *such that $S \neq 0$. If S has no non-zero proper [left, right, two-sided] ideal, then S is either [left, right, two-sided]* 0*-simple or S is a semigroup of order two.*

Proof. We shall consider the case of left ideals. The proof for others are similar. So, assume that S has no proper non-zero left ideal. Then S^2 is left ideal in S and so, either $S^2 = S$ or $S^2 = 0$. In the first case, since $S \neq 0$, $S^2 \neq 0$ and hence S is left 0-simple. If $S^2 = 0$, then for any proper non-empty subset X of S, $X \cup \{0\}$ is a proper non-zero left ideal of S. This is not possible by hypothesis. Hence $S - \{0\}$ contains exactly one element. □

An alternate characterization of 0-simplicity follows as a Corollary to Lemma 2.51.

Corollary 2.54. *A semigroup S is* 0*-simple if and only if $S \neq 0$ and $SaS = S$ for all $0 \neq a \in S$.*

Proof. If S is 0-simple, S is a 0-minimal ideal and so, by Lemma 2.51, $SaS = S$ for all $a \in S$ with $a \neq 0$. Conversely, if $S \neq 0$ and $SaS = S$ for all $0 \neq a \in S$, then for some $a \in S$ with $a \neq 0$, we have $S = SaS \subseteq S^2$ and so, $S^2 \neq 0$. The 0-simplicity of S now follows from the 0-minimality of the ideal S which is a consequence of Lemma 2.51. □

Combining the Corollary above with Lemma 2.51, we obtain

Corollary 2.55. *Let I be an ideal in S with $I^2 \neq 0$. Then I is* 0*-minimal in S if and only if the semigroup I is* 0*-simple. I is minimal if and only if I is simple. Thus if S has kernel, then it is a simple subsemigroup of S* □

Lemma 2.56. *A semigroup S with* 0 *is left* 0*-simple if and only if $S \neq 0$ and the set $T = \{a \in S : a \neq 0\}$ is a left simple subsemigroup of S.*

Proof. Assume that $S \neq 0$ and that T is a left simple subsemigroup of S. Then $T^2 = T$ and so $S^2 = T^2 \cup \{0\} = S$. Hence $S^2 \neq 0$. If L is any non-zero left ideal ideal in S, then $L' = L - \{0\} = L \cap T$ is an ideal in T and so $L' = T$ which implies that $L = S$. Thus S is left 0-simple. *zero-divisors*

Conversely assume that S is left 0-simple. Suppose that $a, b \in T$ and $ab = 0$. Then $L = \{s \in S : sa = 0\}$ is a left ideal and $b \in L$ so that $L \neq 0$. Since S is left 0-simple, $L = S$. But this implies that $L' = \{0, a\}$ is a non-zero left ideal in S and so, $S = L'$. Then $S^2 = \{0, a^2\} = 0$ which contradicts the hypothesis that S is left 0-simple. Therefore T is a subsemigroup of S. If L is any left ideal in T, then $L \cup \{0\}$ is a non-zero left ideal in S and so $S = L \cup \{0\}$ which implies that $L = T$. Hence T is left simple. □

The result above shows that there is no essential difference between the theory of left [right] simple and right 0-simple semigroups in the sense that a right 0-simple semigroup can always be obtained by adjoining a 0 to a left simple semigroup or a left simple semigroup can be obtained by removing the 0 from a right 0-simple semigroup. Thus from any result about left simple semigroups we can obtain a corresponding result about left 0-simple semigroups and vice-versa. However, the situation is entirely different for simple and 0-simple semigroups. For example, a 0-simple semigroup may contain *zero-divisors*; that is there are elements $a \neq 0$, $b \neq 0$ such that $ab = 0$ so that the set of non-zero elements does not form a subsemigroup. Thus the theory of 0-simple semigroups is quite different from that of simple semigroups.

COROLLARY 2.57. *A semigroup S is left 0-simple if and only if $Sa = S$ for all $a \in S - \{0\}$.*

Proof. Let S be left simple. Then by Lemma 2.56, $S = T \cup \{0\}$ where T is a left simple subsemigroup of S. If $a \in S$ and $a \neq 0$, then $a \in T$ and by Corollary 2.54, $Ta = T$. Hence $Sa = Ta \cup \{0\} = T \cup \{0\} = S$> Conversely assume that $Sa = S$ for all $a \in S$ with $a \neq 0$. Let L be a non-zero left ideal in S. If $0 \neq a \in L$, then $S = Sa \subseteq L(a) \subseteq L$ since L is a left ideal. Hence $L = S$ and so, S is 0-simple. □

Corollary 2.55 gives a characterization of 0-minimality of two-sided ideals in terms of 0-simplicity of semigroups. There is no analogous characterization of 0-minimality of one-sided ideals. However, we have the following:

COROLLARY 2.58. *Let L be a left ideal in a semigroup S. If L is a left 0-simple subsemigroup of S, then L is 0-minimal and L is a left simple subsemigroup if and only if L is minimal in S.*

Proof. Suppose that L is left 0-simple. By Corollary 2.57, $La = L$ for all $a \in L$, $a \neq 0$. Let L' be a non-zero left ideal of S contained in L. Then L' is clearly a non-zero left ideal of the semigroup L and by 0-simplicity of L, $L' = L$. Thus L is left 0-minimal.

If L is a left simple subsemigroup of S, it follows from Lemma 2.56 and the proof above that L is minimal. Conversely, if L is minimal and if $L' \subseteq L$ is a left ideal in L, then for any $a \in L'$, La is a left ideals of S contained in L and so $La = L$. Since La is clearly a left ideal in L contained in L', we have $L = La \subseteq L'$. Hence L is minimal. □

Example 2.23 shows that a 0-minimal left ideal L satisfying the condition $L^2 \neq 0$ may not satisfy the condition that $La = L$ for all $a \neq 0 \in L$ and hence L may not be 0-simple.

LEMMA 2.59. *Let I be a 0-minimal ideal of a semigroup S with 0 such that $I^2 \neq 0$. If L is any non-zero left ideal contained in I, then $L^2 \neq 0$.*

Proof. Since LS is an ideal contained in I, by 0-simplicity of I, either $LS = 0$ or $LS = I$. If $LS = 0$, then L is a non-zero ideal contained in I and so $L = I$ and $I^2 = LI \subseteq LS = 0$ which contradicts the hypothesis that $I^2 \neq 0$. Hence $LS = I$. Since $I^2 = I$ by Lemma 2.49, we have $I = I^2 \subseteq LSLS \subseteq L^2S$ which shows that $L^2 \neq 0$. □

LEMMA 2.60. *Let L be a 0-minimal left ideal of a semigroup S with 0 and $x \in S$. Then either $Lx = 0$ or Lx is a 0-minimal left ideal of S. If L is a minimal left ideal of S, then Lx is a minimal left ideal for all $x \in S$.*

Proof. By Lemma 2.50, $L = L_a \cup \{0\}$ for any non-zero $a \in L$. Assume that $Lx \neq 0$. Then there is $a \in L - \{0\}$ such that $ax \neq 0$. If $bx = 0$ for some $b \in L - \{0\}$, since $b \mathscr{L} a$, $a = sb$ for some $s \in S$ and so, $ax = sbx = 0$ which contradicts the choice of a. Hence $ax \neq 0$ for any $a \in L - \{0\}$. It follows that $L_{ax} = L_a x$ is precisely the set of all non-zero elements of Lx which is clearly a left ideal of S. Hence $Lx = L_{ax} \cup \{0\}$ and so by Lemma 2.50, Lx is a 0-minimal left ideal.

If L is minimal in S, $L^0 = L \cup \{0\}$ is 0 minimal in S^0. Since $Lx \neq 0$ for any $x \in S$, it follows from above that $L^0x = Lx \cup \{0\}$ is 0-minimal in S^0 and hence Lx is minimal in S. □

THEOREM 2.61. *Let M be a 0-minimal ideal in a semigroup containing at least one 0-minimal left ideal of S. Then M is the union of all 0-minimal left ideals contained in M. Moreover, if $M^2 \neq 0$, then every left ideal of the semigroup M is also a left ideal of S.*

Proof. Let M_0 be the union of all 0-minimal left ideals contained in M. Then M_0 is clearly a left ideal of S contained in M which, by hypothesis, is non-zero. Let L be a 0-minimal left ideal contained in M and $x \in S$. Then $Lx \subseteq Mx \subseteq M$.

$\boldsymbol{E}(S)$:set of idempotents of S
primitive
primitive! – idempotent

By Lemma 2.60, either $Lx = 0$ or Lx is a 0-minimal left ideal of S; in either case, $Lx \subseteq M_0$. It follows that M_0 is a non-zero ideal contained in M and so, $M_0 = M$ by the 0-minimality of M.

Now suppose that $M^2 \neq 0$. Then by Corollary 2.55, M is a 0-simple subsemigroup of S. Let $K \subseteq M$ be a non-zero left ideal of M. If $a \in K$, $a \neq 0$, then by the above there exists a 0-minimal left ideal L of S with $a \in L \subseteq M$. By Lemma 2.51, $MaM = M$ and so $Ma \neq 0$. Since M is an ideal in S, Ma is a left ideal in S and $Ma \subseteq L$. Hence, by the 0-minimality of L, we have $Ma = L$; in particular, $a \in Ma$. Clearly Ma is a left ideal in M and so, $Ma \subseteq K$. Hence $K = \cup\{Ma : a \in K\}$. Thus K is a union of left ideals in S and so K itself is a left ideal in S (since the lattice of all left ideals in S is complete—see § Subsection 2.1.1). □

For a semigroup S we use the notation

$$\boldsymbol{E}(S) = \{e \in S : e^2 = e\}; \tag{2.42}$$

the set of all idempotents in E. In $\boldsymbol{E}(S)$ define the relation

$$e \;\omega\; f \iff ef = fe = e. \tag{2.43}$$

It is easy to verify that when $\boldsymbol{E}(S) \neq \emptyset$, this defines a partial order on $\boldsymbol{E}(S)$. In the following (in this chapter) $\boldsymbol{E}(S)$ will denote this partially ordered set. (Later in Chapter III, we will define additional properties of $\boldsymbol{E}(S)$.)

Let S be a semigroup with 0. We shall say that $e \in \boldsymbol{E}(S)$ is a *primitive* if for any $f \in \boldsymbol{E}(S) - \{0\}$, $f \;\omega\; e$ implies $f = e$; that is, e is minimal in the partially ordered set of all non-zero idempotent in S. In a semigroup S with out 0, by a primitive idempotent, we shall mean an idempotent which is minimal in $\boldsymbol{E}(S)$

THEOREM 2.62. *Let M be a* 0*-minimal ideal in a semigroup with* 0*. Then the following statements are equivalent.*

(a) $M^2 \neq 0$ *and M contains at least one* 0*-minimal left ideal and at least one* 0*-minimal right ideal.*

(b) M contains a primitive idempotent.

When M satisfies these equivalent conditions, M is a 0*-bisimple and regular subsemigroup of S (see § Subsection 2.6.1) and every non-zero idempotent in M is primitive.*

Proof. Suppose that (a) holds. Let $a, b \in M - \{0\}$ such that $ab \neq 0$; such elements exist since $M^2 \neq 0$. By Theorem 2.61, there is a 0-minimal right ideal R such that $a \in R$. Then $ab \in R$ and since $ab \neq 0$, by Lemma 2.50, $a \mathscr{R} ab$. Dually $b \mathscr{L} ab$. Hence by Theorem 2.44, $L_a \cap R_b$ contains an idempotent.

We now show that any non-zero idempotent e in M is primitive. Let $f \in \mathbf{E}(S)$ with $f\ \omega\ e$ and $f \neq 0$. Then $f = ef \in eS$. By Theorem 2.61, eS is a 0-minimal right ideal and hence Lemma 2.50, $e \mathscr{R} f$. So, by Lemma 2.36(b), $e = fe = f$. Hence e is primitive. This shows that (a) implies (b).

Now assume (b). Let e be a primitive idempotent in M. Then $e \in M^2$ and so, $M^2 \neq 0$. Let $L = Se$. Since $e \in Se$, $L = S^1 e = L(e)$. Let L' be a non-zero left ideal contained in L and $a \in L' - \{0\}$. Then by Lemma 2.51, $MaM = M$ and so there is $s', t' \in M$ with $e = s'at'$. Let $s = es'$ and $t = et'e$. Then

$$e = sat, \quad satsa = esa = as, \quad tsat = te = t.$$

Hence $sa \in \mathscr{V}(t)$ and by Lemma 2.38, $f = tsa$ is an idempotent. Also we have

$$ef = (et)sa = tsa = f, \qquad fe = ts(ae) = tsa = t;$$
$$\text{and} \quad e = e^2 = sa(tsa)t = saft.$$

It follows that $f \leq e$ and $f \neq 0$. Since e is primitive, we have $e = f \in Sa$. Therefore $L = Se = Sa \subseteq L'$. This proves that $L = Se$ is a 0-minimal left ideal. Dually $R = eS$ is a 0-minimal right ideal. Thus (b) implies (a).

Suppose that M satisfies (a) and (b) and $a, b \in M - \{0\}$. Then by (a) and Lemma 2.51, $MaM = M$ and so, $b = sat$ for $s, t \in M$. Since $b \neq 0$, $sa, at \in M - \{0\}$ and so, by Lemma 2.50, $a \mathscr{L} sa$ and $a \mathscr{R} at$. Then $a \mathscr{R} at \mathscr{L} sat = b$ and so, $a \mathscr{D} b$. Hence $M - \{0\}$ is a $\mathscr{D}$-class of S and contains an idempotent by (b). Therefore by Proposition 2.39, every non-zero element, and hence every element in M is regular. We have already shown that any non-zero idempotent in M is primitive. ☐

A semigroup may have kernel (minimal ideal) but may not have minimal left or right ideals (see Example 2.24). The following result for minimal ideals (kernels) due to Clifford Clifford [1948] which corresponds to Theorems 2.61 and 2.62 above for 0-minimal ideals, shows that if a semigroup has minimal right or left ideal, then it has kernel.

THEOREM 2.63. *For a semigroup S, we have the following:*

(a) Suppose that S has at least one minimal left ideal. Then the union K of all minimal left ideals of S is the kernel of S and minimal left ideals of S are $\mathscr{L}$-classes contained in K.

(b) Suppose that S has at least one minimal left ideal and at least one minimal right ideal. Then K is a $\mathscr{D}$-class of S and every $\mathscr{H}$-class contained in K is a subgroup of S.

Proof. Let L be a minimal left ideal in S and $x \in S$. Then by Lemma 2.60, Lx is a minimal left ideal of S. It follows that the union K of all minimal left ideals of S is an ideal in S. Since every $a \in K$ is contained in a minimal left ideal, there is a minimal left ideal L of S containing a. Let $c \in L$. Then $L(c)$ is a left ideal contained in L and so $L(c) = L$. It follows that every element of L generates L as a left ideal and so, $L = L_a$. Hence L is a minimal left ideal of S if and only if it is a $\mathscr{L}$-class of S contained in K. Suppose that $J \subseteq K$ be an ideal. If L is any minimal left ideal, then we have $J.L \subseteq J \cap L \subseteq L$ and so $J \cap L$ is non-empty and is a left ideal. By the minimality of L, we have $J \cap L = L$; that is, $L \subseteq J$. It follows that $K \subseteq J$ and so K is the minimal ideal of S. This proves (a).

To prove (b), let K [K'] be the union of all minimal left [right] ideals of S. Then by (a) and its dual K and K' are minimal ideals of S and so $K = K'$ by Lemma 2.49. Therefore, by (a), minimal left [right] ideals are $\mathscr{L}$-classes [$\mathscr{R}$-classes] contained in K. Let L be an $\mathscr{L}$-class and R be an $\mathscr{R}$-class contained in K. If $a \in L$ and $b \in R$, then $ab \in R(a) = R_a$ and dually, $ab \in L_b$. Hence by Theorem 2.34, $L_a \cap R_b = L \cap R$ contains an idempotent. It follows that $L \cap R$ is nonempty and hence an $\mathscr{H}$-class of K. By Corollary 2.30 and Proposition 2.37, K is a $\mathscr{D}$-class of S and every $\mathscr{H}$-class contained in K is a group. □

Example 2.21: Let $S = \mathscr{T}_X$. Then $K(S)$ is the set of all constant maps on X which is therefore in one-to-one correspondence with X. Also $K(S)$ is a minimal right ideal also and for any $f \in K(S)$, $\{f\}$ is a minimal left ideal. Dually, in S^{op}, $K(S)$ is a minimal left ideal and $\{f\}$ is a minimal right ideal for any $f \in K(S)$.

Example 2.22: Let $\{G_i : i \in I\}$ be a set of groups and let S denote the disjoint union of the groups G_i together with a symbol 0 that does not represent any element in any group G_i. Define product in S as follows: for $s, t \in S$,

$$st = \begin{cases} \text{the product in } G_i & \text{if } s, t \in G_i \text{ for some } i; \\ = 0 & \text{otherwise.} \end{cases}$$

Then S is a semigroup in which $G_i \cup \{0\}$ is a 0-minimal ideal which is also a 0-minimal left as well as a right ideal for each $i \in I$. Hence if $|I| > 1$, then S has more than one 0-minimal [left, right] ideals.

Example 2.23: Let $S = \mathscr{L}\mathscr{T}(V)$ where V is a vector space of dimension n over the field $\Bbbk$ and let W be a subspace spanned by a non-zero vector $v \in V$. Let L_W be the $\mathscr{L}$-class of S corresponding to the subspace W (see Example 2.11). It is easy to check that $L = L_W \cup \{0\}$ is a 0-minimal left ideal of S such that $L^2 \neq 0$. Choose a hyperplane U (subspace U with $\dim U = n-1$) such that $W \subseteq U$. Then $H_{U,W} = R_U \cap L_W$ is a non-empty $\mathscr{H}$-class. Let $f \in H_{U,W}$. Then $f \neq 0$ and $Lf = 0$. In this case $L' = \{0, f\}$ is a proper non-zero left ideal in L and so, the semigroup L is not left 0-simple. Also it is easy to see that $fL = H_{U,W} \cup \{0\}$ is a proper non-zero two-sided ideal in L and so L is also not 0-simple.

Example 2.24: Let $S = \langle p, q; pq = 1\rangle$ be the bicyclic semigroup (see Examples 2.7 and 2.12). It follows from Example 2.16 that S is a bisimple inverse semigroup and hence *semigroup!completely* 0*-simple*

simple. Therefore S has minimal ideal (S itself). Now R is a right ideal in S if and only if for some $r \in \mathbb{N}$, $R = R_r$ and any left ideal of S is L_s for $s \in \mathbb{N}$, where

$$R_r = \{q^{r+m}p^n : m, n \in \mathbb{N}\} \qquad L_s = \{q^m p^{n+s} : m, n \in \mathbb{N}\}.$$

Since $R_r \subseteq R_s$ for $r \geq s$, right ideals in S is an infinite chain and so has no minimum. Thus S does not have minimal right ideals. Similarly S also does not have minimal left ideals. Also, idempotents in S are $e_n = q^n p^n$, $n \in \mathbb{N}$ (see Example 2.16) and it is easy to verify that

$$e_n \ \omega \ e_m \iff n \leq m.$$

It follows from Example 2.7(a) and (c) that $e_n = e_m$ if and only if $n = m$. Hence idempotents in S also form an infinite descending chain and so S does not contain primitive idempotents.

2.7.2 Completely 0-simple semigroups

A semigroup S with 0 is said to be *completely* 0*-simple* if

1. S is 0-simple; and
2. S contains a primitive idempotent.

A semigroup S with out 0 is said to be *completely simple* if S is simple and contains a primitive (minimal) idempotent.

Now S is 0-simple if and only if S is a 0-minimal ideal in S. Hence Theorem 2.62 implies the following result due to Clifford Clifford [1949]

THEOREM 2.64. *The following statements are equivalent for a semigroup S:*

1. *S is completely* 0*-simple;*
2. *S is* 0*-simple and contains a* 0*-minimal left ideal and a* 0*-minimal right ideal;*
3. *S is* 0*-bisimple, regular and every non-zero idempotent in S is primitive.* □

In view of Remark 2.15, from this result, we derive the following characterization of completely simple semigroups. (This also follows from Theorem 2.63).

THEOREM 2.65. *Let S be a semigroup (with out zero). The following statements are equivalent:*

1. *S is completely simple;*
2. *S is simple and contains a minimal left ideal and a minimal right ideal;*
3. *S is bisimple, regular and every non-zero idempotent in S is primitive.*

Moreover, when S is completely simple, every $\mathscr{H}$-class of S is a group. □

If S is completely 0-simple, then by Theorem 2.64, S is 0-bisimple and so $S = D^0 = D \cup \{0\}$ where D is the set of all non-zero elements of S, which is a $\mathscr{D}$-class in S. Now, by Theorem 2.61 and its dual, S is the union of all 0-minimal left ideals and the union of all 0-minimal right ideals. Hence we have

$$D = \cup\{L_a : 0 \neq a \in S\} = \cup\{R_a : 0 \neq a \in S\};$$

where

$$L(a) = L_a \cup \{0\}, \qquad R(a) = R_a \cup \{0\}$$

where $L(a)$ and $R(a)$ are unique 0-minimal left and right ideals of S respectively containing $a \in S$, $a \neq 0$. We now list some important properties of completely 0-simple semigroups that will, later, enable us to construct all such semigroups.

THEOREM 2.66. *Let D denote the set of non-zero elements of a completely 0-simple semigroup S. Then*

(1) For $a, b \in D$, $L_aR_b \neq 0$ if and only if $L_a \cap R_b$ contains an idempotent. If this holds, then $L_aL_b = D$.

(2) For all $a, b \in S$, $H_aH_b = H_{ab}$.

In particular, for any $a \in D$, either $a^2 = 0$ or $a^2 \in H_a$ and H_a is a subgroup of S.

Proof. Suppose that $L_aR_b \neq 0$. Then for $a' \mathscr{L} a$ and $b' \mathscr{R} b$, $a'b' \neq 0$. Since $a'b' \in R(a')$ and since $R(a')$ is the 0-minimal ideal containing a', we have $a' \mathscr{R} a'b'$. Dually $b' \mathscr{L} a'b'$ and so, by Theorem 2.34, $L_{a'} \cap R_{b'} = L_a \cap R_b$ contains an idempotent. Let $e \in L_a \cap R_b$ be the idempotent. Then again by Theorem 2.34, $a''b'' \in D$ and hence $a''b'' \neq 0$ for any $a'' \in L_a$ and $b'' \in R_b$. Therefore $L_aR_b \subseteq D$. Let $c \in D$. By Corollary 2.29, there is an isomorphism $\sigma : L(e) \to L(c)$. Since $L(e) = L_e \cup \{0\}$ and $L(c) = L_c \cup \{0\}$, by Theorem 2.25, σ is an $\mathscr{R}$-class preserving bijection of L_e onto L_c. Therefore if $t = e\sigma$, then we have $e \mathscr{R} t \mathscr{L} c$ and $\sigma = \rho_t$. Let s be the unique element in L_e with $s\sigma = c$. Then $c = s\sigma = st$ which implies that $c \in L_eR_e = L_aR_b$. This proves (1).

To prove (2), suppose that $H_aH_b \neq 0$. Then for some $a' \mathscr{H} a$ and $b' \mathscr{H} b$, $a'b' \neq 0$. Then $L_aR_b = L_{a'}R_{b'} \neq 0$ and so, by (1), $L_aR_b = D$. Hence $a''b'' \neq 0$ for all $a'' \in H_a$ and $b'' \in H_b$. Moreover, $a''b'' \in R_{a''} \cap L_{b''} = H_{ab}$. Since for all $s \in L_a$, $s \mathscr{R} sb$, by Theorem 2.34, $\sigma : s \mapsto sb$ is an $\mathscr{R}$-class preserving bijection of L_a onto L_{ab} and maps H_a onto H_{ab} by Theorem 2.25. Hence

$$H_aH_b \supseteq H_ab = (H_a)\sigma = H_{ab}.$$

Therefore $H_aH_b = H_{ab}$. If $H_aH_b = 0$, then $ab = 0$ and so, in this case also, $H_aH_b = H_{ab}$.

If $a^2 \neq 0$, then by (1) above, H_a contains an idempotent and so, by Proposition 2.37, H_a is a group and hence $a^2 \in H_a$. □

We proceed to prove another characterization of completely 0-simple semigroups. We need some preliminary results. Recall that if x is a regular element in S and $x' \in \mathscr{V}(x)$, then by Lemma 2.38 $e = xx'$ and $f = x'x$ are idempotents in S such that $x \mathrel{\mathscr{R}} e \mathrel{\mathscr{L}} x' \mathrel{\mathscr{R}} f \mathrel{\mathscr{L}} x$. Further for $e \in \boldsymbol{E}(S)$ we denote by $\omega(e) = \{g : g \mathrel{\omega} e\}$ the order ideal of S with respect to the partially ordered set defined by Equation (2.43).

Lemma 2.67. *Let x be a regular element in the semigroup S and let $x' \in \mathscr{V}(x)$. Then the map $\mathfrak{a}(x, x')$ defined by*

$$g\mathfrak{a}(x, x') = x'gx \quad \textit{for all} \quad g \mathrel{\omega} xx' \tag{2.44}$$

is an order isomorphism of $\omega(xx')$ onto $\omega(x'x)$.

Proof. Let $e = xx'$ and $f = x'x$. If $g \mathrel{\omega} e$, we have,

$$\begin{aligned} f\,(g\mathfrak{a}(x, x')) &= x'xx'gx = x'gx = g\mathfrak{a}(x, x'); \\ (g\mathfrak{a}(x, x'))\,f &= x'gxx'x = g\mathfrak{a}(x, x'). \end{aligned}$$

Thus by Equation (2.43), $g\mathfrak{a}(x, x') \mathrel{\omega} f$. If $h \mathrel{\omega} g$ for $h, g \in \omega(e)$, then

$$(g\mathfrak{a}(x, x'))\,(h\mathfrak{a}(x, x')) = x'g(xx')hx = x'gehx = x'hx = h\mathfrak{a}(x, x');$$

and similarly, $(g\mathfrak{a}(x, x'))\,(h\mathfrak{a}(x, x')) = h\mathfrak{a}(x, x')$. Therefore $h\mathfrak{a}(x, x') \mathrel{\omega} g\mathfrak{a}(x, x')$. Thus $\mathfrak{a}(x, x') : \omega(e) \to \omega(f)$ is an order preserving map. Similarly, $\mathfrak{a}(x', x) : \omega(f) \to \omega(e)$ is also an order preserving map. Further. for $g \in \omega(e)$,

$$(g\mathfrak{a}(x, x'))\,\mathfrak{a}(x', x) = x(x'gx)x' = ege = g.$$

Therefore $\mathfrak{a}(x, x') \circ \mathfrak{a}(x', x) = 1_{\omega(e)}$. Similarly $\mathfrak{a}(x', x)\mathfrak{a}(x, x') = 1_{\omega(f)}$ and so $\mathfrak{a}(x, x')$ is an order isomorphism. □

A more details study the map $\mathfrak{a}(x, x')$ defined above will be made later in the chapter on inductive groupoids.

Recall from Proposition 2.28 that two idempotents e and f are $\mathscr{D}$-related if and only if $R_e \cap L_e$ and $L_e \cap R_e$ are non-empty and that, for each $x \in R_e \cap L_e$ there is a unique inverse $x' \in L_e \cap R_e$ (by Proposition 2.40).

Lemma 2.68. *Let $B = \langle p, q; pq = 1\rangle$ be the bicyclic semigroup. Let e and f be $\mathscr{D}$-related idempotents in a semigroup S such that $f \mathrel{\omega} e$ and $f \neq e$. If $x \in R_e \cap L_e$ and y is the unique inverse of x in $L_e \cap R_e$, then for each $n \geq 1$,*

$$y^n \in \mathscr{V}(x^n) \quad \textit{with} \quad x^n y^n = e \tag{2.45a}$$

and for $n, m \in \mathbb{N}$ with $n < m$

$$g_m = y^m x^m \;\omega\; g_n = y^n x^n, \quad g_m \neq g_n. \tag{2.45b}$$

Further, $\{x^n : n \geq 1\}$ and $\{y^n : n \geq 1\}$ are sequences of elements belonging to distinct $\mathscr{H}$-classes in R_e and L_e respectively. Moreover, if $B^* = \langle x, y\rangle$ is the subsemigroup of S generated by x and y, there is an isomorphism $\phi : B \to B^*$ such that $\phi(p) = x$ and $\phi(q) = y$.

Proof. By Lemma 2.38, $e = xy$ and $f = yx$. Also, by Lemma 2.67, $\mathfrak{a}(x, y) : \omega(e) \to \omega(f)$ is an order isomorphism. Since $\omega(f) \subset \omega(e)$, $\mathfrak{a}(x, y)$ is an order embedding of $\omega(e)$ into itself. Hence for each $n \in \mathbb{N}$,

$$\mathfrak{a}(x, y)^n = \mathfrak{a}(x, y) \circ \cdots \circ \mathfrak{a}(x, y) \quad n \text{ factors}$$

is also an order embedding of $\omega(e)$ into itself. Since $e \neq f$, it follows that $e\mathfrak{a}(x, y)^n \neq f\mathfrak{a}(x, y)^n = e\mathfrak{a}(x, y)^{n+1}$. If $g_n = e\mathfrak{a}(x, y)^n$, it follows that

$$g_n \;\omega\; g_m \quad \text{for } n < m \text{ and } \quad g_n \neq g_m.$$

Thus $\{g_n : n \in \mathbb{N}\}$ is a descending infinite sequence of idempotents in D_e. It follows from the Lemma 2.67 that

$$g_n = y^n x^n, \qquad x^n y^n = e \quad \text{and} \quad y^n \in \mathscr{V}(x^n). \tag{$*$}$$

Therefore, by Lemma 2.38,

$$x^n \in R_e \cap L_{g_n}, \qquad y^n \in L_e \cap R_{g_n}.$$

It follows that x and y satisfies Equations (2.45a) and (2.45b). Now since $g_n \;\omega\; g_m$ and $g_n \neq g_m$ for $n < m$, it is not possible that g_n and g_m are $\mathscr{L}$ related or $\mathscr{R}$ related. It follows that $\{x^n : n \in \mathbb{N}\}$ is a sequence of elements belonging to distinct $\mathscr{H}$-classes in in R_e. Similarly, $\{y^n : n \in \mathbb{N}\}$ is a sequence of elements belonging to distinct $\mathscr{H}$-classes of in L_e. Hence $y^n x^m \in R_{g_n} \cap L_{g_m}$. It follows that

$$y^n x^m = y^r x^s \iff n = r, \quad m = s.$$

Also, using ($*$) we have,

$$(y^n x^m)(y^r x^s) = \begin{cases} y^n x^{m-r+s} & \text{if } m \geq r; \\ y^{n+r-m} x^s & \text{if } m < r. \end{cases} \tag{$p*$}$$

It follows that $\{y^n x^m : n, m \in \mathbb{N}\}$ is a subsemigroup of S containing x and y and so $B^* = \{y^n x^m : n, m \in \mathbb{N}\}$. Therefore, compairing the product in B given in Example 2.7(e) and the equation above, it is clear that the map $\phi : B \to B^*$ defined by

$$\phi(q^n p^m) = y^n x^m \quad \text{for all} \quad m, n \in \mathbb{N}$$

is an isomorphism of B onto B^* with $\phi(p) = x$ and $\phi(q) = y$. □

semigroup!group-bound –

We shall say that the semigroup B^* of S constructed above is a bicyclic subsemigroup of S generated by x and y and with identity e.

PROPOSITION 2.69. *Let e be a non-zero idempotent of a 0-simple semigroup S. If $f \neq 0$ is any idempotent with $f \; \omega \; e$ and $f \neq e$, then there exist $x, y \in S$ such that*

$$xy = e \quad \text{and } g = yx \text{ is an idempotent with} \quad g \; \omega \; f.$$

The subsemigroup $B^* = \langle x, y \rangle$ generated by x and y is a bicyclic semigroup with identity e. Therefore if S is not completely 0-simple, then S contains a copy of the bicyclic semigroup.

Proof. By Corollary 2.54, $SfS = S$ and so there is $x', y' \in S$, with $x'fy' = e$. Let $x = ex'f$ and $y = fy'e$. Then we have

$$xy = e, \quad xyx = ex = x, \quad yxy = ye = y.$$

Hence $y \in \mathscr{V}(x)$ and so $g = yx$ is an idempotent. Also

$$fg = fyx = yx = g, \quad \text{and} \quad gf = yxf = yx = g.$$

Hence $g \; \omega \; f \; \omega \; e$. Since $f \neq e$, we have $g \neq e$. Therefore, by Lemma 2.68, x and y generates a bicyclic subsemigroup of S with $e = xy$ as identity.

If S is not completely 0-simple, the idempotent e is not primitive. Hence there is $0 \neq f \in \boldsymbol{E}(S)$ with $f \; \omega \; e$ and $f \neq e$ and so, by the above, S contains a bicyclic semigroup with identity e. □

We may restate the result above as a characterization of those 0-simple semigroups that are not completely 0-simple:

COROLLARY 2.70. *A 0-simple semigroup S is not completely 0-simple if and only if S satisfies one of the following conditions:*

A S does not contain non-zero idempotents;

B S contains a copy of the bicyclic semigroup. □

A semigroup S is said to be *group-bound* if some finite power of each element of S belongs to a subgroup of S. If S is a semigroup with 0 and if $S^2 = 0$, then $a^2 = 0$ for all $a \in S$ and so S is group-bound. A cyclic semigroup is group-bound if and only if it is finite (see § Subsection 2.1.3). Hence any periodic semigroup (§ Subsection 2.1.3) is group bound.

The next theorem is due to Munn Munn [1961]

THEOREM 2.71. *A 0-simple semigroup S is completely 0-simple if and only if it is group-bound.*

Proof. If S is completely 0-simple, then by Theorem 2.66, S is group-bound. *homomorphism!0-restricted –*

Conversely suppose that the 0-simple semigroup S is group-bound. Then for any $a \neq 0$ in S, $SaS = S$ by Corollary 2.54. Then $a \in SaS$ and so there are $x, y \in S$ with $a = xay$. It follows from this that $a = x^n a y^n$ for all $n \in \mathbb{N}$ and since $a \neq 0$, $x^n \neq 0$ for any n. Since S is group-bound, x^n belongs to a subgroup of S; the identity of this group must be a non-zero idempotent in S. Thus S contains non-zero idempotents.

Assume that S is not completely 0-simple. Let e be a non-zero idempotent. Then by Proposition 2.69 there are elements $x, y \in S$ such that $B^* = \langle x, y \rangle$ is a bicyclic semigroup with identity e. Then $xy = e$ and $y \in \mathscr{V}(x)$. So, $yx \mathscr{D} e$, $yx \neq e$ and $yx \ \omega \ e$. Hence by Lemma 2.68, $\{x^n : n \geq 1\}$ is a sequence of elements in R_e belonging to distinct $\mathscr{H}$-classes. This implies that H_{x^n} is not a group for any $n \geq 1$. For if H_{x^n} is a group, we have $x^{2n} = (x^n)^2 \in H_{x^n}$. Then $g_n \mathscr{L} x^n \mathscr{H} x^{2n} \mathscr{L} g_{en}$. Since $g_{2n} \ \omega \ g_n$, this gives $g_n = g_{2n}$ which is impossible by Lemma 2.68. Therefore S is not group bound. □

In view of the discussion preceding the theorem, we have:

COROLLARY 2.72. *Every periodic, in particular, every finite, 0-simple semigroup S is completely 0-simple.* □

Let $\phi : S \to T$ be a homomorphism of a semigroup S with 0 to a semigroup T. We shall say that the homomorphism ϕ is 0-*restricted* if it has the property that $x\phi = 0$ implies $x = 0$.

THEOREM 2.73. *Let $\phi : S \to T$ be a homomorphism of a completely 0-simple semigroup onto a semigroup T. Then T is either $T = 0$, the trivial (one-element) semigroup or T is completely 0-simple and ϕ is 0-restricted.*

Proof. Let $x\phi = 0$ and $x \neq 0$. Since S is regular, there is a non-zero idempotent $e \in S$ with $e \mathscr{R} x$. Then $e\phi \mathscr{R} x\phi = 0$ in T (using Lemma 2.33) which implies that $e\phi = 0$. It follows similarly that $y\phi = 0$ for all $y \in R_e$ and $y \in L_e$. If $z \in D_e$, by Theorem 2.66(1), there is $y_1 \in L_e$ and $y_2 \in R_e$ such that $z = y_1 y_2$ so that $z\phi = (y_1\phi)(y_2\phi) = 0$. Therefore $z\phi = 0$ for all $z \in S$ and since ϕ is surjective, $T = 0$. Hence if $T \neq 0$, then ϕ is 0-restricted.

We now assume that $T \neq 0$. If I is any non-zero ideal in T, clearly, $I\phi^{-1}$ is a non-zero ideal in S and so $I\phi^{-1} = S$ which implies that $I = T$. Hence T is 0-simple. Let $t \in T$ and let $a \in S$ with $a\phi = t$. By Theorem 2.66, either $a^2 = 0$ or $a^2 \in H_a$ and H_a is a group. Then either $t^2 = 0$ or $t^2 \in H_a\phi \subseteq H_t$ and H_t is a group. Therefore T is group-bound and so, by Theorem 2.71, T is completely 0-simple. □

2.7.3 Rees matrix semigroups

Let G^0 be a group with 0 (§ Subsection 2.1.3) and I and Λ be sets. Recall from § Subsection 2.1.3 that a sandwich $\Lambda \times I$-matrix over G^0 is a map $P : \Lambda \times I \to G^0$, $(\lambda, i) \mapsto p_{\lambda i}$. A Rees $I \times \Lambda$-matrix semigroup over a group with 0, G^0 with sandwich $\Lambda \times I$-matrix P is the set

$$\mathcal{M}^0(G; I, \Lambda; P) = (G \times I \times \Lambda) \bigcup \{0\} \tag{2.46a}$$

together with product defined, for any $s, t \in \mathcal{M}^0(G; I, \Lambda; P)$, by

$$st = \begin{cases} (ap_{\lambda j}b, i, \mu) & \text{if } s = (a, i, \lambda),\ t = (b, j, \mu) \text{ and } p_{\lambda j} \neq 0; \\ 0 & \text{if } s = (a, i, \lambda),\ t = (b, j, \mu) \text{ and } p_{\lambda j} = 0; \\ 0 & \text{if either } s = 0,\ t = 0 \text{ or } s = t = 0. \end{cases} \tag{2.46b}$$

By § Subsection 2.1.3, the binary operation defined above is associative and so, $S = \mathcal{M}^0(G; I, \Lambda; P)$ is a semigroup with 0. Again, it follows from § Subsection 2.1.3 that the semigroup is regular if and only if the matrix P is regular in the sense that P satisfies the following:

$$\begin{aligned} &\forall\ i \in I, \quad \exists\ \mu \in \Lambda \quad \text{such that} \quad p_{\mu i} \neq 0 \\ &\forall\ \lambda \in \Lambda, \quad \exists\ j \in I \quad \text{such that} \quad p_{\lambda j} \neq 0. \end{aligned} \tag{2.46c}$$

In particular, if $p_{\mu, i} \neq 0$ for all $\mu \in \Lambda$ and $i \in I$, then P is clearly regular. In this case, it follows from the equation (2.46b) that the set of all non-zero elements of the semigroup $\mathcal{M}^0(G; I, \Lambda; P)$ is a subsemigroup $\mathcal{M}(G; I, \Lambda; P)$. Recall from Example Subsection 2.1.3 that $\mathcal{M}(G; I, \Lambda; P)$ is the Rees matrix semigroup with out zero.

In the following discussion, we will use the notations in introduced above:

Lemma 2.74. *Let $S = \mathcal{M}^0(G; I, \Lambda; P)$ be a regular Rees matrix semigroup. For $\lambda \in \Lambda$ and $i \in I$, let*

$$L_\lambda = \{(b, j, \lambda) : b \in G,\ j \in I\} \quad \text{and} \quad R_i = \{(b, j, \mu) : b \in G,\ \mu \in \Lambda\}.$$

If $x = (a, i, \lambda) \in S$, we have:

$$L_x = L_\lambda \quad \text{and} \quad L(x) = L_\lambda \cup \{0\}; \tag{2.47a}$$

$$R_x = R_i \quad \text{and} \quad R(x) = R_i \cup \{0\}; \tag{2.47b}$$

$$H_x = H_{i\lambda} = \{(b, i, \lambda) : b \in G\}. \tag{2.47c}$$

Consequently there exist bijections $\lambda \mapsto L_\lambda$ and $i \mapsto R_i$ of the set Λ onto the set of non-zero $\mathscr{L}$-classes of S and of the set I onto the set of non-zero $\mathscr{R}$-classes of S respectively. Further, the map $(i, \lambda) \mapsto H_{i\lambda}$ is a bijection of $I \times \Lambda$ onto the set of non-zero $\mathscr{H}$-classes of S.

Proof. . Let $x = (a, i, \lambda) \in S$ and $y \in L(x)$. Since S regular, by Lemma 2.36, $y = sx$ for some $s \in S$ and so, either $y = 0$ or by Equation (2.46b), $y = (b, j, \lambda)$ for some $b \in G$ and $j \in I$. Conversely if $y = (b, j, \lambda)$, then by Equation (2.46c), there is $\mu \in \Lambda$ with $p_{\mu i} \neq 0$ and if $c = ba^{-1}p_{\mu i}$, then we have $y = (c, j, \mu)(a, i, \lambda)$. Hence $y \in L(x)$. By symmetry, we have

$$(a, i, \lambda) \mathscr{L} (b, j, \mu) \iff \lambda = \mu \quad \text{and so,} \quad L(x) = L_\lambda \cup \{0\}$$

where L_λ is the set defined in the statement. Dually, we have

$$(a, i, \lambda) \mathscr{R} (b, j, \mu) \iff i = j \quad \text{and so,} \quad R(x) = R_i \cup \{0\}.$$

It follows that every non-zero $\mathscr{L}$ [$\mathscr{R}$] class is of the form L_λ [R_i] and the mapping $\lambda \mapsto L_\lambda$ [$i \mapsto R_i$] is a bijection. Also for any $(i, \lambda) \in I \times \Lambda$, $H_{i\lambda}$ is a non-empty set consisting of non-zero elements of S and the mapping $(i, \lambda) \mapsto H_{i\lambda}$ is clearly a bijection. Also, by the above,

$$H_{i\lambda} = R_i \cap L_\lambda = R_x \cap L_x = H_x. \quad \square$$

□

THEOREM 2.75. *Every regular Rees matrix semigroup is completely 0-simple.*

Proof. Let $S = \mathcal{M}^0(G; I, \Lambda; P)$ be a regular Rees matrix semigroup. It follows from Lemmas 2.50 and 2.74 that every principal left and right ideal in S is 0-minimal. Further if $x = (a, i, \lambda)$ and $y = (b, j, \mu)$ are non-zero elements in S and $c \in G$, then by Equations (2.47a) and (2.47b)

$$(a, i, \lambda) \mathscr{R} (c, i, \mu) \mathscr{L} (b, j, \mu).$$

Therefore the set of non-zero elements from a $\mathscr{D}$-class in S and so, S is 0-bisimple. By Theorem 2.64, S is completely 0-simple. □

Specializing the areguements above to Rees matrix semigroups with out zero, we obtain:

COROLLARY 2.76. *Every Rees matrix semigroup with out zero is completely simple.* □

The theorem above is a part of the important theorem due to Rees Rees [1940] which asserts that a semigroup is completely 0-simple if and only if it is isomorphic with a regular Rees matrix semigroup. Thus Rees theorem consists of Theorem 2.75 and its converse which we proceed to prove. Here we shall derive the converse from an important result due to Miller and Clifford Miller and Clifford [1956], which applies to regular $\mathscr{D}$-classes of any semigroup.

$x * y$: trace product of x and y
trace!trace product
$D(*)$: trace of the $\mathscr{D}$-class D
trace
trace!– of the $\mathscr{D}$-class D
trace!– of S

Let D denote a $\mathscr{D}$-class of a semigroup S. For $x, y \in D$, let

$$x * y = \begin{cases} xy & \text{if } L_x \cap R_y \text{ contains an idempotent;} \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{2.48a}$$

$x * y$, when it exists, is called the *trace product* of x and y. Then $*$ is a partial binary operation on D. The partial algebra $D(*) = (D, *)$ is called the *trace* of D. Note that, by Theorem 2.34, we may define $*$ equivalently by requiring that $x * y$ is defined if and only if $xy \in R_x \cap L_y$. The partial binary operation $*$ can be extended to a binary operation on $D^0 = D \cup \{0\}$, again denoted by $*$, in the obvious way. For $x, y \in D^0$ let

$$x * y = \begin{cases} xy & \text{if } x, y \in D \text{ and } xy \in R_x \cap L_y; \\ 0 & \text{if } x, y \in D \text{ and } xy \notin R_x \cap L_y; \\ 0 & \text{if either } x = 0,\ y = 0 \text{ or } x = y = 0. \end{cases} \tag{2.48b}$$

The partial algebra

$$S(*) = \bigcup_{D \in S/\mathscr{D}} D(*)$$

is called the *trace of S*.

Note that, the trace product of $x, y \in D$ exists as in Equation (2.48a) if and only if $x * y \neq 0$ in $D^0(*)$. The proof the following statement is quite routine.

LEMMA 2.77. *The binary algebra $D^0(*)$ with operation defined by Equation* (2.48b) *is a semigroup.*

	$L_x \downarrow$	$L_y \downarrow$	$L_z \downarrow$
$R_z \to$		g	z
$R_y \to$	f	y	yz
$R_x \to$	x	xy	xyz

Fig. 4

Proof. Suppose that $x, y, z \in U = D^0(*)$. If $x*y \neq 0$ and $y*z \neq 0$, then it is easy to see that $x*(y*z) \neq 0$ and $(x*y)*z \neq 0$ and the two expressions are equal (see the egg-box diagram on the right). From the diagram it is also clear that if one of $x*y$ and $y*z$ is 0, then $x*(y*z) = 0 = (x*y)*z$. It follows that $*$ is associative. □

The semigroup $D^0(*)$ is called the *trace semigroup* (or simply *trace* if there is no ambiguity). Note that if D is not regular then the trace product is not defined for any pair of elements in D and the semigroup $D^0(*)$ is the null semigroup.

Let D be a regular $\mathscr{D}$-class of the semigroup S and let e be an idempotent in D. By Proposition 2.39 D contains idempotents. Let

$$D/\mathscr{R} = \{R_i : i \in I\} \quad \text{and} \quad D/\mathscr{L} = \{L_\alpha : \lambda \in \Lambda\}$$

be the set of $\mathscr{R}$ and $\mathscr{L}$-classes of S contained in D. Then

$$D/\mathscr{H} = \{H_{i\lambda} = R_i \cap L_\lambda : (i, \lambda) \in I \times \Lambda\}$$

is the set of all $\mathscr{H}$-classes of S contained in D. Since $\mathscr{H}$-classes are disjoint, each $x \in D$ belongs to a unique $\mathscr{H}$-class $H_{i\lambda}$ in the above set. Now with out loss of generality, we may assume that $e \in I \cap \Lambda$ and that $H_{ee} = H_e$ by renaming the index $i_0 \in I$ for the $\mathscr{R}$-class R_e as e and similarly renaming the index for L_e in Λ also as e. We use these notations in the following statement.

THEOREM 2.78. *Let D be a regular $\mathscr{D}$-class of a semigroup S and let I and Λ denote index sets for $\mathscr{R}$ and $\mathscr{L}$-classes of S contained in D. For each $\lambda \in \Lambda$ and $i \in I$, choose*

$$r_\lambda \in H_{e\lambda} = R_e \cap L_\lambda \quad \text{and} \quad q_i \in H_{ie} = R_i \cap L_e$$

and set

$$p_{\lambda i} = \begin{cases} r_\lambda q_i & \text{if } H_{i\lambda} \text{ contains idempotent;} \\ 0 & \text{otherwise.} \end{cases} \tag{2.49}$$

Then the map $P : (\lambda, i) \mapsto p_{\lambda i}$ is a regular $\Lambda \times I$ matrix over H_e^0. For $t \in T = \mathcal{M}^0(H_e; I, \Lambda; P)$, define

$$t\phi = \begin{cases} q_i a r_\lambda & \text{if } t = (a, i, \lambda) \neq 0; \\ 0 & \text{if } t = 0. \end{cases} \tag{2.50}$$

This is an isomorphism $\phi : T \to D^0(*)$. Hence $D^0(*)$ is a completely 0-simple semigroup.

Proof. Let $\lambda \in \Lambda$. Then by Proposition 2.39, there is an idempotent $f \in L_{r_\lambda} = L_\lambda$. Let $R_f = R_i$. Then

$$f \in H_{i\lambda} = R_{q_i} \cap L_{r_\lambda};$$

so, by Theorem 2.34,

$$p_{\lambda i} = r_\lambda q_i = r_\lambda * q_i \in L_{q_i} \cap R_{r_\lambda} = H_e.$$

Hence for each $\lambda \in \Lambda$, there is $i \in I$ with $p_{\lambda i} \in H_e$; in particular, for this i, $p_{\lambda i} \neq 0$. Dually, for each $i \in I$, there is $\lambda \in \Lambda$ with $0 \neq p_{\lambda i} \in H_e$. It follows from Equation (2.46c) that the map $P : (\lambda, i) \mapsto p_{\lambda i}$ is a regular $\Lambda \times I$ matrix over H_e^0. Hence $T = \mathcal{M}^0(H_e^0; I, \Lambda; P)$ is a regular Rees matrix semigroup over H_e^0 (see § Subsection 2.1.3).

We first show that ϕ defined by Equation (2.50) is a bijection of T onto $U = D^0(*)$. If $0 \neq t \in T$, then $t = (a, i, \lambda)$ for some $a \in H_e$, $i \in I$ and $\lambda \in \Lambda$. Also, it is easy to see that

$$t\phi = q_i a r_\lambda = q_i * a * r_\lambda \in H_{i\lambda} \tag{1}$$

(see the diagram below). It follows that ϕ maps T into U such that $t\phi \neq 0$ if and only if $t \neq 0$. Now for any $\lambda \in \Lambda$, by Proposition 2.40, L_e contains at least one inverse of r_λ. Let r'_λ be an inverse of r_λ in L_e so that $r_\lambda r'_\lambda = e$ and $r'_\lambda r_\lambda$ is an idempotent in L_λ. Similarly, for each $i \in I$, choose an inverse q'_i of q_i in R_e. If $x \in H_{i\lambda}$, using Theorem 2.34, we see as above that $a = q'_i x r'_\lambda \in H_e$ (see the diagram below) and so,

$$(q'_i x r'_\lambda, i, \lambda)\phi = q_i q'_i x r'_\lambda r_\lambda = x.$$

Hence ϕ is surjective. If, for $a, b \in H_e$, $q_i a r_\lambda = q_i b r_\lambda$, then, by the choice of q'_i and r'_λ, we have

$$a = eae = q'_i q_i a r_\lambda r'_\lambda = q'_i q_i b r_\lambda r'_\lambda = b.$$

Hence if $(a, i, \lambda)\phi = (b, j, \mu)\phi$, then $H_{i\lambda} = H_{j\mu}$ which implies $i = j$ and $\lambda = \mu$ and this in turn implies, by the above, that $a = b$. Hence $(a, i, \lambda) = (b, j, \mu)$. Thus ϕ is one-to-one.

	$L_e \downarrow$				$L_\lambda \downarrow$
$R_e \to$	$a = q'_i x r'_\lambda$		q'_i		r_λ
	r'_λ				$r'_\lambda r_\lambda$
$R_i \to$	q_i		$q_i q'_i$		$x = q_i a r_\lambda$

Fig. 5

Let $s, t \in T$. If $s = 0$ or $t = 0$, then $s\phi = 0$ or $t\phi = 0$ in U by the definition of ϕ. Hence $(s\phi) * (t\phi) = 0$ in U and so, $(st)\phi = (s\phi)(t\phi)$ in this case. Let $t = (a, i, \lambda), s = (b, j, \mu)$. By Equation (2.46b), $st = 0$ if and only if $p_{\lambda j} \neq 0$. Using Equation (1) we obtain

$$s\phi = q_i * a * r_\lambda \ \mathscr{L} \ r_\lambda \ \mathscr{L} \ p_{\lambda j} \quad \text{and} \quad t\phi = q_j * a * r_\mu \ \mathscr{R} \ q_j \ \mathscr{R} \ p_{\lambda j}.$$

Hence

$$L_{s\phi} \cap R_{t\phi} = H_{j\lambda}.$$

Therefore, by Equations (2.46b), (2.48b) and (2.49), we have

$$s\phi * t\phi \neq 0 \iff p_{\lambda j} \neq 0 \iff st \neq 0.$$

Moreover, using Lemma 2.77, we obtain

$$\begin{aligned}
(s\phi) * (t\phi) &= (q_i * a * r_\lambda) * (q_j * b * r_\mu) && \text{by (1)} \\
&= q_i * (a * p_{\lambda j} * b*) r_\mu && \text{by Equation (2.49)} \\
&= (a p_{\lambda j} b, i, \mu)\phi && \text{by (1)} \\
&= (st)\phi && \text{by Equation (2.46b).}
\end{aligned}$$

Hence $\phi : T \to S$ is a homomorphism. Since ϕ is a bijection, it is an isomorphism. □

We now prove the theorem due to Rees on completely 0-simple semigroups Rees [1940, 1941]. A particular case of this result has been proved earlier by Suschkewitsch in his paper published in 1928 Suschkewitch [1928]. (see also Suschkewitch [1937] where he discuss some further results on this class of semigroups.)

THEOREM 2.79 (REES). *A semigroup S is completely 0-simple if and only if S is isomorphic to to a regular Rees matrix semigroup.*

Proof. If S is isomorphic to a regular Rees matrix semigroup, then by Theorem 2.75, S is completely 0-simple.

Suppose that S is completely 0-simple. By Theorem 2.64 S is 0-bisimple. Let D denote the $\mathscr{D}$-class of non-zero elements of S. Identifying the 0 of S with the 0 of $U = D^0(*)$, the underlying sets of S and U coincide. Let $x, y \in S$. By Theorem 2.66, $xy \neq 0$ in S if and only if $L_x \cap R_y$ contain an idempotent. By Equation (2.48b), this is true if and only if $x * y \neq 0$ in U and, in this case, $xy = x * y$. It follows that binary operations in S and U also coincide. Therefore $S = U$ and by Theorem 2.78, S is isomorphic to a regular Rees matrix semigroup. □

Specializing the arguements above to completely simple semigroups and Rees matrix semigroups with out zero, we obtain:

COROLLARY 2.80. *A semigroup S is completely simple if and only if S is isomorphic to a Rees matrix semigroup with out zero.* □

The isomorphism completely 0-simple semigroups with regular Rees matrix semigroups constructed above is not unique. In fact, from the construction in Theorem 2.78, it is clear that the isomorphism ϕ depends on

(a) the choice of the idempotent e in D; and

(b) the choice of elements $r_\lambda \in H_{e\lambda}$ and $q_i \in H_{ie}$.

For a different choice of these parameters, a different Rees matrix semigroup will result. However, it follows from Theorem 2.78 that these Rees matrix semigroups will be isomorphic. We shall discuss abstract characterization of such isomorphisms (more generally, homomorphisms of Rees matrix semigroups) after we have developed better machinery to analyze structure of regular semigroups and their homomorphisms (see Chapters Chapter 6 and **??**).

2.8 SEMISIMPLICITY OF SEMIGROUP

2.8.1 Principal factors

Recall that an ideal I in a semigroup S is maximal if there is no proper ideal J in S with $I \subset J$ (see § Subsection 2.1.1). If I and J are ideals in S with $I \subseteq J$, then I *is maximal in* J, if A is any ideal in S with $I \subseteq A \subseteq J$, then either $I = A$ or $A = J$; that is, the interval $[I, J] = \{I, J\}$ in the lattice $\mathfrak{I}_S$ of ideals of S (see § Subsection 2.1.1).

Recall that, by the convention adopted in § Subsection 2.1.1, an ideal I in a semigroup with 0 is always non-empty.

LEMMA 2.81. *Let I be an ideal in a semigroup S.*

(1) If J is an ideal in S with $I \subseteq J$, then I is maximal in J if and only if J/I is a minimal or 0-minimal ideal in S/I; if this is the case, then J/I is a simple, 0-simple or null semigroup. J/I is simple if and only if $I = \emptyset$.

(2) I is maximal in S if and only if S/I has no proper non-zero ideal; if this is so, then S/I is either simple, 0-simple or a null semigroup of order two. Again S/I is simple if and only if $I = \emptyset$

Proof. **(1)** Assume that $I \neq \emptyset$. Let $\theta : S \to S/I$ be the quotient mapping. If $\bar{A}$ is a non-zero ideal in S/I contained in J/I, then $A = \bar{A}\theta^{-1}$ is an ideal in S that properly contain I and contained in J. Since I is maximal in J, $A = J$ and so $\bar{A} = A/I = J/I$. Hence J/I is 0-minimal. Hence by Corollary 2.55, the semigroup J/I is 0-simple if $(J/I)^2 \neq 0$.

semigroup!principal factor of the –
$\mathcal{F}_a(S)$: Principal factor of S at a

If $I = \emptyset$, the statement that I is maximal in J is equivalent to the statement that J does not have any proper ideal; that is J is the minimal ideal which is therefore a simple subsemigroup of S by Corollary 2.55. If $I \neq \emptyset$, then clearly J/I has 0 and so is not simple.

(2) If $I \neq \emptyset$ is a maximal ideal in S, then, as above, we see that there is no proper ideal A in S with $I \subset A$ and so S/I has no proper non-zero ideal. By Lemma 2.53, if S/I is null, it is a null semigroup of order two. As in (1), we see that, $I = \emptyset$ if and only if $S = S/I$ is simple. □

Remark* 2.16*: Let I and J be ideals in a semigroup S with $I \subseteq J$. If I is maximal in J in the sense defined above, then I need not be a maximal ideal in the subsemigroup J of S. Consequently, the statement (2) of the Lemma above does not follow from (1) as a particular case of $J = S$. In fact, when I is maximal in J and $J^2 \subseteq I$ (so that J/I is null), the semigroup J/I can cotain more than two elements. The reason for this is that, if A is an ideal of an ideal J in a semigroup S, then A need not be an ideal in S (see Example 2.25 below).

PROPOSITION 2.82. *Let S be a semigroup and $a \in S$. Then*

$$I(a) = J(a) - J_a. \tag{2.51}$$

Then $I(a)$ is an ideal in S which is maximal in $J(a)$ and so, $J(a)/I(a)$ is either a minimal or 0-minimal ideal in $S/I(a)$ and the semigroup $J(a)/I(a)$ is either simple, 0-simple or null. $J(a)/I(a)$ is simple if and only if $J(a)$ is the kernel of S or equivalently, $I(a) = \emptyset$.

Proof. Suppose that $b \in I(a)$ and $s \in S^1$. If $sb \in J_a$, then $usbv = a$ for some $u, v \in S^1$. But this implies that $a \in J(b)$ and so $J(a) = J(b)$ which contradicts the hypothesis. Hence $sb \in I(a)$. Similarly $bs \in I(a)$ for all $s \in S^1$. Hence $I(a)$ is an ideal in S. If A is any ideal in S with $I(a) \subset A \subseteq J(a)$, then $A \cap J_a \neq \emptyset$; if $b \in A \cap J_a$, then $J(a) = J(b) \subseteq A$ and so $A = J(a)$. Thus $I(a)$ is maximal in $J(a)$. By Lemma 2.50, $J(a)$ is minimal if and only if $J(a) = J_a$; that is, $I(a) = \emptyset$. If this is true then $J(a)/I(a) = J(a)$ (see § Subsection 2.1.1) and $I(a)$ is clearly the maximal ideal in $J(a)$. The remaining statements follow from Lemma 2.81(1). □

The semigroup $J(a)/I(a)$, for $a \in S$, is called the *principal factor*of S at a; we denote it by $\mathcal{F}_a(S)$. Thus

$$\mathcal{F}_a(S) = \begin{cases} J(a)/I(a) & \text{if } I(a) \neq \emptyset; \\ J(a) & \text{if } I(a) = \emptyset. \end{cases} \tag{2.52}$$

By the Proposition above, $\mathcal{F}_a(S)$ is either a simple, 0-simple or null semigroup and $\mathcal{F}_a(S)$ is simple if and only if $J(a)$ is the kernel of S.

semigroup!semisimple –

Example 2.25: Let $C = \langle a \rangle$ denote the infinite cyclic group generated by a and let $B = \{b_n : n \in \mathbb{Z}\}$. Let

$$S = A \cup B \cup \{0\} \qquad \text{with product in } S \text{ defined by}$$
$$a^n b_m = b_{n+m}; \quad b_m a^n = b_m b_n = 0 \quad \forall\, m, n \in \mathbb{Z};$$

and 0 is the zero of S. Then S is a semigroup and $B^0 = B \cup \{0\}$ is the unique maximal ideal in S. Further, $J_{b_n} = B$ for all $n \in \mathbb{Z}$. Hence the ideal $\{0\}$ is maximal in B^0, but $\{0\}$ is not maximal in the semigroup B^0 (which is the null semigroup). Also $B^0/\{0\} = B^0$ and the semigroup B^0 has infinitely many non-zero proper ideals; none of these are ideals in S. Thus an ideal of an ideal in a semigroup S may not be an ideal in S.

2.8.2 Semisimple and completely semisimple semigroups

A semigroup S is said to be *semisimple* if its principal factors are either simple or 0-simple. Thus by Proposition 2.82, a semigroup is semisimple if and only if none of its principal factors are null.

The definitions show that simple and 0-simple semigroups are semisimple. The following proposition shows that the class of semisimple semigroups is quite large.

PROPOSITION 2.83. *Every regular semigroup is semisimple.*

Proof. Let S be a regular semigroup and $a \in S$. Then, by Lemma 2.38, J_a contains an idempotent, say, e. Hence $e \in J_a^2$ which implies that $\mathcal{F}_a(S)^2 \neq 0$. Hence, by Proposition 2.82, $\mathcal{F}_a(S)$ is simple or 0-simple. Thus S is semisimple. □

Example 2.25 shows that an ideal of an ideal in a semigroup S need not be an ideal in S. However, we have:

PROPOSITION 2.84. *An ideal of an ideal in a semisimple semigroup S is an ideal in S.*

Proof. Let I be an ideal in S and let A be an ideal in the subsemigroup I. Then clearly $IAI \subseteq A$. Let $b \in A - IAI$ Since S is semisimple, $\mathcal{F}_b(S)$ is simple or 0-simple; in either case, $\mathcal{F}_b(S)^3 = \mathcal{F}_b(S)$. Hence $J(b)^3 \cup I(b) = J(b)$. Now

$$J(b)^3 = S^1bS^1S^1bS^1S^1bS^1 \subseteq S^1bS^1bS^1S^1 = J(b)bJ(b).$$

Since $J(b) \subseteq I$ and $b \in A$, we have

$$J(b)^3 \subseteq J(b)bJ(b) \subseteq IAI.$$

Consequently,

$$J(b) = J(b)^3 \cup I(b) \subseteq IAI \cup I(b).$$

Since $b \notin IAI$ and $b \notin I(b)$, the above Equation implies that $b \notin J(b)$ which is a contradiction. Hence $A = IAI$ and so A is an ideal in S. □

semigroup!completely semisimple –
$\boldsymbol{M}^*_L$*: minimum condition on* $\mathscr{L}$*-classes in a* $\mathscr{J}$*-class*
$\boldsymbol{M}^*_R$*: minimum condition on* $\mathscr{R}$*-classes in a* $\mathscr{J}$*-class*

A semigroup is *completely semisimple* if its principal factors are completely simple or completely 0-simple.

Clearly a simple [0-simple] semigroup S is completely semisimple if and only if S is completely simple [0-simple]. This is an important class of semigroups; we proceed to obtain a number of equivalent characterizations of this class.

Recall (§ Subsection 2.6.1) that Λ [$\mathbf{I}, \mathbf{J}$] (or Λ_S, etc., if necessary) denote the partially ordered set set $S/\mathscr{L}$ of all $\mathscr{L}$-classes [respectively $S/\mathscr{R}$, $S/\mathscr{J}$].

We say that S satisfies the condition $\boldsymbol{M}^*_L$ [$\boldsymbol{M}^*_R$] if for every $a \in S$ the set of $\mathscr{L}$ [$\mathscr{R}$] classes contained in J_a has a minimal element with respect to the ordering in Λ_S [$\mathbf{I}_S$]. We first show that the condition $\boldsymbol{M}^*_L$ (dually $\boldsymbol{M}^*_R$) imply a stronger property.

LEMMA 2.85. *Let S be a semigroup and $a \in S$.*

A If $I(a) \neq \emptyset$, then the $\mathscr{L}$-class L_a is minimal in the set of $\mathscr{L}$-classes contained in J_a if and only if L^0_a is a 0-minimal left ideal of $S/I(a)$.

B If $I(a) = \emptyset$, then the $\mathscr{L}$-class L_a is minimal in the set of $\mathscr{L}$-classes in J_a if and only if L_a is a minimal left ideal of S.

Moreover, if J_a contains a minimal $\mathscr{L}$-class, then every $\mathscr{L}$-class in J_a is minimal.

Proof. Since $I(a)$ is the 0 of $S/I(a)$, by Lemma 2.50 $L^0_a = L_a \cup \{0\}$ is an ideal in $S/I(a)$ if and only if L^0_a is a 0-minimal left ideal in $S/I(a)$. If $\theta : S \to S/I(a)$ is the quotient map, by Theorem 2.5 (see also Remark 2.3) L^0_a is a left ideal in $S/I(a)$ if and only if

$$L_a \cup I(a) = (L^0_a)\theta^{-1}$$

is a left ideal in S. If $L_a \cup I(a)$ is a left ideal and L_c is a $\mathscr{L}$-class contained in J_a with $L_c \leq L_a$, then $c \in L_a \cup I(a)$. Since $c \notin I(a)$, we have $c \in L_a$ and so, $L_a = L_c$. Hence L_a is minimal in the set of all $\mathscr{L}$-classes in J_a. Conversely, suppose that L_a is minimal in the set of all $\mathscr{L}$-classes in J_a. Let $b \in L_a \cup I(a)$ and $s \in S$. If $b \in I(a)$, clearly $sb \in L_a \cup I(a)$ since $I(a)$ is an ideal. If $b \in L_a$ then $sb \in L(a) \subseteq J(a)$. So, if $sb \notin I(a)$, then $sb \in J_a$. Hence L_{sb} is a $\mathscr{L}$-class in J_a with $L_{sb} \leq L_a$ and so $L_{sb} = L_a$ by minimality. Hence $sb \in L_a$ which implies that $L_a \cup I(a)$ is a left ideal. This proves A. Proof of B is similar.

By Proposition 2.82, $J(a)/I(a)$ is a 0-minimal ideal in $S/I(a)$. If J_a contains a minimal $\mathscr{L}$-class L_a, then by A, L^0_a is a 0-minimal left ideal contained in $J(a)/I(a)$. If L_c is any $\mathscr{L}$-class in J_a, by Theorem 2.61, L_c is contained in a 0-minimal left ideal $L \subseteq J(a)/I(a)$. Since L_c consist of non-zero elements of L, by Lemma 2.50, $L = L^0_c$. Hence by A, L_c is minimal in J_a. Therefore every $\mathscr{L}$-class contained in J_a is minimal. □

M_E^*: minimum condition on idempotents in a $\mathscr{J}$-calss

We shall say that the semigroup S satisfies the condition $\boldsymbol{M}_E^*$ (minimum condition on idempotents in a $\mathscr{J}$-class) if for any $e, f \in \boldsymbol{E}(S)$

$$e\ \omega\ f \quad \text{and} \quad e\ \mathscr{J}\ f \Rightarrow e = f. \tag{2.53}$$

We now show that the relation $\mathscr{J}$ in the equation above can be replaced by $\mathscr{D}$.

LEMMA 2.86. *The semigroup S satisfies $\boldsymbol{M}_E^*$ if and only if for any $e, f \in \boldsymbol{E}(S)$,*

$$e\ \omega\ f \quad \textit{and} \quad e\ \mathscr{D}\ f \Rightarrow e = f. \tag{2.53*}$$

Proof. Since $\mathscr{D} \subseteq \mathscr{J}$, the condition $\boldsymbol{M}_E^*$ clearly implies Equation (2.53*). Conversely assume that Equation (2.53*) holds. Suppose that $e, f \in \boldsymbol{E}(S)$ with $e\ \mathscr{J}\ f$ and $e\ \omega\ f$. If $T = J(e)/I(a)$, then clearly $T^2 \neq 0$ and so, by Proposition 2.82, T is simple of 0-simple. Since $f \in J_e$, $f \neq 0$ in T. Hence by Proposition 2.69 there is an idempotent $g\ \omega\ f\ \omega\ e$ such that $g\ \mathscr{D}\ e$. Then by hypothesis, $g = e$ and so $f = e$. □

THEOREM 2.87. *The following conditions are equivalent for a semigroup S.*

(a) S is completely semisimple.

(b) S is regular and satisfies the condition $\boldsymbol{M}_E^*$.

(c) S is regular and satisfies the condition $\boldsymbol{M}_L^*$.

(d) S is regular and satisfies the condition $\boldsymbol{M}_R^*$.

(e) S is semisimple and satisfies both $\boldsymbol{M}_L^*$ and $\boldsymbol{M}_R^*$.

Proof. **(a) ⇒ (b)** Let $a \in S$. By (a), $\mathcal{F}_a(S)$ is completely 0-simple or completely simple. Hence $\mathcal{F}_a(S)$ is 0-bisimple or bisimple. In either case $\mathcal{F}_a(S)$ is regular and the set of non-zero elements is the $\mathscr{D}$-class D_a of S. Hence every $\mathscr{D}$-class is regular and so S is regular. Since every idempotent in D_a is primitive in $\mathcal{F}_a(S)$, S satisfies the condition $\boldsymbol{M}_E^*$.

(b) ⇒ (c) Let L_a and L_b be $\mathscr{L}$-classes in the same $\mathscr{J}$-class such that $L_b \leq L_a$. Since S is regular, by Proposition 2.39, we can find idempotents e and f with $L_a = Le$ and $L_b = L_f$. Since $f \in L(e)$, by Lemma 2.36, $fe = f$. Let $g = ef$. Then

$$g^2 = e(fe)f = ef = g, \qquad gf = g \qquad fg = fef = f.$$

Hence g is an idempotent with $L_g = L_f$, $g\ \mathscr{J}\ e$ and $g\ \omega\ e$. Hence by the condition $\boldsymbol{M}_E^*$, $g = e$ and so $L_f = L_e$. Hence S satisfies $\boldsymbol{M}_L^*$.

(c) ⇒ (e) Since S is regular, it is semisimple by Proposition 2.83. So it is enough to prove that S satisfies $\boldsymbol{M}_R^*$. Let $R_1 \le R_2$ where R_1 and R_2 are $\mathscr{R}$-classes in the same $\mathscr{J}$-class. Since S is regular, by Proposition 2.39, we may assume that $R_1 = R_e$ and $R_2 = R_f$ where $e, f \in \boldsymbol{E}(S)$. Since R_1 and R_2 are contained in the same $\mathscr{J}$-class, we have $e \mathscr{J} f$ and by Lemma 2.36, $fe = e$. If $g = ef$, as above, we find that g is an idempotent with $R_g = R_e$ and $g\ \omega\ f$. Then $L_g \le L_f$. By $\boldsymbol{M}_L^*$, we have $L_g = L_f$. So, $g \mathscr{L} f$ and $g\ \omega\ f$ and these imply that $g = f$. Hence $R_f = R_e$ and this proves $\boldsymbol{M}_R^*$.

The implications (b) ⇒ (d) ⇒ (e) can be proved dually.

(e) ⇒ (a) Let $T = \mathcal{F}_a(S)$. Then by hypothesis, T is simple or 0-simple. Since S satisfies $\boldsymbol{M}_L^*$ and $\boldsymbol{M}_R^*$, by Lemma 2.85 and its dual T contains 0-minimal left and right ideals. By Theorems 2.64 and 2.65, T is completely 0-simple or completely simple. Therefore S is completely semisimple. □

COROLLARY 2.88. *If the semigroup S is completely semisimple, then S is regular and $\mathscr{D}=\mathscr{J}$.*

Proof. Regularity of S follows from the theorem above. So, it is sufficient to show that $\mathscr{D}=\mathscr{J}$. Now the set of non-zero elements of $\mathcal{F}_a(S)$ is J_a. Since $\mathcal{F}_a(S)$ is either completely simple or completely 0-simple any two elements in J_a are $\mathscr{D}$-related in $\mathcal{F}_a(S)$ and hence in S. Thus $J_a = D_a$. □

The Example below shows that the converse of this Corollary is not true. The following gives some further class of semigroups that are completely semisimple.

THEOREM 2.89. *Let S be a semisimple, group bound semigroup. Then S is completely semisimple. In particular any semisimple periodic or semisimple finite semigroup is completely semisimple.*

Proof. If S is group bound any subsemigroup, in particular, any ideal in S is a group bound semigroup. Hence $J(a)$ is group bound for all $a \in S$. It follows that $\mathcal{F}_a(S)$ is a group bound semigroup. If S is also semisimple, then $\mathcal{F}_a(S)$ is a simple or 0-simple, group bound semigroup. Hence by Theorem 2.71 $\mathcal{F}_a(S)$ is completely simple or completely 0-simple for each $a \in S$. When S is periodic or finite, it is clearly group bound. □

***Remark* 2.17:** For a more extensive discussion of the minimal conditions of the set of left, right and two-sided ideals and some related concepts such as stable semigroups, elementary semigroups, etc., we refer the reader to Clifford and Preston [1967], § 6.

Example 2.26: If $B = \langle p, q : pq = 1\rangle$ is the bicyclic semigroup, then it is regular and satisfies the condition $\mathscr{D}{=}\mathscr{J}$; but is not completely semisimple. Similarly the semigroup A of Example 2.13 is simple and hence semisimple but not completely semisimple.

Example 2.27: Let $S = \mathscr{T}_X$ be the semigroup of transformations on the set X (see § Subsection 2.1.3 and Example 2.10). Then S is regular and $\mathscr{D}{=}\mathscr{J}$. If X is infinite, for any infinite subset Y of X, there is $Z \subseteq Y$ such that $|Z| = |Y|$. Then there are idempotents $e, f \in S$ such that $\operatorname{Im} e = Y$, $\operatorname{Im} f = Z$ and $f \;\omega\; e$. Since there is a bijection of Y onto Z, by Example 2.10, $e \;\mathscr{D}\; f$. Hence S is not completely semisimple if X is infinite. If X is finite, then clearly S is a finite regular semigroup and by Theorem 2.89, S is completely semisimple. In a similar way, it can be shown that the semigroup $\mathscr{L}\mathscr{T}(V)$ (cf. § Subsection 2.1.3) of linear transformations on a vector space V is a regular semigroup which satisfies the condition $\mathscr{D}{=}\mathscr{J}$ (see Example 2.11). Also, $\mathscr{L}\mathscr{T}(V)$ is completely semisimple if and only if V is finite dimensional.

Example 2.28 (Baer-Levi semigroups): Let p, q be infinite cardinals such that $p \geq q$ and let X be a set with $|X| = p$. Consider the set S of all one-to-one mappings $\alpha : X \to X$ such that $|X - X\alpha| = q$. If $\alpha, \beta \in S$, then

$$X - X\alpha\beta = (X - X\beta) \cup (X\beta - X\alpha\beta).$$

Since β is one-to-one,

$$|X - X\alpha| = |(X - X\alpha)\beta| = |X\beta - X\alpha\beta| = q.$$

Since $X - X\beta$ and $X\beta - X\alpha\beta$ are disjoint and have the same infinite cardinal q, $(X - X\beta) \cup (X\beta - X\alpha\beta)$ has cardinal q. Hence $|X - X\alpha\beta| = q$. Thus $\alpha\beta \in S$ and so, S is a semigroup under composition (that is, a subsemigroup of $\mathscr{T}_X$); clearly, S does not have 0.

Since S is a semigroup of one-to-one mappings, it is right cancellative. We now show that S is right simple. Accordingly let $\alpha, \beta \in S$. Then $|X - X\alpha| = |X - X\beta| = q$. Since q is infinite, we can find a subset Y of $X - X\beta$ such that Y and its complement in $X - X\beta$ has the same cardinal q. Let $\delta : X - X\alpha \to Y$ be a bijection. Now define

$$x\gamma = \begin{cases} (x\alpha^{-1})\beta & \text{if } x \in X\alpha; \\ x\delta & \text{if } x \in X - X\alpha. \end{cases}$$

Then γ is one-to-one and $X\gamma = X\beta \cup Y$. Hence, by the choice of Y, we have

$$|X - X\gamma| = |X - X\beta - Y| = q.$$

Thus $\gamma \in S$ and clearly, $\alpha\gamma = \beta$. Therefore S is right simple. If $\alpha \in S$, we have $|X - X\alpha| = |X\alpha - X\alpha^2|$. Since q is infinite, $\alpha \neq \alpha^2$ and so, S does not contain idempotents; in particular, S is not regular.

Since S is right simple, it is semisimple and satisfies the condition $\boldsymbol{M}_R^*$. If S satisfies $\boldsymbol{M}_L^*$, then by Theorem 2.87(e), S is completely semisimple and hence regular. This is not possible. Hence S does not satisfy the condition $\boldsymbol{M}_L^*$. It is not difficult to verify directly that S contains an infinite descending chain of $\mathscr{L}$-classes which will also show that it does not satisfy $\boldsymbol{M}_L^*$. The dual construction will give a semigroup which is semisimple, satisfy $\boldsymbol{M}_L^*$, but not $\boldsymbol{M}_R^*$. This shows that the conditions $\boldsymbol{M}_L^*$ and $\boldsymbol{M}_R^*$ are independent.

Example 2.29: Let

$$S = \{(i, j) : 1 \leq i, j \leq \infty, \quad i < j\} \cup \{0\}.$$

translation!linked pair
bitranslation
bitranslations!inner –

Define a product in S by the rule

$$(i, j)(r, s) = \begin{cases} (i, s) & \text{if } j = r; \\ 0 & \text{if } i \neq j; \end{cases}$$

and for all $x \in S$, let

$$0x = 0 = x0.$$

S with this product is a semigroup. Also, for all $(i, j) \in S$, using the definition of product in S, we see that the principal left, right and two-sided ideals and Green's classes are given by

$$\begin{aligned} L(i, j) &= \{0\} \cup \{(r, j) : 1 \leq r \leq i\}, & L_{(i,j)} &= \{(i, j)\}; \\ R(i, j) &= \{0\} \cup \{(i, s) : s \geq j\}; & R_{(i,j)} &= \{(i, j)\}; \\ J(i, j) &= \{0\} \cup \{(r, s) : 1 \leq r \leq i,\ s \geq j\} & J_{(i,j)} &= \{(i, j)\}. \end{aligned}$$

It follows that S satisfies both $\boldsymbol{M}_L^*$ and $\boldsymbol{M}_R^*$. Now $J(i, j)^2 = 0$ and so $\mathcal{F}_{(i,j)}(S)$ is null for all non-zero elements of S. Hence S is not semisimple.

2.9 SOME SPECIAL REPRESENTATIONS OF SEMIGROUPS

In Section 2.5 we had given a general discussion about representations of semigroups. In particular Subsection 2.5.1 discusses representations of semigroups by functions on a set. Given any semigroup S, the right S-system Subsection 2.5.2 S_r affords the specific representation $\boldsymbol{\rho}$ by functions on the set S and the left S-system S_l affords the dual representation $\boldsymbol{\lambda}$. Here we discuss a few such representations by pairs of transformations, partial transformations, matrices over groups with 0, etc. that have proved to be of importance in the structure theory of various classes of semigroups.

2.9.1 Representation by pairs of linked translations

Given a right translation ρ and a left translation λ of a semigroup S, we say that (ρ, λ) is a pair of *linked* translations or that ρ is linked to λ if for all $s, t \in S$,

$$(s\rho)t = s(\lambda t). \tag{2.54}$$

A linked pair of translations is also called a *bitranslation*. For each $a \in S$, (ρ_a, λ_a) is clearly a linked pair and these are called *inner bitranslations*. We can define a right and left action of a bitranslation $\beta = (\rho, \lambda)$ on S as follows: for $s \in S$

$$s\beta = s\rho, \quad \text{and} \quad \beta s = \lambda s.$$

Thus β acts on the right of S as a right translation and on the left as a left translation.

Combining the right regular representation $\boldsymbol{\rho}_S$ and the left regular representation $\boldsymbol{\lambda}_S$, we can obtain a new representation of S by bitranslations. We have the following:

THEOREM 2.90. *Let S be a semigroup and define*

$$\Omega(S) = \{(\rho, \lambda) : \rho \quad \text{is linked to} \quad \lambda\}.$$

Then $\Omega(S)$ is a semigroup with multiplication defined by

$$(\rho, \lambda)(\rho', \lambda') = (\rho\rho', \lambda\lambda').$$

Moreover, for any $a \in S$ and $(\rho, \lambda) \in \Omega(S)$,

$$(\rho, \lambda)(\rho_a, \lambda_a) = (\rho_{\lambda a}, \lambda_{\lambda a}) \quad \text{and} \quad (\rho_a, \lambda_a)(\rho, \lambda) = (\rho_{a\rho}, \lambda_{a\rho}).$$

Consequently the map $\boldsymbol{\pi}_S = \boldsymbol{\pi}$, defined for all $s \in S$, by

$$s\boldsymbol{\pi} = (\rho_s, \lambda_s)$$

is a homomorphism of S onto an ideal of $\Omega(S)$.

Proof. First observe that for $(\rho, \lambda), (\rho', \lambda') \in \Omega(S)$,

$$\begin{aligned}(s)\rho\rho' t &= (s\rho)\rho' t \\ &= (s\rho)(\lambda' t) \ \rho' \text{ is linked to } \lambda'; \\ &= s(\lambda(\lambda' t)) \ \rho \text{ is linked to } \lambda; \\ &= s(\lambda\lambda' t).\end{aligned}$$

Hence $\rho\rho'$ is linked to $\lambda\lambda'$. Since the binary operation defined in the statement is obviousely associative, $\Omega(S)$ is a semigroup. For any $a \in S$ it is clear that

$$a\boldsymbol{\pi} = (\rho_a, \lambda_a) \in \Omega(S)$$

and the map $\boldsymbol{\pi} : S \to \Omega(s)$ defined above is a homomorphism. If $(\rho, \lambda) \in \Omega(S)$ and $s \in S$, we have

$$\begin{aligned}s\rho\rho_a &= (s\rho)a = s(\lambda a) = s\rho_{\lambda a}; \\ \lambda\lambda_a s &= \lambda(as) = \lambda(a)s = \lambda_{\lambda a}s; \\ s\rho_a\rho &= (sa)\rho = s(a\rho) = s\rho_{a\rho}; \\ \lambda_a\lambda s &= a(\lambda s) = (a\rho)s = \lambda_{a\rho}s.\end{aligned}$$

Hence

$$\begin{aligned}(\rho, \lambda)(\rho_a, \lambda_a) &= (\rho_{\lambda a}, \lambda_{\lambda a}); \\ (\rho_a, \lambda_a)(\rho, \lambda) &= (\rho_{a\rho}, \lambda_{a\rho}).\end{aligned}$$

Therefore

$$\operatorname{Im} \boldsymbol{\pi} = \{a\boldsymbol{\pi} : a \in S\}$$

is an ideal of the semigroup $\Omega(S)$. □

The semigroup $\Omega(S)$ is called the *translational hull* of the semigroup S and the homomorphism $\pi_S = \pi$ is called the *regular representation*of S by linked translations. S is said to be *weakly reductive* if the representation π is faithful; that is, if and only if S satisfies the condition

semigroup!translational hull of –
$\Omega(S)$:translational hull of S
π_S: representation of S by bitranslations
representation!regular –
semigroup!weakly reductive

$$sa = sb \quad \text{and} \quad as = bs \quad \text{for all} \quad s \in S \;\Rightarrow\; a = b.$$

Notice that any right or left reductive semigroup is weakly reductive. The following observation implies that the class of weakly reductive semigroups is quite large.

THEOREM 2.91. *Every regular semigroup is weakly reductive.*

Proof. Suppose that S is a regular semigroup and $a\pi = b\pi$ for $a, b \in S$. Then $\rho_a = \rho_b$ which implies in particular that $L(a) = L(b)$. Similarly from $\lambda_a = \lambda_b$, we have $R(a) = R(b)$ and so, by Equations (2.36a), (2.36b) and (2.37c), $a \mathscr{H} b$. Since a is a regular element, by Proposition 2.39 there is an idempotent $e \in R_a$ which by Lemma 2.36, is a left identity of $R_a = R_b$. Therefore,

$$a = e\rho_a = e\rho_b = b.$$

This proves that the representation π is faithful. □

The representation by bitranslations affords a representation by pairs of mappings. Several existing structure theorems for classes of semigroups uses this directly or related representations especially when the semigroup under consideration is weakly reductive. The theorem above suggests that the cost of this assumption is comparatively small.

Theorem 2.90 also shows that π is a representation having some special properties. When S is weakly reductive, it provides an embedding of S as an ideal of its translational hull $\Omega(S)$. We will use this fact in the next section to construct ideal extensions of weakly reductive semigroups.

2.9.2 Lallement's representation

Here we consider a special representation of semigroups by partial tramsformations due to G. Lallement Lallement [1967]. It is shown in Nambooripad and Sitaraman [1979] that various known representations for special classes of semigroups are particular cases of this representation and that it is closely related to the ideal structure of the semigroup.

We begin with a representation closely related to Lallements representation which is also of independent interest.

ϱ^D :Representation of S by partial transformations on the $\mathscr{D}$-class D
ϱ^D :representation of S by partial transformations on the $\mathscr{D}$-class D
representation!partial –
representation!partial dual –
$\boldsymbol{\lambda}^D$: anti-representation of S by partial transformations on D
$\boldsymbol{\varpi}^D$:partial symmetric representation of S on D

Proposition 2.92. *Let D be a $\mathscr{D}$-class of a semigroup S. For each $a \in S$ let*

$$\boldsymbol{D}\varrho_a^D = \{x \in D : x \mathscr{R} xa\} \quad \text{and} \quad \varrho_a^D = \rho_a \mid \boldsymbol{D}\varrho_a^D.$$

Then $\varrho_a^D : \boldsymbol{D}\varrho_a^D \to L(a) \in \mathscr{P}\mathscr{T}_D$ and the map

$$\varrho^D : a \mapsto \varrho_a^D; S \to \mathscr{P}\mathscr{T}_D$$

is a representation of S by partial transformations on D.

Proof. Clearly $\varrho_a^D \in \mathscr{P}\mathscr{T}_D$ for all $a \in S$. For $a, b \in S$ let $x \in A = \operatorname{dom}\left(\varrho_a^D \circ \varrho_b^D\right)$. Then $x \in \boldsymbol{D}\varrho_a^D$ and $xa \in \boldsymbol{D}\varrho_b^D$ and so,

$$x \mathscr{R} xa \mathscr{R} xab$$

which implies that $x \in \boldsymbol{D}\varrho_{ab}^D$. Conversely, if $x \in \boldsymbol{D}\varrho_{ab}^D$, then $x \mathscr{R} xab$ implies, by Theorem 2.26, that $x = xabs$ for some $s \in S^1$. If $t = bs$ then $x = xat$ and so, $x \mathscr{R} xa$. Therefore

$$x \mathscr{R} xa \mathscr{R} xab.$$

Thus $A = \boldsymbol{D}\varrho_{ab}^D$ and for any $x \in \boldsymbol{D}\varrho_{ab}^D$,

$$x\left(\varrho_a^D \circ \varrho_b^D\right) = xab = x\varrho_{ab}^D.$$

Therefore the map $\varrho^D : a \mapsto \varrho_a^D$ is a representation by partial transformations. □

The representation ϱ^D is called the *partial representation* of S on D. Its left-right dual, called *partial anti-representation*, is the homomorphism $\boldsymbol{\lambda}^D : a \mapsto \boldsymbol{\lambda}_a^D$ of S to $\mathscr{P}\mathscr{T}_D^{\text{op}}$ where for each $a \in S$,

$$\boldsymbol{D\lambda}_a^D = \{x \in D : x \mathscr{L} ax\} \quad \text{and} \quad \boldsymbol{\lambda}_a^D = \lambda_a | \boldsymbol{D\lambda}_a^D.$$

Combining these we can get another representation of S as follows:

Corollary 2.93. *For each $a \in S$ let*

$$a\boldsymbol{\varpi}^D = (\varrho_a^D, \boldsymbol{\lambda}_a^D).$$

Then $\boldsymbol{\varpi}^D : S \to \mathscr{P}\mathscr{T}_D \times \mathscr{P}\mathscr{T}_D^{op}$ is a representation of S. Moreover, if D is a regular $\mathscr{D}$-class, then $\boldsymbol{\varpi}^D$ is injective on D.

Proof. The fact that $\boldsymbol{\varpi}^D$ is a representation as claimed, follows readily from Proposition 2.92 and its dual. Th show that $\boldsymbol{\varpi}^D$ is injective on D when D is regular, suppose that $a\boldsymbol{\varpi}^D = b\boldsymbol{\varpi}^D$ for $a, b \in D$. Since D is regular, by

Propositions 2.39 and 2.40, there is an inverse $a' \mathfrak{n} \mathscr{V}(a)$ such that $e = a'a$ and $f = aa'$ are idempotents with

representation!partial symmetric – on D

$$e \mathscr{L} a \mathscr{R} f \mathscr{L} a' \mathscr{R} e.$$

Then $e\varrho_a^D = ea = a$ and since $\varrho_a^D = \varrho_b^D$, $eb = a$. Hence $a \in L(b)$. Similarly $b \in L(a)$ and hence $a \mathscr{L} b$. Similarly, considering the representation $\boldsymbol{\lambda}^D$ we get $a \mathscr{R} b$ dually. Hence $a \mathscr{H} b$. Therefore

$$a = e\varrho_a^D = e\varrho_b^D = b$$

since $e \in R_a = R_b$ and so, by Lemma 2.36, e is a left identity of both a and b. □

ϖ^D is called the *partial symmetric representation* of S on D.

The following result, obtained by considering Proposition 2.92 above for all $\mathscr{D}$-classes simultaneousely, is essentially due to G. Lallement Lallement [1967].

THEOREM 2.94. *Let S be a semigroup and for each $a \in S$, let*

$$\boldsymbol{D}\varrho_a = \bigcup_{D \in D/\mathscr{D}} \boldsymbol{D}\varrho_a^D \quad \text{and} \quad \varrho_a = \bigcup_{D \in D/\mathscr{D}} \varrho_a^D.$$

Then $\boldsymbol{D}\varrho_a$ is a left ideal such that

$$\boldsymbol{D}\varrho_a = \{x \in S : x \mathscr{R} xa\}.$$

Moreover $\varrho_a : \boldsymbol{D}\varrho_a \to L(a)$ is a morphism of ideals and $\varrho : a \mapsto \varrho_a$ is a representation of S by partial transformations on S.

Proof. First notice that, since $S/\mathscr{D}$ is a partetion of S, from the definition of $\boldsymbol{D}\varrho_a^D$ in Proposition 2.92 we see that $\boldsymbol{D}\varrho_a = \{x \in S : x \mathscr{R} xa\}$. Let $x \in \boldsymbol{D}\varrho_a$. If $y \in L(x)$, then $y = tx$ for some $t \in S^1$. If D is the $\mathscr{D}$-class of x then by the definition of $\boldsymbol{D}\varrho_a$, $x \in \boldsymbol{D}\varrho_a^D$ and so $x \mathscr{R} xa$ by Proposition 2.92. Hence $x = xas$ for some $s \in S^1$. Then

$$y = tx = txas = yas$$

and so, $y \in \boldsymbol{D}\varrho_a$. Hence $L(x) \subseteq \boldsymbol{D}\varrho_a$ for all $x \in \boldsymbol{D}\varrho_a$ which shows that $\boldsymbol{D}\varrho_a$ is a left ideal. It is clear from the definition of ϱ_a that

$$\varrho_a = \rho_a \mid \boldsymbol{D}\varrho_a$$

and so, ϱ_a is a morphism of left ideals.

To show that ϱ is a representation, consider $a, b \in S$. For any $x \in S$, let $D = D_x$ be the $\mathscr{D}$-class containing x. Then

$$x\varrho_{ab} = x\varrho_{ab}^D = x\left(\varrho_a^D \circ \varrho_b^D\right) = x\left(\varrho_a \circ \varrho_b\right)$$

by Proposition 2.92. Hence $\varrho_{ab} = \varrho_a \circ \varrho_b$. Therefore ϱ is a representation in $\mathscr{P}\mathscr{T}_S$. □

isodomain
$\boldsymbol{D}\varrho_a$*:isodomain of* ρ_a
ϱ_a*:partial right translation by a*
partial right translation
ϱ*:representation by partial right translations*
$\boldsymbol{D}\boldsymbol{\lambda}_a$*:isodomain of* λ_a
$\boldsymbol{\lambda}_a$*:partial left translation by a*
partial left translation
$\boldsymbol{\lambda}$*:representation by partial left translations*
ϖ*:partial symmetric representation of on S*
representation!partial symmetric –

The ideal $\boldsymbol{D}\varrho_a$ has the important property that restriction of the translation ρ_a (or the partial translation ϱ_a) to any $L(x)$, $x \in \boldsymbol{D}\varrho_a$ induces an isomorphism of $L(x)$ onto $L(xa)$. In fact $\boldsymbol{D}\varrho_a$ is the union of all principal left ideals with this property. We shall call $\boldsymbol{D}\varrho_a$ as the *isodomain* of ρ_a and ϱ_a as *partial right translation* by a. The representation ϱ is called the representation by partial right translations

Again, the left-right dual of ϱ is a representation $\boldsymbol{\lambda} : S \to \mathscr{P}\mathscr{T}_D^{\mathrm{op}}$ (or an anti-representation in $\mathscr{P}\mathscr{T}_S$) where each $\boldsymbol{\lambda}_a : \boldsymbol{D}\boldsymbol{\lambda}_a \to R(a)$ is a morphism of right ideals. Here $\boldsymbol{D}\boldsymbol{\lambda}_a$ is the *isodomain* of λ_a which is the left-right dual of $\boldsymbol{D}\varrho_a$ given by

$$\boldsymbol{D}\boldsymbol{\lambda}_a = \{x \in S : x \mathscr{L} ax\} \quad \text{and} \quad \boldsymbol{\lambda}_a = \lambda_a \mid \boldsymbol{D}\boldsymbol{\lambda}_a.$$

For any $a \in S$ $\boldsymbol{\lambda}_a$ is called the *partial left translation* by a. The representation $\boldsymbol{\lambda}$ is called the representation by partial left translations

We may combine the representations ϱ and $\boldsymbol{\lambda}$ to get a new representation of S in $\mathscr{P}\mathscr{T}_S \times \mathscr{P}\mathscr{T}_S^{\mathrm{op}}$. As a consequence of Theorem 2.94 and its dual, we have:

COROLLARY 2.95. *Let S be a semigroup. For each $a \in S$ let*

$$a\varpi = \left(\varrho_a, \boldsymbol{\lambda}_a\right).$$

Then $\varpi : S \to \mathscr{P}\mathscr{T}_S \times \mathscr{P}\mathscr{T}_S^{op}$ is a representation. □

ϖ is called the *partial symmetric representation* of S. Assume that S is regular. Then for any $a \in S$, by Proposition 2.39, there is an idempotent $e \in S$ with $e \mathscr{R} a$ and e is a left identity of a. Hence Hence $L(e) \subseteq \boldsymbol{D}\varrho_a$ and ϱ_a is an isomorphism of $L(e)$ onto $L(a)$. In particular ϱ_a is surjective from $\boldsymbol{D}\varrho_a$ onto $L(a)$. Similarly $\boldsymbol{\lambda}_a$ is surjective from $\boldsymbol{D}\boldsymbol{\lambda}_a$ onto $R(a)$. Now let $a\varpi = b\varpi$. Then $\varrho_a = \varrho_b$ and $\boldsymbol{\lambda}_a = \boldsymbol{\lambda}_b$. In particular $L(a) = L(b)$ and $R(a) = R(b)$ which implies that $a \mathscr{H} b$. If e is an idempotent in R_a, then we have

$$a = ea = e\varrho_a = e\varrho_b = eb = b.$$

Therefore ϖ is faithful.

COROLLARY 2.96. *If S is a regular semigroup, then the representation ϖ of S is faithful.*

Recall Subsection 2.6.2 that a semigroup is an inverse semigroup if every element of S has a unique inverse. See Theorem 2.44 for various equivalent characterizations of inverase semigroups. In particular, every inverse semigroup is a regular semigroup.

semigroup!weakly inverse –

THEOREM 2.97. *If S is an inverse semigroup, then*

$$\boldsymbol{D}\varrho_a = L(aa^{-1}) = L(e_a) \quad and$$

and $\varrho_a : L(e_a) \to L(f_a)$ *is a is a one-to-one partial transformation. Thus* ϱ *is a faithful representation of S by one-to-one partial transformations of S. Similarly representations* $\boldsymbol{\lambda}$ *and* $\boldsymbol{\varpi}$ *are also faithful.*

Proof. Since S is inverse, by Theorem 2.44, it is regular and by |cor 2.96,the representation ϖ is faithful. To prove that ϱ is faithful, let $\varrho_a = \varrho_b$. By Theorem 2.44, R_a contains a unique idempotent e (say). Then

$$e \in \boldsymbol{D}\varrho_a = \boldsymbol{D}\varrho_b \quad \text{and so,} \quad eb = ea = a \in R_e \cup L_b.$$

By Theorem 2.34, $L_e \cup R_b$ contains an idempotent f. Then $e \mathscr{L} f$ and by Theorem 2.44, $e = f$. Therefore $a \mathscr{R} b$ and so, $a \mathscr{H} b$. Then $a = ea = eb = b$. Therefore ϱ is faithful. Dually $\boldsymbol{\lambda}$ is faithful.

Finally we show that ϱ_a is one-to-one for every $a \in S$. If e is the idempotent in R_a, it is clear that $e \in \boldsymbol{D}\varrho_a$ and $\varrho_a \mid L(e) = \rho_a \mid L(e)$ is an isomorphism of $L(e)$ onto $L(a)$. Suppose that $x \in \boldsymbol{D}\varrho_a$ and f be an idempotent in L_x. Since $s\varrho_a = sa = (se)a$ for all $s \in \boldsymbol{D}\varrho_a$, it follows from the definition of $\boldsymbol{D}\varrho_a$ that $f \mathscr{R} fe$. By Theorem 2.44 fe is an idempotent in R_e and so, again by Theorem 2.44, $f = fe$. Hence $f \in L(e)$ and so, $L(x) \subseteq L(e)$. Hence $L(e) = \boldsymbol{D}\varrho_a$ and so, $\varrho_a = \rho_a \mid L(e)$ which is an isomorphism of $L(e)$ onto $L(a)$. This proves that ϱ_a is a ono-to-one partial transformation of S. □

The representation ϱ for inverse semigroups is known as *Vager-Preston* representation and it was first studied by Vagner Vagner [1953a] and independently by Preston Preston [1954b]. B. R. Srinivasan introduced an studied a class of regular semigroup called *weakly inverse semigroups* **?** which properly contains the class of inverse semigroups and for which the representation ϱ is faithful. This representation need not be faithful for arbitrary regular semigroups. For, let $S = B^1$ where B is an $n \times n$ rectangular band ($n > 1$) and S is obtained by adjoining identity to B. Then the regular representation of S is faithful but the representation ϱ is not faithful. Notice that B is a regular semigroup for which neither the regular representation nor the Lallement representation is faithful.

2.9.3 Schutzenberger representations

Here we shall discuss some representations by matrices over a group with 0 G^0 (see § Subsection 2.1.3 and § Subsection 2.7.3 for relavent definitions).

matrix!row-monomial
matrix!column-monomial –
$M_{row\text{-}mon}$:semigroup of row-monomial matrices

Suppose that $\{g_i : i \in I\}$ is an indexed subset of G^0 indexed by an arbitrary set I. For convenience, we shall write

$$\sum_{i \in I} g_i = \begin{cases} 0 & \text{if } g_i = 0 \text{ for all } i; \\ g_k & \text{if } g_i = 0 \text{ for all } i \text{ with } i \neq k \text{ and} \\ undefined & \text{if there exist } k, l \in I, k \neq l \text{ such that } g_k \neq 0 \text{ and } g_l \neq 0. \end{cases} \tag{$*$}$$

Recall that, for any set I, an $I \times I$-matrix over G^0 is a map $I \times I \to G^0$. Suppose that $m = (g_{ij})$ and $m' = (h_{kl})$ are two $I \times I$ matrices over G^0. The usual (row-column) product of these matrices is

$$(g_{ij})(h_{kl}) = (c_{il}) \quad \text{where} \quad c_{il} = \sum_{j \in I} g_{ij} h_{jl}$$

if the sum is meaningful. Unless an additive structure exists on G^0, the sum should be interpreted as in $(*)$. Hence the product exists if and only if, for each $i, l \in I$, there is exactly one j with $c_{il} = g_{ij} h_{jl}$. This will hold if either every row of m contains exactly one non-zero entry or every column of m' contain exactly one non-zero entry. Thus the product mm' exists if either m is *row-monomial* or if m' is *column-monomial*. If m is row-monomial, so is mm' for any matrix m' and mm' is column-monomial, if m' has this property. Thus the set $M_{\text{row-mon}}$ of all rowmonomial matrices is a semigroup under matrix multiplication abow and similarly, we have the semigroup $M_{\text{col-mon}}$ of all column-monomial matrices. Notice that the set of all monomial matrices § Subsection 2.7.3 is a common subsemigroup of these.

We have discusses representation of completely 0-simple semigroups by monomial matrices in § Subsection 2.7.3. Here we discuss some representations by row-monomial and column-monomial matrices over a group with 0.

Let D be a $\mathscr{D}$-class of S and let $H \subseteq D$ be an $\mathscr{H}$-class contained in D. Let

$$D/\mathscr{R} = \{R_i : i \in I_D\} \quad \text{and} \quad D/\mathscr{L} = \{L_\lambda : \lambda \in \Lambda_D\}$$

denote the set of $\mathscr{R}$-classes and $\mathscr{L}$-classes contained in D respectively. For each $i \in I_D = I$, we denote by $R(i)$ the principal right ideal generated by L_i; similarly $L(\alpha)$ denote the principal left ideal generated by L_λ, $\lambda \in \Lambda_D = \Lambda$. Also, we write $R = R(H)$ and $L = L(H)$. Recall Proposition 2.46 that the automorphism groups of L and of R are isomorphic to the Schützenberger group $\mathfrak{g}(H)$ of the $\mathscr{H}$-class of H. It will be convenient in the sequel to identify these groups. Thus an automorphism $\alpha \in \text{Aut}(L)$ $[\alpha \in \text{Aut}(R)]$ will be identifies with the unique element $\theta \in \mathfrak{g}(H)$ such that $a\alpha = a\theta$ $[\alpha(a) = \theta a]$ for all $a \in H$.

Now if $a, b \in D$, by Proposition 2.28 there is $c \in D$ with $a \mathscr{R} c \mathscr{L} b$ and by Theorem 2.26, there is a unique isomorphism $\sigma : L(a) \to L(b)$ in $\mathbb{L}(S)$ with

$a\sigma = c$. Since σ is an isomorphism in $\mathbb{L}(S)$, there is $s, s' \in S^1$ such that $\sigma = \rho_s \mid L(a)$ and $\sigma^{-1} = \rho_{s'}$. It follows that for each $\lambda \in \Lambda$ we can choose $s_\lambda, s'_\lambda \in S^1$ such that

$$\gamma_\lambda = \rho_{s_\lambda} \mid L \quad \text{is an isomorphism onto } L(\lambda) \text{ and} \quad \rho_{s'_\lambda} \mid L(\lambda) = \gamma_\lambda^{-1}.$$

Suppose that $a \in S$. For any $\lambda \in \Lambda$, $\rho_a \mid L(\lambda)$ is an isomorphism onto $L(\mu)$ for some $\mu \in \Lambda$ if and only if

$$x \mathscr{R}\, xa \in L_\mu \quad \text{for all} \quad x \in L_\lambda.$$

If this is true then

$$h_{\lambda\mu} = \gamma_\lambda \left(\rho_a \mid L(\lambda)\right) \gamma_\mu^{-1} \tag{2.55a}$$

is an automorphism of $L = L(H)$ and so corresponds to a unique element in $\mathfrak{g}(H)$. Now let

$$\mathfrak{m}_{\lambda,\mu}(a) = \begin{cases} h_{\lambda\mu} & \text{if } \rho_a \mid L(\lambda) : L(\lambda) \to L(\mu) \text{ is an isomorphism} \\ 0 & \text{otherwise.} \end{cases} \tag{2.55b}$$

When $\mathfrak{m}_{\lambda,\mu} \neq 0$, it is an automorphism of $L(H) = L$ and so, can be taken to be an element of the Schützenberger group $\mathfrak{g}(H)$ of the $\mathscr{H}$-class H. In either case $\mathfrak{m}_{\lambda,\mu}$ represents a unique element in the group with zero $(\mathfrak{g}(H))^0$. Note that for any $\lambda \in \Lambda$, there exist utmost one $\mu \in \Lambda$ for which $\mathfrak{m}_{\lambda,\mu} \neq 0$. It follows that

$$\mathsf{M}_D(a) = \mathsf{M}(a) = \left(\mathfrak{m}_{\lambda,\mu}(a)\right) \tag{2.55c}$$

is a row monomial $I \times I$-matrix over $(\mathfrak{g}(H))^0$ where $I = I_D$. If $a, b \in S$, by the definition of the product,

$$\mathsf{M}(a)\mathsf{M}(b) = (p_{\lambda\nu}) \quad \text{where} \quad p_{\lambda\nu} = \sum_{\mu\in\Lambda} \mathfrak{m}_{\lambda\mu}(a)\mathfrak{m}_{\mu\nu}(b)$$

By Equation (2.55b), $p_{\lambda\nu} \neq 0$ if and only if there exists a uuique $\eta \in \Lambda$ such that

$$p_{\lambda\nu} = \mathfrak{m}_{\lambda\eta}(a)\mathfrak{m}_{\eta\nu}(b)$$

and so,$p_{\lambda\mu} \neq 0$ if and only if

$$\mathfrak{m}_{\lambda\eta}(a) \neq 0 \quad \text{and} \quad \mathfrak{m}_{\eta\nu}(b) \neq 0.$$

Therefore $x \mathscr{R}\, xa \mathscr{R}\, xab$ for all $x \in L_\lambda$. Hence $x \mathscr{R}\, xab$ and so $\mathfrak{m}_{\lambda\nu}(ab) \neq 0$. Also the automorphism of L corresponding to $p_{\lambda\nu} = \mathfrak{g}(a)_{\lambda\eta}\mathfrak{g}(b)_{\eta\nu}$ in $\mathfrak{g}(H)$ is

$$\begin{aligned} p_{\lambda\nu} &= \mathfrak{m}_{\lambda\eta}(a)\mathfrak{m}_{\eta\nu}(b) \\ &= \gamma_\lambda \left(\rho_a \mid L(\lambda)\right) \gamma_\eta^{-1} \gamma_\eta \left(\rho_b \mid L(\eta)\right) \gamma_\nu^{-1} \\ &= \gamma_\lambda \left(\rho_a\rho_b \mid L(\lambda)\right) \gamma_\nu \\ &= \gamma_\lambda \left(\rho_{ab} \mid L(\lambda)\right) \gamma_\nu \\ &= \mathfrak{m}_{\lambda\nu}(ab). \end{aligned}$$

Therefore $p_{\lambda\nu} = \mathfrak{m}_{\lambda\nu}(ab)$. Conversely, if $\mathfrak{m}_{\lambda\nu}(ab) \neq 0$, $x \mathscr{R} xab$ for all $x \in L_\lambda$ and as in the proof of Proposition 2.92, we have $x \mathscr{R} xa \mathscr{R} xab$. This implies that

$$\mathfrak{m}_{\lambda\eta}(a) \neq 0 \quad \text{and} \quad \mathfrak{m}_{\eta\nu}(b) \neq 0.$$

Thus $p_{\lambda\nu} \neq 0$. Consequently

$$\mathsf{M}(a)\mathsf{M}(b) = \mathsf{M}(ab).$$

Dually, for each $i \in I_D = I$, we can choose an isomorphism $\delta_i : R \to R(i)$ and for each $a \in S$, define

$$\mathsf{M}'(a) = \left(\mathfrak{m}'_{\lambda,\mu}(a)\right) \tag{2.55c*}$$

where

$$\mathfrak{m}'_{\lambda,\mu}(a) = \begin{cases} \delta_i\,(\lambda_a \mid R(i))\,\delta_j^{-1} & \text{if } \lambda_a \mid R(i) : R(i) \to R(j) \text{ is an isomorphism} \\ 0 & \text{otherwise.} \end{cases} \tag{2.55b*}$$

Then it can be verified, using Proposition 2.46 and Theorem 2.47 that $\mathsf{M}' : a \mapsto \mathsf{M}'(a)$ is a dual (anti) representatyion of S by row-monomial matrices over the Schützenberger group with zero $(\mathfrak{g}(H))^0$. If we set

$$\mathsf{M}^*_D(a) = (\mathsf{M}'(a))^t \tag{2.55d}$$

as the transpose of the matrix $\mathsf{M}'(a)$, then $\mathsf{M}^*(a)$ is column-monomial and we have

$$\begin{aligned} \mathsf{M}^*(a)\mathsf{M}^*(b) &= (\mathsf{M}'(a))^t\,(\mathsf{M}'(b))^t \\ &= (\mathsf{M}'(b)\mathsf{M}'(a))^t \\ &= (\mathsf{M}'(ab))^t = \mathsf{M}^*(ab). \end{aligned}$$

Therefore M^*_D is a representation of S by column-monomial $I_D \times I_D$-matrices over $(\mathfrak{g}(H))^0$. We use the notations introduced above in the following statement.

THEOREM 2.98. *Let D be a $\mathscr{D}$-class of a semigroup S and let H be an $\mathscr{H}$-class contained in D. For each $a \in S$, let $\mathsf{M}_D(a)$ be defined by Equation* (2.55c). *Then the map*

$$\mathsf{M}_D : a \mapsto \mathsf{M}_D(a)$$

is a representation of S by row-monomial $I_D \times I_D$-matrix over $(\mathfrak{g}(H))^0$.

Dually for each $a \in S$, let $\mathsf{M}^*_D(a)$ be the matrix defined by 2.55c and* (2.55d). *Then the map*

$$\mathsf{M}^*_D : a \mapsto \mathsf{M}^*_D(a)$$

is a representation of S by $\Lambda_D \times \Lambda_D$ column-monomial matrices over $(\mathfrak{g}(H))^0$.

The representation M_D is called the *Schützenberger representation* of S with respect to the $\mathscr{D}$-class D. Similaely the representation M^*_D is called the *dual Schützenberger representation* of S with respect to the $\mathscr{D}$-class D.

M_D:*The Schützenberger representation of with respect to D representation!Schützenberger –*
M^*_D:*The dual Schützenberger representation of with respect to D representation!dual Schützenberger –*

Suppose that ϕ and ψ are two representations of the semigroup S. We shall say that ϕ and ψ are *equivalent* if $\boldsymbol{\kappa}\phi = \boldsymbol{\kappa}\psi$. If this is the case, it is clear that the semigroups $\operatorname{Im}\phi$ and $\operatorname{Im}\psi$ are isomorphic.

representations!equivalent –

THEOREM 2.99. *Let D be a $\mathscr{D}$-class of a semigroup S then the partial representation $\boldsymbol{\varrho}^D$ and the Schützenberger representation M_D are equivalent. Similarly, the dual representations $\boldsymbol{\lambda}^D$ and M^*_D are also equivalent.*

Proof. Suppose that $a, b \in S$. For brevity, we have write $\mathsf{M}(a)$ for $\mathsf{M}_D D(a)$, etc. Then, by (2.55b) and (2.55c), $\mathsf{M}(a) = \mathsf{M}(b)$ if and only if, for each $\lambda \in \Lambda_D$, ρ_a is an isomorphism on $L(\lambda)$ if and only if ρ_b is an isomorphism on $L(\lambda)$ and the two isomorphisms coincide. Now, by Corollary 2.27, for $x \in D$, $\rho_a \mid L(x)$ is an isomorphism if and only if $x \mathscr{R} xa$. Hence $\rho_a \mid L(x)$ is an isomorphism if and only if $x \in \boldsymbol{D}\boldsymbol{\varrho}^D_a$. It follows that $\mathsf{M}(a) = \mathsf{M}(b)$ if and only if $\boldsymbol{D}\boldsymbol{\varrho}^D_a = \boldsymbol{D}\boldsymbol{\varrho}^D_b$ and the restrictions of ρ_a and ρ_b to this set are equal. Therefore $\mathsf{M}(a) = \mathsf{M}(b)$ if and only if $\boldsymbol{\varrho}^D_a = \boldsymbol{\varrho}^D_b$; that is M_D and $\boldsymbol{\varrho}^D$ are equivalent representations. Dually we can see that $\boldsymbol{\kappa}\mathsf{M}^*_D = \boldsymbol{\kappa}\boldsymbol{\lambda}^D$ and so, these representations are also equivalent. ◻

The representation M_D clearly depends on the choice of the isomorphisms $\gamma_\lambda : L \to L(\lambda)$. However, if M'_D is another representation with respect to D, by the result above, we have

$$\boldsymbol{\kappa}\mathsf{M}_D = \boldsymbol{\kappa}\boldsymbol{\varrho}^D = \boldsymbol{\kappa}\mathsf{M}'_D.$$

Therefore:

COROLLARY 2.100. *Let D be a $\mathscr{D}$-class of a semigroup S. The Schützenberger representation of S with respect to D is unique up to an equivalence.*

For each ω in an index set Ω, let M_ω be a representation of the semigroup S by $\Lambda_\omega \times \Lambda_\omega$-matrices over the group with zero G^0_ω. If $\mathsf{M}_\omega(S) = \operatorname{Im}\mathsf{M}_\omega$, then $\mathsf{M}_\omega(S)$ is a semigroup and M_ω is a homomorphism onto $\mathsf{M}_\omega(S)$. Let $T = \prod_{\omega\in\Omega} \mathsf{M}_\omega(S)$. Each $s \in S$ determine a unique element

$$\mathsf{M}(s) = (\dots, \mathsf{M}_\omega(s), \dots) \in T$$

such that the map

$$\mathsf{M} : s \mapsto \mathsf{M}(s)$$

$\bigoplus_{\omega\in\Omega} \mathsf{M}_\omega$ *:direct sum of representations* M_ω *representations!direct sum*

is a homomorphism of S into T. We write

$$\mathsf{M} = \bigoplus_{\omega\in\Omega} \mathsf{M}_\omega$$

and is called the *direct sum* of representations M_ω. Let $\Lambda = \bigcup_{\omega\in\Omega} \Lambda_\omega$ be the disjoint union of sets Λ_ω and let G^0 be any group with zero containing, for each $\omega \operatorname{Im} \Omega$, G^0_ω as a subgroup with zero (for example, we may take G^0 as the direct product $\prod_{\omega\in\Omega} G^0_\omega$ of all semigroups G^0_ω see Rmk 2.6). Then, for each $s \in S$, $\mathsf{M}(s)$ can be regarded as a $\Lambda \times \Lambda$-matrix over G^0

$$\mathsf{M}(s) = \begin{pmatrix} \ddots & & \\ & \mathsf{M}_\omega(s) & \\ & & \ddots \end{pmatrix} \tag{2.56}$$

in which the matrices $\mathsf{M}_\omega(s)$ form the diagonal blocks along the main diagonal. If each M_ω is a representation by row-monomial (or column-monomial) matrices so is the direct sum M.

Let $S/\mathscr{D} = \Omega$ be the sert of all $\mathscr{D}$-classes of S. By Corollary 2.48, upto isomorphism, there is a unique group associated with each $D \in \Omega$ which is isomorphic to the Schützenberger group of any $\mathscr{H}$-class of D. We shall refer to this group as the Schützenberger group of D. Recall that for each $D \in \Omega$, M_D is a homomorphism of S into the semigroup of all row-monomial matrices over G^0_D where G_D is the Schützenberger group of D. Clearly, sets Λ_D are mutually disjoint and $\Lambda = S/\mathscr{L} = \bigcup_{D\in\Omega} \Lambda_D$. It follows from the remarks above that the direct sum

$$\mathsf{M} = \bigoplus_{D\in\Omega} \mathsf{M}_D \tag{2.57}$$

is a representation by row-monomial $\Lambda \times \Lambda$-matrices over the group with zero G^0 where

$$G^0 = \prod_{D\in\Omega} G^0_D. \tag{2.58}$$

the direct product G^0 of all Schützenberger groups of S. Notice that, by Remark 2.6, G^0 is a group with zero.

Theorem 2.101. *The direct sum* M (2.57) *of all Schützenberger representations of a semigroup S is a representation of S by row-monomial $\Lambda\times\Lambda$-matrices ($\Lambda = S/\mathscr{L}$) over G^0. Morewover, M is equivalent to the representation ϱ by partial right translations (see Theorm 2.94).*

Dually, the direct sum M^* of all dual Schützenberger representations is is a representation by column-monomial $I \times I$-matrices ($I = S/\mathscr{R}$) over G^0 and is equivalent to the representation $\boldsymbol{\lambda}$ by partial left translations. Finally, if $U = \Lambda \cup I$, the direct sum $\mathsf{M}_\sigma = \mathsf{M} \oplus \mathsf{M}^*$ is a representation of S over G^0 and M_σ is equivalent to the by partial symmetric representation $\boldsymbol{\varpi}$ (see Corollary 2.95).

extension
extension!Schreier –

Proof. In view of the discussion preceding the statement, it is only necessary to prove the equivalence of representations M with ϱ. The equivalence of M^* with $\boldsymbol{\lambda}$ will follow by duality and that of M_σ with ϖ from the equivalences mentioned above. To prove that $\boldsymbol{\kappa}\varrho = \boldsymbol{\kappa}\mathsf{M}$, assume that $(a, b) \in \boldsymbol{\kappa}\varrho$. By the definition of ϱ_a and ϱ_b, we obtain, for every $x \in S$,

$$x \mathscr{R}\, xa \quad \text{or} \quad x \mathscr{R}\, xb \Rightarrow xa = xb. \tag{1}$$

Now if $x \mathscr{R}\, ax$ then, by Equation (2.55b), $\mathfrak{m}_{\lambda\mu}(a) \neq 0$ where $L(\lambda) = L(x)$ and $L(\mu) = L(xa)$. If this hold, the condition (1) above implies that

$$\mathfrak{m}_{\lambda\mu}(a) = \mathfrak{m}_{\lambda\mu}(b)$$

When $x \mathscr{R}\, xb$ we similarly see that this equality hold. If neither of these hold then

$$\mathfrak{m}_{\lambda\mu}(a) = \mathfrak{m}_{\lambda\mu}(b) = 0.$$

Therefore

$$\mathfrak{m}_{\lambda\mu}(a) = \mathfrak{m}_{\lambda\mu}(b) \quad \text{for all} \quad \lambda, \mu \in \Lambda = S/\mathscr{L} \tag{2}$$

and so, we have $\mathsf{M}(a) = \mathsf{M}(b)$. Conversely, let $\mathsf{M}(a) = \mathsf{M}(b)$ so that a and b satisfies Equation (2). Assume that $x \in \boldsymbol{D}\varrho_a$ so that $x \mathscr{R}\, xa$. Then, as above, we see that $\mathfrak{m}_{\lambda\mu}(a) \neq 0$ where $L(\lambda) = L(x)$ and $L(\mu) = L(xa)$. By Equation (2), $\mathfrak{m}_{\lambda\mu}(a) = \mathfrak{m}_{\lambda\mu}(b)$. By Equation (2.55b) it follows that $\rho_a \mid L(x) = \rho_b \mid L(x)$ and so, $xa = xb$. Similarly, we see that when $x \in \boldsymbol{D}\varrho_b$, $xa = xb$. It follows that $\varrho_a = \varrho_b$. This completes the proof. □

It follows from Theorem 2.97 and the result above that the representation M_σ is faithful for regular semigroups. For inverse semigroups the representations M, M^* and M_σ are all faithful.

2.10 EXTENSIONS

By an *extension* of a semigroup S we mean a semigroup T containing S as a subsemigroup. The problem of constructing all extensions of a given semigroup is too general to be of much interest (even for groups). A much restricted form of this problem for groups is the following: given two groups N and H construct all groups G having N as a normal subgroup and G/N isomorphic to H. This construction is possible and is given by the *Schreier extension* theory. A direct generalization of this to semigroups is again not possible since, in the case of semigroups, there is no proper replacement for concept of kernel *extension!ideal –*

of homomorphisms. However, in some particulae cases, this construction has been carried out successfully for semigroups (see, for example, Leech **?**, Grillet Grillet, **?** and Clifford Clifford [1949], Clifford and Preston [1961]). Here we shall briefly discuss the later construction due to clifford Clifford and Preston [1961] which is particularly useful in finding structure of several classes of semigroups (in particular, certain classes of finite semigroups).

2.10.1 Ideal extensions

To save repetition we shall assume through out this section that S is a semigroup with zero 0 and U is a semigroup disjoint from S. A semigroup T is an *ideal extension* of a semigroup U if U is isomorphic to an ideal U' of T. Further, we say that T is an ideal extension by a semigroup S with zero if the Rees quotient T/U' is isomorphic to S. For convenience, we may identify U with U' by the given isomorphism and regard U as an ideal of T and $S = T/U$. Theorem 2.90 says that when U is weakly reductive, the translational hull $\Omega(U)$ is an ideal extension of U. Clifford [1949] was first to study ideal extensions (see also Clifford and Preston [1961], Grillet). Petrich and Grillet **?** have also contributed significantly.

Notice that the construction of T from the given semigroups U and S is analogous to the Schreier construction of groups. On the other hand, there are also significant differences between these constructions. For example, given two groups N and H there is always a Schreier extension of N by H; the direct product $N \times H$ is one such extension. However, as shown by the Example 2.30, this is not true for ideal extensions of semigroups.

Let T be an ideal extension of U by S. Then it is clear that

$$T = U \cup S^* \quad \text{where} \quad S^* = S - \{0\}.$$

Also, if $s, t \in T$, the product $s * t$ in T is formed as follws. In the following, products in S or U is indicated by juxtaposition.

$$\begin{array}{lll} (1) & s * t = st \in S^* & \text{if } s, t, st \in S^*; \\ (2) & s * t \in U & \text{if } s, t \in S^* \text{ and } st = 0; \\ (3) & s * t \in U & \text{if } s \in S^* \text{ and } t \in U; \\ (4) & s * t \in U & \text{if } s \in U \text{ and } t \in S^*; \\ (5) & s * t = st \in U & \text{if } s, t \in U. \end{array} \tag{2.59}$$

Therefore an ideal extension T of U by S defines an associative product on $T = U \cup S^*$ satisfying equations (1) – (5). We proceed to discuss some of the consequences of this. Since S and U are given, conditions (1) and (5) can be ensured with out any further work. Other products must be specified in such a way that the resulting product is associative. The products of type (2) defines a map $\phi^T = \phi$ defined by

ramification homomorphism!partial –

$$\phi(s,t) = s * t \quad \text{for all} \quad (s,t) \in Z(S) \tag{2.60a}$$

where

$$Z(S) = \{(s,t) \in S^* \times S^* : st = 0 \quad \text{in} \quad S\}. \tag{2.60b}$$

Following Clifford and Preston [1961], any map $\phi : Z(S) \to U$ is called a *ramification* of S into U. Hence every ideal extension T of U by S *induces* a unique ramification ϕ^T of S into U.

If $s \in S^*$ the products of the form (3) gives a map

$$\lambda_s^U t = s * t \quad \text{for all} \quad t \in U \tag{2.61a}$$

of U into itself which is clearly a left translation of U. In fact, $\lambda_s^U = \lambda_s \mid U$ is the restriction of the inner left translation of T determined bt s to U. Similarly products of the type (4) gives the map, defined for all $t \in S^*$, by

$$s\rho_t^U = s * t \quad \text{for all} \quad s \in U \tag{2.61b}$$

which is a right translation of U. Moreover, for any $s \in S^*$

$$\left(t\rho_s^U\right) u = t\left(\lambda_s^U u\right) \quad \text{for all} \quad t, u \in U.$$

Hence the pair

$$\eta_s^T = \eta_s = (\rho_s^U, \lambda_s^U) \tag{2.62a}$$

satisfies Equation (2.54) and so, this pair belongs to $\Omega(U)$. Associativity of the product $*$ in T imples that

$$\eta_{st} = \eta_s \eta_t \quad \text{for all } s, t \in S^* \text{ such that} \quad st \neq 0. \tag{2.62b}$$

By a *partial homomorphism* of a semigroup S with 0 to a semigroup U we mean is a mapping η of $S^* = S - \{0\}$ into U satisfying Equation (2.62b) above. Thus the discussion above shows that every ideal extension T of U by S induces a partial homomorphism $\eta^T = \eta$ of S to U. η^T is called the partial homomorphism *induced by T*.

Excep for minor changes in notation and terminology, the following result is the same as Proposition 1.1 of Chapter III in Grillet.

THEOREM 2.102. *Suppose that S is a semigroup with zero and U is a semigroup disjoint from S. Let T be an ideal extension of U by S. Then the partial homomorphisn $\eta^T = \eta$ and ramification $\phi^T = \phi$ of S to U satisfy the following: for $s, t, x \in S^*$,*

$$\begin{array}{lll} (1) & (\eta_s u)\eta_t = \eta_s(u\eta_t) & \text{for all } u \in U; \\ (2) & \eta_s\eta_t = \big(\phi(s,t)\big)\boldsymbol{\pi} & \text{if } st = 0 \text{ in } S; \\ (3) & \eta_s\phi(t,x) = \phi(s,t)\eta_x & \text{if } st = 0 = tx; \\ (4) & \eta_s\phi(t,x) = \phi(st,x) & \text{if } st \neq 0,\ tx = 0; \\ (5) & \phi(s,t)\eta_x = \phi(s,tx) & \text{if } st = 0,\ tx \neq 0; \\ (6) & \phi(s,tx) = \phi(st,x) & \text{if } st \neq 0 \neq tx \text{ and } stx = 0. \end{array} \tag{2.63}$$

Conversely, let $\eta : s \mapsto \eta_s$ be a partial homomorphism and ϕ be a ramification of S to U satisfying the conditions (1)...(6) above. On $T = S^* \cup U$ define the binary operation $*$ as follows: For all $s, t \in T$

$$s * t = \begin{cases} st & \text{if } s,t,st \in S^* \text{ or } s,t \in U; \\ \phi(s,t) & \text{if } s,t \in S^* \text{ and } st = 0 \text{ in } S; \\ \eta_s t & \text{if } s \in S^* \text{ and } t \in U; \\ s\eta_t & \text{if } s \in U \text{ and } t \in S^*. \end{cases} \tag{2.64}$$

Then T with this binary operation is the unique ideal extension of U by S such that the partial homomorphism and ramification induced by T coincides with the given maps.

Proof. Let ϕ be the ramification defined by Equation (2.60a) and η be the partial homomorphism defined by Equation (2.62a). The properties listed in Equation (2.63) are immediate consequences of the definitions and the associativity of the product in T. The verification of these are left as exercise.

To prove the converse, we first verify that the product defined by Equation (2.64) is associative. To do this it is necessary to verify the following equality in various cases:

$$(a * b) * c = a * (b * c) \quad \text{for all} \quad a, b, c \in T. \tag{!}$$

Let $s, t, x \in S^*$ and $u, v, w \in U$. The case $a, b, c \in U$ follows from the associativities in the semigroup U. Since η_s acts on the left of U as a left transtlation, we have

$$s * (u * v) = \eta_s(uv) = (\eta_s u)v = (s * u) * v.$$

Dually, we have $(u * v) * s = u * (v * s)$. Since $\eta_s \in \Omega(U)$ the two translations represented by η_s are linked. Hence

$$(u * s) * v = (u\eta_s)v = u(\eta_s v) = u * (s * v).$$

Again by condition (1) above, we have

$$(s * u) * t = (\eta_s u)\eta_t = \eta_s(u\eta_t) = s * (u * t).$$

homomorphism!U-homomorphism
congruence!S-congruence

If $st \neq 0$, since η is a partial homomorphism, we have,

$$(s * t) * u = \eta_{st} u = \eta_s \eta_t u = s * (t * u).$$

If $st = 0$, then by the case 2 of the definition of $*$ and conditions (1) and (2),

$$(s * t) * u = \phi(s, t) u = \boldsymbol{\pi}_{\phi(s,t)} u = \eta_s \eta_t u = s * (t * u).$$

Now if $st \neq 0 \neq tx$ and $stx \neq 0$, then by the case 1 in the definition of $*$, we have $(s * t) * x = s * (t * u)$. If $st \neq 0 \neq tx$ and $stx = 0$ then

$$(s * t) * x = st * x = \phi(st, x) \quad \text{and} s * (t * x) = \phi(s, tx)$$

and so, (!) follows by condition (6). Let $st \neq 0$ and $tx = 0$. Then $(st)x = 0$. Hence

$$(s * t) * x = (st) * x = \phi(st, x) \quad \text{and} \quad s * (t * x) = s * \phi(t, x) = \eta_s \phi(t, x).$$

So, the equality (!) holds by condition (4). Similarly it can be shown that in the case when $st = 0$ and $tx \neq 0$ (!) holds because of condition (5). Finally if $st = 0 = tx$,

$$(s * t) * x = \phi(s, t) * x = \phi(s, t) \eta_x \quad \text{and} \quad s * (t * x) = \eta_s \phi(t, x).$$

Therefore, in this case, (!) holds by condition (3). This completes the proof of associativity of $*$.

Equation (2.64) clearly shows that U is an ideal in T. Since $T = S^* \cup U$, the Rees quotient T/U is clearly in one-to-one correspondance with S. The first and second cases in Equation (2.64) shows that this correspondance is an isomorphism. Thus T is an ideal extension of U by S. Comparing Equation (2.60a) and Equation (2.64), we see that the ramifcation ϕ^T of T (defined by Equation (2.60a)) and the given map coincide. Similarly comparing Equations (2.61a), (2.61b), (2.62a) and (2.64), we see that $\eta^T = \eta$; that is, the partial homomorphism η^T associated with T coincides with the given map η. This proves the uniqueness of the construction of T. □

Let T and T' be ideal extensions of U. A homomorphism $\theta : T \to T'$ is said to be an *U-homomorphism* if $\theta \mid U = 1_U$ (see Grillet, page 65). Similarly a congruence σ on T is an *U-congruence* if the restriction of σ to U is identity; that is $\sigma \cap U \times U = 1_U$. Recall from Theorem 2.90 that $\boldsymbol{\pi} : U \to \Omega(U)$ is a homomorphism which sends $u \in U$ to (ρ_u, λ_u).

THEOREM 2.103. *Let T be an ideal extension of U by S and let $\eta = \eta^T$ be the partial homomorphism induced by T. Then there is a unique homomorphism $\tau = \tau_T : T \to \Omega(U)$ defined for all $s \in T$ by*

$$s\tau = \begin{cases} \eta_s & \text{for all } s \in S^*; \\ s\boldsymbol{\pi} = (\rho_s, \lambda_s) & \text{for all } s \in U. \end{cases}$$

τ is a U-homomorphism if and only if U is weakly reductive.

Proof. Since $S^* \cap U = \emptyset$ and $S^* \cup U = T$, τ is well defined map of T into $\Omega(U)$. We must show that

$$(a * b)\tau = (a\tau)(b\tau) \quad \text{for all} \quad a, b \in T \tag{!!}$$

We need to verify several cases.

1 $s, t \in S^*$ and $st \neq 0$. Then by the deinitionof τ, we have

$$(s * t)\tau = (st)\tau = \eta_{st} = \eta_s\eta_t = (s\tau)(t\tau).$$

2 $s, t \in S^*$ and $st = 0$. Then for any $v \in U$,

$$v(s * t)\tau = v * (s * t) = (v * s) * t = (v\eta_s)\eta_t = v\left((s\tau)(t\tau)\right).$$

Hence the right translation determined by $(s * t)\tau$ and $(s\tau)(t\tau)$ are the same. Similarly

$$(s * t)\tau v = \eta_s(\eta_t v) = ((s\tau)(t\tau))\, v$$

and so, the left translation by $(s * t)\tau$ and $(s\tau)(t\tau)$ are also the same. Therefore (!!) holds in this case.

3 $s \in S^*$ and $u \in U$.

$$\begin{aligned} v\left((s * u)\tau\right) &= v\left((s * u)\boldsymbol{\pi}\right) = v\rho_{s*u} = v * (s * u) = (v * s) * u \\ &= v\left(\rho_s\rho_u\right) = v\left((s\boldsymbol{\pi})(u\boldsymbol{\pi})\right) = v\left((s\tau)(u\tau)\right). \end{aligned}$$

Similarly,

$$((s * u)\tau)\, v = (s * u) * v = s * (u * v) = (\lambda_s\lambda_u)v = ((s\tau)(u\tau))\, v.$$

It follows from these that $(s * u)\tau = (s\tau)(u\tau)$. It can be shown in a similar way that $(u * s)\tau = (u\tau)(s\tau)$.

4 $u, v \in U$. Since $\tau \mid U = \boldsymbol{\pi}$, the equation (!!) is obviousely hold.

Therefore τ is a homomorphism of T into $\Omega(U)$. The uniqueness of τ follows from the fact that, by definition, $\tau \mid S^* = \eta$ and $\tau \mid U = \boldsymbol{\pi}$.

$\mathfrak{i}\mathfrak{C}_U$:The category of ideal extensions of U

If τ is a U-homomorphism then $\tau \mid U = 1_U = \boldsymbol{\pi}$. Therefore U is weakly reductive. Comversely, if U is weakly reductive, then $\tau \mid U = \boldsymbol{\pi}$ is an isomorphism. Identifying U with $\operatorname{Im}\boldsymbol{\pi}$ by $\boldsymbol{\pi}$, $\Omega(U)$ become an ideal extension of U and $\tau : T \to \Omega(U)$ a U-homomorphism. □

It is clear that, given a semigroup U, there is a category $\mathfrak{i}\mathfrak{C}_U$ with ideal extensions of U as objects and U-homomorphisms as morphisms. Recall Subsection 1.2.3 that from a base $F : \mathcal{C} \to \mathcal{D}$ to $d \in \mathfrak{v}\mathcal{D}$ is a natural transformation from F to the constant functor Δ_d from $\mathcal{C}$ to d. A cone from the inclusion functor of $\mathfrak{i}\mathfrak{C}_U$ in the category $\mathfrak{S}$ of semigroups to the constan functor from $\mathfrak{i}\mathfrak{C}_U$ to $\Omega(U)$ will, for convenience, be called a cone from the base $\mathfrak{i}\mathfrak{C}_U$ to the vertex $\Omega(U)$. This is a map $\tau : T \to \tau_T$ from $\mathfrak{v}\,\mathfrak{i}\mathfrak{C}_U$ to the morphism class of $\mathfrak{i}\mathfrak{C}_U$ making the following diagram commute:

$$\begin{array}{ccc} & & \Omega(U) \\ & \overset{\tau_T}{\nearrow} & \uparrow{\scriptstyle \tau_{T'}} \\ T & \xrightarrow[\theta]{} & T' \end{array} \tag{2.65}$$

THEOREM 2.104. *Let T and T' be ideal extensions of U and let $\theta : T \to T'$ is a U-homomorphism. Then the map $\tau : T \mapsto \tau_T$ is a cone from the base $\mathfrak{i}\mathfrak{C}_U$ to the vertex $\Omega(U)$. If U is weakly reductive, then the cone τ is universal and so,*

$$\Omega(U) = \varinjlim \mathfrak{i}\mathfrak{C}_U.$$

Furthermore, in this case,

$$\tau_T = \jmath_T^{\Omega(U)}$$

whenever T is an ideal extension of U which is a subsemigroup of $\Omega(U)$.

Proof. Write $\tau = \tau_T$ and $\tau' = \tau_{T'}$. Since θ is a U-homomorphism,

$$u\theta \circ \tau' = u\boldsymbol{\pi} = u\tau \quad \text{for all} \quad u \in U.$$

Let $a \in S^*$. Then for any $u \in U$, since $\theta \mid U = 1_U$, we have

$$u(a\theta \circ \tau') = u * (a\theta) = u\theta * a\theta = (u * a)\theta = u * a = u\eta_a.$$

Similarly, $(a\theta \circ \tau')u = \eta_a u$. Therefore

$$a\theta \circ \tau' = a\tau \quad \text{for all} \quad a \in S^*$$

idealextension!dense –

Consequently $\theta \circ \tau' = \tau$ and so the diagram 2.65 commutes. Therefore $\tau : T \mapsto \tau_T$ is a cone with base $\mathfrak{i}\mathfrak{C}_U$ and vertex $\Omega(U)$.

Suppose now that U is weakly reductive and let

$$\tau' = \tau_{\Omega(U)}.$$

Then $\Omega(U)$ is an ideal extension of U. Since U is identified with $\pi(U)$, τ' is a U-homomorphism of $\Omega(U)$ onto itself. We first show that $\tau' = 1_U$. If $u \in U$, we have $u\tau' = u\pi = u$. If $a \in S^*$ where $S = \Omega(U)/U$, then a is an outer bitranslation (bitranslation which is not inner). By the definition of $\tau' = \tau_{\Omega(U)}$ in Theorem 2.103 and Equation (2.62a) (definition of $\eta^{\Omega(U)}$), we have

$$u(a\tau') = u\eta_a = u * a = ua \quad \text{for any} \quad u \in U.$$

Similarly $au = (a\tau')u$ for all $u \in U$. Since both a and $a\tau'$ are bitranslations, this implies that $a = a\tau'$ for all $a \in S^*$. Therefore $\tau' = 1_{\Omega(U)}$. To show that the cone τ is universal, let σ be amy cone from the base $\mathfrak{i}\mathfrak{C}_U$ to the vertex V. Tnen $\sigma' = \sigma_{\Omega(U)}$ is a U-homomorphism. Then for any $T \in \mathfrak{v}\,\mathfrak{i}\mathfrak{C}_U$, τ_T is a U-homomorphism and so,

$$\sigma_T = \tau_T \circ \sigma'.$$

This shows that τ is universal and so, $\Omega(U) = \varinjlim \mathfrak{i}\mathfrak{C}_U$. Finally, assume that $U \subseteq T \subseteq \Omega(U)$. Since $j_T^{\Omega(U)}$ is a U-homomorphism, by the above,

$$\tau_T = j_T^{\Omega(U)} \circ \tau_{\Omega(U)} = j_T^{\Omega(U)}$$

because $\tau_{\Omega(U)} = 1_{\Omega(U)}$. □

An ideal extension D of U is said to be *dense* if identity is the only non-trivial U-congruence on D. This is equivalent to the statement that any U-homomorphism of D is injective. When U is weakly reductive, any subsemigroup T of $\Omega(U)$ containing U is dense. For, let $\theta : T \to T'$ be any U-homomorphism. Then $\phi = \theta \circ \tau_{T'}$ is a U-homomorphism of T to $\Omega(U)$. Then by Theorem 2.104, we have

$$\phi = \phi \circ \tau_{\Omega(U)} = \tau_T = j_T^{\Omega(U)}$$

which says that ϕ is injective. Hence θ is also injective. Thus T is dense. In the same way, it can be seen that an ideal extension D is dense if and only if D is isomorphic to an ideal extension $T \subseteq \Omega(U)$.

COROLLARY 2.105. *Let D be an ideal extension of a weakly reductive semigroup U. Then D is dense if and only if it is isomorphic to an ideal extension $T \subseteq \Omega(U)$.*

If S is a semigroup with 0 and U is disjoint from S, then by Theorem 2.102, an ideal extension of U by S is determined by a partial homomorphism η and a ramification ϕ satisfying the conditions in Equation (2.63). The condition (2) in Equation (2.63) shows that the ramification ϕ is uniquely determined by the partial homomorphism when U is weakly reductive. This simplifies the result considerably as the following theorem shows.

For convenience, if U is weakly reductive, we shall assume that U has been identified with $\boldsymbol{\pi}(U) \subseteq \Omega(U)$ so that a statement that the bitranslation $\beta \in U$ will mean that β is an inner bitranslation $s\boldsymbol{\pi}$ for some unique $s \in U$.

THEOREM 2.106. *Suppose that U is weakly reductive and let $\eta : S^* \to \Omega(U)$ be a partial homomorphism such that*

$$\eta_s\eta_t \in U \quad \textit{for all} \quad s,t \in S^* \quad \textit{with} \quad st = 0. \tag{$\triangleright$}$$

Then $T = S^* \cup U$ with product $*$ defined, for all $s,t \in T$, by

$$s * t = \begin{cases} st & \textit{if } s,t,st \in S^* \textit{ or } s,t \in U; \\ \eta_s\eta_t & \textit{if } s,t \in S^* \textit{ and } st = 0 \textit{ in } S; \\ \eta_s t & \textit{if } s \in S^* \textit{ and } t \in U; \\ s\eta_t & \textit{if } s \in U \textit{ and } t \in S^*. \end{cases} \tag{2.66}$$

is the unique ideal extension of U by S such that the partial homomorphism η^T induced by T coincides with η. Conversely if T is any ideal extension of U by S, then the partial homomorphism induced by T satisfies the property ($\triangleright$).

Proof. Assume that $\eta : S^* \to \Omega(U)$ is a partial homomorphism satisfying ($\triangleright$). We proceed to show that we can define a ramification ϕ such that the pair η and ϕ satisfies the conditions in Equation (2.63). Define

$$\phi(s,t) = \eta_s\eta_t \quad \text{for all} \quad (s,t) \in Z(S). \tag{$\triangleleft$}$$

By ($\triangleright$) $\phi(s,t) \in U$ and so defines a ramification of S in U. Since we have identified U with $\operatorname{Im}\boldsymbol{\pi} = \boldsymbol{\pi}(U)$, condition (1) of Equation (2.63) follows from associativity in $\Omega(U)$. Since $\boldsymbol{\pi} = 1_U$, condition (2) is the definition of ϕ. If $st = 0 = tx$, using ($\triangleleft$), we have

$$\eta_s\phi(t,x) = \eta_s(\eta_t\eta_x) = (\eta_s\eta_t)\eta_x = \phi(s,t)\eta_x.$$

This proves condition (3). To prove (4), let $st \neq 0$ and $tx = 0$. Then $\eta_s\eta_t = \eta_{st}$ and $(st)x = 0$. Hence

$$\eta_s\phi(t,x) = \eta_s(\eta_t\eta_x) = (\eta_s\eta_t)\eta_x = \eta_{st}\eta_x = \phi(st,x).$$

The statement (5) is proved in a similar way. To prove (6), assume that $st \neq 0 \neq tx$ and $stx = 0$. Then

$$\phi(s, tx) = \eta_s \eta_{tx} = (\eta_s \eta_t)\eta_x = \eta_{st}\eta_x = \phi(st, x).$$

Hence partial homomorphism η and ramification ϕ satisfy the six conditions of Equation (2.63). Also, in view of (◃), the definition of $*$ in the statement coincide with the product $*$ defined by Equation (2.64). Therefore, by Theorem 2.102, $T = U \cup S^*$ is a semigroup with respect to $*$ which is an ideal extension of U by S. The uniquiness of T also follows from Theorem 2.102.

To prove the converse, let T be an ideal extension of U by S where U is weakly reductive. Since $\pi = 1_U$, by condition (2) of Equation (2.63), the ramification ϕ^T induced by T satisfies (◃) and hence the partial homomorphism η^T induced by T satisfies (▹). □

Remark* 2.18*: The result above can be generalized to arbitrary semigroups by replacing the particular dense extension $\Omega(U)$ by an arbitrary dense extension D. Thus an ideal extension of a semigroup U by a semigroup S can be constructed by considering a partial homomorphism $\theta : S^* \to D$ satisfying the condition (▹). Then we can get a partial homomorphism into $\Omega(U)$ as $\theta \circ \eta^D$. Defining ramification by $\phi(s, t) = (s\theta)(t\theta)$ we can show that this pair satisfies conditions of Equation (2.63). See Grillet for details. Notice that, by Corollary 2.105, this is equivalent to Theorem 2.106 when U is weakly reductive.

Example 2.30: Let $S = \{e, f, 0\}$ be the semilattice with $ef = 0$ and $N = x^+$ be the free cyclic semigroup. Any partial homomorphism θ of S to $\Omega(N)$ must map f to an idempotent in $\Omega(N)$. But $\Omega(N)$ is isomorphic to N^1 and so, any idempotent in $\Omega(N)$ must be identity which is also the only external bitranslation of N. Hence if θ exists, we must have $(e\theta)(f\theta) = 1_N$. So, there cannot exist $\phi(e, f) \in N$ such that

$$(e\theta)(f\theta) = \big(\phi(e, f)\big)\pi.$$

Therefore there cannot exist an ideal extension of N by S.

CHAPTER 3

Biordered sets

In many algebraic systems like semigroups, rings, algebras, etc. idempotents idempotents are important structural elements. To use them effectively in analysing the structure of the abgebraic object under consideration, it is necessary to know the nature of the set of their idempotents. In the case of inverse and orhtodox semigroups the set of idempotents form subsemigroups of known type. Many authors used this fact to determine the structure of semigroups in these classes of semigroups. However, these methods cannot be extend to determine the structure of semigroups in the more general class of semigroups such as the class of regular semigroups, completely regular semigroups, etc. since the set of idempotents $\boldsymbol{E}(S)$ of a regular (or completely regular, etc.) semigroup S is not in general a subsemigroup of S even though the role of $\boldsymbol{E}(S)$ in the structure of S is transparent. T.E. Hall (1973) made an attempt to study the the structure of regular semigroup S in terms of the subsemigroup generated by idempotents. He constructed a universal fundamental representation of S using the subsemigroup $< \boldsymbol{E}(S) >$ of S generated by $\boldsymbol{E}(S)$. The concept of *biordered* set was originally introduced by Nambooripad [1972, 1979] to represent the structure of the set of idempotents of a semigroup in general and that of a regular semigroup in particular. He identified a partial binary operation on $\boldsymbol{E}(S)$ arising from the semigroup product in S. The resulting structure on $\boldsymbol{E}(S)$ involving the partial binary operation is abstracted to the concept of a biordered set.

Historical Background

The idea using of the set $\boldsymbol{E}(S)$ of idempotents of a semigroup S in studying its structure has a long history. In 1941 Clifford [1941] used $\boldsymbol{E}(S)$ to characterize certian semigroups which were semilattice of groups. Later in 1966 W.D. Munn constructed an inverrse semigroup $T(E)$, now called the Munn semigroup, form an arbitrary semilattice E for which $\boldsymbol{E}(T(E))$ is isomorphic to E (see Munn

semigroups!inverse! fundamental
semigroup!orthodox
idempotent generated semigroup
cross-connections
warp
biordered set

[1970]). Morever, if S is any inverse semigroup for which $\boldsymbol{E}(S)$ is isomorphic to E then there is an idempotent separating homomorphism of $\phi_S : S \to T(E)$ to a full subsemigroup of $T(E)$. ϕ_S is an isomorphism onto a full subsemigroup of $T(E)$ if and only if S is fundamental. This implies that the structure an inverse semigroup S is determined by its semilattice of idempotents and a certain family of groups. This turned out to be a landmark contribution and many people tried to extend the results to wider class of semigroups. Recall that a semigroup S is *orthodox* if the set $\boldsymbol{E}(S)$ is a band (a semigroup of idempotents). Hall [1968] and Yamada [1970] observed that when S is a regular orthodox semigroup, the structure of S can be described in terms of $\boldsymbol{E}(S)$. In particular, Hall suitably extended Munn's theory to the class of regular orthodox semigroups [see Hall, 1968].

For an arbitrary regular semigroup S, $\boldsymbol{E}(S)$ is not a subsemigroup of S. Consequently it is not clear how one can extend Munn's theory to this class of semigroups. Three different approaches to the use of the set of idempotents $\boldsymbol{E}(S)$ in the study of the regular semigroup can be traced. T.E.Hall(1973) used the *idempotent generated semigroup* $< \boldsymbol{E}(S) >$ as the basic object in place of the set $\boldsymbol{E}(S)$ of idempotents in studying the structure of the regular semigroup S. Grillet [1974a,b,c] refined Halls results using the theory of *cross-connections*. A.H. Clifford (1974) introduced the concept of *warp* which was the partial algebra W on $\boldsymbol{E}(S)$ with partial binary operation $*$ induced from the semigroup product in S: for $e, f \in \boldsymbol{E}(S)$

$$e * f = \begin{cases} ef & \text{if } ef \in \boldsymbol{E}(S); \\ \text{undefined} & \text{otherwise.} \end{cases}$$

K.S.S. Nambooripad introduced the concept of a *biordered set* in Nambooripad [1972] as an order the structure to represent the set of idempotents of a semigroup; [see also Nambooripad, 1975]. He identified two quasiorders ω^r and ω^l and a set of partial tranofsformations on the set $\boldsymbol{E}(S)$ of idempotents of a semigroup satisfying certain axioms (see the definition below). Later, followiing Clfford's work ([see Clifford, 1974]), he refined the definition of biordered set by showing that biordered sets are cetain partial binary algebras. Nambooripad [1979] showed that any biordered set satisfying the regularity condition (see below) can be embedded as the set of idempotents of a regular semigroup. It is known from Nambooripad [1979] that the partial algebra of idempotents of any semigroup satisfies the axioms in Nambooripad [1979]. David Easdown (1985) proved the converse that any biordered set can be embedded as the biorered set of idempotents of a suitable semigroup and thus showing that the biorder axioms of Nambooripad [1979] are both necessary and sufficient in order that the resulting structure represents the set of idempotents of a semigroup.

biordered set
E:biordered set
ω^r:right quasiorder
quasiorder!right
$\tau^r(e)$:right translation
translation!right
ω^r:left quasiorder
quasiorder!left
$\tau^l(e)$:left translation

3.1 BIORDERED SETS

As observed above, biordered sets can be viewed either as an order structure or as a partial algebra. We give below both versions. The first definition is essentially from Nambooripad [1972] with some rationaizations (see also Nambooripad [1975]).

Recall from Section 1.1 that given any relation R on the set X and $x \in X$, $R(x)$ denote the set $\{x' \in X : x'Rx\}$ (see Equation (1.5a)). Also, 1_X denote the identty map (or relation) on X.

DEFINITION 3.1. Let E be a non empty set and ω^l, ω^r be quasiorders on E. Let

$$\mathscr{R} = \omega^r \cap (\omega^r)^{-1}, \quad \mathscr{L} = \omega^l \cap (\omega^l)^{-1} \quad \text{and} \quad \omega = \omega^l \cap \omega^r. \tag{3.1}$$

Suppose further that

$$T^r = \{\tau^r(e) : e \in E\} \quad \text{and} \quad T^l = \{\tau^l(e) : e \in E\}.$$

are families of partial transformations of E. Here, by the dual of a statement involving the quasiorderes ω^r, ω^l and partial tranesforfmations $\tau^r(e)$, $\tau^l(e)$, $e \in E$, we mean the statement that result by interchanging ω^r with ω^l and $\tau^r(e)$ with $\tau^l(e)$. The structure $< E, \omega^l, \omega^r, T^l, T^r >$ is called a *biordered set* if the following axioms and their duals hold. Here e, f, g, etc. denote arbitrary elements of E.

(BO1) (1) $\omega^r \cap (\omega^l)^{-1} = \omega^l \cap (\omega^r)^{-1} = 1_E$.
(2) For each $e \in E$, $\tau^r(e) : \omega^r(e) \to \omega(e)$ is an idempotent partial transformation.

(BO2) (1) $f \; \omega^r \; e \Rightarrow f \; \mathscr{R} \; f\tau^r(e) \; \omega \; e$.
(2) $g \; \omega^r \; f \; \omega^r \; e \Rightarrow g\tau^r(f) = (g\tau^r(e))\tau^r(f)$.

(BO3) Let $f, g \in \omega^r(e)$ and $g \; \omega^l \; f$. Then
(1) $g\tau^r(e) \; \omega^l \; f\tau^r(e)$ and
(2) $\left(g\tau^l(f)\right)\tau^r(e) = (g\tau^r(e))\,\tau^l\,(f\tau^r(e))'$.

(BO4) Let $g, f \in \omega^r(e)$ and $g\tau^r(e) \; \omega^l \; f\tau^r(e)$. Then there exist $g_1 \in \omega^r(e)$ such that $g_1 \; \omega^l \; f$ and $g_1\tau^r(e) = g\tau^r(e)$.

The data required to specify a biordered set E consist of a pair of quasiorders ω^r and ω^l and two families of partial transformations T^r and T^l. We will refer to ω^r*right quasiorder* of E and, for each $e \in E$, the partial transformation $\tau^r(e)$ as the *right translation* of E. Similarly ω^l is called the *left quasiorder* and $\tau^l(e)$,

translation!left
biordered set!natural partial order
$< E, D_E, * >$*:partial algebra on E with domain* D_E
$D_{\mathbb{E}}$*:domain of the partial operation on* $\mathbb{E}$
dual
T^**:dual of T*

$e \in E$ is called the left translation of E. For brevity, we shall often write $E =< E, \omega^l, \omega^r, T^l, T^r >$ to mean that E is a biordered set with quasiorders ω^l, ω^r and translations T^l, T^r. The relation ω defined by (3.1) is clearly a quasiorder and axiom (BO1) implies in particular that

$$\omega \cap (\omega)^{-1} \subseteq \omega^r \cap (\omega^l)^{-1} = 1_E.$$

Hence the relation ω on E defined by (3.1) is a partial order. We shall call ω, the *natural partial order* of the biordered set E.

A. H. Clifford (1974) observed that the data required to specify a biordered set may be given in terms of a partial binary operation on the underlying set E. This idea simplified the definition of biordered set a great deal. The definition of biordered sets given in Nambooripad [1979] used this idea to simplify the presentation. The following theorem formulates this definition in which we have also taken into account the reordering of axioms suggested by the work of **?**.

Recall from Subsection 1.2.1 that a partial algebra is a set together with a partial binary operation. We write $< E, D_E, * >$ for a partial algebra on the set E with D_E denoting the domain of the binary operation or $< E, D_E >$ if the binary operation is clear from the context. If no confusion is likely, we shall use juxtaposition to denote the product. If $\mathbb{E}$ is a partial algebra, we shall often denote the underlying set by $\mathbb{E}$ itself; and the domain of the partial binary operation on $\mathbb{E}$ will then be denoted by $D_{\mathbb{E}}$. Also, for brevity, we write $ef = g$, to mean $(e, f) \in D_{\mathbb{E}}$ and $ef = g$. The *dual* of a satatement T about a partial algebra $\mathbb{E}$ is the statement T^* obtained by replacing all products ef by its left-right dual fe. When D_E is symmetric, T^* is meaningful whenever T is.

Proposition 3.1. *Let* $E =< E, \omega^l, \omega^r, T^l, T^r >$ *be a biordered set. Define*

$$D_E = \omega^r \cup \omega^l \cup (\omega^r)^{-1} \cup (\omega^l)^{-1} \tag{3.2}$$

and for $(e, f) \in D_E$ *define* $e * f$ *by*

$$e * f = \begin{cases} e\tau^r(f) & \text{if } e \;\omega^r\; f; \\ e & \text{if } e \;\omega^l\; f; \\ f & \text{if } f \;\omega^r\; e; \\ e\tau^l(f) & \text{if } f \;\omega^l\; e. \end{cases} \tag{3.3}$$

Then $\mathbb{E}(E) =< E, D_E, * >$ *is a partial algebra such that, for all* $e, f \in E$*, we have:*

$$\begin{aligned} e \;\omega^r\; f &\iff f * e = e; \\ e \;\omega^l\; f &\iff e * f = e. \end{aligned} \tag{3.4}$$

Proof. From the defenition of D_E it is clear that D_E is reflexive and symmetric. Now we observe that $e * f$ is well-defined. For let $e\ \omega^r\ f$ and $e\ \omega^l\ f$. Then $e\ \omega\ f$. Now by axiom (BO1)(2), $\tau^r(f)$ is identity on $\omega(f)$ and so $e\tau^r(f) = e$. Also by definition of $*$ we have $e * f = e\tau^r(f) = e$ since $e\ \omega^r\ f$ and $e * f = e$ since $e\ \omega^l\ f$. Hence the two assignments coinside and so $e * f$ is well defined in this case. Now suppose that $e\ \omega^r\ f$ and $f\ \omega^r\ e$. Then by (BO21), we have

$$f\ \omega^r\ e\ \mathscr{R}\ e\tau^r(f)\ \omega\ f$$

which gives $f = e\tau^r(f)$ and so, $*$ is welldefined. If $e\ \omega^r\ f$ and $f\ \omega^l\ e$ then $e = f$ by (BO1)(1) and again the definition of $*$ is consistant. In a similar way, the remaining cases can be checked for consistancy. Therefore Equation (3.3) defines a partial binary operation on E with domain D_E.

To prove Equation (3.4), let $e\ \omega^r\ f$. Then $f * e = e$ by Equation (3.3). Conversely if $f * e = e$ then $(e, f) \in D_E$ and so, one of the statemnts $e\ \omega^r\ f$, $e\ \omega^l\ f$, $f\ \omega^r\ e$ or $f\ \omega^l\ e$ holds. If $e\ \omega^l\ f$, by Equation (3.3) and (BO2)(1), $e = f * e = e\tau^r(f)\ \omega\ f$ and so, $e\ \omega^r\ f$. If $f\ \omega^r\ e$, $e = f * e = f\tau^r(e)\ \mathscr{R}\ f$ by (BO2)(1) which gives $e\ \omega^r\ f$. Finally, if $f\ \omega^l\ g$, by (BO1)(1), $g = f$ and the relation $g\ \omega^r\ f$ follows. Therefore, in all cases, the first equation in Equation (3.4) is true. The second equation can be proved similarly. □

The next theorem characterizes those partial algebras that are induced by biordered sets as in the proposition above.

THEOREM 3.2. *Let $\mathbb{E} =< E, D_E >$ be a partial algebra. Define ω^r, ω^l, τ^r and τ^l as follows: for all $e, f \in E$,*

$$\begin{aligned} e\ \omega^r\ f \quad & \text{if} \quad fe = e, \\ e\ \omega^l\ f \quad & \text{if} \quad ef = e; \quad \text{and} \\ f\tau^r(e) = fe, \quad & \text{for all} \quad f \in \omega^r(e), \\ g\tau^l(e) = eg, \quad & \text{for all} \quad g \in \omega^l(e). \end{aligned} \tag{3.5}$$

Let $T^r = \{\tau^r(e) : e \in E\}$ and $T^r = \{\tau^r(e) : e \in E\}$. Then $E =< E, \omega^r, \omega^l, T^r, T^l >$ is a biordered set and the partial algebra $\mathbb{E}(E)$ determined as in Proposition 3.1 coincides with $\mathbb{E}$ if and only if $\mathbb{E}$ satisfies the following axioms and their duals. In the statements below $\mathscr{R}$, $\mathscr{L}$ and ω denote relations defined by Equation (3.1) *and e, f, g, etc., denote arbitrary elements in E.*

(B1) *(1)* *ω^r and ω^l are quasiorders on E.*
(2) $D_E = (\omega^r \cup \omega^l) \cup (\omega^r \cup \omega^l)^{-1}$.

(B2) *(1)* *For all $e\ \omega^r\ f \Rightarrow e\ \mathscr{R}\ ef \omega f$.*
(2) $g\ \omega^r\ f\ \omega^r\ e \Rightarrow gf = (ge)f$.

(B3) *For* $g, f \in \omega^r(e)$, $g \ \omega^l \ f \Rightarrow ge \ \omega^l \ fe$ *and* $(fg)e = (fe)(ge)$.

(B4) *If* $f, g \in \omega^r(e)$ *and* $ge \ \omega^l \ fe$, *then there exists* $g_1 \in \omega^r(e)$ *such that* $g_1 \ \omega^l \ f$ *and* $g_1 e = ge$.

Proof. Suppose that $\mathbb{E} =< E, D >$ satisfies the given axioms. If $e \ \omega^r \ f$ and $f \ \omega^l \ e$, by Equation (3.5), $f * e = e$ and $f * e = f$. Hence $e = f$ and so, E satisfies axiom (BO1)(1). By Equation (3.5) and (B2)(1), $f\tau^r(e) = f * e \in \omega(e)$ for all $f \in \omega^r(e)$ and $f\tau^r(e) = f$ for all $f \in \omega(e)$. Hence E satisfies axiom (BO1)(2). The remaining axioms (BOi) are translations of the corresponding axioms (Bi), $i = 2, 3, 4$ obtained by replacing the elements $f\tau^r(e)$ and $g\tau^r(h)$ by appropriate products given by Equation (3.5). To show that $\mathbb{E} = \mathbb{E}(E)$ it is clear that from Equations (3.3) and (3.5) that the underlying sets of $\mathbb{E}$ and $\mathbb{E}' = \mathbb{E}(E)$ coincide with E. Let D' denote the domain of the partial product on $\mathbb{E}'$. If $(e, f) \in D$ implies, by (B1)(2), that $e \ \omega^r \ f$, $e \ \omega^l \ f$, $f \ \omega^r \ e$ or $f \ \omega^l \ e$. If the first case is true, then by Equation (3.5), $fe = e$ in $\mathbb{E}$ and $fe = e$ in $\mathbb{E}'$. Hence $(e, f) \in D'$. In the same way, this conclusion holds in all cases so that $D \subset D'$ and the products coincide on D in both algebras. Reverse inclusion can be verified in a similar way using Equations (3.4) and (3.5). Therefore $\mathbb{E} = \mathbb{E}'$.

Conversely assume that E is a biordred set and $\mathbb{E} = \mathbb{E}'$. Axiom (B1)(1) holds by hypothesis and (B1)(2) follows from Equation (3.3). The remaining axioms are obtained by replacing the values $f\tau^r(e)$, $f\tau^l(e)$, etc. by products fe, ef, etc. Hence $\mathbb{E}$ satisfies axioms of the statement. □

Definition 3.1 and Theorem 3.2 shows that biordred sets are structures that affords resentation either as an order structure or as a partial algebra. The partial algebra representation simplifies the presentation significantly. On the other hand, any nontrivial discussion of biordered sets will have to deal with the order structure. We shall therefore use a hybrid approch that combine both these representations. Notice also that the empty set can be regarded as a biordered set.

Easdown [1985] proposed yet another way of presenting biorder axioms. He uses two arrow symbols to denote the relations ω^r and ω^l. Combining these arrows suitably he derives arrow symbols to dente other relations ω, $\mathscr{R}$ and $\mathscr{L}$. In this way he is able to exibit complex relations between elements of a biordered set using arrows [see Easdown, 1985, Higins, 1992, Chapter 3].

Since biordered sets are partial algebras, morphisms of biordered sets can be defined as partial algebra homomorphisms. However, we shall find it convenient to adopt a more restrictive definition.

DEFINITION 3.2. A mapping $\theta : E \to E'$ of biordered sets is called a *bimorphism* if

bimorphism!isomorphism biordered set! biordered subset bimorphism!embedding

$$(D_E)\,\theta \subseteq D_{E'}$$

and for all $(e, f) \in D_E$,

$$(ef)\theta = (e\theta)(f\theta).$$

A bijective bimorphism $\theta : E \to E'$ is an *isomorphism* if $\theta^{-1} : E' \to E$ is also a bimorphism. A biordered set $E' =< E', D_{E'} >$ is a *biordered subset* of $E =< E, D_E >$ if $E' \subseteq E$ and

$$D_{E'} = D_E \cap E' \times E'.$$

We write $E' \subseteq E$. A biorder isomorphism $\phi : E \to E'$ of E onto a biordered subset of E' is called an *embedding* of E in E'.

It is clear that the identity maps on biordered sets are bimorphisms and that composit of bimorphisms are again bimorphisms. Hence there is a category $\mathfrak{B}$ of biordered sets with objects as biordered sets and morphisms as bimorphisms. An isomorphism of biordered sets is an isomorphism in $\mathfrak{B}$. The concept of biordered subsets defined above provide a natural choice of subobjects in $\mathfrak{B}$.

***Remark* 3.1:** It may be noted that not all subalgebras of a biordered set are biordered subsets. For, let E be a biordered set containing e, f and g with $f, g \in \omega^r(e)$, $ge\ \omega^l\ fe$ and $(f, g) \notin D_E$. Then $E' = \{e, f, g, fe, ge\}$ is a subalgebra of E which is not a biordered subset.

Also if $E' \subseteq E$, then the inclusion $E' \subseteq E$ is a bimorphism. However, the converse is not true. For, let $E' =< E', D' >$ be the partial algebra with

$$\begin{gathered} E' = \{e, f, z\}, \qquad D' = E' \times E' - \{(e, f), (f, e)\} \\ \text{and with product} \quad ee = e, \quad ff = f, \quad e0 = 0e = 0f = f0 = 0. \end{gathered} \tag{M}$$

It can be seen that E' is a biordered set. Let $E =< E, D >$ be the partial algebra with $E = E'$, $D = E \times E$ and the products in E are those given above together with

$$ef = f \quad \text{and} \quad fe = e.$$

Then E is also a biordered set and identity mapping on E is a bimorphism. But E' is not a biordered subset of E.

Notice that there is a change in the terminilogy from Nambooripad [1979]. A biordered set E' is a biordered subset of E according to our definition above if and only if it is a is a biordered subset that is *relatively regular* in E according to the definition there [see Nambooripad, 1979, page. 3].

As shown in theorem below, the set of idempotents of a semigroup is a biordered set and the restriction of homomorphisms to the biordered set of the domain are bimorphisms. The concept of biordered sets has evolved as an abstraction of the structure of the set of idempotents of a semigroup.

THEOREM 3.3. *For each semigroup S, let* $E(S) = \{e \in S : e^2 = e\}$ *denote the set of idempotents in S and*

$$D_{E(S)} = \omega^r \cup \omega^l \cup \left(\omega^r \cup \omega^l\right)^{-1}$$

where

$$\omega^r = \{(e, f) \in E(S) \times E(S) : fe = e\} \quad \textit{and} \quad \omega^l = \{(e, f) \in E(S) \times E(S) : ef = e\}.$$

Then $\boldsymbol{E}(S) =< E(S), D_{E(S)} >$ *is a biordered set with respect to the restriction of the product in S to* $D_{E(S)}$*. Further, if* $\phi : S \to S'$ *is a homomorphism of semigroups, then* $\boldsymbol{E}(\phi) = \phi \mid E(S)$ *is a bimorphism of* $\boldsymbol{E}(S)$ *to* $\boldsymbol{E}(S')$*. The assignments*

$$S \mapsto \boldsymbol{E}(S) \quad \textit{and} \quad \phi \mapsto \boldsymbol{E}(\phi) \tag{3.6}$$

is a functor $\boldsymbol{E} : \mathfrak{S} \to \mathfrak{B}$ *from the category of semigroups to the category of biordered sets.*

Proof. First, we show that $\boldsymbol{E}(S)$ is a partial algebra; that is, for every $(e, f) \in D_{E(S)}$, $ef, fe \in E(S)$. By the definition of $D_{(S)}$, $e\ \omega^r\ f$, $e\ \omega^l\ f$, $f\ \omega^r\ e$ or $f\ \omega^l\ e$. If the first possibility hold, $fe = e \in E(S)$ by the efinition of ω^r and $(ef)^2 = efef = eef = ef$ so that $ef \in E(S)$. The remaining cases can be verified in the same way. Axioms (Bi), $i = 1, 2, 3$, are consequences of associativity of multiplication in S. To prove (B4), let $e, f, g \in E(S)$ with $f, g \in \omega^r e$ and $ge\ \omega^l\ fe$. Let $g_1 = gf$, the product in S. Then

$$\begin{aligned} g_1^2 = gfgf &= (eg)(ef)(eg)f && \text{since } e, f, g \in E(S), \\ &= (eg)f = gf && \text{since } ge\ \omega^l\ fe. \end{aligned}$$

Hence $g_1 \in E(S)$. Also, by associativity,

$$\begin{aligned} &g_1 f = g_1 && \text{and so, } g_1\ \omega^l\ f. \text{ Again} \\ &g_1 e = (gf)e = (ge)(fe) = ge. \end{aligned}$$

This proves that $\boldsymbol{E}(S)$ is a biordered set. The remaining assertions are routine to verify. □

Easdown [1985] proved the converse of this by showing that each biordered set can be realised as the biordered set of some semigroup.

3.1.1 Regular Biordered Sets

$M(e, f)$:quasiordered set $(\omega^l(e) \cap \omega^r(f), \preceq)$
sandwich set
$\mathscr{S}(e, f)$: Sandwich set
$\simeq$: equivalence relation $\preceq \cap \preceq^{-1}$
maximum
biordered set!regular
bimorphism!regular

We now consider biordered sets arising from regular semigroups. We require the concept of *sandwich sets* of a pair of idempotents.

Definition 3.3. Let E be a biordered set. For $e, f \in E$ let

$$M(e, f) = \left(\omega^l(e) \cap \omega^r(f), \preceq\right)$$

where $\preceq$ is the relation defined by

$$g \preceq h \iff g, h \in \omega^l(e) \cap \omega^r(f) \quad \text{and} \quad eg\ \omega^r\ eh, \quad gf\ \omega^l\ hf.$$

The *sandwich set* of e and f is defined as

$$\mathscr{S}(e, f) = \{h \in M(e, f) : g \preceq h \quad \text{for all} \quad g \in M(e, f)\}$$

Clearly, $\preceq$ is a quasiorder on $\omega^l(e) \cap \omega^r(f)$. Hence $M(e, f)$ is a quasiodered set and

$$\simeq = \preceq \cap \preceq^{-1} \tag{3.7}$$

is an equivalence relation on $M(e, f)$. Therefore, if

$$\tilde{M}(e, f) = \{\tilde{e} : e \in E\}$$

denote the quotient set $M(e, f)/\simeq$, then $\tilde{M}(e, f)$ is a partially ordered set under the induced relation defined by

$$\tilde{g} \preceq \tilde{h} \iff g \preceq h.$$

The sandwich set of e and f, if nonempty, is a $\simeq$ class in $M(e, f)$ and represents the *maximum* element in the partially ordered set $\tilde{M}(e, f)$. It is easy to construct example of a biordered sets E to show that $\mathscr{S}(e, f) = \emptyset$ for some $e, f \in E$. Also, it is clear that $\mathscr{S}(e, f)$ and $\mathscr{S}(f, e)$ are in general not the same.

The distinguishing property of a biordered set arising from regular semigroups can be seen in the sandwich sets.

Definition 3.4. A biordered set E is said to be a *regular* if

(R) $\mathscr{S}(e, f) \neq \emptyset$ for all $e, f \in E$.

A bimorphism $\theta : E \to E'$ is said to be *regular* if it satisfies the following:

(RM1) $\mathscr{S}(e, f)\theta \subseteq \mathscr{S}(e\theta, f\theta)$; and

(RM2) $\mathscr{S}(e, f) \neq \emptyset \iff \mathscr{S}(e\theta, f\theta) \neq \emptyset$.

biordered subset!relatively regular
$\tilde{\mathfrak{B}}$*:subcategory of* $\mathfrak{B}$ *with morphisms as regular bimorphisms*
$\mathfrak{RB}$*:subcategory of* $\tilde{\mathfrak{B}}$ *with objects as regular biordered sets*
$\mathscr{S}_1(e,f)$*:see definition on 172*
$\mathscr{S}_2(e,f)$*:see definition on 172*

We shall say that a biordered subset $E' \subseteq E$ is said to be *relatively regular* in E if the inclusion $j^E_{E'}$ is a regular bimorphism.

Note that the sandwich set of every pair of idempotents in a biordered set need not be non-empty (see the example below). Also we can have regular bimorphisms of nonregular biordered sets. If E is regular, it is clear that any bimorphism of $\theta : E \to E'$ is regular if it satisfies the condition (RM1). Thus axiom (RM2) is relevant only for bimorphisms of nonregular biordered sets.

Clearly compositions of regular bimorphisms are regular and identity on biordered sets are regular. Hence we have a category $\tilde{\mathfrak{B}}$ in which objects are biordered sets and morphisms are regular bimorphisms. Clearly $\tilde{\mathfrak{B}}$ is a subcategory of $\mathfrak{B}$. Moreover, there is a subcategory $\mathfrak{RB}$ of $\tilde{\mathfrak{B}}$ with objects are regular biordered sets.

We proceed to prove that the biordered set of idempotetns of a regular semigroup is a regular biordered set.

First we give a different description of sandwich sets for biordered sets of idempotents of a semigroup. In the following, we write $x \perp y$ for elemente x, y of a semigroup S to mean that $x \in \mathscr{V}(y)$.

Proposition 3.4. *Let $E = \boldsymbol{E}(S)$ be the biordered set of a semigroup S. For $e, f \in E$ define*

$$\mathscr{S}_1(e,f) = \{h \in M(e,f) : ehf = ef\} \quad \text{and} \quad \mathscr{S}_2(e,f) = \{h \in M(e,f) : h \perp ef\}. \tag{3.8}$$

Then we have

$$\mathscr{S}_1(e,f) = \mathscr{S}_2(e,f) \subseteq \mathscr{S}(e,f). \tag{a}$$

Moreover, ef is a regular element in S if and only if

$$\mathscr{S}_1(e,f) = \mathscr{S}_2(e,f) = \mathscr{S}(e,f) \neq \emptyset. \tag{b}$$

Proof. Let $h \in \mathscr{S}_1(e,f)$. Then $h \in M(e,f)$ and so,

$$h(ef)h = (he)(fh) = hh = h; \quad \text{and} \quad (ef)h(ef) = e(fhe)f = ehf = ef.$$

Hence $h \in \mathscr{S}_2(e,f)$. If $h \in \mathscr{S}_2(e,f)$, $ehf = (ef)h(ef) = ef$ and so, $h \in \mathscr{S}_1(e,f)$. Therefore $\mathscr{S}_1(e,f) = \mathscr{S}_2(e,f)$.

Again let $h \in \mathscr{S}_1(e,f)$ and $g \in M(e,f)$. Then

$$(eh)(eg) = (ehe)g = (ehf)g = efg = eg; \quad \text{and}$$
$$(gf)(hf) = (ge)(fhf) = g(ehf) = gef = gf.$$

Thus $g \preceq h$ and so, $h \in \mathscr{S}(e,f)$. Thus Equation (a) follows.

Now suppose that ef is a regular element in S and let $a \in \mathscr{V}(ef)$. If $h = fae$, then

$$h^2 = (fae)(fae) = f(aefa)e = fae = h.$$

Hence $h \in \mathscr{S}_1(e, f) \subseteq \mathscr{S}(e, f)$. To complete the proof on Equation (b), it is sufficient to show that $\mathscr{S}(e, f) \subseteq \mathscr{S}_1(e, f)$. If $g \in \mathscr{S}(e, f)$, by the above, $h, g \in \mathscr{S}(e, f)$. This gives $eg \mathscr{R} eh$ and $gf \mathscr{L} hf$ so that

$$egf = (eg)(ef) = (eg)(ehf) = (eg)(eh)f = (eh)f = ef.$$

Hence $g \in \mathscr{S}_1(e, f)$. This complete the proof. □

Observe that the sandwich set $\mathscr{S}(e, f)$ of $e, f \in E$ is defined entirely in terms of the structure of the biordered set E. On the other hand, the sets $\mathscr{S}_1(e, f)$ and $\mathscr{S}_2(e, f)$ depend on the semigroup product ef non-trivially. However, this distinction is not of any consequence if we are dealing entirely with regular biordered sets and regular semigroups (see Proposition 3.8).

THEOREM 3.5. *The biordered set $\boldsymbol{E}(S)$ of a regular semigroup S is regular. Further, if $\phi : S \to S'$ is a homomorphism of the regular semigroup S to a semigroup S', then $S\phi$ is a regular subsemigroup of S' and $\boldsymbol{E}(\phi) : \boldsymbol{E}(S) \to \boldsymbol{E}(S')$ is a regular bimorphism such that*

$$\boldsymbol{E}(S\phi) = (\boldsymbol{E}(S))\, \boldsymbol{E}(\phi). \tag{3.9}$$

In particular, if ϕ is injective or surjective, so is $\boldsymbol{E}(\phi)$.

Proof. By Theorem 3.3, $\boldsymbol{E}(S)$ is a biordered set. To show that $\boldsymbol{E}(S)$ is regular, consider $e, f \in \boldsymbol{E}(S)$. Then by Proposition 3.4, $\mathscr{S}(e, f) \neq \emptyset$. Hence, by Definition 3.3, $\boldsymbol{E}(S)$ is regular. Next, let $\phi : S \to S'$ be a homomorphism where S is a regular semigroup. If $x \in S$ and if $x' \in \mathscr{V}(x)$, then

$$(x'\phi)(x\phi)(x'\phi) = (x'xx')\phi = x'\phi$$

and

$$(x\phi)(x'\phi)(x\phi) = x\phi.$$

Therefore $x'\phi \in \mathscr{V}_{S'}(x\phi)$. Hence every element of $S\phi$ is regular and so, $S\phi$ is a regular subsemigroup of S'. Let $\theta = \phi \mid \boldsymbol{E}(S)$. By Theorem 3.3, θ is a bimorphism. If $h \in \mathscr{S}(e, f)$, by Proposition 3.4, $h \in \mathscr{S}_1(e, f)$ and so, $h \in M(e, f)$ and $ehf = ef$. Since ϕ is a homomorphism, $h\theta \in M(e\theta, f\theta)$ and $(e\theta)(h\theta)(f\theta) = (e\theta)(f\theta)$. Therefore, by Proposition 3.4, $h\theta \in \mathscr{S}_1(e\theta, f\theta)$. Hence $\mathscr{S}(e, f)\theta \subseteq \mathscr{S}(e\theta, f\theta)$ and by Proposition 3.4, $\theta : \boldsymbol{E}(S) \to \boldsymbol{E}(S')$ is a regular bimorphism.

Clearly $\boldsymbol{E}(S\phi) \supseteq (\boldsymbol{E}(S))\, \theta$ where $\theta = \boldsymbol{E}(\phi)$. To prove Equation (3.9), let $\bar{h} \in \boldsymbol{E}(S\phi)$ so that $x\phi = \bar{h}$ for some $x \in S$. Since S is regular, by Lemma 2.38, there is $x' \in \mathscr{V}(x)$. Let $h \in \mathscr{S}(e, f)$ where $e = x'x$ and $f = xx'$. Since θ is a regular bimorphism $h\theta \in \mathscr{S}(e\theta, f\theta)$. Since $e \mathscr{L} x \mathscr{R} f$, we have $e\theta \mathscr{L} x\phi = \bar{h} \mathscr{R} f\theta$.

local structure

An application of Proposition 3.4 gives $\bar{h} \in \mathscr{S}(e\theta, f\theta)$. Now suppose that $\bar{g} \in \mathscr{S}(e\theta, f\theta)$. Then by Definiton 3.3, we have

$$e\theta\bar{g}\ \omega\ e\theta \quad \text{and} \quad \bar{g}\ \mathscr{L}\ e\theta\bar{g}\ \mathscr{L}\ e\theta\bar{h} = e\theta\ \mathscr{L}\ \bar{h}.$$

Similarly, $\bar{g}\ \mathscr{R}\ \bar{h}$ which gives $\bar{g} = \bar{h}$. Thus $\mathscr{S}(e\theta, f\theta) = \{\bar{h}\}$. Therefore $h\theta = \bar{h}$. This proves Equation (3.9).

It is clear that, if ϕ is injective, so is θ. If ϕ is surjective, by Equation (3.9), θ is surjective. □

Equation (3.9) implies the following important result due to Lallement [see Lallement, 1967, Proposition 3.5].

Corollary 3.6. *Let $\phi : S \to S'$ be a homomorphism of regular semigroups. If $e \in S\phi$ is an idempotent if and only if there is an idempotent $f \in S$ such that $f\phi = e$.* □

It is clear from Theorem 3.5 that there is a functor of the category $\mathfrak{RS}$ of regular semigroups to the category $\mathfrak{RB}$ of regular biordered sets which is the restriction $\boldsymbol{E} \mid \mathfrak{RS}$ of the functor $\boldsymbol{E} : \mathfrak{S} \to \mathfrak{B}$ of Theorem 3.3 to the category $\mathfrak{RS}$ of regular semigroups. We shall denote this restriction also by $\boldsymbol{E}$.

Recall that the trace product $x * y$ of $x, y \in S$ exists if and only if $L_x \cap R_y$ contains an idempotent. If this is the case, $x * y = xy$ (see Equation (2.48a)). The partial algebra $S(*)$ on the set S with respect to the trace product represents the *local structure* of S. The structure of $S(*)$ is known by Theorem 2.78. Next theorem shows that arbitrary products in a regular semigroup S can be reduced to trace products of suitable elements using the structure of the biordered set $\boldsymbol{E}(S)$.

Theorem 3.7. *Let x and y be regular elements of a semigroup S, $x' \in \mathscr{V}(x)$ and $y' \in \mathscr{V}(y)$. If $g \in M(x'x, yy')$, then*

$$xgy = (xg) * (gy), \qquad y'gx' = (y'g) * (gx')$$

where $*$ denote the trace product in S and $xgy \perp y'gx'$. In particular, if $h \in \mathscr{S}_1(x'x, yy')$, then

$$(xh) * (hy) = xy \quad \perp \quad y'hx'$$

where $\mathscr{S}_1(e, f)$ is defined in Proposition 3.4.

Proof. Let $e = x'x$ and $f = yy'$. By Lemma 2.38, $e\ \mathscr{L}\ x$ and $f\ \mathscr{R}\ y$. Since $g \in M(e, f)$ $ge = g = fg$ and so,

$$(xg)(gx')(xg) = xggeg = xg \quad \text{and} \quad (gx')(xg)(gx') = geggx' = gx'.$$

Hence $gx' \in \mathscr{V}(xg)$ and, again by Lemma 2.38,

$$g = gx'xg\ \mathscr{L}\ xg\ \mathscr{R}\ xgx'\ \mathscr{L}\ gx'\ \mathscr{R}\ g.$$

Similarly,

$$g = (gy)(y'g) \mathscr{R}\, gy \mathscr{L}\, y'gy \mathscr{R}\, y'g.$$

Consequently, we have

$$xg \mathscr{L}\, g \mathscr{R}\, gy \quad \text{and} \quad y'g \mathscr{L}\, g \mathscr{R}\, gy$$

It follows by Equation (2.48a) that the trace products $(xg) * (gy)$ and $(y'g) * (gx')$ are defined. A simple computation shows that $y'gx' \in \mathscr{V}(xgy)$.

By the definition of $\mathscr{S}_1(e, f)$, $h \in M(e, f)$ and so $chy \perp y'hy$ by the above. Also, $ehf = ef$ (see Proposition 3.4). Hence

$$xhy = x(ehf)y = x(ef)y = xy.$$

This completes the proof. □

If $S' \subseteq S$ is a regular subsemigroup of S, then the inclusion is a homomorphism of a regular semigroup S' into the semigroup S. Hence by Theorem 3.5 its bimorphism is regular. It is clear that $\boldsymbol{E}(j^S_{S'}) = j^{E(S)}_{E(S')}$. Hence $\boldsymbol{E}(S')$ is a regular biordered subset of $\boldsymbol{E}(S)$ which is relatively regular in $\boldsymbol{E}(S)$. Thus a regular biordered subset E' of $\boldsymbol{E}(S)$ is relatively regular in $\boldsymbol{E}(S)$ if there exist a regular subsemigroup $S' \subseteq S$ such that $\boldsymbol{E}(S') = E'$. The following result shows that the converse of this also holds under an additional condition.

Proposition 3.8. *Let S be a semigroup such that $\boldsymbol{E}(S) \neq \emptyset$ and let E be a regular biordered subset of $\boldsymbol{E}(S)$. Then E is the biordered set of a regular subsemigroup of S if and only if E is relatively regular in $\boldsymbol{E}(S)$ and for all $e, f \in E$, $\mathscr{S}_1(e, f) \neq \emptyset$. In particular, if S is regular and if E' is a regular biordered subset of $\boldsymbol{E}(S)$ then there is a regular subsemigroup S' of S such that $E' = \boldsymbol{E}(S')$ if and only if E' is relatively regular in $\boldsymbol{E}(S)$.*

Proof. If there exist a regular subsemigroup S' of S such that $\boldsymbol{E}(S') = E'$, then by the remark above, E' is relatively regular in $\boldsymbol{E}(S)$. Further, if $e, f \in E'$, then ef is a regular element of S and so $\mathscr{S}_1(e, f) \neq \emptyset$ by Proposition 3.4.

Conversely assume that E' satisfies the given conditions and let S' be the subsemigroup of S generated by idempotents. Consider $e, f \in E'$. Since $\mathscr{S}_1(e, f) \neq \emptyset$, by Proposition 3.4, $\mathscr{S}(e, f) = \mathscr{S}_1(e, f)$. Since E' is regular and the inclusion is relatively regular, there exists $h \in E'$ such that $h \in \mathscr{S}(e, f) = \mathscr{S}_1(e, f)$. It follows from Proposition 3.4 that $h \perp ef$ in S. Since $h, ef \in S'$, we have $eh, hf \in E'$ and $ef \in R_{eh} \cap L_{hf}$. Inductively assume that every product x of n elements in E' has the property that there are $e_x, f_x \in E'$ with $e_x \mathscr{R}\, x \mathscr{L}\, f_x$ and let $x = e_0e_1 \dots e_n$. If $x = ye_n$ where $y = e_0 \dots e_{n-1}$ then the induction hypothesis holds for y and so we can find $f \in E'$ with $y \mathscr{L} f$. As before, we can find $k \in \mathscr{S}_1(f, e_n) \cap E'$. Then $k\ \omega^r\ e_n$ and so, $ke_n \in E'$. By Theorem 3.7, $x \in R_{yk} \cap L_{ke_n}$ and so, $x \mathscr{L} ke_n$. Dually we can show that there exists $g \in E'$ such that $g \mathscr{R} x$. This implies in particular that S' is a regular subsemigroup of S. By definition, $E' \subseteq \boldsymbol{E}(S')$. Let $u \in \boldsymbol{E}(S')$. By the above, there is $e, f \in E'$ such that $e \mathscr{L} u \mathscr{R} f$. Then by Theorem 2.34, $ef \in R_e \cap L_f$. Let $h \in \mathscr{S}_1(e, f) \cap E'$. Then $h\ \omega^r\ f$ and by axiom (B21), $hf\ \omega\ f$. But by Theorem 3.7, $hf \mathscr{L} ef \mathscr{L} f$ which gives $hf = f$. Hence $h \mathscr{R} f$. Duall $h \mathscr{L} e$. This implies that h and u are $\mathscr{H}$-equivalent idempotents in S' and so $u = h$. Therefore $u \in E'$ and so, $E' = \boldsymbol{E}(S')$.

To prove the last statement, we observe that when S is regular, ef is a regular element of S and so, $\mathscr{S}_1(e, f) \neq \emptyset$ for all $e, f \in E'$. Therefore every regular biordered subset of $\boldsymbol{E}(S)$ which is relatively regular is the biordered set of a regular subsemigroup of S. This complete the proof. □

3.1.2 Examples

Now we give some examples of biordered sets.

Example 3.1: The empty set with respect to empty relations and translations is a biordered set. (Observe that if $E = \emptyset$, all axiom remain valid vacuously.)

Example 3.2: Every semilattice is a biordered set. Let (E, ω) be a semilattice. We assume every semilattice to be a lower semilattice; i.e., for every $e, f \in E$, the greatest lover bound $e \wedge f$ exists. It is easy to see that $\wedge$ is a commutative and associative multiplication on E and thus $(E\wedge)$ is a commutative band. We regard E as a biordered set as follows. The quasiorders are $\omega^r = \omega^l = \omega$ on E. The domain D_E of the partial binary operation is

$$D_E = \{(e, f) : e\ \omega\ f \quad \text{or} \quad f\ \omega\ e\}$$

The axioms are easily verfied. We observe that for $e, f \in E$, if $h = e \wedge f$, the set $M(e, f)$ is given by

$$M(e, f) = \{h : h\ \omega\ h\} \quad \text{and} \mathscr{S}(e, f) = \{h\}$$

is singleton.

Example 3.3: Let I, Λ be non-empty sets and $B = I \times \Lambda$ be the rectangular band on I and Λ. That is, define multiplication in B by

$$(i, \lambda)(j, \mu) = (i, \mu) \quad \text{for all} \quad (i, \lambda), (j, \mu) \in I \times \Lambda.$$

This gives B, the structure of a band and by Theorem 3.3, $B = \boldsymbol{E}B$ is a biordered set. Here the domain D_B is given by

$$D_B = \{((i, \lambda)(j, \mu)) : i = j \quad \text{or} \quad \lambda) = \mu\}.$$

In this case

$(i, \lambda)\ \omega^r\ (j, \mu)$ if and only if $i = j$ and

$(i, \lambda)\ \omega^l\ (j, \mu)$ if and only if $\lambda = \mu$.

Also the sandwich set is given as follows. For $e = (i, \lambda)$ and $f = (j, \mu)$ we have $S(e, f) = \{(j, \lambda)\}$.

Example 3.4: Let $E = (X, \leq)$ be a partially ordered set. Then one can verify with out difficulty that E is a biordered set with

$$\omega^r = \omega^l = \omega = \leq \tag{1}$$

Then clearly $D_E = \leq \cup (\leq^{-1})$ and the basic product in E is given by

$$ef = fe = e \quad \text{if and only if} \quad e \leq f.$$

Conversely, if E is any biordered set satisfying (1), then E is the biordered set determined by the partially ordered set (E, ω).

Recall that, in any partially ordered set E and $e, f \in E$, $e \wedge f$ denote the greatest lover bound of e and f in E if it exists. In this case, in the biordered set E determined by the partially ordered set as above, the sandwich set of $e, f \in E$ is

$$\mathscr{S}(e, f) = \begin{cases} \{e \wedge f\} & \text{if } e \wedge f \text{ exists;} \\ \emptyset & \text{otherwise.} \end{cases}$$

Therefore, E is a regular biordered set if and only if $e \wedge f$ exists for every pair of elements $e, f \in E$ in which case, the biordered set E coincides with the biordered set of Example 3.2. It follows that E is a regular biordered set if and only if E is a semilattice.

Example 3.5: Let $E = \{e, f, g\}$ be a biordered set in which $\omega^r = \omega^l = \{(e, g), (f, g)\}$. In this case

$$D_E = \{(e, g), (g, e), (f, g), (g, f)\}$$

and the products are determined by partial order as in the example above. Now $M(e, g) = \{e\}$ and so, $\mathscr{S}(e, g) = \{e\}$. But $M(e, f)$ is empty so that $\mathscr{S}(e, f) = \emptyset$.

Again, let $X = \{e, f\} \cup \mathbb{N}$. Define partial order on X by

$$n \leq e, \ n \leq f \quad \text{for all} \quad n \in \mathbb{N}$$

and the restriction of this partial order to $\mathbb{N}$ coincides with the natural order on $\mathbb{N}$. In this case we have $M(e, f) = \mathbb{N}$ and so, it is not empty. However $\mathscr{S}(e, f) = \emptyset$.

Example 3.6 (Example 1.1 in Nambooripad [1979]): Consider the following bands B_i, $i = 1, 2$ on the same set $B = \{e, f_1, f_2, f_3\}$ with the multiplication table:

B_1	e	f_1	f_2	f_3
e	e	f_1	f_2	f_3
f_1	f_2	f_1	f_2	f_3
f_2	f_2	f_1	f_2	f_3
f_3	f_3	f_1	f_2	f_3

B_2	e	f_1	f_2	f_3
e	e	f_1	f_2	f_3
f_1	f_3	f_1	f_2	f_3
f_2	f_2	f_1	f_2	f_3
f_3	f_3	f_1	f_2	f_3

For each $i = 1, 2$, B_i is a band and hence $\boldsymbol{E}(B_i) = B_i$ is a regular biordered set. It is easy to see that

$$\omega^r(B_1) = \{(e, e), (f_i, e), (f_i, f_j) : i, j = 1, 2, 3\} = \omega^r(B_2)$$
$$\omega^l(B_1) = 1_B \cup \{(f_2, e), (f_3, e)\} = \omega^l(B_2).$$

where $\omega^r(B_i)$ and $\omega^l(B_i)$ denote the quasiorders of the biordered set B_i. However, the basic product $f_1 e = f_2$ in B_1 and $f_1 e = f_3$ in B_2. So $B_1 \neq B_2$. $E_2 = (E, \circ)$ is also a biordered set. Thus B_1 and B_2 are biordered sets with the same underlying quasiorderes and differ only in basic product. It follows that the quasiorders of a biordered set does not completely determine the biordered set.

Example 3.7 (Example 1.2 in Nambooripad [1979]): Let

$$C = \{e, f, h_{11}, h_{12}, h_{21}, h_{22}, g_{11}, g_{12}, g_{21}, g_{22}\}$$

be the band with the following multiplication table.

C	e	f	h_{11}	h_{12}	h_{21}	h_{22}	g_{11}	g_{12}	g_{21}	g_{22}
e	e	h_{11}	h_{11}	h_{12}	h_{11}	h_{12}	g_{21}	g_{22}	g_{21}	g_{22}
f	h_{22}	f	h_{21}	h_{22}	h_{21}	h_{22}	g_{21}	g_{22}	g_{21}	g_{22}
h_{11}	h_{12}	h_{11}	h_{11}	h_{12}	h_{11}	h_{12}	g_{21}	g_{22}	g_{21}	g_{22}
h_{12}	h_{12}	h_{11}	h_{11}	h_{12}	h_{11}	h_{12}	g_{21}	g_{22}	g_{21}	g_{22}
h_{21}	h_{22}	h_{21}	h_{21}	h_{22}	h_{21}	h_{22}	g_{21}	g_{22}	g_{21}	g_{22}
h_{22}	h_{22}	h_{21}	h_{21}	h_{22}	h_{21}	h_{22}	g_{21}	g_{22}	g_{21}	g_{22}
g_{11}	g_{11}	g_{12}	g_{12}	g_{12}	g_{12}	g_{12}	g_{11}	g_{12}	g_{11}	g_{12}
g_{12}	g_{12}	g_{12}	g_{12}	g_{12}	g_{12}	g_{12}	g_{11}	g_{12}	g_{11}	g_{12}
g_{21}	g_{21}	g_{22}	g_{22}	g_{22}	g_{22}	g_{22}	g_{21}	g_{22}	g_{21}	g_{22}
g_{22}	g_{22}	g_{22}	g_{22}	g_{22}	g_{22}	g_{22}	g_{21}	g_{22}	g_{21}	g_{22}

C is band consising of singleton subbands ($\mathscr{D}$-classes) (e) and (f) and rectangular bands

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$

Consider $E = C - \{h_{22}\}$. In the partial algebra determined by E, the products which are not defined are

$$fe, \quad fh_{12}, \quad h_{21}e, \quad h_{21}h_{12}.$$

These are not basic products in C since these pairs are not related by ihe quasi orders ω^r or ω^l in C. Hence the partial algebra E is a biordered subset of $\boldsymbol{E}(C) = C$. However,

$$\mathscr{S}(h_{12}, h_{21}) = \{h_{22}\} \quad \text{in } C \text{ and} \quad \mathscr{S}(h_{12}, h_{21}) = \{g_{22}\} \quad \text{in } E.$$

Hence E is not relatively regular in C.

Example 3.8 (Example 1.3 in Nambooripad [1979]): Let $\Gamma = (\Gamma, \leq)$ be a semilattice and X be a set such that $|X| \geq 1$. Define a partial binary operation on $E = X \times \Gamma$ as follows:

$$(x, e)(y, f) = \begin{cases} (x, ef) & \text{if} \quad e \leq f \quad \text{or} \quad f \leq e; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

It is easy to see that E with this partial product is a regular biordered set such that $\omega^r = \omega \subseteq \omega^l$. Let E^0 denote the biordered set obtained by adjoining zero 0 to E so that $E^0 = E \cup \{0\}$ with basic product extended to E^0 by

$$(x, e)0 = 0(x, e) = 00 = 0 \quad \text{for all} \quad (x, e) \in E.$$

Then the natural partial order E^0_ω of E^0 is a 0-disjoint union of semilattices isomorphic to Γ^0, the semilattice obtained by adjoining 0 to Γ. So E^0_ω is a semilattice and hence a regular biordered set. However, E^0 is not a semilattice since $|X| \geq 1$. Observe that the identity map is a bimorphism of E^0_ω onto E^0 which fail to satisfy axioms (RM1) and (RM2).

3.2 PROPERTIES OF BIORDERED SETS

Except for reordering, axioms (Bi), $i = 1, 2, 3$ of Theorem 3.2 are the same as those in Nambooripad [1979]. However, axiom (B4) here appears as (B4′) which is a part of Proposition 2.4 of Nambooripad [1979]. Now we proove that axiom (B4) for biordered sets in the Theorem 3.2 (or axiom (BO4) in Definition 3.1) can be replaced by an axiom, stated in Theorem 3.11 below as axiom (B4′), involving the sandwich sets. Notice that this is the same as [axiom (B4) Nambooripad, 1979]. We need some elementary properties of biordered sets in the proof of the equivalence. In the following, we use the abbreviation (Bij) for the axiom (Bi)(j) of Theorem 3.2.

In the first three results below, we assume that E denotes a partial algebra that satisfies all axioms of a biordered set except axioms (B4) and (B4)* of Theorem 3.2. All these atatements about biordered sets have their dual whose proof is the dual of the original statements. We shall not usually state or prove these explicitly. All results in this and the next sections are from [Nambooripad, 1979, Section 2].

PROPOSITION 3.9. *If* $(e, f) \in D_E$ *then* $ef \in \mathscr{S}(f, e)$.

Proof. Since $(e, f) \in D_E$, one of the following is true: $e\omega^r f$, $e\omega^l f$, $f\omega^r e$ or $f\omega^l e$. Suppose $e\omega^r f$. Then $e \mathscr{R} ef \omega f$ by axiom (B21). So $ef \in M(f, e)$. Let $g \in M(f, e)$. Then $g\omega^r e \mathscr{R} ef$ and $g, ef \in \omega^l f$. So by axiom (B3)* (ie dual of (B3)), $fg \omega^r f(ef) = ef$. Also, $ge \omega e = (ef)e$. Hence, by Definition 3.2, $g \preceq ef$ and it follows that $ef \in \mathscr{S}(e, f)$. Now, let $e \omega^l f$. Then clearly, $ef = e \in M(f, e)$. If $g \in M(e, f)$ then $g \omega^r e$ and $g, e = ef \in \omega^l(e)$. Hence, by (B3)*, $fg \omega^l fe$ and $ge \omega e = ee$. This gives $g \preceq ef$ and so, $ef \in \mathscr{S}(f, e)$. The result follows in the reaining cases by duality. □

PROPOSITION 3.10. *If* $f \omega^r e$ *then for every* $g \in \omega^r(f)$ *we have* $(gf)e = g(fe) = (ge)(fe)$.

Proof. By axiom (B21) we have $f \mathscr{R} fe$. So $\omega^r(fe) = \omega^r(f)$. Let $g \in \omega^r(f)$. Then by axiom (B21) we get $g \mathscr{R} gf \omega f$. Since $f \omega^r e$ we have $gf \omega^r e$. Also, from $gf \omega f$ we have $gf \omega^l f$. Now by axiom (B3) $(gf)e \omega^l fe$. Again from $gf \omega f$ we have $gf \omega^r f$ and so by (B21) $(gf)e \mathscr{R} gf \omega^r fe$. Thus $(gf)e \omega fe$. Now applying (B3) we get

$$(gf)e = ((gf)e)(fe) = (gf)(fe) = g(fe) = (ge)(fe). \qquad \square$$

Now we prove the equivalence of the two axioms (B4) and (B4′).

THEOREM 3.11. *Let E be an idempotent partial algebra satisfying all the axioms (Bi) of Theorem 3.2 above except (B4) and its dual. Then the following statements are equivalent:*

(B4) Let $e \in E$ and $f, g \in \omega^r(e)$ with $ge\ \omega^l\ fe$. Then there exist $g_1 \in M(f, e)$ such that $g_1 e = ge$.

(B4′) Let $e \in E$. Then for all $f, g \in \omega^r(e)$ we have $\mathscr{S}(f, g)e = \mathscr{S}(fe, ge)$.

Mreover, when these hold, the element g_1 in (B4) is unique.

Proof. Suppose that E satisfies (B4). Let $f, g \in \omega^r(e)$ and $h \in \mathscr{S}(f, g)$. Then by axiom (B21) and (B22) $he \in \omega^l(fe) \cap \omega^r(ge)$. Suppose that $k' \in \omega^l(fe) \cap \omega^r(ge)$. Then $k'\ \omega\ e$ and so k' and f satisfy the hypothesis of axiom (B4). Therefore there exists $k \in \omega^l(f) \cap \omega^r(e)$ such that $ke = k'$. Since $k\ \mathscr{R}\ k'\ \omega^r\ ge\ \mathscr{R}\ g$ we have $k \in \omega^l(f) \cap \omega^r(g)$ and so $k \preceq h$ in $M(f, g)$. Hence

$$
\begin{aligned}
(fe)k' &= (fe)(ke) && \\
&= (fk)e && \text{by (B3)}\\
&\ \omega^r\ (fh)e && \text{by (B21) and since } k \preceq h \text{ in } M(f,g)\\
&= (fe)h' && \text{by (B3);}\\
\text{and}\qquad k'(ge) &= (ke)(ge) && \\
&= (kg)e && \text{by Proposition 3.10}\\
&\ \omega^l\ (hg)e && \text{by (B22) since } k \preceq he \text{ in } M(f,g)\\
&= h'(ge) && \text{by Proposition 3.10.}
\end{aligned}
$$

Therefore $k' \preceq he \in M(fe, ge)$. This proves that $he \in \mathscr{S}(fe, ge)$. Consequently, $\mathscr{S}(f, g)e \subseteq \mathscr{S}(fe, ge)$.

To prove the reverse inclusion consider $h' \in \mathscr{S}(fe, ge)$. Using (B4) we can show as before there exists $h \in M(f, g)$ such that $he = h'$. Let $k \in M(f, g)$. Then using axioms (B21) and (B22) we get that $k' = ke \in M(fe, ge)$ and since $h' \in \mathscr{S}(fe, ge)$ we have $k' \preceq h'$ in $M(fe, ge)$. That is, $(fe)k'\omega^r(fe)h'$ and $k'(ge)\omega^l h'(ge)$.

Therefore

$$\begin{aligned}
fk &= ((fk)e)f && \text{by (B22)}\\
&= ((fe)(ke))f && \text{by (B3)}\\
&=\omega^r\ ((fe)h')f && \text{by (B21) since } k' \preceq h' \text{ in } M(fe, ge)\\
&= ((fh)e)f && \text{by (B3) since } h' = he\\
&= (fh)f && \text{by (B22)}\\
&= fh && \text{since } fh \leq f;\\
\text{and}\quad kg &= ((kg)e)g && \text{by (B22)}\\
&= ((ke)(ge))g && \text{by Proposition 3.10}\\
&\omega^l\ (h'(ge))g && \text{by (B3)}\\
&= ((hg)e)g && \text{by Proposition 3.10}\\
&= hg && \text{by (B31).}
\end{aligned}$$

Thus $k \preceq h$ in $M(f,g)$ and so $h \in \mathscr{S}(f,g)$. Therefore $\mathscr{S}(fe,ge) \subseteq \mathscr{S}(f,g)e$ and we conclude that $\mathscr{S}(f,g)e = \mathscr{S}(fe,ge)$. Thus (B4′) holds.

Conversely suppose that E satisfies (B4′) and let $e,g,h \in E$ satisfy the hypothesis of (B4). By the dual of Proposition 3.9 $ge \in \mathscr{S}(he,ge)$ and so by (B4′) there exists $g_1 \in \mathscr{S}(h,g)$ such that $g_1e = ge$. Clearly $g_1\ \omega^l\ h$. Hence (B4) holds.

Now we prove the uniqueness of g_1 in the statement (B4). Let g_2 also satisfy (B4) so that $g_2 \in M(f,e)$ and $g_2e = ge$. Then $g_1\ \mathscr{R}\ g_1e = ge = g_2e\ \mathscr{R}\ g_2$ by (B21) so that $g_1\ \mathscr{R}\ g_2$. On the other hand

$$\begin{aligned}
fg_1 &= ((fg_1)e)f && \text{by (B22)}\\
&= ((fe)(ge))f && \text{by (B3)}\\
&= ((fe)(g_2)e)f\\
&= fg_2.
\end{aligned}$$

Therefore $g_1\ \mathscr{L}\ fg_1 = fg_2\ \mathscr{L}\ g_2$. Hence by (B1), $g_1 = g_2$. □

For the remainder of this section, we assume that E, E', etc. denote biordered sets.

PROPOSITION 3.12. *Let $e\ \mathscr{L}\ e'$ and $f\ \mathscr{R}\ f'$ where $e,e',f,f' \in E$. Then $M(e,f) = M(e',f')$. Consequently $\mathscr{S}(e,f) = \mathscr{S}(e',f')$.*

Proof. The hypothesis implies that $\omega^l(e) \cap \omega^r(f) = \omega^l(e) \cap \omega^r(f)$. Let $g,h \in \omega^l(e) \cap \omega^r(f)$ and $g \preceq g$ in $M(e,f)$. Then by the definition, $eg\ \omega^r\ eh$ and $gf\ \omega^l\ hf$. Hence by the dual of axiom (B22) and (B3), we have

$$e'g = e'(eg)\ \omega^r\ e'(eh) = e'h.$$

Dually $gf' \ \omega^l \ hf'$. Therefore $g \leq h$ in $M(e', f')$. Interchanging e with e' and f with f' we infer similarly that $g \leq h$ in $M(e', f')$ implies they are so related in $M(e, f)$. Therefore the quasiorders on $M(e, f)$ and $M(e', f')$ are also the same. Thus $M(e, f) = M(e', f')$. The last statement now follows immediately from the definition of sandwich sets. □

Let S be a regular semigroup and $x, y \in S$. In view of the proposition above we may write $\mathscr{S}(x, y)$ for $\mathscr{S}(e, f)$ where $e, f \in \boldsymbol{E}(S)$ with $e \mathscr{L} x$ and $f \mathscr{R} y$.

If E is a biordered set, it is often necessary to verify whether a subset $E' \subseteq E$ is a biordered subset or not. Next proposition simplifies this verification.

PROPOSITION 3.13. *Let E' be a subset of the biordered set E. Then E' is a biordered subset of E if and only if E' satisfies the following conditions and their duals.*

(1) For all $e', f' \in E'$, $(e', f') \in D_E$ implies $e'f' \in E'$.

(2) If $e' \in E'$, $f', g' \in \omega^r(e') \cap E'$ and $g'e' \ \omega^l \ f'e'$ then there exists $g'_1 \in E' \cap M(f', e')$ such that $g'_1 e' = g'e'$.

Moreover, E' is relatively regular in E if and only if for all $e', f' \in E'$

(3) $\mathscr{S}'(e', f') = \mathscr{S}(e', f') \cap E'$; and

(4) $\mathscr{S}'(e', f') = \emptyset$ implies $\mathscr{S}(e', f') = \emptyset$

where $\mathscr{S}'(e', f')$ denote the sandwich set in E'.

Proof. Let E' be a biordered subset of E. Then E' is a partial subalgebra of E and so, $D_{E'} = D_E \cap E' \times E'$. Hence the condition (1) holds. The condition (2) is the same as axiom (B4) stated for E'. Conversely let E' be a subset of E satisfying (1) and (2). Then by (1) E' is a subalgebra of E so that the domain of the partial product on E' is $D_E \cap E' \times E'$. It can be verified that axioms (Bi), $i = 1, 2, 3$ hold. Statement (2) is precisely axiom (B4) stated for E'. Hence E' is a biordered subset of E.

Suppose that $E' \subseteq E$ is relatively regular so that $j^E_{E'}$ is a regular bimorphism. Let $\mathscr{S}'(e', f') \neq \emptyset$. By (RM1) $\mathscr{S}(e', f') \subseteq \mathscr{S}(e', f') \cap E'$. Since $\mathscr{S}(e', f')$ contain an element of $\mathscr{S}'(e', f')$, it is follows from Definition 3.3 that $\mathscr{S}'(e', f') = \mathscr{S}(e', f') \cap E'$. Thus the statement (3) holds. If $\mathscr{S}'(e', f') = \emptyset$, by axiom (RM2) the statement (4) also holds. Conversely, if statements (3) and (4) holds, then the map $j^E_{E'}$ satisfies axioms (RM1) and (RM2) of Definition 3.4 and so $j^E_{E'}$ is a regular bimorphism. Therefore E' is a relatively regular biordered subset of E. □

As an immediate application, we have:

biorder!right ideal
biorder!left ideal
biorder!ω-ideal

COROLLARY 3.14. *Let $\{E_i : i \in I\}$ be a family of biordered subsets of E. Then*

$$E' = \bigcap_{i \in I} E_i$$

is a biordered subset of E.

Proof. Let $e', f' \in E'$ and $(e', f') \in D_E$. Then $e', f' \in E_i$ and since E_i is a biordered subset, by Proposition 3.13(1), $e'f' \in E_i$ for every $i \in I$. Hence $e'f' \in E'$ and so, E' satisfies condition (1) of Proposition 3.13. Let e', f' and g' satisfy the hypothesis of the statement (2) of Proposition 3.13. Then by axiom (B4), there is $g_1 \in M(f', e')$ such that $g_1 e' = g' e'$. Since E_i is a biordered subset, by (2), $g_1 \in E_i$ for every i. Hence $g_1 \in E'$. Thus E' satisfies (2). □

3.2.1 Biorder ideals

For $e \in E$, the biordered subsets $\omega^r(e)$ will be called the principal *biorder right ideal*, $\omega^l(e)$ is called the principal *left* ideal and $\omega(e)$, the ω-ideal of E generated by e. A biorder isomorphism $\alpha : \omega(e) \to \omega(f)$ is called an *ω-isomorphism* of E. Since ω is a partial order, each ω-ideal has unique generator. So, if α is an ω-isomorphism, there is a unique $e_\alpha \in E$ such that $\operatorname{dom}\alpha = \omega(e_\alpha)$. Similarly there is a unique $f_\alpha \in E$ with $\operatorname{cod}\alpha = \omega(f_\alpha)$.

PROPOSITION 3.15. *For every $e \in E$,*

$$\omega^r(e), \quad \omega^l(e) \quad \text{and} \quad \omega(e)$$

are relatively regular biordered subsets of E and the translations

$$\tau^r(e) : f \mapsto fe \quad \text{and} \quad \tau^l(e) : g \mapsto eg$$

are regular idempotent bimorphisms of $\omega^r(e)$ and $\omega^l(e)$ respectively onto $\omega(e)$.

Proof. Let $f, g \in \omega^r(e)$ and $(f, g) \in D_E$. Then either

$$f \;\omega^r\; g, \quad g \;\omega^r\; f, \quad f \;\omega^l\; g, \quad \text{or} \quad g \;\omega^l\; f.$$

In the first case, $f \mathscr{R} fg \;\omega\; g \;\omega^r\; e$ by axiom (B21). Hence $fg \in \omega^r(e)$. If $g \;\omega^l\; f$, then by (B21)*, $fg \;\omega\; f \;\omega^r\; e$ and hence $fg \in \omega^r(e)$. In the remaining cases the conclusion $fg \in \omega^r(e)$ follows from Equation (3.5). Therefore $\omega^r(e)$ satisfies condition (1) of Proposition 3.13. Let $f, g, h \in \omega^r(e)$, $g, h \in \pm[r](f)$ and $gf \;\omega^l\; hf$. Then by axiom (B4), there exist $g_1 \in M(h, f)$ such that $g_1 f = gf$. Then $g_1 \mathscr{R} g_1 f = gf \;\omega\; f \;\omega^r\; (e)$ which implies that $g_1 \in \omega^r(e)$. Therefore by Proposition 3.13, $\omega^r(e)$ is a biordered subset of E. Now, for any $g, h \in \omega^r(e)$, we

$T^*(E)$:ordered groupoid of ω-isomorphisms of E

have $M(g,h) \subseteq \omega^r(e)$ and so, $\mathscr{S}(g,h) \subseteq \omega^r(e)$. This proves that $\omega^r(e)$ is relatively regular in E. Proofs for $\omega^l(e)$ and $\omega(e)$ are entirely similar.

By definition (see Equation (3.5)) $\tau^r(e) : \omega^r(e) \to \omega(e)$ is an idempotent map. To probe that $\tau^r(e)$ is a bimorphisms, let $f, g \in \omega^r(e)$ and $(f,g) \in D_E$. If $f\ \omega^r\ g$, by Proposition 3.10, we have

$$(fg)\tau^r(e) = (fg)e = (fe)(ge) = (f\tau^r(e))(g\tau^r(e)).$$

If $g\ \omega^r\ f$, then $fg = g$ and $ge\ \omega^r\ fe$ by axiom (B21). Hence we have $(fg)\tau^r(e) = (f\tau^r(e))(g\tau^r(e))$. If $f\ \omega^l\ g$, $fg = f$ and $fe\ \omega^l\ ge$ by axiom (B3). Thus $(fg)e = (fe)(ge)$. Finally, if $g\ \omega^l\ f$ then by (B3), $(fg)e = (fe)(ge)$. This proves, by definition 3.2, that $\tau^r(e)$ is a bimorphism. By condition (B4′) of Theorem 3.11 $\mathscr{S}(f,g)e = \mathscr{S}(fe,ge)$ for all $f,g \in \omega^r(e)$. Hence $\tau^r(e)$ satisfies (RM1) and (RM2) and so $\tau^r(e)$ is a regular bimorphism. Proof for $\tau^l(e)$ is dual. □

COROLLARY 3.16. *For $(e,f) \in \mathscr{L} \cup \mathscr{R}$ and $g\omega e$, define*

$$g\tau(e,f) = \begin{cases} fg & \text{if } e\ \mathscr{L}\ f; \\ gf & \text{if } e\ \mathscr{R}\ f. \end{cases}$$

Then $\tau(e,f) : \omega(e) \to \omega(f)$ is a biorder isomorphism.

Proof. Let $e\ \mathscr{R}\ f$. Then $\tau(e,f) = \tau^r(f)|\omega(e)$ and hence it is a bimorphism. Also, $\tau(e,f)^{-1} = \tau(f,e)$ and so $\tau(e,f)$ is a biorder isomorphism. Dually, $\tau(e,f) : \omega(e) \to \omega(f)$ is a biorder isomorphism when $e\ \mathscr{L}\ f$. □

Let T_E^* denote the collection of all ω-isomorphisms of E. It is easy to see that T_E^* is a groupoid under the groupoid composition:

$$\alpha \cdot \beta = \begin{cases} \alpha\beta & \text{the usual composition, if } f_\alpha = e_\beta; \\ \text{undefined} & \text{if } f_\alpha \neq e_\beta. \end{cases} \tag{3.10}$$

(see Examples 1.21 and 1.22.) Also the usual restriction of ω-isomorphisms defined by:

$$g \centerdot \alpha = \alpha|\omega(g) \quad \text{for all} \quad g\ \omega\ e_\alpha \tag{3.11}$$

is a partial order on T_E^*. With respect to this order, T_E^* satisfies axioms of Definition 1.6 and hence T_E^* is an ordered groupoid.

Since $\mathscr{R}$ is an equivalence relation $\mathscr{R}$ is also a groupoid (called the simplecial groupoid; see Example 1.20) in which $\mathfrak{v}\mathscr{R} = E$ and morphisms are pairs (e,f) with $e\ \mathscr{R}\ f$ and composition is defined by

$$(e,f)(f,g) = (e,g) \quad \text{if} \quad e\ \mathscr{R}\ f\ \mathscr{R}\ g. \tag{3.12}$$

Define restriction of $(e, f) \in \mathscr{R}$ to $g\ \omega\ e$ as follows:

τ_R: order preserving functor from $\mathscr{R}$ to T^*_E
τ_L: order preserving functor from $\mathscr{L}$ to T^*_E

$$g \centerdot (e, f) = (e, f)|g = (g, gf). \tag{3.13}$$

With respect to the partial order on $\mathscr{R}$ induced by this restriction, $\mathscr{R}$ is an ordered groupoid. Further if $e\ \mathscr{R}\ f\ \mathscr{R}\ g$, and $h\ \omega\ e$, then $h\ \omega^r\ g\ \omega^r\ f$. Hence by axiom (B22),

$$h\tau(e, f)\tau(f, g) = (hf)g = hg = h\tau(e, g).$$

Hence $\qquad \tau(e, f)\tau(f, g) = \tau(e, g).$

Also, for all $k\ \omega\ h$, again by (B22),

$$k\tau(h \centerdot (e, f)) = k\tau(h, hf) = k(hf) = (kf)(hf) = (kh)f = kf = k\tau(e, f).$$

Thus $\qquad \tau(h \centerdot (e, f)) = \tau(e, f)|\omega(h).$

Since $\tau(e, e) = 1_{\omega(e)}$, the assignments

$$\tau_R : e \mapsto 1_{\omega(e)}, \quad \text{and} \quad (e, f) \mapsto \tau(e, f) \tag{3.14}$$

is an order pteserving functor $\tau_R : \mathscr{R} \to T^*_E$.

Dually, the simplecial groupoid $\mathscr{L}$ is an ordered groupoid in which restriction of $(e, f) \in \mathscr{L}$ to $g\omega e$ is

$$g \centerdot (e, f) = (e, f)|g = (g, fg) \tag{3.13*}$$

and the assignments

$$\tau_L : e \mapsto 1_{\omega(e)}, \quad \text{and} \quad (e, f) \mapsto \tau(e, f) \tag{3.14*}$$

is an order preserving functor functor $\tau_L : \mathscr{L} \to T^*_E$.

Finally, since $\tau_R(e) = \tau_L(e)$ for all $e \in E$, the following diagram of ordered groupoids (in the category $\mathfrak{OG}$) commutes:

$$\begin{array}{ccc} \mathscr{R} & \xrightarrow{\tau_R} & T^*_E \\ \uparrow{\scriptstyle \jmath_r} & & \uparrow{\scriptstyle \tau_L} \\ 1_E & \xrightarrow[\jmath_l]{} & \mathscr{L} \end{array} \tag{3.15}$$

Here $\jmath_r : 1_E \subseteq \mathscr{R}$ is the inclusion of 1_E in $\mathscr{R}$. Observe that 1_E is trivially an ordered groupoid and the inclusion $\jmath_r$ is an order preserving functor. Dually $\jmath_l : 1_E \subseteq \mathscr{L}$ is an order preserving functor of 1_E into $\mathscr{L}$. We summorise these ideas for convenience of later reference:

E-array
E-square
E-square!degenerate
E-square!singular!column
E-square!singular!row
E-square!singular

PROPOSITION 3.17. *Let E be a biordered set. Then the set T^*_E of all ω-isomorphisms of E is an ordered groupoid with respect to the composition and restriction defined by Equations* (3.10) *and* (3.11). *Also simplecial groupoids $\mathscr{R}$ and $\mathscr{L}$ are ordered groupoids with respect to restriction defined by Equations* (3.13) *and* (3.13*) *respectively. Finally, the assignments of Equations* (3.14) *and* (3.14*) *define order preserving functors $\tau_R : \mathscr{R} \to T^*_E$ and $\tau_L : \mathscr{L} \to T^*_E$ such that the diagram 3.15 commutes in the category $\mathfrak{OG}$.* □

By an *E-array* we mean a marix

$$A = (e_{i\lambda})_{I\times\Lambda} \quad \text{over } E \text{ such that} \quad e_{i\lambda} \mathrel{\mathscr{L}} e_{j\lambda} \quad \text{and} \quad e_{i\lambda} \mathrel{\mathscr{R}} e_{i\sigma}$$

for all $i, j \in I$ and $\lambda, \sigma \in \Lambda$. The elements $e_{i\lambda}$ are called vertices of A. If $X \subseteq E$, A is an array in X if vertices of A belong to X. An E-subarray B of an E-array A is an E-array whose vertex set is a subsets of that of A. A 2×2 E-array is called an *E-square*. An E-square of one of the following type

$$\begin{pmatrix} e & f \\ e & f \end{pmatrix}, \quad \begin{pmatrix} e & e \\ f & f \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} e & e \\ e & e \end{pmatrix}$$

is said to be *degenerate*. If $g, h \in \omega^r(e)$ and $g \mathrel{\mathscr{L}} h$, then by axioms (B21) and (B3), $g \mathrel{\mathscr{R}} ge \mathrel{\mathscr{L}} he \mathrel{\mathscr{R}} h$. Hence we have the E-square

$$\begin{pmatrix} g & ge \\ h & he \end{pmatrix} \quad \text{whenever} \quad g, h \in \omega^r(e), \quad \text{and} \quad g \mathrel{\mathscr{L}} h.$$

Such E-squares are said to be *column-singular*. Dually, we have the E-square

$$\begin{pmatrix} g & h \\ eg & eh \end{pmatrix} \quad \text{whenever} \quad g, h \in \omega^l(e) \quad \text{and} \quad g \mathrel{\mathscr{R}} h.$$

An E-square of this form is said to be *row-singular*. A *singular E*-square is either clumn-singular, row-singular or degenerate.

An E-square $\left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right)$ is said to be τ-commutative if the following diagram commute:

$$\begin{array}{ccc} \omega(e) & \xrightarrow{\tau(e,f)} & \omega(f) \\ {\scriptstyle \tau(e,g)}\downarrow & & \downarrow{\scriptstyle \tau(f,h)} \\ \omega(g) & \xrightarrow[\tau(g,h)]{} & \omega(h) \end{array} \tag{3.16}$$

Every degenerate E-square is obviousely τ-commutative. Also, we say that an E-array A is τ-commutative if every 2×2-subsquare of A is τ-commutative. We have:

PROPOSITION 3.18. *Every singular E-square in a biordered set E is τ-commutative.*

Proof. Let $g, h \in \omega^r(e)$ and $g \mathscr{L} h$. To show that $\begin{pmatrix} g & ge \\ h & he \end{pmatrix}$ is τ-commutative, let $k \omega g$. Then

$$\begin{aligned} k\tau(g,h)\tau(h,he) &= (hk)(he) = ((hk)h)e && \text{by Proposition 3.10} \\ &= (hk)e = (he)(ke) && \text{by axiom (B3).} \end{aligned}$$

Also
$$\begin{aligned} k\tau(g,ge)\tau(ge,he) &= (he)(k(ge)) \\ &= (he)((kg)e) = (he)(ke) && \text{by Proposition 3.10.} \end{aligned}$$

It follows that every column-singular E-square is τ-commutative. Dually every row-singular E-square is

τ-commutative. The proof is now complete in view of the remark preceeding the statement of the proposition. □

The following proposotion derives some important consequences of axiom (B4) (and/or condition (B4′) of Theorem 3.11).

PROPOSITION 3.19. *Let $g, h \in \omega^r(e)$ and $ge \, \omega^l \, he$. Then there exists a unique E-square $G = \begin{pmatrix} g & g_1 \\ g_2 & h' \end{pmatrix}$ such that*

(a) $h' \omega h$;

(b) $ge = g_1 e$;

(c) $g_2 e = h'e = (he)(ge)$.

When G satisfies these conditions, then G is commutative and we have

(d) $h(kg_1) = (g_2 k)h$ for all $k \in \omega(g)$.

Mpreover, $h' = h$ if and only if $ge \mathscr{L} he$.

Proof. Since g, h and e satisfies the hypothesis of axiom (B4), there is $g_1 \in M(h, e)$ satisfying the condition (b). Let $h' = hg_1$ so that h' satisfies (a). Since $ge \, \omega^l \, he$, by proprefpr:3bs, $(he)(ge) \in \mathscr{S}(ge, he)$ and by Theorem 3.11(B4′), there is $g_2 \in \mathscr{S}(g, h)$ such that $g_2 e = (he)(ge)$. By axiom (B3), we have $h'e = (hg)e = (he)(ge)$ and so, g_2 and h' satisfies (c).

We next show that G is an E-square. By axiom (B21) and (b), we have $g \mathscr{R} ge = g_1 e \mathscr{R} g_1$. Similarly from (B21) and (c), we see that $g_2 \mathscr{R} h'$ and by the definition of h' and (B21)*, we have $h' \mathscr{L} g_1$. Since $g_2 \in \mathscr{S}(g, h)$, we have $g_2 \, \omega^l \, g$ and

$$\begin{aligned} gg_2 \mathscr{R} (gg_2)e &= (ge)(g_2 e) && \text{by axiom (B3);} \\ &= (ge)\,((he)(ge)) && \text{by (c);} \\ &= ge \mathscr{R} g && \text{by axioms (B21) and (B21)*.} \end{aligned}$$

Since $gg_2\ \omega\ g$, we have $gg_2 = g$. Therefore $g_2\ \mathscr{L}\ gg_2 = g$ and this proves that G is an E-square. To prove the uniquiness, let $G' = \left(\begin{smallmatrix} g & g_1' \\ g_2' & h'' \end{smallmatrix}\right)$ be another E-square satisfying conditions (a), (b) and (c). From (b) and (B4′) it follows that $g_1 = g_1'$. Now $h''\ \mathscr{L}\ g_1' = g_1\ \mathscr{L}\ h'$ and by (c), $h''\ \mathscr{R}\ h''e = h'e\ \mathscr{R}\ h'$. Therefore $h'' = h'$ and this forces $g_2 = g_2'$. Hence $G = G'$.

By Proposition 3.18 the column-singular E-squares $A = \left(\begin{smallmatrix} g & ge \\ g_2 & g_2e \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} g_1 & g_1e \\ h' & h'e \end{smallmatrix}\right)$ are commutative. Since $ge = g_1e$ and $g_2e = h'e$, we obtain

$$\begin{aligned}
(g, g_1)\tau(g_1, h') &= \tau(g, ge)\tau(ge, g_1)\tau(g_1, h') \\
&= \tau(g, ge)\tau(g_1e, h'e)\tau(h'e, h') && \text{from } B; \\
&= \tau(g, ge)\tau(ge, g_2e)\tau(h'e, h') && \text{by (b) and (c);} \\
&= \tau(g, g_2)\tau(g_2, g_2e)\tau(g_2e, h') && \text{by (c);} \\
&= \tau(g, g_2)\tau(g_2, h').
\end{aligned}$$

Hence G is commutative. To prove (d) we first verify a particular case:

$$g_2h = (g_2e)h = (h'e)h = h'h = h' = hg_1. \tag{d*}$$

Let $k \in \omega(g)$. Then

$$\begin{aligned}
k\tau(g, g_1)\tau(g_1, h') &= h'(kg_1) = (hg_1)(kg_1) && \text{by the definition of } h'; \\
&= h(g_1(kg_1)) && \text{by Proposition 3.10}^*; \\
&= h(kg_1).
\end{aligned}$$

Similarly
$$\begin{aligned}
k\tau(g, g_2)\tau(g_2, h') &= (g_2k)h' = (g_2k)(g_2h) && \text{by definition of } h' \text{ and (d}^*\text{);} \\
&= ((g_2k)h)(g_2h) && \text{by (B22);} \\
&= (g_2k)h && \text{since } (g_2k)h\ \omega\ g_2h.
\end{aligned}$$

This proves (d).

If $h' = h$ then $g_1\ \mathscr{L}\ h' = h$ and so, $ge = g_1e\ \mathscr{L}\ he$. On the other hand, if $ge\ \mathscr{L}\ he$, then by (c), $h'e = (he)(ge) = he$ and so, $h'\ \mathscr{R}\ h$. Since $h'\ \omega\ h$ by (a), it follows that $h' = h$. □

The following is a self-dual form of the proposition above. Part of it appeared as axiom (B5) in Nambooripad [1972]. Recall that $M(e, f)$ is the quasiordered set $(\omega^l(e) \cap \omega^r(f), \preceq)$ where $g \preceq h$ if and only if $eg\ \omega^r\ eh$ and $gf\ \omega^l\ hf$ (see Definition 3.3). Recall also that $\simeq$ which is an equivalence relation on $M(e, f)$ (see (3.7)).

PROPOSITION 3.20. *Let $g, h \in M(e, f)$ and $g \preceq h$. Then there exists a unique E-square $G = \left(\begin{smallmatrix} g & g_1 \\ g_2 & h' \end{smallmatrix}\right)$ in $M(e, f)$ such that*

(a) $h'\ \omega\ h$;

(b) $eg = eg_2 \mathscr{R} eh' = eg_1, \quad gf = g_1 f \mathscr{L} h'f = g_2 f;$

(c) $h(kg_1) = (g_2 k)h$ *for all* $k \in \omega(g)$.

In particular G is commutative and

$$g \simeq g_1 \simeq h' \simeq g_2.$$

Moreover, $g \simeq h$ *if and only if* $h' = h$.

Proof. The given conditions imply that g, h and f satisfies the hypothesis of Proposition 3.19 and g, h and e satisfy the dual hypothesis. Hence by Proposition 3.19 and its dual there exists uniqye E-squares $G = \begin{pmatrix} g & g_1 \\ g_2 & h' \end{pmatrix}$ and $K = \begin{pmatrix} g & k_1 \\ k_2 & k' \end{pmatrix}$ such that G satisfies (a), (b) and (c) of Proposition 3.19 with respect to g, h and f and K satisfies (a)*, (b)* and (c)* with respect to g, h and e. We show that $G = K$ thereby proving the proposition; we shall prove that G satisfies the conditions (a)*=(a),

(b)* $eg_2 = eg$; and

(c)* $eg_1 = eh' = (eg)(eh)$.

Since $eg_1 \mathscr{R} eg \ \omega^r \ eh$ and $g_1 \ \omega^l \ h$, we have $eg_1 \ \omega \ eh$. Hence

$$eh' = e(hg_1) = (eh)(eg_1) = eg_1 \quad \text{and so,} \quad eg_2 \mathscr{R} eh' = eg_1 \mathscr{R} eg.$$

Since $g_2 \mathscr{L} g$, we have $eg_2 = eg$. This proves (b)*. Again $eh' = e(g_2 h) = (eg_2)(eh) = (eg)(eh)$ and so, (c)* follows. Thus G satisfies (a)*, (b)* and (c)* with respect to g, h and e and by the uniqueness in Proposition 3.19*, $G = K$. Since $g, h \in M(e, f)$ $g_i \in M(e, f)$ for $i = 1, 2$ and so, G is an E-square in $M(e, f)$. Commutativity of G follows from Proposition 3.19 and relations $g \simeq g_1 \simeq h' \simeq g_2$ follow from (b). Finally, if $g \simeq h$, then by the definition of $\simeq$, $gf \mathscr{L} hf$ and so, $h' = h$ by Proposition 3.19. □

COROLLARY 3.21. *If* $e, f \in E$ *and* $\mathscr{S}(e, f) \neq \emptyset$, *then* $\mathscr{S}(e, f)$ *is a* τ*-commutative E-array.*

Proof. If $g, h \in \mathscr{S}(e, f)$ then $g \simeq h$. So $eg \mathscr{R} eg$ and $gf \mathscr{L} hf$. By Proposition 3.20 there is a unique commutatyive E-square $G = \begin{pmatrix} g & g_1 \\ g_2 & h \end{pmatrix}$ in $M(e, f)$ such that $g \simeq g_1 \simeq h \simeq g_2$. It follows that $g, g_1, h, g_2 \in \mathscr{S}(e, f)$ and hence G is a commutative E-square contained in $\mathscr{S}(e, f)$. Therefore $\mathscr{S}(e, f)$ is a commutative E-array. □

The following proposition is the biordered set analogue of [Clifford, 1974, Proposition 2.14] and is a crusial in associativity proofs.

PROPOSITION 3.22. *Let $g \in \mathscr{S}(e,f)$ and $h\ \omega^r\ f$. Then*

$$\mathscr{S}(g,h) \subseteq \mathscr{S}(e,h) \quad \text{and} \quad \mathscr{S}(g,h) \neq \emptyset \iff \mathscr{S}(e,h) \neq \emptyset.$$

Proof. Suppose that $k \in \mathscr{S}(g,h)$ and $i \in M(e,h)$. Since $k\ \omega^l\ g\ \omega^l\ e$, $i, k \in M(e,h) \subseteq M(e,f)$. Hence $i \preceq g$ and $k \preceq g$ in $M(e,f)$. Hence there is a unique E-square $I = \begin{pmatrix} i & i_1 \\ i_2 & g' \end{pmatrix}$ in $M(e,f)$ satisfying the conditions (a), (b) and (c) of Proposition 3.20. Hence $u \preceq g$ for all vertices of G. Now $i_1\ \mathscr{R}\ i\ \omega^r\ h$ and $i_1\ \omega^l\ g$. Hence $i_1 \in M(g,h)$ and so $i_1 \preceq k$ in $M(g,h)$. Since $gi_1\ \omega\ g\ \omega^l\ e$ and $i_1 \preceq g$, we have $gi_1\ \omega^l\ e$ and $ei_1\ \omega^r\ eg$. Hence $e(gi_1) = (eg)(ei_1) = ei_1$. Since $k \preceq g$, $ek\ \omega^r\ eg$ so that $e(gk) = (eg)(ek) = ek$. Therefore

$$ei\ \mathscr{R}\ ei_1 = e(gi_1)\ \omega^r\ e(gk) = ek \qquad \text{since } i_1 \preceq k \text{ in } M(g,h)$$

and so, $gi_1\ \omega^r\ gk$. Since $i\ \omega^r\ h\ \omega^r\ f$, using (B22) we obtain

$$ih = (if)h = (i_1 f)h = i_1 h\ \omega^l\ kh \qquad \text{since } i \preceq k \text{ in } M(g,h).$$

Therefore $i \preceq k$ in $M(e,h)$. This proves that $k \in \mathscr{S}(e,h)$ and that $\mathscr{S}(g,h) \neq \emptyset$ implies $\mathscr{S}(e,h) \neq \emptyset$.

Now let $u \in \mathscr{S}(e,h)$. Then $u \in M(e,h) \subseteq M(e,f)$ and so $u \preceq g$. By Proposition 3.20, there exista an E-square $H = \begin{pmatrix} u & u_1 \\ u_2 & g' \end{pmatrix}$ satisfying conditions

(a) $g'\ \omega\ g$;

(b) $eu = eu_2\ \mathscr{R}\ eg' = eu_1, \quad uf = u_1 f\ \mathscr{L}\ g'f = u_2 f$;

(c) $g(ku_1) = (u_2 k)g \quad \text{for all} \quad k \in \omega(u)$.

Since $u\ \mathscr{R}\ u_1$, $eu\ \mathscr{R}\ eu_1$ and $uh = (uf)h = (u_1 f)h = u_1 h$. So $u_1 \in \mathscr{S}(e,h)$. Since $u_1\ \omega^l\ g$, $u_1 \in M(g,h)$. If $v \in M(g,h)$ then $v \in M(e,h)$ and so, $v \preceq u_1$ in $M(e,h)$. Hence $ev\ \omega^r\ eu_1$ and so, $gv = g(ev)\ \omega^r\ g(eu_1) = gu_1$. Since $vh\ \omega^l\ u_1 h$, we conclude that $v \preceq u_1$ in $M(g,h)$. Therefore $u_1 \in \mathscr{S}(g,h)$. This also shows that if $\mathscr{S}(e,h) \neq \emptyset$, then $\mathscr{S}(g,h) \neq \emptyset$. □

As an immediate corollary we have the following [see Nambooripad, 1972, lemma 3.9].

COROLLARY 3.23. *Let $e, g \in E$ and $\alpha : \omega(f) \to \omega(f')$ be an ω-isomorphism of E. Let $h_1 \in \mathscr{S}(e,f)$, $h_2 \in \mathscr{S}(f',g)$, $h_1' = (h_1 f)\alpha$ and $h_2' = (f'h_2)\alpha^{-1}$. Then we have*

$$\mathscr{S}(h_1,h_2') \subseteq \mathscr{S}(e,h_2'),\ \mathscr{S}(h_1',h_2) \subseteq \mathscr{S}(e,h_2'),\ \text{and}\ \big(\mathscr{S}(h_1,h_2')f\big)\alpha = f'\mathscr{S}(h_1',h_2).$$

Proof. Clearly $h_2'\ \omega\ f$ and $h_1'\ \omega\ f'$. Hence from Proposition 3.22 and its dual we have $\mathscr{S}(h_1,h_2') \subseteq \mathscr{S}(e,h_2')$ and $\mathscr{S}(h_1',h_2) \subseteq \mathscr{S}(e,h_2')$. By axiom (B4′) $\mathscr{S}(h_1,h_2')f = \mathscr{S}(h_1 f,h_2' f) = \mathscr{S}(h_1 f,h_2')$. Since $\alpha : \omega(f) \to \omega(f')$ is a biorder isomorphism, it preserves basic products and by Definition 3.3 α induces an isomorphism of $M(h_1 f, h_2')$ onto $M((h_1 f))\alpha, (h_2')\alpha = M(h_1', f'h_2)$. Therefore

order-reflecting
order-reflecting!weakly

$$\left(\mathscr{S}(h_1, h_2')f\right)\alpha = \mathscr{S}(h_1', f'h_2) = f'\mathscr{S}(h_1', h_2). \qquad \square$$

3.2.2 Bimorphisms and biorder congruences

Here we propose to discuss cetain properties of bimorphisms. We shall be mostly concerned with regularity properties. We shall also give an intrinsic characterization of regular congruences on regular biordered sets.

Let (X, ρ) and (Y, σ) be quasi-ordered sets. Recall that a mapping $f : X \to Y$ is order-preserving if for all $x, y \in X$ with $x\rho y$, we have $xf\sigma yf$. f is said to *reflect* the quasiorders if for all $x, y \in X$, $x\rho y$ if $xf\sigma f$; θ reflect the quasiorders *weakly* if for all $y, y' \in Y$, $y'\sigma y$ and $x \in X$, $xf = y$, there exists $x' \in X$ with $x'\rho x$ and $x'f = y'$.

Next proposition establish some important properties of regular bimorphisms and shows that the category $\tilde{\mathfrak{B}}$ of biordered sets with morphisms as regular bimorphisms has images.

PROPOSITION 3.24. *Let $\theta : E \to E'$ be a regular bimorphism. Then satisfies the following conditions:*

(RM31) For all $e, f \in E$, the map $\theta : M(e, f) \to M_1(e\theta, f\theta) = M(e\theta, f\theta) \cap E\theta$ is surjective and quasiorder-preserving.

(RM32) $E\theta$ is a biordered subset of E'.

In particular, θ weakly reflects ω^r and ω^l.

Proof. By Definition 3.2 θ maps $M(e, f)$ into $M(e\theta, f\theta)$ and it preserves $\preceq$. To show that θ maps $M(e, f)$ onto $M_1(e\theta, f\theta) = M(e\theta, f\theta) \cap E\theta$, consider $g' \in M_1(e\theta, f\theta)$. Choose $g_1 \in E$ with $g_1\theta = g'$. Since $g' \ \omega^l \ e\theta$, by Proposition 3.9,

$$h' = (e\theta)g' = (eg_1)\theta \in \mathscr{S}(g', e\theta) = \mathscr{S}(g_1\theta, e\theta).$$

Hence, by axiom (RM2) of Definition 3.2 $\mathscr{S}(g_1, e) \neq \emptyset$. Let $h \in \mathscr{S}(g_1, e)$. Then by (RM1), $h\theta \in \mathscr{S}(g_1\theta, e\theta) = \mathscr{S}(g', e\theta)$. Then $h\theta \ \omega^l \ g' \ \omega^l \ e\theta$ and $h\theta \ \omega^r \ e\theta$. Therefore $h\theta \ \omega \ e\theta$ so that $(he)\theta = (h\theta)(e\theta) = h\theta$. Thus $g' \ \mathscr{L} \ h' \ \mathscr{L} \ h\theta$. By Proposition 3.12

$$\mathscr{S}((he)\theta, g_1\theta) = \mathscr{S}((he)\theta, g') = \mathscr{S}(g', g') = \{g'\}$$

and by (RM2), $\mathscr{S}(he, g_1) \neq \emptyset$. If $k_1 \in \mathscr{S}(he, g_1)$, than, as above, $k_1\theta = g'$ and $k_1 \ \omega^l \ he \ \omega \ e$. Dually there exists $k_2 \in \omega^r(f)$ such that $k_2\theta = g'$. Then

$\mathscr{S}(k_1\theta, k_2\theta) = \{g'\}$, and again by (RM2), $\mathscr{S}(k_1, k_2) \neq \emptyset$. If $g \in \mathscr{S}(k_1, k_2)$, $g \in M(k_1, k_2) \subseteq M(e, f)$ and $g\theta = g'$. This proves (RM31).

Before verifying (RM32), we shall show that θ weakly reflects ω^r and ω^l. If $e', f' \in E_i = E\theta$, and if $e' \ \omega^l \ f'$, then $e' \in M(f', e')$. Hence if $e, f \in E$ with $e\theta = e'$, $f\theta = f'$, then there exists $e_1 \in M(f, e)$ such that $e_1\theta = e'$. Therefore θ weakly reflects ω^l. Dually θ weakly reflects ω^r.

Let $e', f' \in E_1 = E\theta$ such that $(e', f') \in D_{E'}$. Then either $e' \ \omega^r \ f'$, $e' \ \omega^l \ f'$, $f' \ \omega^r \ e'$ or $f' \ \omega^l \ e'$. In all cases, we can find $e, f \in E$ with $e\theta = e'$, $f\theta = f'$ and (e, f) satisfies the same relation as (e', f'). Then $(ef)\theta = e'f' \in E_i$. Therefore E_1 satisfies condition (1) of Proposition 3.13. To prove (2), let $g', h', e' \in E_1$ with $g', h' \in \omega^r(e')$ and $g'e' \ \omega^l \ h'e'$. Then by Proposition 3.19 there is an E-square $G = \left(\begin{smallmatrix} g' & g'_1 & g'_2 & h'_1 \end{smallmatrix}\right)$ such that $h'_1 \ \omega \ h'$ and $h'_1e' = (h'e)(g'e')$. Therefore $h'_1 = (h'e')(g'e')h'$ and so, $h_{1'} \in E_1$ by (1). Also $\mathscr{S}(h'_1, g') = \{g'_1\} \neq \emptyset$. Now if $e \in E$ with $e\theta = e'$, since θ weakly reflects ω^r and ω^l, there exists $g, h_1 \in \omega^r(e)$ such that $g\theta = g'$, $h_1\theta = h'_1$. By (RM2), $\mathscr{S}(h_1, g) \neq \emptyset$. If $g_1 \in \mathscr{S}(h_1, g)$, $g_1\theta \in \mathscr{S}(h_1\theta, g\theta) = \mathscr{S}(h'_1, g') = \{g'_1\}$. Hence $g_1\theta = g'_1 \in E_1$. This proves (2). Since the proof for (2)* is dual, the statement (RM32) follows from Proposition 3.13. □

COROLLARY 3.25. *A bijective bimorphism θ is an isomorphism if and only if θ is regular.*

Proof. If θ is regular, by the proposition above θ reflects ω^r and ω^l weakly and since θ is bijective, it reflects the quasiorders. Hence if $(e', f') \in D_{E'}$ there exists $x, y \in E$ such that $(x, y) \in D_E$ is of the same type as (x', y') and $(xy)\theta = x'y'$. Therefore

$$(x'\theta^{-1})(y'\theta^{-1}) = xy = (x'y')\theta^{-1}.$$

It follows that $\theta^{-1} : E' \to E$ is the inverse of θ and so, θ is an isomorphism. Conversely, if θ is an isomorphism, it is clear that θ is regular. □

A partial converse of the above statement is also true: if θ is any bimorphism that satisfies (RM31) and the following,

(RM33) $E\theta$ is a relativelu regular biordered subset of E'.

then θ satisfies (RM1) [See Nambooripad, 1979, Proposition 2.14 for a proof].

Example 3.9 shows that conditions (RM31), (RM32) and/or (RM33) are neither necessary nor sufficient for regularity of a bimorphism.

However, if E is regular, conditions (RM31) and (RM32) completely characterises regularity.

PROPOSITION 3.26. *Let $\theta : E \to E'$ be a bimorphism of the regular biordered set E to E'. Then θ is regular if and only if it satisfies (RM31) and (RM33).*

Proof. Assume that θ is regular. Then by Proposition 3.24, θ satisfies (RM31) and (RM32) so that $E_1 = E\theta$ is a biordered subset of E'. Therefore, to prove (RM33), it is sufficient to show that

κθ:the biorder congruence of the bimorphism θ
biorder congruence
biorder congruence!regular–

$$\mathscr{S}_1(e\theta, f\theta) = \mathscr{S}'(e\theta, f\theta) \cap E_1 \quad \text{for all} \quad e, f \in E$$

where $\mathscr{S}_1$ denote the sandwich set in E_1 and $\mathscr{S}'$ denote the sandwhich set in E'. Let $e', f' \in E_1$ and choose $e, f \in E$ with $e' = e\theta$ and $f' = f\theta$. Since E is regular, $\mathscr{S}(e, f) \neq \emptyset$. Let $h \in \mathscr{S}(e, f)$. The regularity of θ implies that $h' = h\theta \in \mathscr{S}'(e', f')$. Cearly

$$\mathscr{S}'(e\theta, f\theta) \cap E_1 \subseteq \mathscr{S}_1(e\theta, f\theta)$$

Therefore $h' \in \mathscr{S}_1(e', f')$. If $g' \in \mathscr{S}_1(e', f')$, we have $g' \simeq h'$ in E_1 and so $e'g' \mathscr{R} e'h'$ and $g'f' \mathscr{L} h'f'$. Since these relations hold in E', $g' \in \mathscr{S}'(e', f')$. Therefore the desired equality holds.

Conversely assume that θ satisfies (RM31) and (RM33). In particular, θ satisfies (RM32) and so, by Proposition 3.24, θ satisfies (RM1). Since E is regular, axiom (RM32) is automatically satisfied. Therefore θ is regular. ☐

Let $\theta : E \to E'$ be a bimorphism. Then

$$\boldsymbol{\kappa}\theta = \theta \circ (\theta)^{-1} = \{(f, g) : f\theta = g\theta\} \tag{3.17}$$

is clearly an equivalence relation on E. $\boldsymbol{\kappa}\theta$ is called the *biorder congruence* of the bimorphism θ. If θ is regular, $\boldsymbol{\kappa}\phi\theta$ is called a *regular* biorder congruence on E.

PROPOSITION 3.27. *Let $\rho = \boldsymbol{\kappa}\theta$ be the congruence of a bimorphism $\theta : E \to E'$. For every $e \in E$, $e\rho$ is a biordered subset of E. If E is regular then $e\rho$ is a regular biordered subset of E and is relatively regular in E.*

Proof. Clearly $e\rho$ satisfies condition (1) of Proposition 3.13. To prove (2), let $f, g, h \in e\rho$, $g, h \in \omega^r(f)$ and $gf \ \omega^l \ hf$. Then by (B4), there is $g_1 \in M(h, f)$ such that $g_1 f = gf$. Let $e' = e\theta$. Since $gf \in e\rho$, we have

$$(g_1\theta)(f\theta) = (g_1\theta)e' = (g_1 f)\theta = (gf)\theta = e'$$

and

$$g_1\theta = (fg_1)\theta = e'(g_1\theta)$$

and so, $g_1\theta \ \mathscr{R} \ e'$. Since $g_1 \ \omega^l \ h$, $g_1\theta \ \omega^l \ e'$. This gives $g_1\theta = e'$ so that $g_1 \in e\rho$. Thus $e\rho$ satisfies condition (2). By duality (2)* also follows and so, $e\rho$ is a biordered subset of E.

Suppose that θ is regular and $f \in e\rho$. Then

$$\mathscr{S}(e, f)\theta \subseteq \mathscr{S}'(e\theta, f\theta) = \mathscr{S}'(e', e') = \{e'\}.$$

Then $\mathscr{S}(e, f) \neq \emptyset$. Therefore $e\rho$ is a regular biordered subset. Also $\mathscr{S}(e, f) \subseteq e\rho$ which implies that $e\rho$ is relatively regular in E. ☐

Next theorem characterises regular biorder congroences on a regular biordered set. Since we will not have occation to deal with the more general type of congruences, for brevity, we shall call these as biorder congroences (or simply congruences if no confusion is likely).

THEOREM 3.28. *Let ρ be an equivalence relation on a regular biordered set E. Then ρ is a congruence on E if and only if ρ satisfies the following conditions and their duals. In these statements $e, f, g \ldots$ etc. denote arbitrary elements of E.*

(BC1) $e\rho e', f\rho f'$ *and* $(e, f), (e', f') \in D_E \quad \Rightarrow \quad ef\rho e'f'$.

(BC2) $e' \in \rho(e) \quad \Rightarrow \quad \mathscr{S}(e', e) \subseteq \rho(e)$.

(BC3) $g, h \in \omega^r(e)$ *and* $\rho(ge) \cap \omega^l(he) \neq \emptyset \quad \Rightarrow$ *there exists* $g_1 \in M(h, g)$ *such that* $g_1 g \in \rho(g)$ *and* $g_1 e \in \rho(ge)$.

(BC4) $g \in M(e, f), e'\rho e$ *and* $f'\rho f \quad \Rightarrow \quad M(e', f') \cap \rho(g) \neq \emptyset$.

Proof. We observ that, since axioms for biordered sets and the axioms for comgruences above are self dual, the duality principle applies in this case. We shall use this observation in the following proof.

Let $E_1 = E/\rho$ where ρ is an equivalence relation on E satisfying the given conditions. Define a partial binary operation on E_1 by

$$\rho(e)\rho(f) = \begin{cases} \rho(e'f') & \text{if there exist } e'\rho e,\ f'\rho f \text{ with } (e', f') \in D_E; \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{3.18a}$$

If $e\rho e'\rho e''$ and $f\rho f'\rho f''$, and $(e', f'), (e'', f'') \in D_E$ then by (BC1), $\rho(e'f') = \rho(e''f'')$. Hence the equation above defines a single-valued partial binary operation on E_1.

Let ω_1^r and ω_1^l denote the relations on E_1 defined by Equation (3.5) with respect to this partial binary operation. We show that

$$\rho(f)\ \omega^r{}_1\ \rho(e) \iff \text{there exists } f' \in \rho(f) \text{ such that } f'\ \omega^r\ e; \tag{3.18b}$$

$$\rho(f)\ \omega^l{}_1\ \rho(e) \iff \text{there exists } f' \in \rho(f) \text{ such that } f'\ \omega^l\ e. \tag{3.18c}$$

If $f' \in \rho(f)$ with $f'\ \omega^r\ e$, then $(e, f') \in D_E$ and $ef' = f'$. Hence by Equation (3.18a), $\rho(e)\rho(f) = \rho(e)\rho(f') = \rho(f') = \rho(f)$. Thus $\rho(f)\ \omega^r{}_1\ \rho(e)$. On the other hand if $\rho(f)\ \omega^r{}_1\ \rho(e)$, there exist $e' \in \rho(e)$, $f' \in \rho(f)$ with $(e', f') \in D_E$ and $\rho(e'f') = \rho(f)$ so that $e'f' \in \rho(f)$. Since the basic product $e'f'$ exists, by axiom (B1) of Theorem 3.2, one of the following relation must hold:

$$e'\ \omega^r\ f', \quad e'\ \omega^l\ f', \quad f'\ \omega^r\ e', \quad f'\ \omega^l\ e'.$$

Let $e'\ \omega^r\ f'$. Then $e' \in M(e', f)$ and $f'\rho f$. Hence by (BC4), $M(e', f) \cap \rho(e') \neq \emptyset$. Let $e'' \in M(e', f) \cap \rho(e')$. Then $\rho(e'') = \rho(e)$ and $e''\ \omega^r\ f$ so that $e''f\ \mathscr{R}\ e''$. Also $\rho(e''f) = \rho(e'f') = \rho(f)$. Hence $e''f \in M(e'', f)$ and $e''\rho e$. Therefore, again by (BC4), there exist $f_1 \in M(f, e) \cap \rho(e''f)$. Then $f_1\ \omega^r\ e$ and $f_1 \in \rho(e''f) = \rho(f)$ which proves Equation (3.18b) in this case. If $e'\ \omega^l\ f'$, then $e'f' = e'$ and so $\rho(e) = \rho(e') = \rho(e'f') = \rho(f)$. Hence Equation (3.18b) holds in this case if we let $f' = e$. If $f'\ \omega^r\ e'$, then $f' \in M(f', e'), e\rho e'$ and $f\rho f'$. Hence by (BC4), there exists $f_1 \in M(f, e) \cap \rho(f')$. Therefore Equation (3.18b) holds with $f' = f_1$. Finally, let $f'\ \omega^l\ e'$ so that $e'f'\ \omega\ e'$ and $\rho(e'f') = \rho(f)$. Again, the desired result follows if we take $e'f'$ as f' in Equation (3.18b). Therefore Equation (3.18b) holds in all cases. Equation (3.18c) follows by duality.

It obvious from Equations (3.18b) and (3.18c) that ω^r_1 and ω^l_1 are quasiorders on E_1. Let $(\rho(e), \rho(f)) \in D_{E_1}$. Then by Equation (3.18a) there exist $e'\rho e$, $f'\rho f$ such that $(e', f') \in D_E$ and $\rho(e)\rho(f) = \rho(e'f')$. We can see using Equation (3.18b) and Equation (3.18b) that $\rho(e)$ and $\rho(f)$ are related by ω^r_1 and/or ω^l_1 in the same way e' and f' are related in terms of ω^r and ω^l. This implies that E_1 satisfies axiom (B1) of Theorem 3.2. It also follows that the quotient map $\rho^{\#} : e \mapsto \rho(e)$ of E onto E_1 preserve and weakly reflect the quasiorders ω^r to ω^r_1 and ω^l to ω^l_1 respectively. To prove (B21), assume that $\rho(e)\ \omega^r{}_1\ \rho(f)$. By Equation (3.18a) we may assume that $e\ \omega^r\ f$ and so, $e\ \mathscr{R}\ ef\omega f$ by (B21). Since $\rho^{\#}$ preserves quasiorders and their inverses, it follows that $\rho(e)\ \mathscr{R}_1\ \rho(ef)\omega_1\rho(f)$. Similar arguements can be used to prove axiom (B22) for E_1. Let $\rho(g), \rho(f) \in \omega^r_1(\rho(e))$ and $\rho(g)\ \omega^l{}_1\ \rho(f)$. By Equation (3.18b) we may assume that $g, f \in \omega^r(e)$. By Equation (3.18c), there is $g_1 \in \rho(g)$ with $g_1\ \omega^l\ g$. By (BC2), $\mathscr{S}(g_1, g) \subseteq \rho(g)$. Let $g' \in \mathscr{S}(g_1, g)$ Then $g' \in M(g_1, g) \subseteq M(f, e)$ and so, $g', f \in \omega^r(e)$. Therefore $g'e\ \omega^l\ fe$. Further, by Equation (3.18a) and Equation (3.18c) we have

$$\rho(g)\rho(e) = \rho(g'e)\ \omega^l{}_1\ \rho(fe) = \rho(f)\rho(e);$$

and
$$\begin{aligned}(\rho(f)\rho(e))\,(\rho(g)\rho(e)) &= \rho(fe)\rho(g'e) = \rho((fe)(g'e)) && \text{by (3.18a);}\\ &= \rho((fg')e) = \rho(fg')\rho(e) && \text{by (B3);}\\ &= (\rho(f)\rho(g))\,\rho(e).\end{aligned}$$

This proves axiom (BC3). To prove (B4), let $\rho(g), \rho(h) \in \omega^r_1(\rho(e))$ and $\rho(g)\rho(e)\ \omega^l{}_1\ \rho(h)\rho(e)$ where $g, h \in \omega^r(e)$. Then $\rho(ge)\ \omega^l\ \rho(he)$ and by Equation (3.18c), $\rho(ge) \cap \omega^l(he) \neq \emptyset$. By (BC3) there exists $g_1 \in M(g, h)$ such that $g_1g \in \rho(g)$ and

$g_1e \in \rho(ge)$. This implies that $\rho(g_1)\rho(g) = \rho(g_1g) = \rho(ge) = \rho(g)\rho(e)$. Since $g_1 \; \omega^l \; h$ if follows that $\rho(g_1) \; \omega^l{}_1 \; \rho(h)$ and axiom (B4) follows. Since duals of these axioms follow, we have shown that E_1 is a biordered set and $\rho^\# : E \to E_1$ is a bimorphism.

We proceed to show that $\rho^\# : E \to E_1$ is a regular bimorphism. Since $\rho^\#$ is surjective, (RM33) holds. Since $\rho^\#$ is a bimorphism, $\rho^\#$ is a map of $M(e, f)$ into $M(\rho(e), \rho(f))$ that preserve the quasiorder $\preceq$. Now suppose that $G \in M(\rho(e), \rho(f))$. By Equations (3.18b) and (3.18c) we can find $g_1, g_2 \in G$ such that $g_1 \; \omega^l \; e$ and $g_2 \in \omega^r \; f$. If $g \in \mathscr{S}(g_1, g_2)$ then by (BC2), $g \in G$ and so $\rho(g) = G$. Further $g \in M(g_1, g_2) \subseteq M(e, f)$. Hence $\rho^\#$ maps $M(e, f)$ onto $M(\rho(e), \rho(f))$ and thus $\rho^\#$ satisfies (RM31). Therefore by Proposition 3.26, $\rho^\#$ is a regular bimorphism. In particular, E_1 is a regular biordered set.

Comversely, assume that $\rho = \boldsymbol{\kappa}\theta$ where $\theta : E \to E'$ is a surjective regular bimorphism of the regular biordered set E. Then by Definition 3.2, (RM1) and (RM31), ρ satisfies (BC1), (BC2) and (BC4). Let $g, h \in \omega^r(e)$ and $\rho(ge) \cap \omega^l(he) \neq \emptyset$. Then $g\theta, h\theta \in \omega^r(e\theta)$ and $g\theta e\theta = (ge)\theta \; \omega^l \; (he)\theta = f\theta e\theta$. Hence, by (B4), there exists $G \in M(h\theta, e\theta)$ such that $G\rho(e) = \rho(g)\rho(e)$. Thus $G \; \mathscr{R} \; \rho(g)$. Hence $G \in M(\rho(h), \rho(g))$. Then by (RM31), there exist $g_1 \in M(h, g)$ such that $G = g_1\theta = \rho(g_1)$. Therefore we have

$$\rho(g_1g) = \rho(g_1)\rho(g) = G\rho(g) = \rho(g);$$
$$\rho(g_1e) = G\rho(e) = \rho(g)\rho(e) = \rho(ge).$$

Therefore ρ satisfies axiom (BC3) also. □

If $\theta : E \to E'$ is a regular bimorphism of a regular biordered set, then $E\theta = E_1$ is a relatively regular biordered subset of E' (by Proposition 3.26). If $\rho = \boldsymbol{\kappa}\theta$ then $\rho^\# : E \to E/\rho$ is a surjective regular bimorphism of E onto the quotient E/ρ. Also the map $\psi : E/\rho \to E_1; \; \rho(e) \mapsto e\theta$ is a bijection. Now if $\rho(e)\rho(f)$ exists in E/ρ, by Equation (3.18a) there exist $e' \in \rho(e)$, $f' \in \rho(f)$ such that $(e', f') \in D_E$ and $\rho(e)\rho(f) = \rho(e'f')$. Then

$$(e\theta)(f\theta) = (e'f')\theta$$

so that the product $(e\theta)(f\theta)$ exists and

$$(\rho(e))\psi(\rho(f))\psi = (e\theta)(f\theta) = (e'f')\theta = (\rho(e'f'))\psi.$$

Similarly, one can see that if $(e\theta)(f\theta)$ exists in E_1 then $(\rho(e))(\rho(f))$ exists in E/ρ and we have the equality above. Therefore $\psi : E/\rho \to E_1$ is a biorder isomorphism. We have the following isomorphism theorem.

THEOREM 3.29. *Let E be a regular biordered set and let $\theta : E \to E'$ be a regular bimorphism. Then there exists an isomorphism $\psi : E/\boldsymbol{\kappa}\theta \to E\theta$ such that the following diagram commute:*

$$\begin{array}{ccc} E & \xrightarrow{\boldsymbol{\kappa}\theta^{\#}} & E/\boldsymbol{\kappa}\theta \\ & {}_{\theta}\searrow & \downarrow{\scriptstyle\psi} \\ & & E' \end{array} \tag{3.19}$$

where $\boldsymbol{\kappa}\theta^{\#} : E \to E/\boldsymbol{\kappa}\theta$ is the quotient bimorphism.

3.3 EMBEDDING OF BIORDERED SETS IN SEMIGROUPS

We have seen that the set of idempotents of any semigroup is a biordered set and the map induced by a homomorphism $\phi : S \to S'$ on the biordered set $\boldsymbol{E}(S)$ is a bimorphism $\boldsymbol{E}(\phi) : \boldsymbol{E}(S) \to \boldsymbol{E}(S')$ (see Theorem 3.3). In this section we consider the converse problem of ebedding a given biordered set E as biordered set of some semogroup S so that E is isomorphic to $\boldsymbol{E}(S)$ and thus showing that the original set of axioms for biordered sets [see Nambooripad, 1979, Definition 1.1] are both necessary and sufficient inorder that a biordered set represent the set of idempotents of a semigroup. It may be noted that this problem was solved for the particular case of biordered sets of regular semigroups in Nambooripad [1979] itself using the theory of inductive groupoids which will be considered elsewhere in this work. Results in this section are due to Easdown [1985]. In presenting the results we have followed (except for Easdown's arrow notations) [Higins, 1992] which provide a good account of Easdown's theory.

3.3.1 A representation

We begin by constructing a representation of a given biordered set as biordered subset of a semigroup of partial transformations. This is the principal tool Easdown uses to get the desired embbeding [see Easdown, 1985].

Let E be a biordered set and assume that

$$\begin{aligned} I^{\circ} &= I \cup \{\infty\} \qquad \text{where } I = E/\mathscr{R}; \text{ and} \\ \Lambda^{\circ} &= \Lambda \cup \{\infty\} \qquad \text{where } \Lambda = E/\mathscr{L} \end{aligned} \tag{3.20a}$$

where ∞ does not represent an element in either I or Λ. For $e \in E$, we write R_e [L_e] for the $\mathscr{R}$-class [$\mathscr{L}$-class] of E. Hence for any $R \in I^{\circ}$, either $R = R_e$ for some $e \in E$ or $R = \infty$; similar remarks hold for elements of Λ°. Now, for $e \in E$, define $\rho(e)$ as follows. For any $L \in \Lambda^\circ$,

$$L\rho(e) = \begin{cases} L_{ge} & \text{if } L \in \Lambda \text{ and } g \in L \cap \omega^r(e); \\ \infty & \text{if } L \in \Lambda \text{ and } L \cap \omega^r(e) = \emptyset; \\ \infty & \text{if } L = \infty. \end{cases} \tag{3.20b}$$

The map $\rho(e) : \Lambda^\circ \to \Lambda^\circ$ is single-valued. For, if $L \in \Lambda$ and if $g, h \in L \cap \omega^r(e)$ then by axiom (B3), $L_{ge} = L_{he}$. $\rho(e)$ is clearly single-valued in other cases. Notice that $L\rho(e)$ takes values in Λ if and only if L intersects the right ideal $\omega^r(e)$. Moreover, $\rho(e)$ is an idempotent in $\mathscr{T}_{\Lambda^\circ}$ and so, this gives a map

$$\rho : E \to \boldsymbol{E}(\mathscr{T}_{\Lambda^\circ}), \quad e \mapsto \rho(e).$$

Dually we define $\lambda(e) : I^\circ \to I^\circ$: For any $R \in I^\circ$,

$$R\lambda(e) = \begin{cases} R_{eg} & \text{if } R \in I \text{ and } g \in R \cap \omega^l(e); \\ \infty & \text{if } R \in I \text{ and } R \cap \omega^l(e) = \emptyset; \\ \infty & \text{if } R = \infty. \end{cases} \tag{3.20b°}$$

As above, $\lambda(e) : I^\circ \to I^\circ$ is single-valued and $R\lambda(e)$ takes value in I if and only if R intersects $\omega^l(e)$. Again $\lambda(e)$ is an idempotent in $\mathscr{T}^*_{I^\circ}$, the left-right dual of $\mathscr{T}_{I^\circ}$ and we have the map

$$\lambda : E \to \boldsymbol{E}\left(\mathscr{T}^*_{I^\circ}\right), \quad e \mapsto \lambda(e).$$

We now set

$$\varphi_E(e) = \varphi(e) = (\rho(e), \lambda(e)) \tag{3.20c}$$

which defines a map

$$\varphi : E \to \boldsymbol{E}\left(\mathscr{T}_{\Lambda^\circ} \times \mathscr{T}^*_{I^\circ}\right).$$

We proceed to show that the map φ is a biorder embedding (see Definition 3.2) of E into $\boldsymbol{E}\left(\mathscr{T}_{\Lambda^\circ} \times \mathscr{T}^*_{I^\circ}\right)$. We divide the proof into lemmas some of which will be of interest later.

Lemma 3.30. *For $(e, f) \in D_E$, we have $\rho(ef) = \rho(e)\rho(f)$ and $\lambda(ef) = \lambda(e)\lambda(f)$.*

Proof. By (B12), it is sufficient to prove the following equations. If $e\ \omega^r\ f$ then

$$\rho(e) = \rho(fe) = \rho(f)\rho(e) \tag{i}$$

and
$$\rho(ef) = \rho(e)\rho(f). \tag{ii}$$

If $e\ \omega^l\ f$ then

$$\rho(e) = \rho(ef) = \rho(e)\rho(f) \tag{iii}$$

and
$$\rho(fe) = \rho(f)\rho(e). \tag{iv}$$

To prove (i), assume that $L\rho(e) \neq \infty$ for $L \in \Lambda$. Then there is $g \in L \cap \omega^r(e)$ such that $L\rho(e) = L_{ge}$. Since $g \; \omega^r \; e \; \omega^r \; f$, $gf \; \omega^r \; e$ and we have $L\rho(f) \neq \infty$,

$$L\rho(f)\rho(e) = L_{gf}\rho(e) = L_{(gf)e} \neq \infty.$$

Since $(gf)e = ge$ by (B22), it follows that $L\rho(f)\rho(e) = L\rho(e)$ for all $L \in \Lambda$ such that $L \cap \omega^r(e) \neq \emptyset$. Next assume that $L\rho(f)\rho(e) \neq \infty$. Then by Equation (3.20b), $L\rho(f) \neq \infty$ and so, there is $g \in L \cap \omega^r(e)$ with $L\rho(f) = L_{gf}$ and there is $h \in L_{gf} \cap \omega^r(e)$ with

$$(L\rho(f))\,\rho(e) = L_{gf}\rho(e) = L_h\rho(e) = L_{he}.$$

Now $h \; \mathscr{L} \; gf \; \omega \; f$ and $h \; \omega^r \; e \; \omega^r \; f$ which gives $h \; \omega \; f$. Hence $g, h \in \omega^r(f)$ and $h = hf \; \mathscr{L} \; gf$ and by (B4), there is $h_1 \in L \cap \omega^r(f)$ such that $h_1 f = h$. By (B22), $(h_1 f)e = h_1 e$ which gives

$$(L\rho(f))\,\rho(e) = L_{he} = L_{(h_1 f)e} = L_{h_1 e} = L\rho(e)$$

so that $L\rho(e) \neq \infty$. Consequently, $L\rho(e) = \infty$ if and only if $L\rho(f)\rho(e) = \infty$, so that equation (i) holds in all cases.

Proof of (ii). Assume that $L\rho(ef) \neq \infty$ so that there exists $g \in L \cap \omega^r(ef)$ such that $L\rho(ef) = L_{g(ef)}$. Then

$$g \; \mathscr{R} \; g(ef) \; \omega \; ef \; \mathscr{R} \; e \quad \text{and} \quad ge \; \omega \; e \; \omega^r \; f.$$

Therefore

$$(L\rho(e))\,\rho(f) = (L_{ge})\rho(f) = L_{(ge)f} \neq \infty.$$

On the other hand, if $(L\rho(e))\,\rho(f) \neq \infty$, there is $g \in L \cap \omega^r(e)$ with $L\rho(e) = L_{ge}$. Then $ge \; \mathscr{R} \; e \; \omega^r \; f$ and so, $L_{ge}\rho(f) = L_{(ge)f}$. Since $g \; \omega^r \; e \; \mathscr{R} \; ef$, $L\rho(ef) = L_{g(ef)}$. By Proposition 3.10, $g(ef) = (ge)f$. Therefore $L\rho(ef) \neq \infty$ and $L\rho(ef) = (L\rho(e))\,\rho(f)$. Again we have $(L\rho(e))\,\rho(f) = \infty$ if and only if $L\rho(ef) = \infty$ and equation (ii) holds in all cases.

Proof of (iii). If $L\rho(e) \neq \infty$ there exists $g \in L \cap \omega^r(e)$ such that $L\rho(e) = L_{ge}$. Now we have

$$g \; \mathscr{R} \; ge \; \omega \; e \; \omega^l \; f \quad \text{so that} \quad ge \; \mathscr{L} \; f(ge) \; \omega \; f.$$

Therefore

$$L\rho(e)\rho(f) = L_{ge}\rho(f) = L_{f(ge)}\rho(f) = L_{(f(ge))f} = L_{f(ge)} = L_{ge} = L\rho(e).$$

If $L\rho(e)\rho(f) \neq \infty$, clearly $L\rho(e) \neq \infty$. It follows that equation (iii) holds in all cases.

Proof of (iv). Again suppose that $L\rho(fe) \neq \infty$ so that there is $g \in L \cap \omega^r(fe)$ with $L\rho(fe) = L_{g(fe)}$. Then $g \;\omega^r\; fe \;\omega\; f$ and so, $gf \;\omega\; f$. Then $gf, e \in \omega^l(f)$ and $f(gf) = gf \;\omega^r\; fe$. Hence by (B4*), there exists $g_1 \in \omega^l(f)$ such that $g_1 \;\omega^r\; e$ and $fg_1 = gf$. Then $g_1 \mathscr{L} gf$ and $g_1 e \;\omega\; e \;\omega^l\; f$. Therefore

$$\begin{aligned} f(ge) &= (fg_1)(fe) && \text{by axiom (B3*)} \\ &= (gf)(fe) = g(fe) && \text{by Proposition 3.10.} \end{aligned}$$

Therefore
$$\begin{aligned} (L)\rho(f)\rho(e) &= (L_{gf})\rho(e) = (L_{g_1})\rho(e) \\ &= L_{g_1 e} = L_{f(g_1 e)} = L_{g(fe)} \\ &= (L)\rho(fe). \end{aligned}$$

Let $(L)\rho(f)\rho(e) \neq \infty$. Then there exists $g \in L$ with $g \;\omega^r\; f$ and there exists $h \in L_{gf} \cap \omega^r(e)$ with

$$(L)\rho(f)\rho(e) = (L_{gf})\rho(e) = (L_h)\rho(e) = L_{he}.$$

Now $h, e \in \omega^l(f)$ and $h \;\omega^r\; e$ and so, by (B3*), $fh \;\omega^r\; fe$. Also $fh, g \in \omega^r(f)$ and $gf \mathscr{L} fh = (fh)f$. Hence by Proposition 3.19, there is $h_1 \in M(g, f)$ such that $h_1 \mathscr{L} g$ and $h_1 f = (fh)f = fh$. Therefore $h_1 \;\omega^r\; fe$ and so, $(L)\rho(fe) = L_{h_1(fe)} \neq \infty$. We conclude that the equation (iv) holds.

We have thus shown that for all $(e, f) \in D_E$, $\rho(ef) = \rho(e)\rho(f)$. The statement that for all $(e, f) \in D_E$, $\lambda(ef) = \lambda(e)\lambda(f)$ follows by duality. □

Lemma 3.31. *For $e, f \in E$, $e \;\omega^l\; f$ if and only if $\rho(e)\rho(f) = \rho(e)$ and $e \;\omega^r\; f$ if and only if $\lambda(f)\lambda(e) = \lambda(e)$.*

Proof. The 'if' part of the above statement follows from Lemma 3.30 and so, it is sufficient to prove the 'only if' part. So assume that $\rho(e)\rho(f) = \rho(e)$. Then

$$L_e = (L_e)\rho(e) = (L_e)\rho(e)\rho(f) = (L_e)\rho(f).$$

Hence there exists $g \in L_e$ such that $g \;\omega^r\; f$ and

$$(L_e)\rho(f) = L_{gf} = L_e.$$

Then $e \mathscr{L} gf \;\omega\; f$ so that $e \;\omega^l\; f$. If $\lambda(f)\lambda(e) = \lambda(e)$ then by dual reasoning, we have $e \;\omega^r\; f$. □

The following theorem is one of the fundamental results in Easdown's theory of biordered sets [see **?**].

Theorem 3.32. *Let E be a biordered set and $\varphi : E \to E^*$ be the map defined by Equation* (3.20c) *where*

$$E^* = \boldsymbol{E}\left(\mathscr{T}_{\Lambda^\circ} \times \mathscr{T}^*_{I^\circ}\right). \qquad (\star)$$

Then φ is a biorder embedding of E into E^*. Consequently $E\varphi$ is a biordered subset of E^* isomorphic to E.

φ_E: Fundamental embedding of the biordered set E
fundamental embedding

Proof. We first show that φ is injective. If $\varphi(e) = \varphi(e')$, then $\rho(e) = \rho(e')$ and $\lambda(e) = \lambda(e')$. Hence $\rho(e)\rho(e') = \rho(e)$ so that $e \; \omega^l \; e'$ by Lemma 3.31. Similarly $e' \; \omega^l \; e$ and so $e \mathscr{L} e'$. Dually, we have $e \mathscr{R} e'$. Consequently $e = e'$.

Let $(e, f) \in D_E$. Then by Lemma 3.30, we have

$$\begin{aligned}\varphi(e)\varphi(f) &= (\rho(e), \lambda(e))(\rho(f), \lambda(f)) = (\rho(e)\rho(f), \lambda(e)\lambda(f)) \\ &= (\rho(ef), \lambda(ef)) = \varphi(ef).\end{aligned}$$

Hence $\varphi : E \to E^*$ is a bimorphism. Moreover, $(\varphi(e), \varphi(f)) \in D_{E^*}$ and these are related in the same way as e and f. On the other hand, if $(\varphi(e), \varphi(f)) \in D_{E^*}$, by Lemma 3.31 $(e, f) \in D_E$ and the relation between e and f is the same as the relation between $\varphi(e)$ and $\varphi(f)$. It follows that $\varphi : E \to E\varphi$ is a biorder isomorphism.

For convenience, let us write $E' = E\varphi$. Since

$$\varphi(e)\varphi(f) = \varphi(ef) \quad \text{for all} \quad (e, f) \in D_E,$$

E' is a partial subalgebra of E^*. Hence E' satisfies the condition (1) of Proposition 3.13. We now verify condition (2). Let $e, f, g \in E$ such that $\varphi(f), \varphi(g) \in \omega^r(\varphi(e))$ and $\varphi(g)\varphi(e) \; \omega^l \; \varphi(f)\varphi(e)$. By Lemma 3.31, we have $f, g \in \omega^r(e)$ and $ge \; \omega^l \; fe$. Hence by (B4), there is $g_1 \in M(f, e)$ such that $g_1 e = ge$. Then by Lemma 3.30, $\varphi(g_1) \in M(\varphi(f), \varphi(e))$ and $\varphi(g_1)\varphi(e) = \varphi(g)\varphi(e)$. This proves, by Proposition 3.13, that E is a biordered subset of E^*. □

The theorem above gives an embedding $\varphi : E \to E^*$ where E^* is the biordered set defined by Equation (★) above. We shall call $\varphi = \varphi_E$ as the *fundamental embedding* of the biordered set E. By Lemma 3.30 the projections

$$\rho : E \to \mathscr{T}_{\Lambda^\circ}, \quad e \mapsto \hat{e} = \rho(e);$$

and

$$\lambda : E \to \mathscr{T}^*_{I^\circ}, \quad e \mapsto \tilde{e} = \lambda(e)$$

are bimorphisms which preserve and reflect basic products. Consequently, as in the last paragraph of the above proof, we can show that $E\rho$ is a biordered subset of $\boldsymbol{E}(\mathscr{T}_{\Lambda^\circ})$. Dually, $E\lambda$ is a biordered subset of $\boldsymbol{E}(\mathscr{T}^*_{I^\circ})$.

3.3.2 Easdown's theorem

According to Easdown the following result is due to Hall [see Easdown, 1985, ?]. The following proof is essentially from Higins [1992].

$\langle E\varphi\rangle$:The fundamental semiband of E
letters
words
cover

LEMMA 3.33. *Let $\langle E\varphi\rangle$ denote the subsemigroup of $\mathscr{T}_{\Lambda^\circ} \times \mathscr{T}^*_{I^\circ}$ generated by $E\varphi$. If α is an idempotent in $\langle E\varphi\rangle$ and if $\varphi(e) \mathscr{L} \alpha \mathscr{R} \varphi(f)$ in the semigroup $\langle E\varphi\rangle$ for $e, f \in E$, then $\alpha \in E\varphi$.*

Proof. The given condition implies by Theorem 2.34 that

$$\varphi(e) \mathscr{R} \varphi(e)\varphi(f) \mathscr{L} \varphi(f)$$

in the semigroup $\langle E\varphi\rangle$. Taking projections into $\mathscr{T}_{\Lambda^\circ}$, we have $\rho(e) \mathscr{R} \rho(e)\rho(f) \mathscr{L} \rho(f)$. Since $\rho(e) \mathscr{R} \rho(e)\rho(f)$ these transformations determine the same partition of Λ° (see Example 2.10). Now $L_e\rho(e) = L_e \neq \infty$. Hence the set U in the partition $\pi_{\rho(e)}$ that contain L_e does not contain ∞. Since $(\infty)\rho(e)\rho(f) = \infty$ and $U \in \pi_{\rho(e)} = \pi_{\rho(e)\rho(f)}$, $L_e\rho(e)\rho(f) \neq \infty$. Therefore there is $g \in L_e$ such that $g\ \omega^r\ f$ and $L_e\rho(f) = L_{gf}$. Again since $\rho(e)\rho(f) \mathscr{L} \rho(f)$ in $\langle E\rho\rangle$, we have $\rho(g)\rho(f)\ \omega\ \rho(f)$. On the other hand, by Theorem 2.34 $\rho(f) \mathscr{L} \rho(g)\rho(f)$. Hence $\rho(gf) = \rho(f)$ and so, $\rho(e) \mathscr{L} \rho(g) \mathscr{R} \rho(f)$. This implies, by Lemma 3.31, that

$$\varphi(e) \mathscr{L} \varphi(g) \mathscr{R} \varphi(f).$$

Therefore $\alpha \mathscr{H} \varphi(g)$. Since both α and $\varphi(g)$ are idempotents in $\langle E\varphi\rangle$, $\alpha = \varphi(g)$. Thus $\alpha \in E\varphi$. □

Suppose that E is a biordered set and let E^+ [E^*]denote the free semigroup [monoid] on the set E. Elements of E are called *letters* and those of E^* are called *words*. Symbols e, f, g, h, etc. [u, v, w,etc.] with or without subscripts and superscripts will denote letters [words]. If h is a letter of the word u, the rank of h is the position of h in u when we count letters of u from left. The length $l(u)$ of a word u is the number of letters in u.

Multiplication in E^* will be denoted by juxtaposition and the basic product in E will be denoted by $\cdot$. Thus if $(e, f) \in D_E$, ef denote a word of length two while $e \cdot f$ denotes a single letter. We shall say that v is a subword of w if $w = uvu'$ for some (possibly empty) $u, u' \in E^*$. Words $w_1, w_2, \ldots, w_n$ *cover* a word w if there are subwords w'_i of w_i, $i = 1, 2, \ldots, n$ such that $w = w'_1 w'_2 \ldots w'_n$. We may identify E with a subset of E^+ by identifying every $e \in E$ with the word having the only letter e so that $E \subseteq E^+$. Notice that elements of E are not idempotents in E^+; in particular, E is not a biordered subset of E^+.

Define the relation σ on E^+ by

$$\sigma = \{(fg, f \cdot g) : (f, g) \in D_E\}.$$

Let $\sigma^\#$ denote the smallest congruence on E^* containing σ (see Proposition 2.7). Let

$$\boldsymbol{B}_0(E) = E^+/\sigma^\#. \tag{3.21a}$$

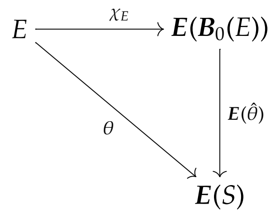
and
$$\chi_E : E \to E(\boldsymbol{B}_0(E)), \quad e \mapsto \sigma^\#(e) \tag{3.21b}$$

The semigroup $\boldsymbol{B}_0(E)$ is called the *E-free semigroup* (or the free semigroup generated by the biordered set E). Easdown's theorem asserts that (see the theorem below) that the map χ_E is an isomorphism of biordered sets. χ_E will be called the *universal isomorphism* of the biordered set E.

$\boldsymbol{B}_0(E)$: Free semigroup generated by E
free semigroup!E-free sewmigroup
χ_E: Universal isomorphism of E
biordered set!universal isomorphism
elementary σ transition
elementary σ transition! of type

The following statement is equivalent to Easdown's theorem [see Easdown, 1985, Theorem 3.3]. Except for minor differences in notation and arrangement, the proof below is the same as the proof from [Easdown, 1985].

THEOREM 3.34. *Let E be a biordered set and let $\boldsymbol{B}_0(E)$ denote the free semigroup generated by E. Then the universal isomorphism $\chi_E : E \to E(\boldsymbol{B}_0(E))$ is a biorder isomorphism such that, given any simigroup S and bimorphism $\theta : E \to \boldsymbol{E}(S)$ there exists a unique homomorphism $\hat{\theta} : \boldsymbol{B}_0(E) \to S$ making the following diagram commute:*



(3.22)

For clarity, we divide the proof into a number of lemmas. Notation established in this section so far is taken into account below.

Recall from Proposition 2.7 that $(w, w') \in \sigma^{\#}$ if and only if there exists a finite sequence w_i, $i = 0, 1, \ldots, n$ of words in E^+ with $w_0 = w$, $w_n = w'$ and for each $i = 1, 2, \ldots, n$ there exist $u_i, v_i \in E^*$ such that either

$$w_{i-1} = u_i(fg)v_i, \quad w_i = u_i(f \cdot g)v_i \quad \text{or} \quad w_{i-1} = u_i(f \cdot g)v_i, \quad w_i = u_i(fg)v_i.$$

The passage from w_{i-1} to w_i is called an *elementary σ-transition* and is indicated as $T : w_{i-1} \mapsto w_i$. In case when $f\ \omega^r\ g$ or $f\ \omega^l\ g$, we have $f\ \mathscr{R}\ f \cdot g$ and the corresponding elementary σ-transition T is called *type* (1). If either $g\ \omega^r\ f$ or $g\ \omega^l\ f$ then $g\ \mathscr{L}\ f \cdot g$ and T is said to be of type (2). For brevity we shall write

$$w_\varphi = \varphi(f_1)\varphi(f_2)\ldots\varphi(f_n) \tag{3.23}$$

for any word $w = f_1 f_2 \ldots f_n \in E^+$.

LEMMA 3.35. *Let $f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_m \in E$ and $u = f_1 \ldots f_n$, $v = g_1 \ldots g_m$. If $\sigma^{\#}(u) = \sigma^{\#}(v)$ then $u_\varphi = v_\varphi$.*

Proof. Since $\varphi : E \to E\varphi$ is a border isomorphism, we have $\varphi(f)\varphi(g) = \varphi(f \cdot d)$ for all $(f, g) \in D_E$. Consequently if $T : w \mapsto w'$ is an elementary σ-transition, then $w_\varphi = w'_\varphi$. It follows from the above remarks that $w_\varphi = w'_\varphi$ whenever $\sigma^{\#}(w) = \sigma^{\#}(w')$. □

The following lemma is also due to T. E. Hall.

LEMMA 3.36. *Suppose that* $\sigma^{\#}(w) \in \boldsymbol{E}(\boldsymbol{B}_0(E))$ *and that* $\sigma^{\#}(w)\ \mathscr{D}\ \sigma^{\#}(e)$ *for some letter* e*. Then the congruence class* $\sigma^{\#}(w)$ *contains a letter.*

Proof. Let $w = e_1e_2\ldots e_N$. Since $\sigma^{\#}(w)$ and $\sigma^{\#}(e)$ are $\mathscr{D}$-related idempotents there exist $u = f_1f_2\ldots f_n$ and $v = g_1g_2\ldots g_m$ such that $\sigma^{\#}(u)$ is an inverse of $\sigma^{\#}(v)$ and

$$\sigma^{\#}(w) = \sigma^{\#}(u)\sigma^{\#}(v) = \sigma^{\#}(f_1\ldots f_ng_1\ldots g_m)$$

and
$$\sigma^{\#}(e) = \sigma^{\#}(v)\sigma^{\#}(u) = \sigma^{\#}(g_1\ldots g_nf_1\ldots f_m).$$

It follows from Lemma 3.35 that w_φ and $\varphi(e)$ are idempotents in $\langle E\varphi\rangle$ and

$$w_\varphi = u_\varphi v_\varphi = (uv)_\varphi$$

and
$$\varphi(e) = v_\varphi u_\varphi.$$

Hence
$$\rho(e) = \rho(g_1)\rho(g_2)\ldots\rho(g_m)\rho(f_1)\rho(f_2)\ldots\rho(f_n).$$

Since $L_e\rho(e) = L_e \neq \infty$, it follows that

$$(L_e)\rho(g_1)\rho(g_2)\ldots\rho(g_m) \neq \infty.$$

By Equation (3.20b), there exist $k_1, k_2, \ldots k_m$ such that

$$k_1 \in L_e \cap \omega^r(g_1)$$
$$\text{and}\quad k_i \in L_{k_{i-1}\cdot g_{i-1}} \cap \omega^r(g_i), \quad 1 < i \leq m.$$

Then in the semigroup $\boldsymbol{B}_0(E)$ we have

$$\sigma^{\#}(k_1\cdot g_1) = \sigma^{\#}(k_1)\sigma^{\#}(g_1)\ \mathscr{L}(\boldsymbol{B}_0(E))\ \sigma^{\#}(e)\sigma^{\#}(g_1) = \sigma^{\#}(eg_1).$$

Similarly

$$\sigma^{\#}(k_2\cdot g_2) = \sigma^{\#}(k_2)\sigma^{\#}(g_2)\ \mathscr{L}(\boldsymbol{B}_0(E))\ \sigma^{\#}(k_1\cdot g_1)\sigma^{\#}(g_2)$$
$$\mathscr{L}(Bo)\ \sigma^{\#}(eg_1)\sigma^{\#}(g_2) = \sigma^{\#}(eg_1g_2)$$

Repeating the process, we finally arrive at

$$\sigma^{\#}(k_m\cdot g_m) = \sigma^{\#}(k_m)\sigma^{\#}(g_m)\ \mathscr{L}(\boldsymbol{B}_0(E))\ \sigma^{\#}(k_{m-1}\cdot g_{m-1})\sigma^{\#}(g_m)$$
$$\mathscr{L}(Bo)\ \sigma^{\#}(eg_1\ldots g_{m-1})\sigma^{\#}(g_m) = \sigma^{\#}(eg_1g_2\ldots g_m)$$

Since

$$\sigma^{\#}(eg_1\ldots g_m) = \sigma^{\#}(e)\sigma^{\#}(g_1\ldots g_m) = \sigma^{\#}(g_1\ldots g_m)\ \mathscr{L}(\boldsymbol{B}_0(E))\ \sigma^{\#}(w),$$

we have $\sigma^\#(k_m \cdot g_m)\ \mathscr{L}(\boldsymbol{B}_0(E))\ \sigma^\#(w)$. Since $k = k_m \cdot g_m$ is a letter, there is a letter k with $\sigma^\#(k)\ \mathscr{L}(\boldsymbol{B}_0(E))\ \sigma^\#(w)$. Dually, there exists a letter $l \in E$ such that $\sigma^\#(l)\ \mathscr{R}(\boldsymbol{B}_0(E))\ \sigma^\#(w)$. It follows from Lemma 3.35 that

$$\varphi(k)\ \mathscr{L}\langle E\varphi\rangle\ w_\varphi\ \mathscr{R}\langle E\varphi\rangle\ \varphi(l).$$

Hence by Lemma 3.33, $\varphi(z) = w_\varphi$ for some $z \in E$. It now follows from Theorem 3.32 that

$$k\ \mathscr{L}\ z\ \mathscr{R}\ l$$

in E. Hence by the definition of $\sigma^\#$, we have $\sigma^\#(k)\ \mathscr{L}(\boldsymbol{B}_0(E))\ \sigma^\#(z)\ \mathscr{R}(\boldsymbol{B}_0(E))\ \sigma^\#(l)$. Therefore $\sigma^\#(z)$ and $\sigma^\#(w)$ are $\mathscr{H}$-related idempotents in $\boldsymbol{B}_0(E)$. Consequently $\sigma^\#(z) = \sigma^\#(w)$. □

We next show that $\chi_E : E \to \boldsymbol{E}(\boldsymbol{B}_0(E))$ is surjective. The lemma above proves that any idempotent $\mathscr{D}$-related to an idempotent $\sigma^\#(e)$, $e \in E$ is again of the same type. Consequently, to prove that χ_E is surjective, it is sufficient to show that every idempotent $\sigma^\#(w)$, $w = e_1e_2\dots e_n$ in $\boldsymbol{B}_0(E)$ is $\mathscr{D}$-related to an idempotent $\sigma^\#(z)$, $z \in E$.

Since $\sigma^\#(w)$ is an idempotent, we have $\sigma^\#(w) = \sigma^\#(w^n)$. Hence there exist words w_k, $k = 1, 2, \dots, N$ with $w_1 = w$, $w_N = w^n$ and elementary transitions $T_k : w_k \mapsto w_{k+1}$, for $1 \le k < N$. For each k, $1 \le k \le N$, we shall construct a cover w_k^i, $i = 1, 2, \dots, n$ of w_k such that each $\sigma^\#(w_k^i)\ \mathscr{D}(\boldsymbol{B}_0(E))\ \sigma^\#(f)$ for some $f \in E$.

We define the subwords w_k^i inductively in terms of the position of letters from 1 to $l(w_k)$. For this purpose, we define three finite sequences of positive integers $\{\alpha_k^i, \beta_k^i, \gamma_k^i : 1 \le i \le n,\ 1 \le k \le N\}$ as follows:

$$\alpha_1^i = \beta_1^i = \gamma_1^i = i \quad \text{for} \quad i = 1, 2, \dots, n; \tag{1}$$

For each i, $1 \le i \le n$, define inductively in k:

$$\beta_{k+1}^i = \begin{cases} \beta_k^i & \text{if } T_k : ufgv \mapsto uf\cdot gv \text{ where } l(u) \ge \beta_k^i - 1; \\ & \text{or } T_k : uf\cdot gv \mapsto ufgv \text{ where } l(u) \ge \beta_k^i; \\ & \text{or } l(u) = \beta_k^i - 1 \text{ and } T_k \text{ is of type (1)}; \\ \beta_k^i - 1 & \text{if } T_k : ufgv \mapsto uf\cdot gv \text{ where } l(u) \le \beta_k^i - 2; \\ \beta_k^i + 1 & \text{if } T_k : uf\cdot gv \mapsto ufgv \text{ where } l(u) \le \beta_k^i - 2; \\ & \text{or } l(u) = \beta_k^i - 1 \text{ and } T_k \text{ is not of type (1)}; \end{cases} \tag{2}$$

$$\alpha^i_{k+1} = \begin{cases} \alpha^i_k & \text{if } T_k : ufgv \mapsto uf \cdot gv \text{ where } l(u) \geq \alpha^i_k - 1; \\ & \text{or } l(u) = \alpha^i_k - 2, \alpha^i_k < \beta^i_k, \text{ and } T_k \text{ is not of type (2);} \\ & \text{or } T_k : uf \cdot gv \mapsto ufgv \text{ where } l(u) \geq \alpha^i_k - 1; \\ \alpha^i_k - 1 & \text{if } T_k : ufgv \mapsto uf \cdot gv \text{ where } l(u) \leq \alpha^i_k - 3; \\ & \text{or } l(u) = \alpha^i_k - 2 \text{ and either } \alpha^i_k = \beta^i_k \text{ or } T_k \text{ is of type (2);} \\ \alpha^i_k + 1 & \text{if } T_k : uf \cdot gv \mapsto ufgv \text{ where } l(u) \leq \alpha^i_k - 2; \end{cases} \tag{3}$$

$$\gamma^i_{k+1} = \begin{cases} \gamma^i_k & \text{if } T_k : ufgv \mapsto uf \cdot gv \text{ where } l(u) \geq \gamma^i_k; \\ & \text{or } l(u) = \gamma^i_k - 1, \text{ and either } \gamma^i_k = \beta^i_k, \text{ or } T_k \text{ is of type (1);} \\ & \text{or } T_k : uf \cdot gv \mapsto ufgv \text{ where } l(u) \geq \gamma^i_k; \\ \gamma^i_k - 1 & \text{if } T_k : ufgv \mapsto uf \cdot gv \text{ where } l(u) \leq \gamma^i_k - 2; \\ & \text{or } l(u) = \gamma^i_k - 1, \beta^i_k < \gamma^i_k \text{ and } T_k \text{ is not of type (1);} \\ \gamma^i_k + 1 & \text{if } T_k : uf \cdot gv \mapsto ufgv \text{ where } l(u) \leq \gamma^i_k - 1; \end{cases} \tag{4}$$

For natural numbers i and j with $i \leq j$, let $[i, j]$ denote the set of all integers k with $i \leq k \leq j$. We now show that

LEMMA 3.37. *Let $\{\alpha^i_k, \beta^i_k, \gamma^i_k\}$ be finite sequences defined by Equations (1), (2), (3) and (4) above. Then for all $k \in [1, N]$,*

$$\beta^1_k \leq \beta^2_k \leq \cdots \leq \beta^n_k, \tag{5}$$

and

$$\alpha^i_k \leq \beta^i_k \leq \gamma^i_k \qquad \textit{for all} \quad i \in [1, n]. \tag{6}$$

Further,

$$[1, l(w_k)] = \bigcup_{i=1}^{n} [\alpha^i_k, \gamma^i_k] \qquad \textit{for all} \quad k \in [1, N]. \tag{7}$$

Proof. Let us say, for brevity that the elementary transition $T_k : w_k \mapsto w_{k+1}$ is expanding if it is of the type $u(f \cdot g)v \mapsto u(fg)v$ so that $l(w_{k+1}) = l(w_k) + 1$. Otherwise, T_k will be called reducing.

To prove (5), notice that, by Equation (1), the desired relations hold for $k = 1$. Assume inductively that the relations (5) hold for $k < N$. We consider two cases and several subcases under each.

T_k **is expanding**. If $l(u) \geq \beta^i_k$ then by hypothesis, $l(u) \geq \beta^{i-1}_k$ and so,

$$\beta^i_{k+1} = \beta^i_k \geq \beta^{i-1}_k = \beta^{i-1}_{k+1}$$

by Equation (2). Suppose that $l(u) = \beta^i_k - 1$ and that T_k is of type (1). If $\beta^i_k = \beta^{i-1}_k$, then we have

$$\beta^i_{k+1} = \beta^i_k = \beta^{i-1}_k = \beta^{i-1}_{k+1}.$$

If $\beta^i_k > \beta^{i-1}_k$, then $l(u) > \beta^{i-1}_k - 1$. So $l(u) \geq \beta^{i-1}_k$. Hence

$$\beta^i_{k+1} = \beta^i_k > \beta^{i-1}_k = \beta^{i-1}_{k+1}.$$

On the other hand, if $l(u) = \beta^i_k - 1$ and that T_k is not of type (1), then $\beta^i_{k+1} = \beta^{i-1}_{k+1}$ if $\beta^i_k = \beta^{i-1}_k$ and if $\beta^i_k > \beta^{i-1}_k$, $l(u) \geq \beta^{i-1}_k$ and so,

$$\beta^i_{k+1} = \beta^i_k + 1 > \beta^{i-1}_k + 1 = \beta^{i-1}_{k+1} + 1 > \beta^{i-1}_{k+1}.$$

Let $l(u) \leq \beta^i_k - 2$. If $l(u) \leq \beta^{i-1}_k$, then

$$\beta^i_{k+1} = \beta^i_k + 1 \geq \beta^{i-1}_k + 1 = \beta^{i-1}_{k+1}.$$

If $l(u) = \beta^{i-1}_k - 1$, then $\beta^{i-1}_{k+1} = \beta^{i-1}_k$ so that

$$\beta^i_{k+1} = \beta^i_k + 1 \geq \beta^{i-1}_k + 1 = \beta^{i-1}_{k+1} + 1 \geq \beta^{i-1}_{k+1}$$

if T_k is of type (1) and if T_k is not of type (1) then

$$\beta^i_{k+1} = \beta^i_k + 1 \geq \beta^{i-1}_k + 1 = \beta^{i-1}_{k+1}.$$

T_k **is reducing.** If $l(u) \geq \beta^i_k - 1$ we clearly have $l(u) \geq \beta^i_k - 1$ and so, $\beta^i_{k+1} \geq \beta^{i-1}_{k+1}$. If $l(u) \leq \beta^i_k - 2$ then again the desired inequality follows if $l(u) \leq \beta^{i-1}_k - 2$. Otherwise, we have $\beta^i_k - 2 \geq l(u) \geq \beta^{i-1}_k - 1$ so that $\beta^i_k - 1 \geq \beta^{i-1}_k$. Hence

$$\beta^i_{k+1} = \beta^i_k - 1 \geq \beta^{i-1}_k = \beta^{i-1}_{k+1}.$$

We have now shown that $\beta^i_{k+1} \geq \beta^{i-1}_{k+1}$ in all cases. Since this holds for all $i = 1, 2, \ldots, n$ the proof of (5) is complete.

To prove (6), we again consider two cases.

T_k **is expanding**. Let $l(u) \geq \beta^i_k$. By inductive hypothesis, $l(u) \geq \alpha^i_k - 1$ and it follows from Equation (3) that

$$\alpha^i_{k+1} = \alpha^i_k \leq \beta^i_k = \beta^i_{k+1}.$$

If we also have $l(u) \geq \gamma^i_k$, then by Equation (4)

$$\beta^i_{k+1} = \beta^i_k \leq \gamma^i_k = \gamma^i_{k+1}.$$

If $\beta^i_k \leq l(u) < \gamma^i_k$ then $l(u) \leq \gamma^i_k - 1$ and so

$$\beta^i_{k+1} = \beta^i_k \leq \gamma^i_k = \gamma^i_{k+1} - 1 < \gamma^i_{k+1}.$$

Next, let $l(u) = \beta^i_k - 1$ and T_k be of type (1). Then $l(u) \geq \alpha^i_k - 1$ and $l(u) \leq \gamma^i_k - 1$. Hence from Equations (3) and (4) we have

$$\alpha^i_{k+1} = \alpha^i_k \leq \beta^i_k = \beta^i_{k+1} \leq \gamma^i_k = \gamma^i_{k+1} - 1 < \gamma^i_{k+1}.$$

If $l(u) = \beta_k^i - 1$ and T_k is not of type (1) then

$$\alpha_{k+1}^i = \alpha_k^i \le \beta_k^i = \beta_{k+1}^i - 1 < \beta_{k+1}^i \le \gamma_k^i = \gamma_{k+1}^i - 1 < \gamma_{k+1}^i.$$

If $l(u) \le \beta_k^i - 2$ then $l(u) \le \gamma_k^i - 2$. If $l(u) \le \alpha_k^i - 2$ then

$$\alpha_{k+1}^i = \alpha_k^i + 1 \le \beta_k^i + 1 = \beta_{k+1}^i \le \gamma_k^i + 1 < \gamma_{k+1}^i.$$

On the other hand, if $l(u) \ge \alpha_k^i - 1$, then, as above we have

$$\alpha_{k+1}^i = \alpha_k^i \le \beta_k^i = \beta_{k+1}^i - 1 < \beta_{k+1}^i \le \gamma_k^i = \gamma_{k+1}^i - 1 < \gamma_{k+1}^i.$$

T_k is reducing. Let $l(u) \ge \beta_k^i - 1$. Then $l(u) \ge \alpha_k^i - 1$. If $l(u) \ge \gamma_k^i$ then from (3) and (4),

$$\alpha_{k+1}^i = \alpha_k^i \le \beta_k^i = \beta_{k+1}^i \le \gamma_k^i = \gamma_{k+1}^i.$$

If $l(u) = \gamma_k^i - 1$ and either $\gamma_k^i = \beta_k^i$ or T_k is of type (1), the equation above remain valid. Again if $l(u) = \gamma_k^i - 1$ and either $\gamma_k^i > \beta_k^i$ or T_k is not of type (1), then

$$\alpha_{k+1}^i = \alpha_k^i \le \beta_k^i = \beta_{k+1}^i \le \gamma_k^i = \gamma_{k+1}^i - 1 < \gamma_{k+1}^i.$$

Assume that $l(u) \le \beta_k^i - 2$ so that $l(u) \le \gamma_k^i - 2$. If $l(u) \ge \alpha_k^i - 1$, then $\beta_k^i - 2 \ge l(u) \ge \alpha_k^i - 1$. Thus $\beta_k^i - 1 \ge \alpha_k^i$. Therefore

$$\alpha_{k+1}^i = \alpha_k^i \le \beta_k^i - 1 = \beta_{k+1}^i \le \gamma_k^i - 1 = \gamma_{k+1}^i.$$

If $l(u) = \alpha_k^i - 2$, $\alpha_k^i < \beta_k^i$ and T_k is not of type (2) then $\beta_k^i \ge l(u) = \alpha_k^i - 2$

$$\alpha_{k+1}^i = \alpha_k^i \le \beta_k^i - 1 = \beta_{k+1}^i \le \gamma_k^i - 1 = \gamma_{k+1}^i.$$

If $l(u) = \alpha_k^i - 2$ and either $\alpha_k^i = \beta_k^i$ or T_k is of type (2), then

$$\alpha_{k+1}^i = \alpha_k^i - 1 \le \beta_k^i - 1 = \beta_{k+1}^i \le \gamma_k^i - 1 = \gamma_{k+1}^i.$$

Again, if $l(u) \le \alpha_k^i - 3$ the desired inequality holds as in the last case above.

To prove (7), first observe that (7) holds for $k = 1$. Inductively assume that (7) holds for $k < N$. This implies that for all $t \in [1, l(w_k)]$, there is some $i \in [1, n]$ with $\alpha_k^i \le t \le \gamma_k^i$; in particular, $l(w_k) = \gamma_k^j$ for some j.

(a) Suppose that $T_k : u(f \cdot g)v \mapsto u(fg)v$ and let s be the rank of $h = f \cdot g$ in w_k. Then every letter in w_k with rank less than s appears in w_{k+1} with the same rank, f and g has ranks s and $s+1$ respectively in w_{k+1} and all letters in w_k with rank $t > s$ appears in w_{k+1} with rank $t+1$. Since by inductive hypothesis $\alpha_k^i \in [1, l(w_k)]$ for all i, and since α_{k+1}^i is either α_k^i or $\alpha_k^i + 1$, $\alpha_{k+1}^i \in [1, l(w_{k+1})]$ for all i. Similarly $\gamma_{k+1}^i \in [1, l(w_{k+1})]$ for all i. Therefore

$$[1, l(w_{k+1})] \supseteq \bigcup_{i=1}^{n} [\alpha_{k+1}^i, \gamma_{k+1}^i]. \qquad (\star)$$

It remains to prove the reverse inclusion. Observe that, for all $t \in [1, l(w_k)]$, $t \in [\alpha^i_k, \gamma^i_k]$.

i) Let $\gamma^i_k < s$. Then we have $l(u) \geq \gamma^i_k > \alpha^i_k - 1$ and so, $\alpha^i_{k+1} = \alpha^i_k$ and $\gamma^i_{k+1} = \gamma^i_k$ by Equations (3) and (4). Therefore, $\alpha^i_{k+1} = \alpha^i_k \leq t \leq \gamma^i_k = \gamma^i_{k+1}$.

ii) Let $\gamma^i_k = s$. Then $l(u) = \gamma^i_k - 1 \geq \alpha^i_k - 1$ and so, $\alpha^i_{k+1} = \alpha^i_k$ and $\gamma^i_{k+1} = \gamma^i_k + 1$. Therefore $\alpha^i_{k+1} \leq s < s + 1 \leq \gamma^i_{k+1}$.

iii) Let $\gamma^i_k > s$. Then $l(u) \leq \gamma^i_k - 2$ and so, $\gamma^i_{k+1} = \gamma^i_k + 1$. If $l(u) \geq \alpha^i_k - 1$, then $\alpha^i_{k+1} = \alpha^i_k$. Hence if $\alpha^i_k \leq t \leq \gamma^i_k$, then $\alpha^i_k \leq t + 1 \leq \gamma^i_{k+1}$. If $l(u) \leq \alpha^i_k - 2$, and if $\alpha^i_k \leq t \leq \gamma^i_k$ then $\alpha^i_{k+1} \leq t + 1 \leq \gamma^i_{k+1}$.

It follows that

$$[1, l(w_{k+1})] \subseteq \bigcup_{i=1}^{n} [\alpha^i_{k+1}, \gamma^i_{k+1}]. \tag{**}$$

From Equations ($\star$) and (**) we see that (7) holds in this case.

(b) Suppose that $T_k : u(fg)v \mapsto u(f \cdot g)v$ and let s be the rank of f in w_k. Then letters in w_k with rank $t < s$ appears in w_{k+1} with the same randk, $h = f \cdot g$ has rank s in w_{k+1} and letters in w_k with rank $t > s + 1$ appears in w_{k+1} with rank $t - 1$ so that $l(w_{k+1}) = l(w_k) - 1$. Again, as in the proof of the case (a), we see that Equation ($\star$) holds. Also by inductive hypothesis, for all $t \in [1, l(w_k)]$, $t \in [\alpha^i_k, \gamma^i_k]$ for some $i \in [1, n]$.

i) Let $t \in [1, s]$. If $\gamma^i_k \leq s$ then $l(u) \geq \gamma^i_k - 1 \geq \alpha^i_k - 1$ and so by Equations (3) and (4), $t \in [\alpha^i_k, \gamma^i_k] = [\alpha^i_{k+1}, \gamma^i_{k+1}]$. If $\gamma^i_k > s$, then $l(u) \leq \gamma^i_k - 2$ so that $\gamma^i_{k+1} = \gamma^i_k - 1 \geq s$. Since $\alpha^i_k \leq s$, we have $\alpha^i_{k+1} = \alpha^i_k$. Hence $t \in [\alpha^i_{k+1}, \gamma^i_{k+1}]$.

ii) Let $t > s$. By induction hypothesis, $t + 1 \in [\alpha^i_k, \gamma^i_k]$ for some $i \in [1, n]$. Then $\gamma^i_k \geq t + 1 \geq s + 2$ and so, $l(u) \leq \gamma^i_k - 3 < \gamma^i_k - 2$. Then by (4), $\gamma^i_{k+1} = \gamma^i_k - 1$. We consider the following cases:

1. $\alpha^i_k \leq s$. In this case, $\alpha^i_{k+1} = \alpha^i_k$ and so, $t \in [\alpha^i_k, \gamma^i_k - 1] = [\alpha^i_{k+1}, \gamma^i_{k+1}]$.
2. $\alpha^i_k = s + 1$. If this holds, we have $t \geq s + 1$. Therefore $t \in [\alpha^i_{k+1}, \gamma^i_{k+1}]$ since $\alpha^i_{k+1} = \alpha^i_k$ or $\alpha^i_k - 1$.
3. $\alpha^i_k \geq s + 2$. If this is the case, $\alpha^i_{k+1} = \alpha^i_k - 1$ $\gamma^i_{k+1} = \gamma^i_k - 1$ and hence $t \in [\alpha^i_{k+1}, \gamma^i_{k+1}]$.

Again we see that Equation (**) holds which proves Equation (7). This completes the proof of the lemma. □

LEMMA 3.38. *Let $w = e_1 e_2 \ldots e_n$ be a word such that $\sigma^\#(w)$ is an idempotent in $\boldsymbol{B}_0(E)$. Then there is $e \in E$ such that $\sigma^\#(w)$ $\mathscr{D}(\boldsymbol{B}_0(E))$ $\sigma^\#(e)$.*

Proof. We continue to use all notations established so far in this section. In particular, w_k, $k \in [1, N]$ are words in E^+ such that $w_1 = w$, $w_N = w^n$ and $T_k : w_k \to w_{k+1}$ is an elementary σ-transition. Also $\{\alpha_k^i, \beta_k^i, \gamma_k^i\}$ denote finite sequences defined by Equations (1), (2), (3) and (4). Moreover, we will denote by $\mathscr{R}$, $\mathscr{L}$, $\mathscr{D}$, etc. Green's relations of the semigroup $\boldsymbol{B}_0(E)$ in the following proof.

For exch $k \in [1, N]$ and $i \in [1, n]$, let w_k^i denote the subword of w_k obtained by removing all letters in w_k to the left of α_k^i-th letter and all letters to the right of the γ_k^i-th letter. By Equation (7), the subwords w_k^i, $1 \le i \le n$ cover w_k for each $k \in [1, N]$. In particular, w_N is covered by $w_N^1, w_N^2, \dots w_N^n$. We now claim that

$$\text{for some } i \quad e_1 e_2 \dots e_n = w_1 \quad \text{is a subword of} \quad w_N^i. \tag{\#}$$

If this is false, w_1 is not a subword of w_N^1 (or w_N^1 does not cover w_1). Inductively assume that $w_N^1, w_N^2, \dots w_N^i$ does not cover $(w_1)^i$. Since w_N^{i+1} does not cover w_1, $w_N^1, \dots w_N^{i+1}$ does not cover $(w_1)^{i+1}$. By induction $w_N^1, w_N^2, \dots w_N^n$ does not cover $(w_1)^n = w_N$. This contradicts the assertion proved in the last paragraph. Therefore (#) must be true.

Let e_k^i be the β_k^i-th letter in w_k. By Equation (6), e_k^i is a letter of w_k^i and so, we can write w_k^i as

$$w_k^i = u_k^i e_k^i v_k^i \quad \text{for some (possibly empty) words} \quad u_k^i \quad \text{and} \quad v_k^i. \tag{\#1}$$

We now prove that, for all k and i.

$$\sigma^\#(e_k^i v_k^i) \;\mathscr{R}\; \sigma^\#(e_k^i) \tag{\#2}$$

and

$$\sigma^\#(u_k^i e_k^i) \;\mathscr{L}\; \sigma^\#(e_k^i) \tag{\#3}$$

by induction on k. If $k = 1$, by Equation (1), $w_1^i = e_i$ for each i and so (#2) holds. Assume that (#2) holds for $k < N$. To prove (#2) for $k + 1$, we need only to verify the assertion in the cases in which the elementary transition

$$T_k : w_k = u' w_k^i v' \mapsto w_{k+1} = u' w_{k+1}^i v'$$

has one of the following forms. In the following, the word shown in the bracket on the left is w_k^i and on the right is w_{k+1}^i.

(*a*) $u'(ufe_k^i v)v' \mapsto u'(uf \cdot e_k^i v)v'$;

(*b*) $u'(fe_k^i v)v' \mapsto u'(f \cdot e_k^i v)v'$;

(*c*) $u'(ue_k^i fv)v' \mapsto u'(ufe_k^i \cdot fv)v'$;

(*d*) $u'(ue_k^i f)v' \mapsto u'(ue_k^i \cdot f)v'$;

(e)	$u'(ue^i_k vf)gv' \mapsto u'(ue^i_k vf\cdot g)v'$	and T_k is of type (1);
	$\mapsto u'(ue^i_k v)f\cdot gv'$	and T_k is not of type (1);
(f)	$u'(ue^i_k v)v' \mapsto u'(ufgv)v'$	where $e^i_k = f\cdot g$.

Cases (a) and (b): For these cases, we have $e^i_{k+1} = f\cdot e^i_k$ and $v^i_{k+1} = v$. Hence (#2) follows since $\mathscr{R}$ is a left congruence.

Case (c): We again have $e^i_{k+1} = e^i_k\cdot f$ and $v^i_{k+1} = v$. Hence

$$\sigma^\#(e^i_{k+1}v^i_{k+1}) = \sigma^\#(e^i_k fv)\ \mathscr{R}\ \sigma^\#(e^i_k).$$

Also

$$\sigma^\#(e^i_k)\ \mathscr{R}\ \sigma^\#(e^i_k f) = \sigma^\#(e^i_{k+1})$$

giving the desired result.

Case (d): (#2) follows immediately since v^i_{k+1} is the empty word in trhis case.

Case (e): If T_k is not of type (1),

$$\sigma^\#(e^i_{k+1}v^i_{k+1}) = \sigma^\#(e^i_k v)\ \mathscr{R}\ \sigma^\#(e^i_k) = \sigma^\#(e^i_{k+1}).$$

If T_k is of type (1) then $f\ \mathscr{R}\ f\cdot g$ and so,

$$\sigma^\#(e^i_{k+1}v^i_{k+1}) = \sigma^\#(e^i_k vf\cdot g)\ \mathscr{R}\ \sigma^\#(e^i_k vf)\ \mathscr{R}\ \sigma^\#(e^i_k) = \sigma^\#(e^i_{k+1}).$$

Case (f): If T_k is of type (1) then $e^i_{k+1} = f$ and $e^i_{k+1}\ \mathscr{R}\ e^i_k$. Therefore

$$\sigma^\#(e^i_{k+1}v^i_{k+1}) = \sigma^\#(f\cdot gv)\ \mathscr{R}\ \sigma^\#(e^i_k)\ \mathscr{R}\ \sigma^\#(e^i_{k+1}).$$

If T_k is not of type (1) then $e^i_{k+1} = g$ and $e^i_{k+1}\ \mathscr{L}\ e^i_k$. Hence

$$\sigma^\#(e^i_{k+1}v^i_{k+1}) = \sigma^\#(gv) = \sigma^\#(ge^i_k v)\ \mathscr{R}\ \sigma^\#(ge^i_k) = \sigma^\#(g) = \sigma^\#(e^i_{k+1}).$$

This proves (#2) by induction. Proof of (#3) is dual.

By (#) there exists $i \in [1, n]$ such that

$$w^i_N = e_j \dots e_n(w_1)^p w_1(w_1)^q e_1 \dots e_k$$

for some integers i, j, p and q. Hence, since $\sigma^\#$ is a homomorphism of : E^+ to $\boldsymbol{B}_0(E)$

$$\sigma^\#(w^i_N) = \sigma^\#(e_j \dots e_n w_1 e_1 \dots e_k)\ \mathscr{R}\ \sigma^\#(e_j \dots e_n w_1)\ \mathscr{L}\ \sigma^\#(w_1);$$

that is,

$$\sigma^{\#}(w_N^i) \mathscr{D} \sigma^{\#}(w_1).$$

On the other hand, from (#2) and (#3), we have

$$\sigma^{\#}(u_N^i e_N^i) \mathscr{L} \sigma^{\#}(e_N^i) \mathscr{R} \sigma^{\#}(e_N^i v_N^i),$$

so that $\quad \sigma^{\#}(u_N^i e_N^i) \mathscr{R} \sigma^{\#}(u_N^i e_N^i)\sigma^{\#}(e_N^i v_N^i) = \sigma^{\#}(u_N^i e_N^i v_N^i) = \sigma^{\#}(w_N^i);$

that is, $\quad \sigma^{\#}(w_N^i) \mathscr{D} \sigma^{\#}(e_N^i).$

Hence $\quad \sigma^{\#}(w_1) \mathscr{D} \sigma^{\#}(e_N^i)$

which proves the lemma. □

We can now prove Easdown's theorem [see Easdown, 1985].

Proof of Theorem 3.34. We first show that $\chi_E = \sigma^{\#} \mid E$ is a surjective bimorphism of E onto $\boldsymbol{E}(\boldsymbol{B}_0(E))$. If $(e, f) \in D_E$ then $(ef, e \cdot f) \in \sigma$ and so,

$$(e\chi_E)(f\chi_E) = \sigma^{\#}(e)\sigma^{\#}(f) = \sigma^{\#}(ef) = \sigma^{\#}(e \cdot f) = (e \cdot f)\chi_E.$$

It follows that χ_E preserve basic products in E and so, it is a bimorphism of E into $\boldsymbol{E}(\boldsymbol{B}_0(E))$. Now if w is a word in E^+ such that $\sigma^{\#}(w)$ is an idempotent in $\boldsymbol{B}_0(E)$, then by Lemma 3.38, $\sigma^{\#}(w) \mathscr{D} \sigma^{\#}(e)$ for some $e \in E$. If this is the case, by Lemma 3.37 there is $g \in E$ such that $\sigma^{\#}(w) = \sigma^{\#}(g)$. Hence every idempotent in $\boldsymbol{B}_0(E)$ is of the form $\sigma^{\#}(e)$ for $e \in E$. Therefore the map $\chi_E : e \mapsto \sigma^{\#}(e)$ is a surjective bimorphism of E onto $\boldsymbol{E}(\boldsymbol{B}_0(E))$.

Now let S be a semigroup and $\theta : E \to \boldsymbol{E}(S)$ be a bimorphism. Then θ extends to a homomorphism θ^+ of E^+ into S such that $\theta^+|E = \theta$. If $(e, f) \in D_E$ then $(ef, e \cdot f) \in \sigma$ and so,

$$\begin{aligned}(ef)\theta^+ &= (e\theta^+)(f\theta^+) = (e\theta)(f\theta)\\ &= (e \cdot f)\theta = (e \cdot f)\theta^+.\end{aligned}$$

since θ is a bimorphism. Hence $\sigma \subseteq \boldsymbol{\kappa}\phi(\theta^+)$. Consequently by Theorem 2.5 there is a unique homomorphism $\hat{\theta} : \boldsymbol{B}_0(E) \to S$ such that

$$\theta^+ = \sigma^{\#} \circ \hat{\theta}.$$

Therefore

$$\theta = \theta^+|E = (\sigma^{\#}|E) \circ (\hat{\theta}|E(\boldsymbol{B}_0(E))) = \chi_E \circ E(\hat{\theta})$$

which shows that the diagram 3.22 commutes.

In particular, by Theorem 3.32, φ is a biorder isomorphism of E onto $E\varphi$ which is a biordered subset of $\boldsymbol{E}(\langle E\varphi\rangle)$. Hence from diagram 3.22 we see that

$$\varphi = \chi_E \circ \boldsymbol{E}(\hat{\varphi}).$$

Since φ is injective, so is χ_E. Since χ_E is surjective, it is a bijection. Therefore the equation above shows that $\boldsymbol{E}(\hat{\varphi}) : \boldsymbol{E}(\boldsymbol{B}_0(E)) \to E\varphi$ is a bijective bimorphism and since $\varphi : E \to E\varphi$ is a biorder isomorphism,

$$(\chi_E)^{-1} = \boldsymbol{E}(\hat{\varphi}) \circ (\varphi)^{-1}$$

is a bimorphism. Thus $\chi_E : E \to \boldsymbol{E}(\boldsymbol{B}_0(E))$ is a biorder isomorphism. □

Recall (from Theorem 3.3) that the assignments in Equation (3.6) is a functor from the category $\mathfrak{S}$ of semigroups to the category $\mathfrak{B}$ of biordered sets. To avoid ambiguity regarding the notation for this functor, for the remainder of this section, we will use the notations E, E', etc. for arbitrary biordered sets. Recall also (from Subsection 1.2.3) that a universal arrow from $d \in \mathfrak{v}\mathcal{D}$ to the functor $F : \mathcal{C} \to \mathcal{D}$ is a pair (c, g) where $c \in \mathfrak{v}\mathcal{C}$ and $g \in \mathcal{D}(d, F(c))$ such that given any pair (c', g') with $g' \in \mathcal{D}(d, F(c')), c' \in \mathfrak{v}\mathcal{C}$, there is a unique $f \in \mathcal{C}(c, c')$ such that $g' = g \circ F(f)$. The last statement of Theorem 3.34 can be interpreted as follows.

COROLLARY 3.39. *Let E be a biordered set. Then $\chi_E : E \to \boldsymbol{E}(\boldsymbol{B}_0(E))$ is a universal arrow from E to the functor $\boldsymbol{E}$.*

Suppose that $\theta : E \to E'$ be a bimorphism. Then $\theta' = \theta \circ \chi_{E'} : E \to \boldsymbol{B}_0(E')$ is a bimorphism. Hence, by Theorem 3.34, there is a unique homomorphism $\phi : \boldsymbol{B}_0(E) \to \boldsymbol{B}_0(E)(E')$ such that the diagram 3.22 commutes. Since ϕ is uniquely determined by θ, we may denote ϕ as $\boldsymbol{B}_0(E)(\theta)$. Then 3.22 becomes

$$\begin{array}{ccc} E & \xrightarrow{\theta} & E' \\ \big\downarrow{\scriptstyle \chi_E} & & \big\downarrow{\scriptstyle \chi_{E'}} \\ \boldsymbol{E}(\boldsymbol{B}_0(E)) & \xrightarrow[\boldsymbol{E}(\boldsymbol{B}_0(\theta))]{} & \boldsymbol{E}(\boldsymbol{B}_0(E')) \end{array} \tag{3.24}$$

The uniqueness of the homomorphism $\boldsymbol{B}_0(\theta)$ in Theorem 3.34 implies that

$$\boldsymbol{B}_0(1_E) = 1_{\boldsymbol{B}_0(E)} \quad \text{and that} \quad \boldsymbol{B}_0(\theta \circ \theta') = \boldsymbol{B}_0(\theta) \circ \boldsymbol{B}_0(\theta')$$

for composable bimorphisms $\theta : E \to E'$ and $\theta' : E' \to E''$. Thus the assignments

$$E \mapsto \boldsymbol{B}_0(E), \quad \text{and} \quad \theta \mapsto \boldsymbol{B}_0(\theta) \tag{3.25a}$$

semiband

is a functor $\boldsymbol{B}_0 : \mathfrak{B} \to \mathfrak{S}$. The diagram 3.24 above shows that

$$\chi : E \mapsto \chi_E; 1_{\mathfrak{B}} \xrightarrow{n} \boldsymbol{B}_0(E) \circ \boldsymbol{E} \tag{3.25b}$$

is a natural isomorphism. Thus from Corollary 3.39 and Theorem 1.6 (ii) we have the following which shows that the construction represented by $\boldsymbol{B}_0$ satisfies the fundamental property of the construction of free objects in a category. Again [see Nambooripad, 1979, Theorem 6.10] for the particular case of this result for regular biordered sets.

THEOREM 3.40. *The assignments in Equation* (3.25a) *defines a functor* $\boldsymbol{B}_0 : \mathfrak{B} \to \mathfrak{S}$ *which is a left adjoint of the functor* $\boldsymbol{E} : \mathfrak{S} \to \mathfrak{B}$ *given by the assignments 3.6. Moreover,* χ *defined by Equation* (3.25b) *is a natural isomorphism which is the unit of the adjunction.*

3.3.3 The fundamental semiband

Following fairly widespread use we shall say that a semigroup S is a *semiband* if S is idempotent generated. Given any biordered set E we have constructed two semibands $\boldsymbol{B}_0(E)$ and $\langle E\varphi\rangle$. The semiband $\boldsymbol{B}_0(E)$ is uniquely determined by E as the free semiband generated by E (see Theorem 3.34) having E as its biordered set. We wish to obtain a similar characterization of $\langle E\varphi\rangle$ also. Notice that, in general E is only (isomorphic to) a biordered subset of $\boldsymbol{E}(\langle E\varphi\rangle)$ and the embedding may be proper (see Example 3.10). However, if E is regular, then by Theorem 3.42 below, $\langle E\varphi\rangle$ is a semiband with $\boldsymbol{E}(\langle E\varphi\rangle)$ isomorphic to E.

So we now consider regular biordered sets. We need the following result due to Easdown [1985]. Recall that $\mathscr{S}_1(e, f) = \{h \in M(e, f) : ehf = ef\}$ (see Proposition 3.4).

LEMMA 3.41. *Let* $e, f \in E$ *and* $h \in \mathscr{S}(e, f)$. *Then*

$$\varphi(e)\varphi(h)\varphi(f) = \varphi(e)\varphi(f).$$

in $\langle E\varphi\rangle$.

Proof. We shall show that $\rho(e)\rho(h)\rho(f) = \rho(e)\rho(f)$. Suppose that $L\rho(e)\rho(f) \neq \infty$. Then for some $a \in L \cap \omega^r(e)$, $L\rho(e) = L_{ae}$ and for $g \in L_{ae} \cap \omega^r(f)$, $L\rho(e)\rho(f) = L_{gf}$. Then $g \in M(ae, f) \subseteq M(e, f)$ so that $g \preceq h$. Hence by Proposition 3.20 there is a commutative E-square $\left(\begin{smallmatrix} g & g_1 \\ g_2 & h' \end{smallmatrix}\right)$ in $M(e, f)$ such that $h' \;\omega\; h$, $hg_1 = g_2h = h'$ and $g_2f = h'f$. Then

$$\begin{aligned}(L)\rho(e)\rho(f) &= (L_g)\rho(f) = L_{gf} \\ &= L_{g_2f} = L_{h'f} = L_{g_2hf} \\ &= (L)\rho(e)\rho(h)\rho(f) \neq \infty.\end{aligned}$$

On the other hand, if $(L)\rho(e)\rho(h)\rho(f) \neq \infty$, then for some $a \in L \cap \omega^r(e)$ and $g \in L_{ae} \cap \omega^r(h)$, $g \in L_g \cap \omega^r(f) \subseteq M(e, f)$. Then $gf = (gf)(hf) = (gh)f$ by axiom (B3) since $g \leq h$ and so, $gf \; \omega^l \; hf$. Therefore

$$(L)\rho(e)\rho(f) = (L_{ae})\rho(f) = L_{gf} = L_{(gf)(hf)} = L_{(gh)f} = (L)\rho(e)\rho(h)\rho(f)$$

which implies that $(L)\rho(e)\rho(f) \neq \infty$. This also shows that $(L)\rho(e)\rho(f) = \infty$ if and only if $(L)\rho(e)\rho(h)\rho(f) \neq \infty$. Thus $\rho(e)\rho(h)\rho(f) = \rho(e)\rho(f)$. Dually $\lambda(e)\lambda(h)\lambda(f) = \lambda(e)\lambda(f)$ and hence $\varphi(e)\varphi(h)\varphi(f) = \varphi(e)\varphi(f)$. □

Theorem 3.5 shows that the biordered set of a regular semigroup is regular. The following result shows that every regular biordered set arises in that way.

THEOREM 3.42. *Let E be a regular biordered set. Then $\langle E\varphi\rangle$ is a regular semigroup such that $\varphi : E \to \boldsymbol{E}(\langle E\varphi\rangle)$ is a biorder isomorphism.*

Proof. Let $S = \langle E\varphi\rangle$ so that, by Theorem 3.32, $E\varphi = \bar{E}$ is a regular biordered subset of $\boldsymbol{E}(S)$ isomorphic to E. If $e, f \in E$ and $h \in \mathscr{S}(e, f)$, then by Lemma 3.41 and Proposition 3.4 $\varphi(h) \in \mathscr{S}_1(\varphi(e), \varphi(f))$. Therefore $\bar{E}$ is a regular biordered subset of $\boldsymbol{E}(S)$ which is relatively regular in $\boldsymbol{E}(S)$ and such that $\mathscr{S}_1(\varphi(e), \varphi(f)) \neq \emptyset$ for all $e, f \in E$. Hence, by Proposition 3.8 there is a regular subsemigroup $S' \subseteq S$ such that $\boldsymbol{E}(S') = \bar{E}$. Since S is generated by $\bar{E}$, we must have $S' = S$. Therefore S is a regular idempotent generated semigroup and $\varphi : E \to \boldsymbol{E}(S)$ is a biorder isomorphism. □

The result above shows that when E is a regular biordered set, $E\varphi = \boldsymbol{E}(\langle E\varphi\rangle)$; that is, φ does not create any new idempotents in $\langle E\varphi\rangle$. If E is any finite biordered set, the sets I° and Λ° are also finite. Therefore the semigroup $\langle E\varphi\rangle$ must be finite. Consequently if $E\varphi = \boldsymbol{E}(\langle E\varphi\rangle)$ then E is the biordered set of a finite semigroup. In particular, bythe theorem above, this holds if E is a finite regular biordered set.

COROLLARY 3.43. *Every finite regular biordered set is the biordered set of a finite regular semigroup.*

The equality $E\varphi = \boldsymbol{E}(\langle E\varphi\rangle)$ may not true if E is not regular (see Example 3.10). We can also see from the theorem above that $\langle E\varphi\rangle$ is a regular semigroup when E is a regular biordered set. Example 3.10 shows that $\boldsymbol{B}_0(E)$ need not be regular even if $\langle E\varphi\rangle$ is regular. However, when E is regular, an application of Proposition 3.8 shows that $\boldsymbol{B}_0(E)$ is indeed regular. Consequently the restriction of the functor $\boldsymbol{B}_0$ to the category $\mathfrak{RB}$ of regular biordered sets is a functor to the category $\mathfrak{RS}$ of regular semigroup. By Theorem 3.5 $\boldsymbol{E}(S)$ is a regular biordered set for all regular semigroup S and so the restriction of the functor $\boldsymbol{E}$ to the category $\mathfrak{RS}$ is a functor to the category of regular biordered sets. Thus, as a corollary to Theorem 3.42, we have [see Nambooripad, 1979, Theorem 6.10].

COROLLARY 3.44. *Let $\boldsymbol{B}_0$ and $\boldsymbol{E}$ be functors of Equations* (3.25a) *and* (3.6) *respectively. Then $\boldsymbol{B}_0 \mid \mathfrak{RB}$ is a functor to the category $\mathfrak{RS}$ of regular semigroups and $\boldsymbol{E} \mid \mathfrak{RS}$ is a functor to the category $\mathfrak{RB}$ of regular biordered sets. Moreover, $\boldsymbol{B}_0 \mid \mathfrak{RB}$ is a left adjoint of the functor $\boldsymbol{E} \mid \mathfrak{RS}$.*

Let S be a semigroup with $E = \boldsymbol{E}(S) \neq \emptyset$. If Λ° denote the sets defined by Equation (3.20a) then each $L \in \Lambda$ gives a unique regular $\mathscr{L}$-class $L^\cdot$ of S such that $L^\cdot \cap E = L$. Let Λ_r denote the set of all regular $\mathscr{L}$-classes in S so that $\cdot : L \mapsto L^\cdot$ is a bijection of Λ onto Λ_r. There is an obvious identification of $\Lambda^\circ = \Lambda \cup \{\infty\}$ with $\Lambda^\cdot = \Lambda_r \cup \{\infty\}$. For $e \in E$ let $\rho^\cdot$ be the map defined as follows: for $L^\cdot \in \Lambda_r$

$$(L^\cdot)\rho^\cdot(e) = \begin{cases} L^\cdot_{xe} & \text{if } L^\cdot = L^\cdot_x \in \Lambda_r \text{ and } x \mathrel{\mathscr{R}} xe; \\ \infty & \text{otherwise; and} \end{cases} \tag{3.20b*}$$

$$(\infty)\rho^\cdot(e) = \infty.$$

Dually there is a bijection $\cdot : R \mapsto R^\cdot$ sending each $R \in I$ to the unique $\mathscr{R}$-class in S containing R. We define the set $I^\cdot = I_r \cup \{\infty\}$ where I_r denote the set of all regular $\mathscr{R}$-classes in S and for each $e \in E$, the map $\lambda^\cdot(e) \in \mathscr{T}^*_{I^\cdot}$ as in Equation (3.20b).

LEMMA 3.45. *For each $L \in \Lambda^\circ$ and $R \in I^\circ$, we have*

$$(L^\cdot)\rho^\cdot(e) = (L\rho(e))^\cdot \quad \textit{and} \quad (R^\cdot)\lambda^\cdot(e) = (R\lambda(e))^\cdot$$

for all $e \in E$.

Proof. We prove the first statement. The second follows by duality. Suppose that $L\rho(e) \neq \infty$. Then by Equation (3.20b) there is $g \in L$ with $g \mathrel{\omega^r} e$ and so, $g \mathrel{\mathscr{R}} ge$ by axiom (B21). Then $g \mathrel{\mathscr{R}} ge$ in S and so,

$$(L^\cdot)\rho^\cdot(e) = (L^\cdot_g)\rho^\cdot(e) = L^\cdot_{ge} = (L\rho(e))^\cdot.$$

Hence, if $L\rho(e) \neq \infty$, then $(L^\cdot)\rho^\cdot(e) \neq \infty$. On the other hand if $(L^\cdot)\rho^\cdot(e) \neq \infty$, $L^\cdot \in \Lambda_r$, then $x \mathrel{\mathscr{R}} xe$ for some $x \in L^\cdot$. Let $f \in L = L^\cdot \cap E$. Since $x \mathrel{\mathscr{R}} xe$ and since $x \in L^\cdot$, x is regular and so, xe is regular. Therefore $\mathscr{S}(f, e) \neq \emptyset$. Let $g \in \mathscr{S}(f, e)$. Then $xe \mathrel{\mathscr{L}} fe = (fg)(ge) \mathrel{\mathscr{L}} ge$ and $f \mathrel{\mathscr{R}} fe \mathrel{\mathscr{R}} fg \mathrel{\omega} f$ which implies that $fg = f$. Therefore $f \mathrel{\mathscr{L}} g \mathrel{\omega^r} e$ and so, $L\rho(e) = L_{ge} \neq \infty$ and

$$(L\rho(e))^\cdot = L^\cdot_{ge} = (L^\cdot)\rho^\cdot(e).$$

This completes the proof. ∎

semigroup!fundamental

The result above shows that we may replace the maps $\rho(e)$, $\lambda(e)$ and $\varphi(e)$ respectively by $\rho^{\cdot}(e)$, $\lambda^{\cdot}(e)$ and $\varphi^{\cdot}(e)$ and vice-versa when $E = \boldsymbol{E}(S)$ for some semigroup S. The advantage of this replacement is that the map $\rho^{\cdot}(e)$ is induced by the right translation ρ_e of S whereas $\rho(e)$ is completely determined by the biordered set. The lemma above ensures that these identifications only amount to a change in notation. Consequently, henceforth, it will be convenient to identify $\Lambda^{\cdot}$ and Λ° by the bijection $\cdot$ which will identify the map $\rho^{\cdot}(e)$ with $\rho(e)$. Dually we identify $\lambda^{\cdot}(e)$ with $\lambda(e)$.

Let S be a semiband with $E = \boldsymbol{E}(S)$. Extend the map $\rho : E \to \boldsymbol{E}(\mathscr{T}_{\Lambda^{\circ}})$ to $\rho : S \to \mathscr{T}_{\Lambda^{\circ}}$ by setting

$$(L)\rho(w) = (L)\rho(e_1)\rho(e_2)\ldots\rho(e_n) \tag{3.26}$$

for all $w = e_1e_2\ldots e_n \in S$, and $L \in \Lambda^{\circ}$. Let us write $L_i = (L)\rho(e_1)\ldots\rho(e_i) = (L)\rho(w_i)$, $i = 1, 2, \ldots, n$, for $L \in \Lambda$. If $x \in L$, then by (3.20b*) and Theorem 2.26, the map $\rho_{e_{i+1}} : y \mapsto ye_{i+1}$ is an isomorphism of the leftideal $L(xw_i)$ generated by L_i onto the leftideal $L(xw_{i+1})$ such that $y \mathscr{R} ye_{i+1}$. Hence by the above, ρ_w is an isomorphism of the left ideal $L(x)$ generated by $L = L_x$ onto $L(xw)$. So, by Theorem 2.25, we have $x \mathscr{R} xw$ for all $x \in L$. Consequently, for any $L \in \Lambda^{\circ}$, we have

$$(L)\rho(w) = \begin{cases} L_{xw} & \text{if } L \in \Lambda \text{ and } x \mathscr{R} xw \text{ for some } x \in L; \text{ and} \\ \infty & \text{otherwise.} \end{cases} \tag{3.27}$$

This implies that $\rho : S \to \mathscr{T}_{\Lambda^{\circ}}$ is a homomorphism. Dually the map $\lambda : e \mapsto \lambda(e)$ extends to a homomrphism $\lambda : S \to \mathscr{T}^*_{I^{\cdot}}$. Therefore

$$\varphi_S : w \mapsto (\rho(w), \lambda(w)) \quad \text{for all} \quad w \in S \tag{3.20c*}$$

is a homomorphism (representation) of S to $\langle E\varphi\rangle \subseteq \mathscr{T}_{\Lambda^{\circ}} \times \mathscr{T}^*_{I^{\circ}}$ which extends the biorder embedding φ_E of Theorem 3.32. Since S is a semiband, $\varphi_S : S \to \langle E\varphi\rangle$ is surjective.

A semigroup S is called *fundamental* if the congruence $\mathscr{H}_{(c)} = 1_S$ (see Proposition 2.7(b)).

PROPOSITION 3.46. *For any semigroup S, let*

$$\mu(S) = \mathscr{H}_{(c)}.$$

Then $S/\mu(S)$ is fundamental.

Proof. Let $S' = S/\mu$ where $\mu = \mu(S)$ and let $\phi = \mu^{\#}$ denote the quotient homomorphism of S onto S'. If $x \mathscr{R} y$, $x, y \in S$ then clearly $x\phi \mathscr{R} y\phi$ in S'. On the other hand, if $y\phi \in (x\phi)(S')^1$ then $y\phi = (x\phi)(r\phi)$ for some $r \in S^1$. Hence congruence! idempotent separating

$\mu(y) = \mu(xr)$ which implies $(y, xr) \in \mu \subseteq \mathscr{H}$. It follows that $y \in xS^1$. Similarly $x\phi \in (y\phi)(S')^1$ implies $x \in yS^1$. Therefore $x \mathscr{R} y$ if and only if $x\phi \mathscr{R} y\phi$. Dually $x \mathscr{L} y$ if and only if $x\phi \mathscr{L} y\phi$ and so, $x \mathscr{H} y$ if and only if $x\phi \mathscr{H} y\phi$.

Suppose that $\mu' = \mathscr{H}^{S'}{}_{(c)}$ where $\mathscr{H}^{S'}$ denote the $\mathscr{H}$-relation on S'. Suppose $a, b \in S$ with $a\phi\mu' b\phi$. Then by Proposition 2.7(b), $(xay)\phi \mathscr{H} (xby)\phi$ for all $x, y \in S^1$. Therefore, by the above remarks, $xay \mathscr{H} xby$ for all $x, y \in S^1$ and so $a\mu b$. This proves that $\mu' = 1_{S'}$. □

A congruence σ on a semigroup S is *idempotent separating* if each congruence class of σ contain atmost one idempotent. This is equivalent to requiring that the quotient homomorphism $\sigma^\# : S \to S/\sigma$ is injective on $\boldsymbol{E}(S)$; that is the bimorphism $\boldsymbol{E}(\sigma^\#)$ is injective.

Some authors define a fundamental semigroup S as those for which, the only idempotent separating congruence is 1_S. The following result shows that the two definitions agree on a regular semigroup.

PROPOSITION 3.47. *Let S be a regular semigroup. A congruence σ on S is idmpotent separating if and only if $\sigma \subseteq \mathscr{H}$. In particular, the congruence $\mu(S)$ is idempotent separating. Moreover, a regular semigroup S is fundamental if and only if the only idempotent separating congruence on S is the trivial (identity) congruence.*

Proof. If $\sigma \subseteq \mathscr{H}$ then $\sigma(x) \subseteq H_x$ for all $x \in S$ and since no $\mathscr{H}$ can contain more than one idempotent (see Proposition 2.40) σ is idempotent separating.

Conversely, assume that σ is idempotent separating. Let $\phi : S \to S/\sigma = S'$ be the quotient homomorphism. Then by Theorem 3.5 $\theta = \boldsymbol{E}(\phi)$ is a regular bimorphism of $E = \boldsymbol{E}(S)$ onto $E' = \boldsymbol{E}(S')$. Since σ is idempotent separating, by Corollary 3.25, θ is an isomorphism. Let $(x, y) \in \sigma$. If $x' \in \mathscr{V}(x)$ then $e = xx'\sigma yx' = u$. Let $f, g \in E = \boldsymbol{E}(S)$ with $f \mathscr{L} u \mathscr{R} g$ and let $h \in \mathscr{S}(f, g)$. Then $h\theta \in \mathscr{S}(f\theta, g\theta) = \{e\theta\}$ since $f\theta \mathscr{L} e\theta \mathscr{R} g\theta$. So $h\theta = e\theta$ which gives $h = e$ since θ is an isomorphism. Then $fe\ \omega\ f$ and so, $(fe)\theta = (f\theta)(e\theta) = f\theta$ which gives $fe = f$. Hence $f \mathscr{L} e$ and similarly, $e \mathscr{R} g$. Therefore $e \mathscr{H} u$. Now by Theorem 2.26, $\rho_x : L(e) \to L(x)$ is an isomorphism of left ideals and hence, by Theorem 2.25, $x \mathscr{H} ux = yx'x$. Hence $x \in yS$. Interchanging x and y, we get $y \in xS$ and so, $x \mathscr{R} y$. Dually $x \mathscr{L} y$. Therefore $x \mathscr{H} y$.

Since $\mu(S) \subseteq \mathscr{H}$ by definition (see Proposition 3.46), $\mu(S)$ is idempotent separating. The last statement follows from Proposition 3.46. □

THEOREM 3.48. *Let S be a regular semiband with $E = \boldsymbol{E}(S)$. Then*

$$\mu(S) = \{(w, w') \in S \times S : \varphi_S(w) = \varphi_S(w')\}.$$

Consequently, $\langle E\varphi\rangle$ is a fundamental semiband.

fundamental representation
fundamental image

Proof. We have observed that $\varphi = \varphi_S$ is a surjective homomorphism of S onto $\langle E\varphi\rangle$ which extends the biorder embedding φ_E of Theorem 3.32. Hence, by definition, $\sigma = \boldsymbol{\kappa}\varphi$ is idempotent separating and so, $\sigma \subseteq \mu = \mu(S)$ by Proposition 3.47. Let τ be a congruence on S with $\tau \subseteq \mathscr{H}$. We show that $\tau \subseteq \sigma$. Since the quotient homomorphism $\psi = \tau^{\#} : S \to S' = S/\tau$ is idempotent separating, $\boldsymbol{E}(\psi) = E \to E' = \boldsymbol{E}(S')$ is an isomorphism. Hence, identifying E and E' by ψ, we have $\varphi_S(e) = \varphi_{S'}(\tau(e))$ for all $e \in E$. Hence if $a = e_1e_2 \dots e_n$ then by Equation (3.20c*)

$$\begin{aligned}\varphi_S(a) &= \varphi_S(e_1)\varphi_S(e_2)\dots\varphi_S(e_n)\\ &= \varphi_{S'}(\tau(e_1))\varphi_{S'}(\tau(e_2))\dots\varphi_{S'}(\tau(e_n))\\ &= \varphi_{S'}(\tau(a)).\end{aligned}$$

Therefore, if $(a, b) \in \tau$, then $\tau(a) = \tau(b)$ and so,

$$\varphi_S(a) = \varphi_{S'}(\tau(a)) = \varphi_{S'}(\tau(b)) = \varphi_S(b).$$

Hence $(a, b) \in \boldsymbol{\kappa}\varphi = \sigma$ and thus $\tau \subseteq \sigma$. Therefore σ is the largest idempotent separating congruence on S so that $\sigma = \mu(S)$ by Proposition 3.47. Since $\langle E\varphi\rangle = \operatorname{Im}\varphi_S$, $\langle E\varphi\rangle$ is a semiband isomorphic to $S/\mu(S)$ and so, $\langle E\varphi\rangle$ is fundamental semiband by Proposition 3.46. ☐

When E is a regular biordered set, we shall use the notation $\boldsymbol{B}_\tau(E)$ to denote the fundamental semiband of E so that $\boldsymbol{B}_\tau(E)$ is ismorphic to $\langle E\varphi\rangle$.

By a *fundamental representation* of a semigroup is a homomorphism $\phi : S \to T$ such that $\boldsymbol{\kappa}\phi = \mathscr{H}_{(c)}$; the semigroup $\operatorname{Im}\phi = \phi(S)$ is called the *fundamental image* of S. Clearly the fundametal image of S is unique up to isomorphism and so, we may refer to the fundamental image of S. Also the fundamental image of a semiband S is uniquely determined by its biordered set $E = \boldsymbol{E}(S)$; in this case, the fundamental image of S will be referred to as the fundamental semiband of E. The theorem above shows that φ_S is a fundamental representation of a regular semiband S. In particular, if E is any regular biordered set then $\varphi_{\boldsymbol{B}_0(E)}$ is the fundal representation of $\boldsymbol{B}_0(E)$ onto $\boldsymbol{B}_\tau(E)$. Therefore

COROLLARY 3.49. *For any regular biordered set E, $\boldsymbol{B}_\tau(E)$ is the fundamental regular semiband of E.*

We shall return to fundamental regular semigroups again in the chapter on inductive groupoids where we will discuss Munn's theory and various fundamental representations.

biorder property
P-biordered set
biorder property!strict –
strict P-biordered set
semigroup!completely regular –

3.4 BIORDER CLASSIFICATION OF SEMIGROUPS

Since biordered sets are nontrivial invariants of semigroups it is natural to consider classification of semigroups in terms of their biordered set of idempotents. Several classes of semigroups can be characterised in this way [see Higins, 1992]. Such classification of the class of regular semigroups will be of particular interest since structure of regular semigroups are closely related to their biordered sets of idempotents [see Nambooripad, 1979, §7, page 103–114]

Suppose that P is a property of a class of semigroups. If there is a property P^* for biordered sets such that, whenever a biordered set E has P^* *there exists* a semigroup S with $\boldsymbol{E}(S) = E$ having P, then P^* will be called a *biorder property* and E will be called a *P-biordered set*. We shall say that the biorder property P is *strict* and E a *strict P-biordered set* if whenever a biordered set E has P^* *every* semigroup S with $\boldsymbol{E}(S) = E$ has P.

3.4.1 Completely semisimple biordered sets

Let E be a biordered set and

$$\delta_0 = (\mathscr{L} \cup \mathscr{R})^t. \tag{3.28}$$

If there exists a completely semisimple semigroup (see Subsection 2.8.2) with $\boldsymbol{E}(S) = E$, then for $e, f \in E$, it follows from Theorem 2.87 that

$$(e, f) \in \delta_0 \quad \text{and} \quad e\ \omega\ f \Rightarrow e = f. \tag{P_1}$$

Conversely, if E satisfiies the condition above, then any semiband S with $\boldsymbol{E}(S) = E$ is completely semisimple. For any semiband S we have

$$\delta_0 = \mathscr{D} \cap (E \times E)$$

and so, the desired result again follows from Theorem 2.87. Thus if we define a completely semisimple biordered set as one that satisies condition (P_1) above, then we see that completely semi-simplicity is a biorder property.

THEOREM 3.50. *A biordered set is completely semisimple if and only if there is a compleletly semisimple semigroup S with $\boldsymbol{E}(S) = E$.*

3.4.2 Solid and orthodox biordered sets

Recall [see **?**] that a semigroup is *completely regular* if it is a union of groups. If S is completely regular then every $\mathscr{H}$-class of S is a group and so, S is a disjoint union of its group-$\mathscr{H}$-classes. To characterise the biordered sets of this class of semiroups, we introduce the following definition. A biordered set E is *solid*[see **?**] if E is regular and *biordered set!solid –*

$$\mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}. \tag{3.29}$$

Then $\delta_0 = \mathscr{L} \circ \mathscr{R}$. This condition is equivalent to the fact that if $e \mathrel{\mathscr{L}} f \mathrel{\mathscr{R}} g$ in E, there is $h \in E$ so that $A = \left(\begin{smallmatrix} e & h \\ f & g \end{smallmatrix}\right)$ is an E-square. We have [see Nambooripad, 1979, Theorem 7.2]

Theorem 3.51. *The following conditions are equivalent for a biordered set E.*

(1) E is solid.

(2) Each δ_0-class is an E-array.

(3) There exists a completely regular semigrop S such that $\boldsymbol{E}(S) = E$.

Proof. (1) ⇒ (2): By (1), $\mathscr{L}$ and $\mathscr{R}$ are commuting equivalences and so $\mathscr{L} \circ \mathscr{R}$ is an equivalence relation which implies by Equation (3.28) that $\delta_0 = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$. Hence if $(e, f) \in \delta_0$ there exists $g, h \in E$ such that $e \mathrel{\mathscr{L}} h \mathrel{\mathscr{R}} f \mathrel{\mathscr{L}} g \mathrel{\mathscr{R}} e$ which means that $\left(\begin{smallmatrix} e & g \\ g & f \end{smallmatrix}\right)$ is an E-square. It fllows that $\delta_0(e)$ is an E-array for all $e \in E$.

(2) ⇒ (3): Assume that S is a regular semiband with $\boldsymbol{E}(S) = E$. By Theorem 3.34 S exists (for example we may choose $S = \boldsymbol{B}_0(E)$). Inductively assume that everyproduct of fewer than n idempotents in S belongs to a group and let $a = e_1 e_2 \dots e_n$. Suppose that $b = e_1 e_2 \dots e_{n-1}$ and $k \in E$ with $b \mathrel{\mathscr{L}} k$. If $h \in \mathscr{S}(k, e_n)$ then $h \mathrel{\omega^l} e_{n-1}$ and so $\bar{h} = e_{n-1} h$ is a basic product and hence $\bar{h} \in E$. Therefore

$$c = bh = e_1 e_2 \dots e_{n-2} \bar{h}$$

is a product of $n - 1$ idempotents in S. Hence by induction hypothesis there is a $g \in E$ such that $g \mathrel{\mathscr{H}} c$. Now $c = bh \mathrel{\mathscr{L}} h$ and so, $g \mathrel{\mathscr{L}} h \mathrel{\mathscr{R}} he_n$. Hence $(g, he_n) \in \delta_0$ and so, by (2), there exists $l \in E$ with $g \mathrel{\mathscr{R}} l \mathrel{\mathscr{L}} he_n$ so that $l \mathrel{\mathscr{H}} g(he_n)$. Then $c(he_n) = (bh)(he_n) \mathrel{\mathscr{H}} l$. It follows by Theorem 3.7 that $a = be_n \mathrel{\mathscr{H}} l$. The induction hypothesis clearly holds for $n = 2$. Therefore every finite product of idempotents in S belongs to a subgroup of S. Since S is a semiband it is completely regular.

(3) ⇒ (1): If S is completely regular with $\boldsymbol{E}(S) = E$ and if $e \mathrel{\mathscr{L}} f \mathrel{\mathscr{R}} g$, then here is $h \in E$ such that $h \mathrel{\mathscr{H}} eg$ and so, $A = \left(\begin{smallmatrix} e & h \\ f & g \end{smallmatrix}\right)$ is an E-square. Hence E is solid. □

A more detailed account of completely regular seigroups will be given later in the next chapter (see Subsection 4.3.2).

biordered set!orthodox – band!left regular – band!right unipotent –

Recall that an E-square A is τ-commutative if

$$\tau(e,f)\tau(f,g) = \tau(e,h)\tau(h,g)$$

(see Diagram 3.15) where $\tau(e,f) : \omega(e) \to \omega(f)$ is the ω-isomorphism defined in Corollary 3.16. It is readily verified that A is τ-commutative if and only if A is a 2×2-rectangular subband of any fundamental semigroup S for which $\boldsymbol{E}(S) = E$. Again S exists since, by Theorem 3.48, we may take $S = \boldsymbol{B}_\tau(E)$. We say that a biordered set E is *orthodox* if the fundamental semiband $\boldsymbol{B}_\tau(E)$ is a band.

Corollary 3.52. *A biordered set E is orthodox if and only if E is solid and every E-square in E is τ-commutative.*

Proof. If $E = \boldsymbol{E}(B)$ where B is a band, and if $e \mathscr{L} f \mathscr{R} g$, then $eg = h \in E$ and so, $A = \begin{pmatrix} e & h \\ f & g \end{pmatrix}$ is an E-square in E. By the remarks preceeding the statement of the corollary, A is τ-commutative. On the other hand, assume that E is solid in which every E-square is τ-commutative. By Theorem 3.48 $\boldsymbol{B}_\tau(E)$ is a fundamental semiband with $\boldsymbol{E}(\boldsymbol{B}_\tau(E)) = E$. If $e, f \in E$ and $h \in \mathscr{S}(e,f)$, $eh \mathscr{L} h \mathscr{R} hf$. Since E is solid, there is $k \in E$ such that $A = \begin{pmatrix} eh & k \\ h & hf \end{pmatrix}$ is an E-square. Since A is τ-commutative, we have

$$(eh)(hf) = k \quad \text{in } \boldsymbol{B}_\tau(E) \text{ so that} \quad ef = k$$

in $\boldsymbol{B}_\tau(E)$ by Theorem 3.7. Therefore product any two idempotents in $\boldsymbol{B}_\tau(E)$ is an idempotent. It follows that $\boldsymbol{B}_\tau(E)$ is a band with $\boldsymbol{E}(\boldsymbol{B}_\tau(E)) = E$. □

The biordered set E_6 of Example 3.15 is solid but not orthdox.

Several properties of bands may be described in terms of their biordered sets. For example, define a band B to be *left regular* if $fef = fe$ for all $e, f \in B$ [see **???**]. The condition for left reularity of B clearly implies that, if $e, f \in B$ and $e \mathscr{R} f$ then $e = f$; thus every $\mathscr{R}$-class of B contain exactly one idempotent. A left regular band is also said to be *right unipotent*. The later condition is clearly a biorder condition. Consequently, we may define a biordered set E to be left regular or right unipotent if $\boldsymbol{B}_\tau(E)$ is a left regular band. A regular semigroup S is right unipotent if $\boldsymbol{E}(S)$ is right unipotent.

Theorem 3.53. *The following conditions are equivalent for a regular biordered set E.*

(1) E is right unipotent;

(2) $\omega^r \subseteq \omega^l$;

(3) If S is any regular semigroup with $\boldsymbol{E}(S) = E$, then S is right unipotent.

In particular if S is any semiband with $\boldsymbol{E}(S) = E$, then S is isomorphic to $\boldsymbol{B}_\tau(E)$. pseudo-semilattice

Proof. Condition (1) implies that the relation $\mathscr{R} = 1_E$ which implies that $\omega^r = \omega$. Hence (2) holds.

Now suppose that E satisfies (2) and that S is a regular semigroup with $\boldsymbol{E}(S) = E$. If $e, f \in E$, and $e \mathscr{R} f$ then $e \mathscr{R} f$ and $e\ \omega^l\ f$ and so, $e = f$. Hence S is right unipotent; thus (2) implies (3).

If S is right unipotent, then by definition $\boldsymbol{E}(S) = E$ has this property and so, (3) implies (1).

Finally assume that S is any semiband with $\boldsymbol{E}(S) = E$. If $e, f \in E$ and $h \in \mathscr{S}(e, f)$ then $h\ \omega\ f$ and so $hf = h$. By Theorem 3.7, the product ef in S is

$$ef = (eh)(hf) = (he)h = he.$$

It follows that S is a band and hence fundamental. Therefore S is isomorphic to $\boldsymbol{B}_\tau(E)$. □

3.4.3 Pseudo-semilattices

Similar to the concept of a biordered set, Schein [1972] defined a *pseudo-semilattice* $E = (E, \omega^l, \omega^r)$ as an order structure determined by two quasiorders ω^l and ω^r on the set E such that for all $e, f \in E$ there is unique element $h \in E$ satisfying

$$\omega^l(e) \cap \omega^r(f) = M(e, f) = \omega(h) \quad \text{where} \quad \omega = \omega^l \cap \omega^r. \tag{3.30}$$

The uniqueness of h implies that ω is a partial order and that the map

$$(e, f) \mapsto h = f \wedge e \tag{3.31}$$

is a binary operation on E. The binary algebra $E = (E, \wedge)$ obtained in this way is also called a pseudo-semilattice. Given the binary algebra, define the relations ω^l_1 and ω^r_1 as follows:

$$e\omega^l_1 f \iff e \wedge f = e \quad \text{and} \quad e\omega^r_1 f \iff f \wedge e = e. \tag{3.32}$$

Then ω^l_1 and ω^r_1 are again quasiorders satisfying Equation (3.30) and it can be shown [see Schein, 1972] that the binary operation defined by Equation (3.31) with respect to (E, ω^l, ω^r) and $(E, \omega^l_1, \omega^r_1)$ coincide. We shall therefore assume that in all pseudo-semilattices under consideration, the quasiorders and the binary operation $\wedge$ are related by Equation (3.32). Schein [1972] has shown that pseudo-semilattices form a class binary algebras defined by a set of equations (identities). [see Nambooripad, 1981, Schein, 1972] for relevant definitions *local semilattices*

and results. Some authors call pseudo-semilattices as *local semilattices*. The statement (d) of the following theorem shows the relevance of this terminology.

If E is a semilattice with the partial order ω, then $\omega^l(e) \cup \omega^l(f) = \omega(f \wedge e)$ for all $e, f \in E$ and so, (E, ω, ω) is a pseudo-semilattice in which the binary operation of Equation (3.31) coincides with the meet $\wedge$ of the semilattice. Thus every semilattice is a pseudo-semilattice. It is easy to see that the biordered set of any completely 0-simple semigrup is a pseud-semilattice which is not a semilattice. However, not all pseudo-semilattices are biordered sets (see Example 3.16). The reader should refer to Nambooripad [1981, 1982a,b] for characterisation of the varieties of pseudosemilattices, structure of various classes of pseudo-inverse semigroups, etc.

We proceed to discuss the exact relations between biordered sets and pseudo-semilattices. We shall say that a pseudo-semilattice E is a biordered set if the restriction of the binary operation $\wedge$ to the relation

$$D_E = (\omega^l \cup \omega^r) \cap (\omega^l \cup \omega^r)^{-1}$$

is the basic product of a biordered set. Conversely a biordered set $E =< E, \omega^l, \omega^r, T^l, T^r >$ (see Definition 3.1)is a pseudo-semilattice if the quasiorders ω^l and ω^r satisfy Equation (3.30). If this is the case, it follows from Definition 3.3 that

$$\mathscr{S}(e, f) = \{f \wedge e\} \quad \text{for all} \quad e, f \in E. \tag{3.33}$$

The following theorem characterizes those biordered sets that are psedo-semilattices [see Nambooripad, 1979, Theorem 7.6].

THEOREM 3.54. *The following conditions are equivalent for a biordered set E.*

(a) (E, ω^l, ω^r) *is a pseudo-semilattice.*

(b) For all $e, f \in E$, $\mathscr{S}(e, f)$ *contains exactly one element.*

(c) For all $e \in E$, $\omega^l(e)$ *is left regular and* $\omega^r(e)$ *is right regular.*

(d) For all $e \in E$, $\omega(e)$ *is a semilattice.*

Proof. (a) ⇒ (b): Follows from Equation (3.33).

(b) ⇒ (c): Let $e \in E$. To show that $\omega^l(e)$ is left regular, by Theorem 3.53(2), it is sufficient to show that the relation $\mathscr{R} \mid \omega^l(e)$ is identity on $\omega^l(e)$. So, let $f, g \in \omega^l(e)$ and $f \mathscr{R} g$. By Proposition 3.9, $fe = f \in \mathscr{S}(e, f)$ and $g \in \mathscr{S}(e, g)$. By Proposition 3.12, $\mathscr{S}(e, f) = \mathscr{S}(e, g)$ and so, $f = g$ by (b). Dually $\omega^r(e)$ is right regular.

(c) ⇒ (d): By (c) the relations $\mathscr{L}$ and $\mathscr{R}$ are identity on $\omega(e)$. Hence, by Proposition 3.15 $\omega(e)$ is a biordered subset of E on which the relations ω^l and ω^r coincide. Hence $\omega(e)$ is a semilattice (see Example 3.2).

(d) ⇒ (a): Let $e, f \in E$ and $h \in \mathscr{S} + (e, f)$. If $g \in M(e, f)$, then $g \leq h$. Then $eg\ \omega^r\ eh$ and since $\omega(e)$ is a semilattice, we have $eg\ \omega\ eh$. Hence $g\ \mathscr{L}\ eg\ \omega\ eh\ \mathscr{L}\ h$ and so, $g\ \omega^l\ h$. Dually, $g\ \omega^r\ h$ and so, $g\ \omega\ h$. Therefore Equation (3.30) holds. □

Next theorem characterises those pseudo-semilattices that are biordered sets [see Nambooripad, 1981, Theorem 2].

THEOREM 3.55. *Let $E = (E, \omega^r, \omega^l)$ be a pseudo-semilattice. Then E is a biordered set if and only if E satisfies the following conditions and their duals: for all $f, g \in \omega^r(e)$,*

(PA1) $(g \wedge e) \wedge f = g \wedge f$;

(PA2) $(f \wedge e) \wedge (g \wedge e) = f \wedge (g \wedge e) = (f \wedge g) \wedge e$.

Proof. First assume that E is a biordered set. Then by Theorem 3.54(c), $\omega^r(e)$ is right regular for all $e \in E$. Then by the definition of right regular biordered sets, the basic products in $\omega^r(e)$ can be extended in such a way that $\omega^r(e)$ be comes a right regular band B_e. Then for any $f, g \in B_e$, from Proposition 3.4 that the product fg in B_e belongs to $\mathscr{S}(g, f)$. By Equation (3.33), $\mathscr{S}(g, f) = \{f \wedge g\}$. Hence $fg = f \wedge g$ for all $f, g \in B_e$. Identities (PA1) and (PA2) now follows from the associativity of $\wedge$ in B_e. Duals of these identities are proved similarly.

Conversely let E be a pseudo-semilattice satisfying (PA1), (PA2) and their duals. Define basic product in E as the restriction of $\wedge$ to D_E. Axioms (B11) and (B12) are clearly satisfied. If $f\ \omega^r\ e$ then by Equation (3.32), $e \wedge f = f$ and

$$e \wedge (f \wedge e) = (e \wedge f) \wedge e = f \wedge e;$$
$$(f \wedge e) \wedge e = f \wedge (e \wedge e) = f \wedge e; \qquad \text{and}$$
$$f \wedge (f \wedge e) = (f \wedge f) \wedge e = f \wedge e$$

by (PA2). Again, by (PA1), we have

$$(f \wedge e) \wedge f = f \wedge f = f.$$

This proves axxiom (B21). (B22) follows from (PA1). To prove (B3) let $f, g \in \omega^r(e)$ and $g\ \omega^l\ f$. Then by (PA2) we have

$$(g \wedge e) \wedge (f \wedge e) = g \wedge (f \wedge e) = (g \wedge f) \wedge e = g \wedge e;$$
$$(g \wedge f) \wedge e = (g \wedge e) \wedge (f \wedge e)$$

which gives (B3). Again assume that $f, g \in \omega^r(e)$ and $g \wedge e\ \omega^l\ f \wedge e$. Let $g_1 = g \wedge f$. Then $g_1\ \omega^l\ f$ (by Equation (3.30)) and

$$g_1 \wedge e = (g \wedge f) \wedge e = (g \wedge e) \wedge (f \wedge e) = g \wedge e.$$

Hence axiom (B4) follows and the proof is cmplete. □

band!normal – normal biordered set

If E is a semilattice, the binary operatin specified by the associated pseudo-semilattice E is the meet $\wedge$ which is associative. Schein observed that the binary operation $*$ on E need not be associative. [In Nambooripad, 1981] a pseudo-semilattice E is said to be partially associative if E satisfies (PA1) and (PA2). Example 3.16 gives a pseudo-semilattice which is not partially associative. Example 3.17 is a pseudo-semilattice which is partially associative, but not associative. Schein [1972] shows that a pseudo-semilattice E is associative if and only if $(E, \wedge)$ is a *normal band*; that is, E is a band with respect to $\wedge$ and satisfies the identity

$$x \wedge y \wedge z \wedge u = x \wedge z \wedge y \wedge u \quad \text{for all} \quad x, y, z, u \in E. \tag{3.34}$$

As above, we shall say that a biordered set E is *normal biordered set* if $\boldsymbol{B}_\tau(E)$ is a normal band.

Corollary 3.56. *A biordered set E is normal if and only if it is an orthodox pseudo-semilattice.*

Proof. Let E be an orthdox pseudo-semilattice. Since E is orthodox, $B = \boldsymbol{B}_\tau(E)$ is a band. Hence for all $e, f \in E$, the product fe in B belongs to $\mathscr{S}(e, f)$. Hence by Theorem 3.54, $\mathscr{S}(e, f) = \{fe\}$. If $f \mathscr{R} g$, by Proposition 3.12, $\mathscr{S}(e, f) = \mathscr{S}(e, g)$ and so $fe = ge$. Dually, if $f \mathscr{L} g$, we have $ef = eg$. Now let $e, f, g, h \in E$. Then $gf \ \omega^r \ g$ and so, $gfg \mathscr{R} gf$. Hence $(gfg)h = (gf)h$ and dually, $e(fg) = e(gfg)$. Therefore

$$\begin{aligned} efgh &= (e(fg))h = (e(gfg))h \\ &= e((gf)h) = egfh. \end{aligned}$$

Hence B is normal.

Conversely assume that E is normal so that $B = \boldsymbol{B}_\tau(E)$ is a normal band. Then for $e, f \in B$, $fe \in \mathscr{S}(e, f)$. If $g \in M(e, f)$ then

$$g = fge = ffge = fgfe = gfe \quad \text{and} \quad g = fgee = feg$$

and so, $g \ \omega \ fe$. Thus $M(e, f) = \omega(fe)$ and so, E is a pseudo-semilattice. ☐

Succeeding chapters we will characterise a number of additional classes of regular semigroups in terms of biorder properties.

3.4.4 Examples

Here we list a number of examples and counter exaples. Most of these are due to Easdown [see **?**] which appeared in **?**.

Example 3.9: Let $E = \{e, f\} \cup \mathbb{N}$ where $\mathbb{N} = \{0, 1, 2, \dots\}$ denote the set of natural numbers (with usual order) and let $\leq$ be a partial order on E with

$\mathbb{N}$:*system of natural numbers*

$$x \leq x, \quad \text{and} \quad 0 \leq x \quad \text{for all} \quad x \in E;$$
$$n \leq e \quad \text{and} \quad n \leq f \quad \text{for all} \quad n \in \mathbb{N}$$

and the usual order between natural numbers. We shall denote by E the biordered set determined by the partial ordered set $(E, \leq)$ as in Example 3.4. Let $\theta : E \to E$ be defined by

$$e\theta = e, \quad f\theta = f, \quad \text{and} \quad n\theta = 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

Then $\theta : E \to E$ is a regular bimorphism such that $E\theta = E_1 = \{e, f, 0\}$. The surjective bimorphism $\theta^\circ : E \to E_1$ determined by θ and the inclusion $\jmath : E_1 \subseteq E$ are not regular even though $\theta = \theta^\circ \jmath$. Thus θ satisfies (RM31) and (RM32) but not (RM33). Also θ° satisfies (RM31) and (RM32) but not regular.

Example 3.10 (?): Let $E_0 = \{e, f : e^2 = e, \ f^2 = f\}$. This is a biordered set with $D_{E_0} = \{(e, e), (f, f)\}$. The free idempotent generated semigroup $B_0 = \boldsymbol{B}_0(E_0)$ consists of words of the form $(ef)^n$, $(fe)^n$, $f(ef)^n$ and $(ef)^n e$ where $n = 1, 2, \dots$. Also $\boldsymbol{E}(B_0) = \{e, f\}$. In this case, we can identify Λ° and I° with the set $\{e, f, \infty\}$. The maps $\rho(e)$ and $\lambda(e)$ send $e \mapsto e$ and $x \mapsto \infty$ for $x \neq e$. $\rho(f) = \lambda(f)$ is defined similarly and the map

$$\rho(f)\rho(e) = \rho(e)\rho(f) = \lambda(e)\lambda(f) = \lambda(f)\lambda(e)$$

is the constant map, denoted by 0, with value ∞. Then $S_0 = \langle E_0\varphi \rangle = \{\varphi(e), \varphi(f), 0\}$ which is a semilattice having three elements so that $E_0\varphi \neq \boldsymbol{E}(S_0)$.

Example 3.11 (?): Let $E = \{e, f, g, h, k\}$ be a set and define quasiorders ω^r and ω^l on X by:

$$\omega^r = (\{e\} \times E - \{f\}) \cup (\{f\} \times E - \{e\}) \cup (\{g, h, k\} \times \{g, h, k\});$$
$$\omega^l = \{(e, g), (e, k), (f, h), (f, g)\} \cup 1_E.$$

Suppose that D_E is the relation on E defined by Equation (3.2). A partial binary operation $\cdot$ with domain D_E satisfying Equation (3.4) is specified if we specify products $h \cdot e$ and $k \cdot f$. Let $E_1 = (E, D_E, \cdot)$ be the partial algebra where $\cdot$ is obtained as specified above by setting

$$h \cdot e = g = k \cdot f.$$

Verify that E_1 is a biordered set. Completion of the partial binary operation $\cdot$ on E_1 to a binary operation by setting $ef = fe = g$ gives a band B_1 with $\boldsymbol{E}(B_1) = E_1$. Hence, E_1 is, in particular, a regular biordered set. Also the representation φ of Theorem 3.32 extends to an isomorphism of B_1 to $S_1 = \langle E_1\varphi \rangle$.

Example 3.12 (?): Let E and D_E be as in Example 3.10. Let let $\cdot$ be the partial binary operation specified as in Example 3.10 with

$$h \cdot e = k; \quad \text{and} \quad k \cdot f = h.$$

Verify that $E_2 = (E, D_E, \cdot)$ is a biordered set which is not regular (show that $\mathscr{S}(e, f) = \emptyset$). Let B_0^0 be the semigroup B_0 of Example 3.9 with 0 adjoined. Let B_2 denote the ideal extension of B_0^0 by the right-zero semigroup $R = \{h, g, k\}$ (see Subsection 2.10.1). Then $E_2 = \boldsymbol{E}(B_2)$. Since B_0 is an infinite semigroup, so is B_2. In fact E_2 is not the biordered set of a finite semigroup. For if it is, $a = ef$ has finite order and there exist an integer n

such that $u = a^n$ is an idempotent. It is clear that $f \neq u \neq e$ since $u = e$ would imply that $ef = e$ which is impossible. Similarly $u \neq f$. If $u = k$, then $h = kf = uf = u = k$ which is not possible. If $u = g$, then

$$g = hg = h(ef)^n = (he)f(ef^{n-1}) = h(ef)^{n-1} = \cdots = h$$

and if $u = h$, we similarly have

$$h = gh = g(ef)^n = g$$

and both these are false. Thus E_2 is a finite biordered set which is not embeddable as the biordered set of a finite semigroup. However, verify that

$$S_2 = \langle E_2\varphi\rangle = E_2 \cup \{ef, fe\}$$

where $(ef)e = fe$ and $(fe)f = ef$ is a finite band containing $E_2\varphi$ as a biordered subset, but $E_2\varphi \neq \boldsymbol{E}(S_2)$. (Here we identify $\varphi(e)$ with e for brevity.)

Example 3.13 (?): Again let E and D_E be as in Example 3.10 and assume that $\cdot$ is the partial binary operation specified as in Example 3.10 with

$$h \cdot e = g; \quad \text{and} \quad k \cdot f = h. \tag{3*}$$

Verify that $E_3 = (E, D_E, \cdot)$ is a non-regular biordered set. Let φ denote the representation of E_3 of Theorem 3.32 and $B_3 = \langle (E_3)\varphi\rangle$. Since E_3 is finite it is immediate from the definition that B_3 is a finite semigroup. All products of elements in $(E_3)\varphi$ except for $\varphi(e)\varphi(f)$ belong to $(E_3)\varphi$. Hence B_3 is a semigroup with six elements in which five elements are idempotents and $\varphi(e)\varphi(f)$ is not regular. Notice that in this case we have $E\varphi = \boldsymbol{E}(\langle E\varphi\rangle)$. Also, E_3 cannot be embedded as a biordered subset of a band. For if $E_3 \subseteq E$ where E is a band then $ef \in E$ and so

$$g = gf = (he)f = h(ef) = (kf)(ef) = k(ef)(ef) = k(ef) = h$$

which is not possible. Also E_3 is the smallest non-regular biordered set which is the biordered set of a finite semigroup.

Example 3.14 (?): Let

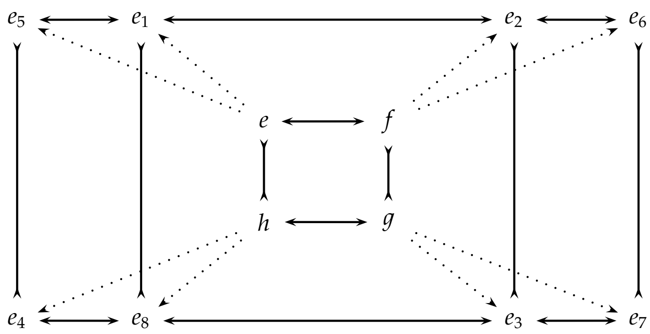
$$E_5 = \{e, f, g, 0 : e \mathscr{L} f \mathscr{R} g;\ x0 = 0x = 0 \quad \text{for all} \quad x = e, f, h\}.$$

Then E_5 is a regular biordered set such that $\boldsymbol{B}_\tau(E_5)$ is a completely 0-simple semigroup with the non-zero $\mathscr{D}$-class containing four elements including the non-identity element $a = eg$ with $e \mathscr{R} a \mathscr{L} g$. Show that E_5 is the smallest regular biordered set which is not the biordered set of a union of groups.

Example 3.15 (Easdown): Let $E_6 = \delta_1 \cup \delta_2$ be the biordered with two δ_0-classes $\delta_1 = \{e, f, g, h\}$ and $\delta_2 = \{e_i : 1 \leq i \leq 8\}$. The relations in E_6 are shown in the figure below; the horizondal arrows denote $\mathscr{R}$-relations, vertical arrows denote $\mathscr{L}$-relations and doted arrows shows ω-relatins. Notice that every element in δ_1 has two elments in δ_2 which is ω-related to it. Basic products are specified by the relatins shown in the diagram and the following equatins:

$$ee_4 = e_5,\ ee_8 = e_1,\ e_3h = e_4,\ e_7h = e_8,$$
$$e_1f = e_2,\ e_5f = e_6,\ ge_2 = e_3,\ ge_6 = e_7.$$

It is easy to verify that E_6 is a biordered set which is clearly solid. However $\boldsymbol{B}_\tau(E_6)$ is not a band. For if $\bar{e}, \bar{f}$, etc. denote idempotents in $\langle E_6\varphi\rangle$ corresponding to e, f etc. in E_6, then we see that $\overline{fh} \in H_{\bar{e}}$ in $\langle E_6\varphi\rangle$ but $\bar{e} \neq \overline{fh}$. In fact, we have $H_{\bar{e}} = \{\bar{e}, \overline{fh}\}$ and so $H_{\bar{e}}$ is a group of order 2. It follows that $\langle E_6\varphi\rangle$ and hence $\boldsymbol{B}_\tau(E_6)$ is not a band. Thus E_6 is solid but not orthodox. E_6 is the smallest biordered set with this property.



Example 3.16: Let $E = \{a, b, c, d\}$. Define ω^r and ω^l on E by

$$\omega^r(a) = \omega^l(a) = \{a\};$$
$$\omega^r(b) = \omega^l(b) = \{a, b\};$$
$$\omega^r(c) = \omega^l(c) = \{a, b, c\};$$
and $$\omega^r(d) = \{a, b, d\}; \quad \omega^l(d) = \{a, d\}.$$

Then ω^r and ω^l are quasiorderes on E and (E, ω^r, ω^l) is a pseudo-semilattice. Let $\wedge$ denote the binary operation on E determined by the pseudo-semilattice. Then

$$(b \wedge d) \wedge b = a \quad \text{and} b \wedge (d \wedge b) = b.$$

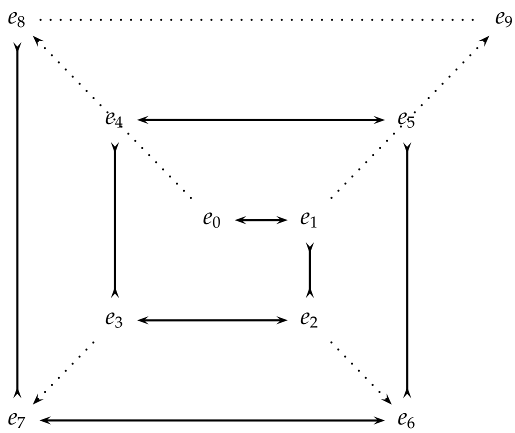
Hence $\omega^r(d)$ is not associative. Hence by Theorem 3.54, E cannot be a biordered set.

Example 3.17 (?): Let $\mathbb{N} = \{0, 1, \dots\}$ denote the set all non-negative integers and let $E^{sp} = \{e_n : n \in \mathbb{N}\}$. Define relations $\mathscr{R}$, $\mathscr{L}$ and ω on E^{sp} as follows:

$$\mathscr{L} = \{(e_n, e_n) : n \in \mathbb{N}\} \cup \{(e_n, e_m) : m = n + (-1)^{n+1}, \quad \text{for all} \quad n > 0\};$$
$$\mathscr{R} = \{(e_n, e_n) : n \in \mathbb{N}\} \cup \{(e_n, e_m) : m = n + (-1)^n, \quad \text{for all} \quad n \in \mathbb{N}\};$$
and $$\omega = \{(e_n, e_m) : n \geq m \quad \text{and} \quad n = m \pmod 4).$$

Moreover, let $\omega^r = \mathscr{R} \circ \omega$ and $\omega^l = \mathscr{L} \circ \omega$.

Then show that $E^{sp} = (E^{sp}, \omega^r, \omega^l)$ is a pseudo-semilattice and a biordered set (see the figure below where horizondal arrows dente $\mathscr{R}$-relations, vertical arrows denote *lrs*-relations and doted arrows denote the ω relations.



Compute the semigroup $\boldsymbol{B}_\tau(E^{sp})$ and show that it is bisimple. Moreover if $a = e_1e_3$, then a^n does not belong to a subgroup of semigroup $\boldsymbol{B}_\tau(E^{sp})$ for any $n > 0$ so that it is not group-bound.

natural partial order
semilattice union of groups
partial order!compatible –
semigroup!ordered –

CHAPTER 4

Regular Semigroups

In Chapter Chapter 2 we had given a general discussion of properties of semigroups. In this chapter, our aim is to discuss certain properties of regular semigroups that are of interest in the later development of ’>the theory of regular semigroups. We begin with a study of a partial order on semigroups which is called, following Mitsch Mitsch [1986], the *natural partial order*. This relation has particular relevance for regular semigroups and we pay particular attention to this case. We then proceed to a discussion of certain properties of congruences on regular semigroups and decompositions of regular semigroups. These naturally lead to the classical theorem of Clifford on *semilattice union of groups*. Many of the results given here are quite classical or refinements of classical results. Wherever proofs can be simplified or results can be refined using biordered set and other advanced technique, we have not hesitated to use the same, even though, often, proofs with out using them may be available in literature.

4.1 THE NATURAL PARTIAL ORDER ON A SEMIGROUP

Let $\leq$ be a partial order on a semigroup S. We shall say that $\leq$ is *compatible* if

$$a \leq b,\ c \leq d \Rightarrow ac \leq bd. \tag{4.1}$$

If $\leq$ is compatible, we say $(S, \leq)$ is an *ordered semigroup* or that S is an ordered semigroup with respect to $\leq$.

Remark 4.1: Every semigroup S can be endowed with a partial order so that S becomes an ordered semigroup. For, if ρ is any partial order on S, then the relation

$$\rho_{(c)} = \{(x, y) : (axb, ayb) \in \rho\ \forall\ a, b \in S^1\}.$$

can be seen to be the largest compatible partial order contained in ρ. Further, as observed in Remark 2.8, S^1 is a faithful left S-set and so, the representation of S by right translations of S^1 is faithful. Hence the semigroup S can be embedded as a subsemigroup of $\boldsymbol{B}_S$. Since inclusion is a compatible partial order on $\boldsymbol{B}_S$, (see Example Subsection 2.1.3) it induces a compatible partial order on S (via the embedding).

A systematic account of ordered semigroups is not in the scope of this book. However, there are partial orders on semigroups (which may not be compatible) whose study throw considerable light on the structure of the semigroups. Our aim here is to study one such partial order, called the *natural partial order*.

The natural partial order was first studied for the class of inverse semigroups by Vagner Vagner [1953a]. It has proved to be of great importance in every area of the theory of inverse semigroups. Later Nambooripad [1980] extended it to the class of regular semigroups. Finally Mitsch [1986] extended the concept to arbitrary semigroups ([see also Bingjun]). While the natural partial order on an inverse semigroup is compatible, this is not the case for arbitrary semigroups. Even so, the natural partial order is related closely to the structure of regular semigroups (see Theorem 4.10 below). Our treatment here is based mainly on Nambooripad [1980] and will emphasize regular case since we shall find the concept extremely useful in what follows.

4.1.1 Definition and properties

Most of the results in this section is due to Mitsch and Yu Bingjun Bingjun, Mitsch [1986].

LEMMA 4.1. *Let $\leq$ be the relation on a semigroup S defined as follows: for $a, b \in S$*

$$a \leq b \iff a \leq_r b \quad \text{and for some } x \in S^1 \quad a = xa = xb \tag{4.2}$$

where $\leq_r$ is the quasiorder on S defined by Equation (2.36a). *Then $\leq$ is a partial order on S whose restriction to $\boldsymbol{E}(S)$ coincides with the natural partial order ω of $\boldsymbol{E}(S)$.*

Proof. The relation $\leq$ is clearly reflexive. Suppose that $a \leq b$ and $b \leq c$. Then there exists $x, y \in S^1$ such that

$$a = xa = xb, \qquad b = yb = yc.$$

Since $a \leq_r b$, there is $s \in S^1$ with $a = bs$. This gives

$$xyc = xb = a \quad xya = xybs = xbs = xa = a.$$

Since $\leq_r$ is transitive, this shows that $\leq$ is transitive. Now assume that $a \leq b$ and $b \leq a$. As before there is $x, y, s \in S^1$ with $a = xa = xb$, $b = yb = ya$ and $a = bs$. Hence

$$b = ya = ybs = bs = a.$$

partial order!natural – $\leq_S$: The natural partial order on S

Therefore $\leq$ is anti-symmetric.

Let $e, f \in \boldsymbol{E}(S)$. If $e\ \omega\ f$, then Equation (4.2) clearly holds with $x = e$. Conversely if $e \leq f$, then, by the definition, $e\ \omega^r\ f$ and $e = xe = xf$ for some $x \in S^1$. Then $ef = xf^2 = xf = e$ and so $e\ \omega^l\ f$ and so $e\ \omega\ f$. □

The partial order $\leq$ on S defined in the lemma above is called the *natural partial order* on the semigroup S. In the following $\leq_S$ (or just $\leq$ if there is no ambiguity) will denote the natural partial order on the semigroup S.

The definition of natural partial order above is one-sided; but we show below that the dual definition also gives rise to the same relation.

PROPOSITION 4.2. *Let $\leq$ denote the natural partial order on a semigroup S. The following statements are equivalent for all $a, b \in S$.*

(1) $a \leq b$;

(2) $a \leq_l b$ and $a = ay = by$ for some $y \in S^1$;

(3) $a = xa = xb = ay = by$ for some $x, y \in S^1$.

Proof. By Lemma 4.1, (1) implies that there is $x, s \in S^1$ such that $a = bs$ and $a = xa = xb$. Then $a = xb \in Sb$ and so, $a \leq_l b$. Also $a = xa = xbs = as$. Thus (2) holds.

The proof of (2) implies (1) is dual; thus (1) and (2) are equivalent. Therefore it is clear that if (1) holds, then (3) also holds. On the other hand, if (3) holds, then from $a = by$ we have that $a \leq_r b$ and so (1) holds. □

If b is a regular element of a semigroup S, by Proposition 2.39, both $E(R_b)$ and $E(L_b)$ contains idempotents. We use this fact in the following characterization of natural partial order on regular elements. Clearly, the following proposition is valid, in particular, for natural partial order on regular semigroups.

PROPOSITION 4.3. *Suppose that b is a regular element of a semigroup S and $a \in S$. Then following statements are equivalent.*

(1) $a \leq b$;

(2) for any $f \in E(R_b)$ there is $e \in E(R_a)$ such that $e\ \omega\ f$ and $a = eb$;

(3) for any $f' \in E(L_b)$ there is $e' \in E(L_a)$ such that $e'\ \omega\ f'$ and $a = be'$;

(4) $a \leq_h b$ and $a = ab'a$ for some [for all] $b' \in \mathscr{V}(b)$;

(5) $a = be = fb$ for some $e, f \in \boldsymbol{E}(S)$.

Proof. We shall prove the following:

$$(1) \Rightarrow (2) \Rightarrow (4) \text{ for all } b' \in \mathscr{V}(b);$$
$$(4) \text{ for some } b' \in \mathscr{V}(b) \Rightarrow (5) \Rightarrow (1).$$

Since the proof of the implications

$$(1) \Rightarrow (3) \Rightarrow (4) \text{ for all } b' \in \mathscr{V}(b)$$

are dual to the implications in the first line above, it will follow that all statements above are equivalent.

(1) ⇒ (2): Let $f \in E(R_b)$. Since $fb = b$, by Corollary 2.27 and Lemma 2.36, $\rho_b|L(f) = Sf$ is an isomorphism onto Sb; let $\rho_t : Sb \to Sf$ be its inverse. If $a \leq b$, then by Proposition 4.2, there exist $x, y \in S^1$ with $a = xa = xb = ay = by$. Hence $a \in Sb$. Let $e = a\rho_t = at$. Then by Theorem 2.25,

$$e \mathscr{R} a \quad \text{and} \quad e^2 = atat = xbtbyt = xbyt = xat = at = e.$$

Since

$$e \mathscr{R} a \leq_r b \mathscr{R} f \quad \text{and} \quad ef = atbt = at = e,$$

e is an idempotent with $e \ \omega \ f$. Also $eb = atb = a$. This proves (2).

(2) ⇒ (4) for all $b' \in \mathscr{V}(b)$: Let $b' \in \mathscr{V}(b)$ and $f = bb'$. Then by Lemma 2.38, $f \in E(R_b)$. Also, by Corollary 2.27, $\rho_b|L(f) : L(f) \to L(b)$ is the unique isomorphism sending f to b and $\rho_{b'}$ is its inverse. By (2), there is $e \in E(R_a)$ such that $e \ \omega \ f$ and $a = eb = e\rho_b$. Hence

$$ab' = a\rho_{b'} = e\rho_b\rho_{b'} = e.$$

Therefore $ab'a = ea = a$. Since $a \mathscr{R} e \leq_r f \mathscr{R} b$ and $a = eb \in Sb$, $a \leq_h b$. Thus (4) holds for all $b' \in \mathscr{V}(b)$.

(4) for some $b' \in \mathscr{V}(b)$ **⇒ (5)**: Assume (4) for some $b' \in \mathscr{V}(b)$. By Lemma 2.38, $f = bb' \in E(R_b)$ and $g = b'b \in E(L_b)$. From $a \leq_h b$, we get $a \in fS \cap Sg$. Also $e = ab'$ and $h = b'a$ are idempotents such that

$$a = fa = b(b'a) = bh \quad \text{and} \quad a = ag = (ab')b = eb$$

which shows that (5) holds.

(5) ⇒ (1): By (5), there exists $e, f \in \boldsymbol{E}(S)$ with $a = be = fb$. Then we have

$$a = ae = be = fa = fb.$$

Hence the statement Proposition 4.2(3) holds for a and b. Hence by Proposition 4.2 we have $a \leq b$. □

The following are some of the consequences of the proposition above frequently needed in the sequel.

COROLLARY 4.4. *For semigroups S and T, we have:*

(a) Let $\phi : S \to T$ be a homomorphism. If $x \leq_S y$, then $x\phi \leq_T y\phi$.

(b) Let T be a subsemigroup of S. For $x, y \in T$, if $x \leq y$ in T then $x \leq y$ in S; the converse holds if y is a regular element of T.

In particular, the natural partial order on a regular subsemigroup T of a semigroup S is the restriction of the natural partial order of S to T.

Proof. Since the natural partial order on a semigroup is defined in terms equations, it is clear that it is preserved under homomorphisms. Thus (a) holds. The direct part of (b) follows from the fact that the inclusion is a homomorphism of T into S. To prove the converse assume that $x \leq y$ in S and that $y \in \operatorname{Reg} T$. Then y has an inverse y' in T. Since y' is an inverse of y in S, by Proposition 4.3(4), $x = xy'x$. Hence, again by the same result, we conclude that $x \leq y$ in T. □

If S is an inverse semigroup, by Theorem 2.44, conditions in Proposition 4.3 can be simplifies considerably. For example, we have the following which is useful in applications.

COROLLARY 4.5. *Let S be an inverse semigroup. The following statement are equivalent for $x, y \in S$:*

(1) $x \leq y$;

(2) $x = ey$ for some $e \in \boldsymbol{E}(S)$;

(3) $x = yf$ for some $f \in \boldsymbol{E}(S)$;

(4) $x \leq_h y$ and $x = xy^{-1}y$.

Proof. By Proposition 4.3(2), the statement (1) implies (2). If (2) holds, then $x = ee_y y$ and by Theorem 2.44, $ee_y \in \boldsymbol{E}(S)$ and $ee_y \leq e_y$. Hence (2) implies (1) by Proposition 4.3. By left-right symmetry (1) and (3) are equivalent. The statement (3) above is equivalent to the atatement Proposition 4.3(4) in an inverse semigroup and so the proof is complete. □

PROPOSITION 4.6. *Let b be an element of a semigroup S and let $e \in \boldsymbol{E}(S)$. Then*

$$e \leq_r b \Rightarrow e \mathscr{R} eb \leq b. \tag{4.3}$$

Moreover, if a is a regular element of S such that $a \leq b$ if and only if there is a idempotent $e \in E(R_a)$ such that $e \leq_r b$ and $a = eb$.

Proof. If $e \leq_r b$, then $e = bu$ for some $u \in S^1$ so that $e = e^2 = bubu \in bubS = ebS$. Since $eb \in eS$, we have $e \mathscr{R} eb$. Hence $eb \leq_r b$. Also, if $a = eb$, $a = ea = eb$ and so, by the definition of natural partial order, $eb = a \leq b$.

The 'if' part of the remaining statement follows from the above. Conversely, assume that a is regular such that $a \leq b$. Let $a' \in \mathscr{V}(a)$. Then $h = aa' \in E(R_a)$ and so $h \leq_r b$. Since $a \leq b$, there exists $x \in S^1$ such that $a = xa = xb$. Then $xh = xaa' = aa' = h$ and so $e = hx$ is an idempotent such that $he = e$ and $eh = hxh = h$. Hence $e \mathscr{R} h \mathscr{R} a$ and $eb = hxb = ha = a$. □

Recall Equation (1.11b) that a subset Y of a partially ordered set X is an order ideal if for all $y \in Y$, every $z \leq y$ also belongs to Y.

PROPOSITION 4.7. *The natural partial order on semigroup S has the following properties:*

(a) *The set* Reg S *of regular elements and the set $\boldsymbol{E}(S)$ of idempotents of S are order ideals with respect to the natural partial order on S.*

(b) *Let $a, b \in S$ with $a \leq b$. If either $a \mathscr{R} b$ or $a \mathscr{L} b$, then $a = b$.*

(c) *Let $b \in S$ and $a_i \leq b$, $i = 1, 2$. If $a_1 \leq_h a_2$ then $a_1 \leq a_2$. In particular, if $a \leq_h b$ there exist utmost one $c \in H_a$ such that $c \leq b$.*

Proof. If $a \leq b$ and if b is regular, by Proposition 4.3 a is also regular. Hence Reg S is an order ideal. If f is an idempotent and if $x \leq f$, it follows from Proposition 4.3(2) that x is also an idempotent; this implies that $\boldsymbol{E}(S)$ is an order ideal.

To prove (b), suppose that $a \leq b$ and $a \mathscr{R} b$. Then by definition of the natural partial order, there is $x, s \in S^1$ such that $a = xa = xb$ and $b = as$. Then $b = as = xas = xb = a$. If $a \mathscr{L} b$, dually, we have $a = b$.

The conditions given in the statement (c) implies that there exist $x_i, y_i \in S$, $i = 1, 2$ such that

$$a_i = x_i a_i = x_i b = a_i y_i = b y_i, \qquad i = 1, 2$$

and since $a_1 \leq_h a_2$, there is $s \in S^1$ with $a_1 = sa_2$. Hence

$$x_1 a_2 = x_1 b y_2 = a_1 y_2 = s a_2 y_2 = s a_2 = a_1 = x_1 a_1.$$

Since $a \leq_r b$, it follows that $a_1 \leq a_2$. In particular, if $a_1 \mathscr{H} a_2$, then $a_1 \leq_h a_2$ and $a_2 \leq_h a_1$ and so, by the above $a_1 \leq a_2$ and $a_2 \leq a_1$; by antisymmetry of natural partial order, we conclude that $a_1 = a_2$. This completes the proof of (c). □

Notice that every [left, right, two-sided] ideal of a semigroup S is an order ideal with respect to the natural partial order on S. If $a \in S$, we denote by $S(a)$ the principal order ideal of S (with respect to the natural partial order) generated by a. Clearly, $S(a) \subseteq L(a) \cap R(a)$. Recall (Subsection 2.6.1) also that a morphism $\sigma : L \to L'$ of left ideals is an inner right translation of S^1 restricted to L; that is $\sigma = \rho_t|L$ for $t \in S^1$.

Proposition 4.8. *Let $\sigma = \rho_t|L(a) \to L(b)$ be an isomorphism. Then σ is an order isomorphism of $L(a)$ onto $L(b)$. Dually an isomorphism of principal right ideals is an order isomorphism. Consequently, if $a \mathscr{D} b$, then there is an order isomorphism $\theta : S(a) \to S(b)$ such that for all $x \leq a$, $x \mathscr{D} x\theta$.*

Proof. Let $c, d \in L(a)$ and $c \leq d$. By Theorem 2.25, $c\sigma = ct \mathscr{R} c$ and $d\sigma = dt \mathscr{R} d$. Since $c \leq_r d$, we have $c\sigma \leq_r d\sigma$. Also, there exists $x \in S^1$ with $c = xc = cd$ and so $c\sigma = ct = xct = xdt = x(d\sigma)$. Hence $c\sigma \leq d\sigma$. This proves that σ preserves natural partial order. Similarly, σ^{-1} also preserves natural partial order and so σ is an order isomorphism. Clearly this induces an order isomorphism of $S(a)$ onto $S(a\sigma)$. The proof for right ideals is dual.

If $a \mathscr{D} b$, then by Proposition 2.28, there is $c \in S$ with $a \mathscr{L} c \mathscr{R} b$. Also, by Green's lemma (Theorem 2.26) there is an isomorphism $\sigma : L(a) \to L(c)$ with $a\sigma = c$ and so an order isomorphism of $S(a)$ onto $S(c)$ by the observation in the previous paragraph. Further by Theorem 2.25, $x \mathscr{R} x\sigma$ for all $x \in L(a)$ and hence for all $x \in S(a)$ in particular. Dually there exists an order isomorphism $\tau : S(c) \to S(b)$ such that $y \mathscr{L} y\tau$ for all $y \in S(c)$. Hence if $\theta = \sigma \circ \tau$, then $\theta : S(a) \to S(b)$ is an order isomorphism such that $x \mathscr{D} x\theta$ for all $x \in S(a)$. □

Let $f \in E(R_b)$. Recall from Proposition 2.40 that L_f contains an inverse of b. We use this below.

Proposition 4.9. *Let $e \ \omega \ f$, $e, f \in \boldsymbol{E}(S)$. Then for each $(b, b') \in R_f \times L_f$ with $b' \in \mathscr{V}(b)$, there is a unique pair $(a, a') \in R_e \times L_e$ with $a' \in \mathscr{V}(a)$ such that $a'a = b'eb$, $a \leq b$ and $a' \leq b'$.*

Proof. Assume that $(b, b') \in R_f \times L_f$ with $b' \in \mathscr{V}(b)$. Then clearly $e \mathscr{R} eb$ and $e \mathscr{L} b'e$. Also

$$(eb)(b'e)(eb) = e(bb')eb = efeb = eb, \qquad (b'e)(eb)(b'e) = b'efe = b'e$$

and so, $b'e \in \mathscr{V}(eb)$. By Proposition 4.3(2) and (3), $eb \leq b$ and $b'e \leq b'$ and clearly $(b'e)(eb) = b'eb$. Thus the pair $(eb, b'e)$ satisfies the requirements. To prove the uniqueness, let (a, a') be any pair satisfying the given conditions. Then by Lemma 2.38,

$$aa' = e = (eb)(b'e), \qquad a'a = b'eb = (b'e)(eb)$$

and so $a \mathscr{H} eb$ and $ba' \mathscr{H} b'e$. Since $a \leq b$ and $eb \leq b$ by Proposition 4.7(c), $a = eb$. Similarly (dually), $a' = b'e$. □

***Remark* 4.2:** The definition of the natural partial order on a semigroup S implies certain properties for the categories $\mathbb{L}(S)$ and $\mathbb{R}(S)$ 2.1 of principal left and right ideals of S. If $a \leq b$, then by Proposition 4.2(3), there exists $x, y \in S^1$ with $a = xa = xb = ay = by$. Then $\tau_x = \lambda_x|R(b)$ is clearly a retraction of $R(b)$ onto $R(a)$ such that $a = \tau_x b$ (see Subsection 1.3.2). Thus in this case the inclusion $R(a) \subseteq R(b)$ splits. Similarly $\sigma_y = \rho_y$ is a retraction of $L(b)$ onto $L(a)$ with $a = b\sigma_y$ and the inclusion $L(a) \subseteq L(b)$ splits. Conversely if $\tau : R(b) \to R(a)$ is a refraction, it is easy to see that $\tau b \leq b$ and dually for left ideals. Note that, in case $R(a)$ has an idempotent generator e, then $\tau_e : R(b) \to R(a)$ is a retraction. By Proposition 4.6 every retraction of $R(b)$ onto $R(a)$ is induced in this way by an idempotent generator of $R(a)$. Therefore if S is regular, then every inclusion in $\mathbb{R}(S)$ and every inclusion in $\mathbb{L}(S)$ splits.

Example 4.1: Let $S = \mathscr{T}_X$ be the semigroup of all transformations on a set X (see Subsection 2.1.3). Then $f \leq g$ in $\mathscr{T}_X$ ($\leq$, being the natural partial order on $\mathscr{T}_X$) if and only if $\pi_g \subseteq \pi_f$ and for some cross-section Y of π_f, $f|Y = g|Y$. Similarly $f \leq g$ in $S = \mathscr{L}\mathscr{T}(V)$ if and only if $\boldsymbol{N}(g) \subseteq \boldsymbol{N}(f)$ and $f|U = g|U$ for some complement U of $\boldsymbol{N}(f)$ in V. It is easy to see that the natural partial order is not compatible on $\mathscr{T}_X$ or $\mathscr{L}\mathscr{T}(V)$.

Example 4.2: Let S be an inverse semigroup. Then S is regular and the conditions of Proposition 4.3 simplifies considerably in this case. For example, one of the equations in Proposition 4.3(5) is sufficient to characterize natural partial order on S. For, let $a, b \in S$. If $a = eb$ for some $e \in \boldsymbol{E}(S)$ then since idempotents in S commute, $a = ebb^{-1}b = bb^{-1}eb = a = bf$ where $f = b^{-1}eb \in \boldsymbol{E}(S)$. Similarly, if $a = bf$ there is an idempotent e with $a = eb$. Hence by Proposition 4.3(5),

$$a \leq b \iff \text{either} \quad a = eb,\ e \in \boldsymbol{E}(S), \quad \text{or} \quad a = bf,\ f \in \boldsymbol{E}(S). \tag{$*$}$$

It follows as a consequence of ($*$) that the natural partial order is compatible (which is also a consequence of Theorem 4.23 below).

Example 4.3: If S is a semigroup with involution (see Subsection 2.1.2) $a \mapsto a^*$, then it follows from Proposition 4.2(3) that $a \leq b$ if and only if $a^* \leq b^*$; that is the involution is an order isomorphism. In particular, if S is an inverse semigroup, then the map $a \mapsto a^{-1}$ is an involution (which is a consequence of the fact that idempotents in S commute) and so $a \leq b$ if and only if $a^{-1} \leq b^{-1}$.

Example 4.4: If S is the additive semigroup of positive real numbers then the usual order on S is compatible; however, it is not the natural partial order on S (which is, in fact, the identity relation). Similarly, the inclusion is a compatible partial order on the semigroup $\boldsymbol{B}_X$ of relations on the set X which is not the natural partial order. On the other hand, the inclusion is the natural partial order on the symmetric inverse semigroup $\boldsymbol{I}_X$ of all one-to-one partial transformations on X and it is compatible.

Example 4.5: The natural partial order on the free semigroup X^+ or the free monoid X^* on a set X is the identity relation. Note that X^+ contains no regular element and identity (empty word) is the only regular element of X^*.

4.1.2 Trace products and natural partial order

Recall that the trace product $x * y$ Equation (2.48a) of two elements x and y of a semigroup S is defined if and only if $L_x \cap R_y$ contains an idempotent or equivalently, $xy \in R_x \cap L_y$. This definition can be extended to the trace product $x_0 * \cdots * x_n$ of a finite sequence $x_0, x_1, \ldots, x_n \in S$ if the trace product $x_{i-1} * x_i$ exists for all $i = 1, 2, \ldots, n$. By Lemma 2.77, $D^0_{x_0}$ is a semigroup with respect to the product defined by Equation (2.48b). Therefore the extended trace product exists and is independent of the grouping of elements. Observe that trace products exist only for regular elements so that a statement that the trace product $x_0 * \cdots * x_n$ exists would imply in particular that x_i is a regular element in S for all i.

The following theorem generalizes Theorem 3.7 of Chapter Chapter 3 as well as Theorem 1.6 of Nambooripad [1980]. It also shows how one can use the natural partial order to reduce an arbitrary product in a regular semigroup S to the trace product in $S(*)$.

THEOREM 4.10. *Let $x_0, x_1, \ldots, x_n$ be elements of a semigroup S auch that their product $u = x_0 x_1 \ldots x_x$ is regular. Then there exist regular elements $y_i \in S$, $i = 0, 1, \ldots, n$ such that*

$$y_i \le x_i, \quad i = 0, 1, \ldots, n; \quad \textit{and} \tag{1}$$

$$u = x_0 x_1 \ldots x_n = y_0 * y_1 * \cdots * y_n. \tag{2}$$

Further if the trace product $x_0 * \cdots * x_n$ exists in $S(*)$ and if $y_0, \ldots, y_n$ are elements in S satisfying (1) *and* (2)*, then $x_i = y_i$ for all $i = 0, 1, \ldots, n$.*

Proof. The proof is by induction on n. We first prove the case for $n = 2$. Let $u = x_0 x_1$ be regular. Then by Proposition 2.39 there exists idempotents $e, f \in \boldsymbol{E}(S)$ with $e \mathscr{R} u \mathscr{L} f$. Then $e \in uS \subseteq x_0 S$ and so $e \le_r x_0$. Hence by Proposition 4.6. $u \mathscr{R} e \mathscr{R} ex_0 \le x_0$. Dually $u \mathscr{L} f \mathscr{L} x_1 f \le x_1$. Hence

$$u \in R_{ex_0} \cap L_{x_1 f} \quad \text{and} \quad (ex_0)(x_1 f) = e(x_0 x_1) f = u.$$

Therefore, if $y_0 = ex_0$ and $y_1 = x_1 f$, we have

$$y_0 \le x_0, \quad y_1 \le x_1 \quad \text{and} \quad u = y_0 * y_1.$$

If the trace product $x_0 * x_1$ exists and if y_0 and y_1 satisfy the above relations, then $y_0 \mathscr{R} u \;\; x_0$ and so, $y_0 = x_0$ by Proposition 4.7(b). Similarly $y_1 = x_1$.

$\leq_\sigma$:quotient of $\leq$ by σ
relation!quotient –

Now assume, inductively, that the theorem holds for all $r \leq n$ and that $u = x_0 \dots x_n$ is regular. Let $z = x_1 x_2 \dots x_n$. Then by the above there is $y_0 \leq x_0$ and $z_0 \leq z$ such that $u = y_0 * z_0$. By Proposition 4.6, there exists $g \in \boldsymbol{E}(S)$ with $g \mathscr{R} z_0 = gz$. Then $g \leq_r z \leq_r x_1$ so that $g \mathscr{R} gx_1 \leq x_1$. Let $y_1 = gx_1$. Then $z_0 = gz = y_1 x_2 \dots x_n$ is a regular element which is a product of n elements in S. Then by induction hypothesis,

$$z_0 = y_1' * y_2 * \cdots * y_n \quad \text{where} \quad y_1' \leq y_1,\ y_i \leq x_i,\ 1 < i \leq n.$$

Since $y_1' \mathscr{R} z_0 \mathscr{R} g \ \ y_1$, by Proposition 4.7, $y_1' = y_1$. Hence $z_0 = y_1 * y_2 * \cdots * y_n$ and so

$$u = y_0 * z_0 = y_0 * y_1 * \cdots * y_n, \quad \text{where} \quad y_i \leq x_i \quad i = 0, 1, \dots, n.$$

Assume that the trace product $x_0 * x_1 * \cdots * x_n$ exists in $S(*)$ and that $y_0, \dots, y_n$ satisfies conditions (1) and (2) of the statement. From (2), we have

$$x_0 \mathscr{R} x_0 * x_1 * \cdots * x_n = y_0 * y_1 * \cdots * y_n \mathscr{R} y_0.$$

Since $y_0 \leq x_0$ by (1), we have $y_0 = x_0$ by Proposition 4.7. Assume that $y_{k-1} = x_{k-1}$ for $k \geq 2$. Let e_{k-1} and f_{k-1} be idempotents such that $x_{k-1} \mathscr{L} e_{k-1} \mathscr{R} x_k$ and $y_{k-1} \mathscr{L} f_{k-1} \mathscr{R} y_k$. Then $e_{k-1} \mathscr{L} f_{k-1}$. Since $y_k \leq x_k$ we have $f_{k-1} \mathscr{R} y_k \leq_r x_k \mathscr{R} e_{k-1}$. Hence $f_{k-1} \ \omega^r \ e_{k-1}$ and so, $e_{k-1} = e_{k-1} f_{k-1} = f_{k-1}$. This implies that $y_k \mathscr{R} x_k$. Therefore by Proposition 4.7, $x_k = y_k$. □

4.1.3 Green's relations, congruences and natural partial order

Let X be a partially ordered set. An equivalence relation σ is said to reflect the partial order $\leq$ on X (or simply, σ is reflective, if $\leq$ is clear from the context) if for all $x, y, \in X$,

$$x \leq y \ \sigma \ z \Rightarrow x \ \sigma \ y' \leq z \quad \text{for some} \quad y' \in X; \tag{4.4a}$$

or equivalently, $\quad \leq \circ \sigma \ \subseteq \ \sigma \circ \leq .$

This is again equivalent to the statement that given $x \leq y$ there exists a map $\theta : \sigma(y) \to \sigma(x)$ with $\theta(z) \leq z$ for all $z \in \sigma(y)$.

Given the equivalence relation σ on X, let

$$\leq_\sigma = \{(\sigma(x), \sigma(y)) : \text{for some} \quad x', y' \in X,\ x \sigma x' \leq y' \sigma y\}. \tag{4.4b}$$

Then $\leq_\sigma$ is a relation on the quotient set X/σ; $\leq_\sigma$ is called the *quotient relation* of $\leq$ by σ. Note that the relation $\leq_\sigma$ is the image of $\leq$ by the quotient map $\sigma^\# : X \to X/\sigma$. Recall that an order preserving map $f : X \to Y$ of partially ordered sets weakly reflects the partial order on Y in the sense of Chapter relation!convex – relation!disjoint –

Chapter 3 if $y' \leq y$ in Y and $x \in X$ with $xf = y$, there exist $x' \leq x$ in X with $x'f = y'$. It is easy to see that σ reflects $\leq$ if and only if $\sigma^{\#} : X \to X/\sigma$ weakly reflects the relation $\leq_\sigma$; if this is the case, then $\leq_\sigma$ is clearly a quasi-order on X/σ. Again a reflective equivalence relation σ is said to *convex* with respect to the partial order $\leq$ if $\leq_\sigma$ is a partial order on X/σ. Thus σ is convex if and only if it satisfies the following.

$$x \leq y, \quad \text{and} \quad x\sigma y \Rightarrow [x, y] \subseteq \sigma(y) \tag{4.4c}$$

where $[x, y] = \{u \in X : x \leq u \leq y\}$ is the interval with endpoints x and y (see Equation (1.11a)). We shall say that σ is *disjoint* (from $\leq$) if it is reflective and every element $x \in X$ is minimal in its σ-class $\sigma(x)$; that is,

$$\forall \; x, y \in X, \quad x \leq y, \quad \text{and} \quad x\sigma y \Rightarrow x = y. \tag{4.4d}$$

Note that if σ is disjoint, then it is convex.

LEMMA 4.11. *Let X be a partially ordered set and let σ be a reflective equivalence relation on X. Then the relation*

$$\bar{\sigma} = \{(x, y) : \textit{there exist } x', y' \in X \textit{ with } \quad x\sigma x' \leq y, \; y\sigma y' \leq x\}$$

is the finest convex equivalence relation on X containing σ.

Proof. It is clear that $\bar{\sigma}$ is a reflexive and symmetric relation containing σ. Let $x\bar{\sigma}y$ and $y\bar{\sigma}z$. Then there exist x' and y' such that

$$x\sigma x' \leq y\sigma y' \leq z.$$

Since σ is reflective there exists x'' with

$$x\sigma x'\sigma x'' \leq y' \leq z.$$

Similarly, here is z'' with $z\sigma z'' \leq x$. Therefore $x\bar{\sigma}z$ and so, $\bar{\sigma}$ is an equivalence relation.

Suppose that $x \leq y\bar{\sigma}z$. Then, by definition, there exist z' such that $x \leq y\sigma z' \leq z$. Since σ is reflective, there is x' with $x\sigma x' \leq z' \leq z$. Hence $x\bar{\sigma}x' \leq z$ and so, $\bar{\sigma}$ is reflective. To prove that $\bar{\sigma}$ is convex, let $x \leq u \leq y$ and $x\bar{\sigma}y$. Then $u \leq y\bar{\sigma}x$ and so $u\bar{\sigma}u' \leq x$ for some $u' \in X$. Since $x \leq u$, it follows that $x\bar{\sigma}u$.

Finally, let ρ be a convex equivalence relation containing σ. If $x\bar{\sigma}y$, there exist x' with $x\sigma x' \leq y$ and so $x'\bar{\sigma}y$. Hence there exists y' such that $y\sigma y' \leq x' \leq y$. Since ρ is convex, $\rho^{\#} : X \to X/\rho$ is an order preserving map of partially ordered sets. Therefore, since $\sigma \subseteq \rho$, we have

$$\rho^{\#}(y) = \rho^{\#}(y') \leq \rho^{\#}(x') = \rho^{\#}(x) \leq \rho^{\#}(y).$$

Consequently, $\rho^{\#}(x) = \rho^{\#}(y)$; that is, $x\rho y$. ☐

Let S be a semigroup. An equivalence relation σ on S is said to be *reflective, convex or disjoint* if σ has the corresponding property with respect to the natural partial order $\leq$ on S. In the following, we write $x < y$ if $x \leq y$ and $x \neq y$.

PROPOSITION 4.12. *Let $\mathcal{K}$ denote one of the relations $\mathscr{L}$, $\mathscr{R}$ or $\mathscr{D}$. Then for $x, y, z \in S$,*

$$x < y \mathrel{\mathcal{K}} z \Rightarrow x \mathrel{\mathcal{K}} y' < z \quad \text{for some} \quad y' \in S. \tag{4.5}$$

Consequently, $\mathscr{L}$ $\mathscr{R}$ and $\mathscr{D}$ are reflective equivalence relations on S.

Proof. Assume that $x < y \mathrel{\mathcal{K}} z$. It follows from Proposition 4.8 that there is an order isomorphism $\theta : S(y) \to S(z)$ so that for all $c \in S(y)$, $c \mathrel{\mathcal{K}} c\theta \in S(z)$. Hence $x \mathrel{\mathcal{K}} x\theta < z$. The last statement is clear from the definition of reflective relations. □

From the proposition above and Proposition 4.7, we have the following.

COROLLARY 4.13. *The Green's relations $\mathscr{L}$ and $\mathscr{R}$ are disjoint.* □

Recall (from Subsection 1.1.2) that an element x in a subset X of S (with respect to the natural partial order) is minimal in X if $y \in X$ and $y \leq x$ implies $y = x$.

COROLLARY 4.14. *Let D be a $\mathscr{D}$-class of a semigroup S. If D contains a minimal element, then every element of D is minimal.*

Proof. Let $x, y \in D$ and $x < y$. If z is an arbitrary element of D, then by Equation (4.5), there is $z' \in D$ with $z' < z$. Hence z is not minimal in D and so D does not contain minimal elements. □

For regular semigroups we have the following relation between Green's relations $\mathscr{D}$ and $\mathscr{J}$.

THEOREM 4.15. *Let S be a regular semigroup and $x, y \in S$. Then*

$$x \in J(y) \iff x \mathrel{\mathscr{D}} y' \leq y \quad \text{for some} \quad y' \in S.$$

Consequently $\bar{\mathscr{D}} = \mathscr{J}$.

Proof. If $x \in J(y)$ then there exists $u, v \in S^1$ such that $x = uyv$. By Theorem 4.10, there exist $u_1 \leq u$, $y_1 \leq y$ and $v_1 \leq v$ such that $x = u_1 * y_1 * v_1$. By the definition of trace product, the element y_1 belongs to D_x. Hence $x \mathrel{\mathscr{D}} y_1 \leq y$. Conversely if y' exists with $x \mathrel{\mathscr{D}} y' \leq y$, then $x \mathrel{\mathscr{J}} y' \leq_j y$ and so, $x \in J(y)$.

Now $x \mathrel{\mathscr{J}} y$ if and only if $x \in J(y)$ and $y \in J(x)$. By the above, this is true if and only if there exist $x', y' \in S$ with

$$x \mathrel{\mathscr{D}} y' \leq y \quad \text{and} \quad y \mathrel{\mathscr{D}} x' \leq x.$$

By Lemma 4.11, the statement above holds if and only if $x\bar{\mathscr{D}}y$. Hence $\bar{\mathscr{D}} = \mathscr{J}$. □

The result above may not hold if S is not regular. For example let $S = A$ where A is the semigroup of Example 2.13. Then on A $\mathscr{D} = 1_A$ and so $\bar{\mathscr{D}} = \mathscr{D}$. Since A is simple $\mathscr{J}$ is the universal relation $(A \times A)$. Thus $\bar{\mathscr{D}} \neq \mathscr{J}$ on A. However, it is always true that $\bar{\mathscr{D}} \subseteq \mathscr{J}$.

It follows from Lemma 4.11 and Theorem 4.15 that $\mathscr{D}$ is convex if and only if $\mathscr{D} = \mathscr{J}$; thus:

COROLLARY 4.16. *For a regular semigroup S, the equality $\mathscr{D} = \mathscr{J}$ holds if and only if $\mathscr{D}$ is convex.* □

COROLLARY 4.17. *Let D be a $\mathscr{D}$-class of a regular semigroup S. If D contains a minimal element x, then $D = J_x$ and every element of J_x is minimal.*

Proof. Let $y \ \mathscr{J} \ x$. By Theorem 4.15, for some $x' \in S$, $y \ \mathscr{D} \ x' \leq x$. Then $x' \ \mathscr{J} \ x$ and so there is $x'' \in S$ with $x \ \mathscr{D} \ x'' \leq x'$. Hence $x'' \leq x' \leq x$. By Corollary 4.14, every element of D is minimal and so, $x'' = x' = x$. Therefore $y \in D$ and so, $J_x \subseteq D$. Hence $D = J_x$. □

Recall that a semigroup S is [0-]simple if and only if the set of all [non-zero] elements form a $\mathscr{J}$-class of S. Hence from Theorem 4.15 we have:

COROLLARY 4.18. *A regular semigroup S is [0-]simple if and only if for any $x, y \in S$ [$x, y \in S - \{0\}$] there is $x' \in S$ [$x' \in S - \{0\}$] such that $x \ \mathscr{D} \ x' \leq y$.* □

Recall from Equation (2.53) and Lemma 2.86, a semigroup satisfies the condition $\boldsymbol{M}_E^*$ if and only if every idempotent $e \in \boldsymbol{E}(S)$ is minimal in $E(D_e)$ with respect to the partial order ω on $\boldsymbol{E}$. By Lemma 4.1, this is true if and only if e is minimal in D_e with respect to the natural partial order. Hence by Corollary 4.14, every element of D_e is minimal with respect to the natural partial order. If S is regular, by Proposition 2.39 every $\mathscr{D}$-class of S contain idempotents and so a regular semigroup S satisfies $\boldsymbol{M}_E^*$ if and only if every element in S is minimal in its $\mathscr{D}$-class or equivalently, the Green's relation $\mathscr{D}$ is disjoint. Therefore, by Theorem 2.87 we have:

THEOREM 4.19. *A regular semigroup S is completely semisimple if and only if the Green's relation $\mathscr{D}$ is disjoint.* □

We next consider the relation between congruences and natural partial order on regular semigroups.

By Corollary 4.4(a), homomorphisms of semigroups preserve natural partial orders. If S is also regular we have:

THEOREM 4.20. *A homomorphism $\phi : S \to T$ of a regular semigroup S into T preserves and weakly reflect natural partial orders.*

Proof. In view of Corollary 4.4(a), it is sufficient to verify that ϕ weakly reflects natural partial orders. Since $\operatorname{Im}\phi$ is a regular subsemigroup of T by Theorem 3.5 and since, by Corollary 4.4(b), the natural partial order of $\operatorname{Im}\phi$ is the restriction of the natural partial order of T to $\operatorname{Im}\phi$, we may assume with out loss of generality that ϕ is surjective. Let $u, v \in T$ and $u \leq v$. Choose $y \in S$ with $y\phi = v$. If $f \in E(R_y)$, then $f' = f\phi \in E(R_v)$. By Proposition 4.3(2), there exists $e' \in E(R_u)$ with $e' \ \omega \ f'$ and $u = e'v$. By Proposition 3.24, $\boldsymbol{E}(\phi)$ weakly reflects ω^r. Hence we can find $e \in \boldsymbol{E}(S)$ with $e \ \omega \ f$ and $e\phi = e'$. If $x = ey$, then, again by Proposition 4.3(2), $x \leq y$ and we have $x\phi = (e\phi)(y\phi) = e'v = u$. □

Reformulating the result above in terms of congruences, we have:

COROLLARY 4.21. *Every congruence on a regular semigroup S is convex.*

Proof. Let σ be a congruence on S and let $\phi = \sigma^\# : S \to S/\sigma$ be the quotient homomorphism. If $x \leq y\sigma z$ in S, then $x\phi \leq y\phi = z\phi$. Hence by the theorem above, there is $z' \in S$ with $z' \leq z$ and $x\phi = z'\phi \leq z\phi = x\phi$. Then $x\sigma z' \leq z$. Hence σ is reflective. If $x\sigma y$ and $x \leq u \leq y$, then $x\phi \leq u\phi \leq y\phi$ and $x\phi = y\phi$. These imply that $x\phi = u\phi = y\phi$ and so, $x\sigma u\sigma y$. Thus σ is convex. □

The theorem above and the corollary may not hold for semigroups that are not regular. For by Corollary 2.19 any semigroup S is a homomorphic image of a free semigroup X^+ for a suitable set X and by Example 4.5, the natural partial order is the identity relation on a free semigroup. It is therefore clear that if $y' < y$ in S, it is not possible to find $x, x' \in X^+$ with $x' < x$ which is mapped to y and y' respectively. The corollary above also shows that, if σ is any congruence on S, the natural partial order on S/σ coincides with the quotient order $\leq_\sigma$ defined by Equation (4.4b).

Those congruences on regular semigroups that are disjoint can be characterized as follows.

THEOREM 4.22. *A congruence σ on the regular semigroup S satisfies the condition*

$$x \leq y \quad \text{and} \quad x \ \sigma \ y \Rightarrow x = y$$

(that is, σ is disjoint) if and only if, for all $e \in \boldsymbol{E}(S)$, $\sigma(e)$ is a completely simple subsemigroup of S.

Proof. First suppose that σ is disjoint and let $\phi = \sigma^\# : S \to S/\sigma = T$ be the quotient homomorphism. Let $e \in \boldsymbol{E}(S)$ and $x \in \sigma(e)$. If $f \in E(R_x)$ and $g \in E(L_x)$, then $f\theta \mathscr{R} x\phi = e\theta \mathscr{L} g\theta$. Hence by Proposition 3.12

$$\mathscr{S}(g\theta, f\theta) = \mathscr{S}(e\theta, e\theta) = \{e\theta\}.$$

Since by Theorem 3.5 $\theta = \phi|\boldsymbol{E}(S) : \boldsymbol{E}(S) \to \boldsymbol{E}(T)$ is a regular bimorphism (see Definition 3.4), we have

$$\mathscr{S}(g, f)\theta \subseteq \mathscr{S}(g\theta, f\theta) = \{e\theta\}$$

and so, $\mathscr{S}(g, f) \subseteq \sigma(e)$. Hence if $h \in \mathscr{S}(g, f)$, then $hx, xh \in \sigma(e)$ since $\sigma(e)$ is a subsemigroup of S. By Proposition 4.3(2) and (3), $h \mathscr{R} hx \leq x$ and $h \mathscr{L} xh \leq x$. Since σ is disjoint, we have $xh = x = hx$ and so $x \in H_h$. Therefore $\sigma(e)$ completely simple.

Conversely assume that $\sigma(e)$ is completely simple for each $e \in \boldsymbol{E}(S)$. By Theorem 2.65 $\sigma(e)$ bisimple, regular and every idempotent in $\sigma(e)$ is minimal in $\sigma(e)$ with respect to the natural partial order of $\sigma(e)$. Hence by Corollary 4.14, every element of $\sigma(e)$ is minimal. Since, by Corollary 4.4, the natural partial order of $\sigma(e)$ is the restriction of the natural partial order of S to $\sigma(e)$ every element in $\sigma(e)$ is minimal in $\sigma(e)$ with respect to the natural partial order of S. Therefore σ is disjoint. □

4.1.4 Compatibility on the natural partial order

We have noted that the natural partial order is not, in general compatible with the multiplication in the semigroup. We proceed to characterize the class of regular semigroups for which the natural partial order is compatible.

Recall that a pseudonverse (locally inverse) semigroup is a regular semigroup S such that $\boldsymbol{E}(S) = E$ is a pseudo-semilattice. If this holds, by Theorem 3.54, $\omega(e)$ is a semilattice for all $e \in E$. Since $\boldsymbol{E}(eSe) = \omega(e)$, by Theorem 2.44, $\omega(e)$ is a semilattice if and only if eSe is an inverse subsemi group of S. By Theorem 3.54, this is equivalent to the fact that $\mathscr{S}(e, f)$ contains a unique element for all $e, f \in \boldsymbol{E}(S)$. Recall also that for $x, y \in S$, $\mathscr{S}(x, y)$ denotes $\mathscr{S}(e, f)$ for some [for all] $e \in E(L_x)$ and $f \in E(R_x)$ (see Proposition 3.12).

THEOREM 4.23. *The following statements are equivalent for a regular semigroup S.*

(a) S is locally inverse.

(b) If $x, y, u, v \in S$, $x \leq u$ and $y \leq v$, then $xy \leq uv$.

(c) If $x, y \in S$, $y' \in \mathscr{V}(y)$ and $x \leq y$, then there is a unique $x' \in \mathscr{V}(x)$ such that $x' \leq y'$.

Proof. (a) ⇒ (b): Let $f \in E(L_u)$ and $e \in E(R_v)$. Since $x \leq u$, by Proposition 4.3, there is $f' \; \omega \; f$ such that $x = uf'$. Similarly there exists $e' \; \omega \; e$ with $y = e'v$. By primitive!– element

(a), we have $\mathscr{S}(u,v) = \{h\}$ and $\mathscr{S}(x,y) = \{g\}$ for some $h, g \in \boldsymbol{E}(S)$. Then by (), ref ch3 $g \in M(f,e) = \omega(h)$ and so, $g\ \omega\ h$. Also, by (a), $\omega(f)$ is a biordered subset of $\boldsymbol{E}(S)$ which is a semilattice. Hence the relations $\mathscr{L}$ and $\mathscr{R}$ coincide with the identity on $\omega(f)$. Now, $f'g\ \mathscr{L}\ g\ \mathscr{L}\ fg$ and $f'g, fg \in \omega(f)$. Therefore $f'g = fg$ and so, $xg = uf'g = ufg = ug$. Dually, $gy = gv$. By Proposition 2.40 we can find $u' \in \mathscr{V}(u) \cap R_f$ so that $u'u = f$. Then $uhu' \in E(R_{uv})$ and $ugu'\ \omega\ uhu'$. Also by Theorem 3.7

$$xy = (xg)(gy) = (ug)(gv) = (ugu')uv \leq uv$$

by statement (2) of Proposition 4.3.

(b) ⇒ (c): Let $x \leq y$ and $y' \in \mathscr{V}(y)$. Then by Proposition 4.3(2), there is $e\ \omega\ f = yy'$ such that $x = ey$. Since $y' \in L_f$, by Proposition 4.9, there is $x' = y'e \in \mathscr{V}(x)$ such that $x' \leq y'$. If $x'' \in \mathscr{V}(x)$ with $x'' \leq y'$, then by (b),

$$e = xx' \leq yy' = f, \quad \text{and} \quad e' = xx'' \leq yy' = f.$$

Since $e, e' \in \omega(f) \cap E(R_x)$, it follows by (b) that

$$e' = ee' \leq ef = e \quad \text{and similarly,} e' \leq e.$$

Hence $e = e'$ and so $x'\ \mathscr{L}\ x''$. Dually $x'\ \mathscr{R}\ x''$. Hence by Proposition 2.40, $x' = x''$. This proves the uniqueness of x'.

(c) ⇒ (a): It is sufficient to show that for every $e \in \boldsymbol{E}(S)$, the biordered subset $\omega(e)$ is a semilattice (see Theorem 2.44); this will follow if we show that the relations $\mathscr{L}$ and $\mathscr{R}$ coincide with the identity on $\omega(e)$. Let $f, g \in \omega(e)$ and $f\ \mathscr{R}\ g$. Then $f, g \in \mathscr{V}(f)$, $f \leq e$ and $g \leq e$. Then by (c), we have $f = g$. Similarly, if $f\ \mathscr{L}\ g$, then also $f = g$ by (c). Hence $\omega(e)$ is a semilattice. □

Remark* 4.3*: Compatibility of natural partial order on arbitrary semigroups have been considered in literature Bingjun, Blyth and Gomes [1983], Mitsch [1986]. Also some generalizations of the concept of compatibility has also been discussed by Bingjun.

Example 4.6: Recall that a band B is normal if and only if the biordered set of B is a local semilattice (see Corollary 3.56). Theorem 4.23 gives another characterization of normal bands: the band B is a normal if and only if the natural partial order (in this case, the relation ω) on B is compatible.

4.1.5 Primitive semigroups

An element x in a semigroup S is said to be *primitive* if x is a minimal element in the set of non-zero elements of S. If S has no zero, this means that primitive elements of S are minimal elements of S. Since the restriction of the natural partial order to $\boldsymbol{E}(S)$ coincides with the relation ω, this agrees with the earlier definition of primitive idempotents (see Subsection 2.7.1); that is, an idempotent which is a primitive element according to this definition if and only if it is a primitive idempotent as defined earlier. A semigroup S is said to be *primitive* if every non-zero element of S is primitive. semigroup!primitive –

A semigroup S is called a 0-disjoint union of semigroups S_α, $\alpha \in \Omega$, if S is obtained by taking the disjoint union of all semigroups S_α and identifying all zeros. That is, we take S to be the set given by

$$S = \left(\overset{*}{\bigcup_{\alpha \in \Omega}} (S_\alpha - \{0\}) \right) \cup \{0\} \tag{4.6a}$$

where $\overset{*}{\cup}$ denote disjoint union, and define binary operation in S by

$$xy = \begin{cases} xy, & \text{the product in } S_\alpha \text{ if } x, y \in S_\alpha \text{ for some } \alpha \in \Omega; \\ 0, & \text{otherwise.} \end{cases} \tag{4.6b}$$

It is easy to varify that the set S, with the binary operation above is a semigroup. Observe that in the semigroup S, each S_α is an ideal.

If S is completely 0-simple, by Theorem 2.64 it contains primitive idempotents. These are minimal in the $\mathscr{D}$-class of non-zero elements of S. Then by Corollary 4.14, every non-zero element in S is minimal in the $\mathscr{D}$-class of non-zero elements. This implies that every non-zero element in S is primitive. Hence every completely 0-simple semigroup S is a primitive regular semigroup. More generally we have:

THEOREM 4.24. *A regular semigroup S is primitive if and only if S is either a completely simple semigroup or a 0-disjoint union of completely 0-simple semigroups.*

Proof. Suppose that S does not have zero and let $x, y \in S$. If S is completely simple, then it follows from Theorem 2.65 (as in the remarks preceeding the statement of the theorem), that S is primitive. Conversely assume that S is primitive. Now by Theorem 4.10, $xy = x_1 * y_1$ where $x_1 \leq x$ and since S is primitive, $x = x_1$ and $y = y_1$. Hence the trace product $x * y$ exists in S. Therefore $x \mathscr{D} y$ and $L_x \cap R_y$ contains an idempotent. Consequently, S is completely simple.

Let $S = S^0$. If S is a 0 disjoint union of completely simple semigroups $\{S_\alpha : \alpha \in \Omega\}$, then by Equations (4.6a) and (4.6b), each S_α is a maximal ideal in S. Hence if $x \leq y$ in S, then $x, y \in S_\alpha$ for some $\alpha \in \Omega$. Also by Corollary 4.4(b), the natural partial order on S_α is the restriction of the natural partial order on

subsemigroup!naturally embedded –
semigroup!regular-free –
extension!primitive –

S to S_α. Since every element in S_α is primitive in S_α by the remarks preceeding the statement, it follows that every element in S is primitive.

Conversely, assume that $S = S^0$ is primitive and that $x, y \in S - \{0\}$. If $xy \neq 0$, it follows from Theorem 4.10, as in the first paragraph of the proof, that $xy = x * y$; in particular, $x \mathscr{D} y$. If D is a non-zero $\mathscr{D}$-class of S, then it follows from Theorem 2.64 and this remark that D^0 is a completely 0-simple subsemigroup of S and that S is the 0-disjoint union the semigroups D^0 as D varies over non-zero $\mathscr{D}$-classes of S. □

Let T be a subsemigroup of a semigroup S. Then we say that T is *naturally embedded* in S if the natural partial order on T is the restriction of the natural partial order of S to T. Note that, by Corollary 4.4, every regular subsemigroup T of S is naturally embedded in S.

A semigroup N is said to be *regular-free* if N has no non-zero regular element. A *primitive extension* S of a primitive regular-free semigroup N by a primitive regular semigroup T is an ideal extension of N by T such that N is naturally embedded in S.

LEMMA 4.25. *A primitive extension S of a primitive regular-free semigroup N by a primitive regular semigroup T is primitive.*

Proof. Let $x \leq y$ in S, and $x \neq 0$. Assume that $y \in T$. By Proposition 4.7(a), x is also regular. Since N is an ideal in S, any element $u \in N$ which is regular in S must be regular in N and so $u = 0$. Hence $x \notin N$. Since $T = (S - N) \cup \{0\}$, it follows that x is a non-zero element of T. Since T is primitive, we have $x = y$. If $y \in N$, since N is an ideal, $x \in N$. Hence $x, y \in N$ and $x \leq y$ in S. Since N is naturally embedded in S, $x \leq y$ in N. Since N is primitive, we have $x = y$. Since $S = (T - \{0\}) \cup N$, it follows that every non-zero element in S is primitive. □

We now proceed to give a classification of primitive semigroups. The following theorem is due to Bingjun.

THEOREM 4.26. *A semigroup S is primitive if and only if S is one of the following types of semigroups:*

(a) a primitive regular semigroup;

(b) a primitive regular-free semigroup;

(c) a primitive extension of a regular-free semigroup by a primitive regular semigroup.

Proof. If S is one of the type (a), (b) or (c), by definitions and Lemma 4.25, S is primitive. Hence it is sufficient to show that, if S is primitive and if S is not primitive regular or primitive regular-free, then it is of type (c).

Accordingly assume that S is primitive and that S is not regular, but $\operatorname{Reg} S$ contains non-zero elements. Let

$$N = \begin{cases} S - \operatorname{Reg} S, & \text{if } S \text{ has no zero;} \\ (S - \operatorname{Reg} S) \cup \{0\}, & \text{the 0 of } S, \text{ if } S = S^0. \end{cases}$$

Suppose that $a \in N$ and $b \in S$. If $ab \in S - N$, then ab is a non-zero regular element of S and so, by Theorem 4.10 there exist regular elements $a', b' \in S$ with $a' \leq a$, $b' \leq b$ and $ab = a' * b'$. Since $a \in N$ this implies that $a' = 0$ and so, $ab = 0$ which contradicts the hypothesis. Hence $ab \in N$ and so, N is a right ideal. Similarly, N is a left ideal and hence N is an ideal. Since S is primitive, N is primitive regular-free subsemigroup and is naturally embedded in S. Let $T = S/N$ be the Rees quotient. Then, it follows from the definition of Rees congruences (and Rees quotients)Subsection 2.2.1 that $T - \{0\}$ can be identified with $\operatorname{Reg} S - \{0\}$. Since every non-zero element in $\operatorname{Reg} S$ is regular in S, it is regular element of T and so T is a regular semigroup. If $a, b \in T - \{0\}$ and $a \leq b$, it it follows from statement (3) of Proposition 4.2, that $a = xa = xb = ay = by$ for some $x, y \in T - \{0\}$. These elements satisfy the same equations in S also. Since S is primitive, $a = b$. This implies that T is a primitive regular semigroup. Therefore S is of type (c). □

Example 4.7: Let X be a set. Then it is clear that the free semigroup X^+ is a primitive regular-free semigroup and is a naturally embedded ideal in the monoid X^*. Also the Rees quotient $X^*/X^+ = H$ the trivial (one-element) group with 0 adjoined. H is clearly a primitive regular semigroup. Hence X^* is a primitive extension of the primitive regular-free semigroup X^+ by the primitive regular semigroup H.

Example 4.8: We give an example, due to Bingjun, to show that an ideal extension of a primitive regular-free semigroup by a primitive regular semigroup need not be a primitive extension. Let $A = \langle a \rangle$ and $B = \langle b \rangle$ be infinite cyclic semigroups and let $C = \langle c; c^2 = 1 \rangle$ be a cyclic group of order 2. Let $S = A \cup B \cup C$ be the disjoint union. Define product in S as follows:

$$c^k a^m = a^m c^k = c^k b^m = b^m c^k = a^m, \quad \text{and} \quad a^m b^n = b^n a^m = a^{m+n}$$

where $k = 1, 2$, $m \geq 1$ and $n \geq 1$. It is easy to verify that S is a semigroup and that $T = A \cup B$ is an ideal in S. Now for any $x \in T$, $xy = x$ for $y \in T^1$ if and only if $y = 1$. Hence it follows from the definition of natural partial order that T is primitive. It is clearly regular-free and S/T is isomorphic to the group with zero, C^0. Since C^0 is a primitive regular semigroup, S is an ideal extension of a primitive regular-free semigroup by a primitive regular semigroup. But, since $a = ca = ac = cb = bc$, $a < b$ in S by Proposition 4.2(3). Hence S not primitive. Notice that T is an ideal of S which is not naturally embedded in S. This also gives an example of a subsemigroup of a semigroup which is not naturally embedded in it.

admissible family

4.2 CONGRUENCES ON REGULAR SEMIGROUPS

In this section we discuss some properties of congruences that applies mainly to regular semigroups as well as certain basic representations of regular semigroups (see Subsection 2.2.1 and Subsection 2.5.1 for general definitions). These results are of interest in their own right. Furthermore they are also needed in our development structure theory of regular semigroups.

4.2.1 Admissible and normal families

Let $\mathcal{A} = \{A_i : i \in I\}$ be a family of pairwise disjoint subsets of a semigroup S. We say that $\mathcal{A}$ is an *admissible family* of subsets of S if there is a congruence ρ on S such that for each $i \in I$, A_i is a ρ-class of S; that is, for each $i \in I$, there is $s_i \in S$ with $A_i = \rho(s_i)$; in this case we also say that the congruence ρ *admits* $\mathcal{A}$.

LEMMA 4.27. *Let $\mathcal{A}$ be an admissible family of subsets of S. Then the set of all congruences that admits $\mathcal{A}$ is an interval in the lattice $\mathfrak{L}$ of all congruences on S.*

Proof. As in Proposition 2.7, let $R^{(c)}$ denote the smallest congruence on S containing the relation R. Consider the relation

$$\Theta = \bigcup\{A_i \times A_i : i \in I\}; \quad \text{and let} \quad \alpha = \Theta^{(c)}.$$

By definition, for any $A_i \in \mathcal{A}$ and $x \in A_i$ $A_i \subseteq \alpha(x)$. Since $\mathcal{A}$ is admissible, there is a congruence ρ which admits $\mathcal{A}$ and so $\Theta \subseteq \rho$. Then $\alpha \subseteq \rho$ and so $\alpha(x) \subseteq \rho(x) = A_i$. Hence α admits $\mathcal{A}$ and so is the smallest congruence admitting $\mathcal{A}$.

Let C denote the set of all congruences that admits $\mathcal{A}$ and let $\beta = \vee C$, the join of C in $\mathfrak{L}$. Then it follows from Proposition 2.6 that

$$\beta = \left(\bigcup_{\rho \in C} \rho\right)^{(t)}.$$

Clearly, $A_i \subseteq \beta(x)$ for any $A_i \in \mathcal{A}$ and $x \in A_i$. If $y \in \beta(x)$, by definition, there exists $n \in \mathbb{N}$, $\rho_j \in C$ for $j = 1, 2, \ldots, n$ and $u_j \in S$ for $j = 0, 1, 2, \ldots, n$ with $u_0 = x$, $u_n = y$ such that $(u_{j-1}, u_j) \in \rho_j$, $j = 1, \ldots, n$. Since ρ_1 admits $\mathcal{A}$, $u_1 \in A_i$. If $u_{j-1} \in A_i$, we similarly have $u_j \in A_i$, $j = 1, \ldots, n$. By induction, it follows that $y \in A_i$ and so $\beta(x) = A_i$. Hence β admits $\mathcal{A}$ and is clearly the largest congruence that admits $\mathcal{A}$. If ρ is any congruence on S such that $\alpha \subseteq \rho \subseteq \beta$ then for any $x \in A_i$, we have

$$A_i = \alpha(x) \subseteq \rho(x) \subseteq \beta(x) = A_i.$$

Thus ρ also admits $\mathcal{A}$. Therefore $C = [\alpha, \beta]$. ☐

We say that a family $\mathcal{A}$ of subsets of S is *normal* in S if there is a unique congruence ρ that admits $\mathcal{A}$. In this case, the interval $[\alpha, \beta]$ of congruences that admits $\mathcal{A}$ reduces to a single congruence so that $\alpha = \rho = \beta$. For example, if ρ is a congruence on a group G, so that ρ is the coset decomposition of G with respect to a normal subgroup of G — see Example 2.2. Then any congruence class of ρ is normal.

normal
idempotent!– ρ-class

In the following, we use the following notation: if T is a subsemigroup of S, $\operatorname{Reg} T$ denotes the set of elements of T that are regular in T; that is, $u \in \operatorname{Reg} T$ if and only if T contains at least one inverse of u. Note that $\operatorname{Reg} T$ need not be a subsemigroup of T. If ρ is a congruence on S we will refer to those ρ-classes that are idempotents in S/ρ as *idempotent ρ-classes*. Note that any idempotent ρ-class is a subsemigroup of S. The following lemma shows that these are precisely ρ-classes of the form $\rho(e)$ for $e \in \boldsymbol{E}(S)$ (see also (Theorem 3.5). We need the following lemma.

LEMMA 4.28. *Assume that ρ is a congruence on the regular semigroup S and let A be an idempotent ρ-class. If $x \in A$, $x' \in \mathcal{V}(x)$ and $h \in \mathcal{S}(x'x, xx')$, then $h, hxh \in \operatorname{Reg} A$.*

Proof. Let $\phi = \rho^{\#} : S \to S/\rho$ be the quotient homomorphism, and let $A\phi = e' \in \boldsymbol{E}(S/\rho)$ be the idempotent represented by the idempotent ρ-class A. If $f = x'x$ and $g = xx'$ clearly,

$$f' = f\phi \mathrel{\mathscr{L}} x\phi = e' \mathrel{\mathscr{R}} g\phi = g'$$

and so, by Theorem 3.5, $h\phi \in \mathcal{S}(f', g') = \{e'\}$. Hence $h \in A$ and since h is an idempotent, $h \in \operatorname{Reg} A$. Also, $(hxh)\phi = e'$ and so $hxh \in A$. Let $k \in \mathcal{S}(x'hx, h)$. Now,

$$(x'hx)\phi = (x'\phi)(hx\phi) = (x'\phi)(x\phi) = (x'x)\phi = f\phi = f'.$$

Hence we have $k\phi \in \mathcal{S}(f', e') = \{e'\}$ by Proposition 3.9 since $f' \mathrel{\mathscr{L}} e'$. Also, since h is an inverse of itself and hx' is an inverse of xh, by Theorem 3.7, $u = hx'kh$ is an inverse of $hk(xh) = hxh$. Since

$$u\phi = (h\phi)(x'\phi)(kh\phi) = (e')(x'\phi)(e'),$$

by Proposition 2.40(a), $u\phi$ is an inverse of e' in $H_{e'}$. Since e' is an idempotent, it is an inverse of itself. Hence by Proposition 2.40(b), $u\phi = e'$ and so $u \in A$. This proves that hxh is a regular element of A. ☐

THEOREM 4.29. *Let ρ be a congruence on the regular semigroup S and let A be an idempotent ρ-class of S. Then $\operatorname{Reg} A$ is a regular subsemigroup of A.*

Proof. As above we write $\phi = \rho^{\#}$. Let $e' = A\phi$ and $T = \operatorname{Reg} A$. If $x, y \in T$, then T contains inverses x' and y' of x and y respectively. Let $h \in \mathcal{S}(f, g)$ where $f = x'x$

and $g = yy'$. Then by Theorem 3.5 $\theta = \boldsymbol{E}(\phi)$ is a regular bimorphism so that θ satisfies (RM1). Since $f' = f\phi =\mathscr{L}\ x\phi = e'\ \mathscr{R}\ g\phi = g'$, by Proposition 3.12, we have

$$h\phi \in \mathscr{S}(f', g') = \mathscr{S}(e', e') = \{e'\}.$$

Hence $h\phi = e'$ and by so $y'hx' \in A$. By Theorem 3.7, $y'hx'$ is an inverse of xy and so $xy \in T$. Hence T is a subsemigroup of A. □

For many interesting classes of congruences on a regular semigroup S the congruence classes containing idempotents are regular subsemigroups. For example, idempotent separating congruences (see the § Subsection 4.2.2 below), Rees congruences, etc have this property. Also, many subclasses of the class of regular semigroups have the property that for any congruence ρ on a regular semigroup S belonging to one of these class, all idempotent ρ-classes are regular. For example, we have:

Corollary 4.30. *Let ρ be a congruence on the regular semigroup S.*

(a) *If S is primitive then a non-zero idempotent ρ-class is completely simple and the* 0 *ρ-class is an ideal in S.*

(b) *If S/ρ is an inverse semigroup, then any idempotent ρ-class is a regular subsemigroup of S. In particular, if S is an inverse semigroup, then every idempotent ρ-class is an inverse subsemigroup of S.*

Proof. Again, we write $\phi = \rho^{\#}$ in the following.

Assume that S is primitive and let A be an idempotent ρ-class. If $0 \in A$, it is clear that A is an ideal in S. So, assume that $0 \notin A$. Let $x \in A$, $x' \in \mathscr{V}(x)$ and $h \in \mathscr{S}(f, g)$ where $f = x'x$ and $g = xx'$. Then by Lemma 4.28, $h \in A$ and so, $h \neq 0$. Since S is primitive and $h\ \omega^{L}\ f$, we have $h\ \mathscr{L}\ f$. Similarly $h\ \mathscr{R}\ g$ and so, $h\ \mathscr{H}\ x$. Let x' be the inverse of x in H_h. Then $x'\phi$ is the inverse of $x\phi = h\phi$ in the $\mathscr{H}$-class $H_{h\phi}$ in S/ρ and hence, by Proposition 2.40(b), $x'\phi = h\phi$. Therefore $x' \in A$. This implies that A is a primitive regular semigroup with out zero and hence A is completely simple. This proves (a).

To prove (b), assume that A is an idempotent ρ-class in the regular semigroup S and $x \in A$. If $x' \in \mathscr{V}(x)$, since $x\phi$ is an idempotent, $x\phi$ and $(x')\phi$ are inverses of $x\phi$ in S/ρ. Since, S/ρ is an inverse semigroup, we have $x\phi = x'\phi$. Hence $x' \in A$ and so A is regular. If S is an inverse semigroup, by Theorem 2.44, S/ρ is an inverse semigroup and so, by the above, A is a regular subsemigroup of an inverse semigroup. Therefore A is an inverse subsemigroup of S. □

Example at the end of this section shows that shows that idempotent congruence classes may not be regular for congruences on arbitrary regular semigroups.

Next theorem show that any congruence ρ on a regular semigroup S is uniquely determined by the set $\{\operatorname{Reg}\rho(e) : e \in \boldsymbol{E}(S)\}$ of regular subsemigroups of idempotent congruence classes of ρ; in particular, the set of all idempotent congruence classes of ρ form a normal family of subsets in S.

THEOREM 4.31. *Let ρ and σ be congruences on a regular semigroup S. The follow statements are equivalent.*

(a) For all $e \in \boldsymbol{E}(S)$, $\operatorname{Reg}\rho(e) = \operatorname{Reg}\sigma(e)$.

(b) For all $e \in \boldsymbol{E}(S)$, $\rho(e) = \sigma(e)$.

(c) $\rho = \sigma$.

Consequently, given any congruence ρ on S, the set

$$\mathcal{A}_\rho = \{\rho(e) : e \in \boldsymbol{E}(S)\}$$

is a normal family of subsets of S.

Proof. Observe that implications (b) ⇒ (a) and (c) ⇒ (b) are obvious. So it is sufficient to prove the implications: (a) ⇒ (b) ⇒ (c).

(a) ⇒ (b) For convenience, let $\phi = \rho^\#$ and $\psi = \sigma^\#$. Choose $e \in \boldsymbol{E}(S)$ and $x \in \rho(e)$. Also let $h \in \mathscr{S}(f,g)$ where $f \in E(L_x)$ and $g \in E(R_x)$. Then, by Lemma 4.28, $h, hxh \in \operatorname{Reg}\rho(e)$. Then by (a), $h, hxh \in \sigma(e)$. Now, (a) implies that ρ and σ induces the same biorder congruence on $E = \boldsymbol{E}(S)$. Therefore

$$f\phi \mathrel{\mathscr{L}} x\phi = e\phi \mathrel{\mathscr{R}} g\phi \Rightarrow f\psi \mathrel{\mathscr{L}} e\psi \mathrel{\mathscr{R}} g\psi.$$

Since $f\psi \mathrel{\mathscr{L}} x\psi \mathrel{\mathscr{R}} g\psi$ by the choice of f and g, it follows that $e\psi \mathrel{\mathscr{H}} x\psi$. Hence

$$(exe)\psi = (e\psi)(x\psi)(e\psi) = x\psi$$

and so $exe \mathrel{\sigma} x$. Since h and hxh are regular elements of $\rho(e)$, we have $h \mathrel{\sigma} e$ and $hxh \mathrel{\sigma} e$ by (a). Therefore

$$x \mathrel{\sigma} exe \mathrel{\sigma} hxh \mathrel{\sigma} e$$

which implies that $x \in \sigma(e)$. Hence $\rho(e) \subseteq \sigma(e)$. Interchanging ρ and σ we obtain $\sigma(e) \subseteq \rho(e)$ and hence $\rho(e) = \sigma(e)$.

(b) ⇒ (c) Let $x\rho y$. Suppose that $a \in \mathscr{V}(x)$ and $b \in \mathscr{V}(y)$. Then, using the fact that ρ is right compatible, we get $xa\rho ya$. Since $xa \in \boldsymbol{E}(S)$, by (b), we have $xa\sigma ya$. Similarly, $bx\sigma by$. Using these and the fact that σ is a congruence, we have

$$\begin{aligned} x = xax \quad &\sigma \quad yax = ybyax \quad \sigma \quad ybxax \\ = ybx \quad &\sigma \quad yby = y. \end{aligned}$$

kernel normal system
$\mathcal{A}_\rho$: Kernel normal system of ρ

Thus $(x, y) \in \sigma$ so that $\rho \subseteq \sigma$. The arguements can be repeated with ρ and σ interchanged giving $\sigma \subseteq \rho$. Hence (c) follows.

The last statement is a consequence of the definition of normal families. Whence the theorem. □

Aa in Clifford and Preston [1967], normal family $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$ of subsets of the regular semigroup S is called a *kernel normal system* (a KN-system or even KNS for short) on S if there is a congruence ρ on S such that

$$\mathcal{A} = \mathcal{A}_\rho = \{\rho(e) : e \in \boldsymbol{E}(S)\}. \tag{4.7a}$$

Given the congruence ρ, the family $\mathcal{A}_\rho = \{\rho(e)\}$ will be called the kernel normal system of ρ. The kernel normal system $\mathcal{A}_\phi$ of any homomorphism $\phi : S \to T$ is the kernel system of the congruence $\boldsymbol{\kappa}\phi$ of ϕ. Thus

$$\mathcal{A}_\phi = \{\big((\phi \circ \phi^{-1})(e)\big) : e \in \boldsymbol{E}(S)\}. \tag{4.7b}$$

We shall consider the problem of characterization of KN-systems of inverse semigroups in the next chapter. The characterization of KN-systems on regular semigrous will be considered later later in the chapter on inductive groupoids (Chapter 6).

***Remark* 4.4:** The last statement (as well as the statement (b)) of the theorem above is classical ([see Clifford and Preston, 1967, Theorem 7.38]). However, the statement (a) is considerably stronger. An alternate approach for its proof is using inductive groupoids; in fact, it is a consequence of the equivalence of the category $\mathfrak{RG}$ of regular semigroups and the category $\mathfrak{IG}$ of inductive groupoids (see Chapter 6).

In this context, there is considerable variation in terminologies used by various authors. In Clifford and Preston [1967] the term kernel normal system is used to denote to denote a family of subsemigroups satisfying the conditions in Equation (4.7a), especially in the case when S is an inverse semigroup. However, the KNS of a congruence ρ on S is called the kernel of ρ in Clifford and Preston [1967]. On the other hand **?** and Pastijn and Petrich [1985, 1986] uses the term *kernel* for the union of all congruence classes that contain idempotents. We will not use these here. We shall define kernels later so that they are functors on an appropriate domain category (See Equation (4.8b) for definition of kernels of idempotent separating congruences.)

congruence!idempotent separating

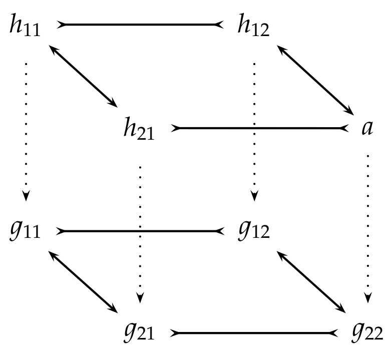
Example 4.9: Consider the regular semigroup

$$S = \{h_{11}, h_{12}, h_{21}, a, g_{11}, g_{12}, g_{21}, g_{22}\}$$

indicated by the $\mathscr{D}$-class diagram on the right in which all elements, except a, are idempotents. In the diagram slanted arrows ($\leftrightarrow$) represent $\mathscr{L}$-relation, horizontal arrows represent $\mathscr{R}$-relation and the dotted (vertical) arrows represent the natural partial order. It is easy to see that

$$\rho = \{(x, y) : x \leq y \text{ or } y \leq x\}$$

where $\leq$ denote the natural partial order, is a congruence on S such that S/ρ is a rectangular band. Here the congruence class $\rho(g_{22}) = \{g_{22}, a\}$ is not a regular subsemigroup of S.



4.2.2 Idempotent separating congruences

A congruence ρ on a semigroup S is said to be *idempotent separating* if any ρ-class contain utmost one idempotent.

THEOREM 4.32. *The following statements are equivalent for a congruence ρ on a regular semigroup S.*

(1) ρ is idempotent separating;

(2) $\rho \subseteq \mathscr{H}$;

(3) for each $e \in \boldsymbol{E}(S)$, $\rho(e)$ is a subgroup of H_e;

(4) the bimorphism $\boldsymbol{E}(\rho^\#) : \boldsymbol{E}(S) \to \boldsymbol{E}(S/\rho)$ is a biorder isomorphism.

When ρ satisfies these equivalent conditions, then for all $x \in S$ we have

$$\rho(x) = \begin{cases} \rho(e)x, & \text{if } e \in E(R_x); \\ x\rho(f), & \text{if } f \in E(L_x). \end{cases}$$

Proof. **(1) $\Rightarrow$ (2).** This follows by Proposition 3.47.

(2) $\Rightarrow$ (3). If $e \in \boldsymbol{E}(S)$, (2) implies that $\rho(e)$ is a subsemigroup of H_e. If $u \in \rho(e)$ and if u' is the inverse of u in H_e, then $u'\phi$ is the inverse of $u\phi = e\phi$ in $H_{e\phi}$, where $\phi = \rho^\#$. Hence $u'\phi = e\phi$ which implies that $u' \in \rho(e)$. Therefore $\rho(e)$ is a subgroup of H_e.

ω-partial functor
F_e:ω-partial functor of F on $\omega(e)$

(3) ⇒ (4). Since $\phi = \rho^{\#}$ is a homomorphism of regular semigroups, by Theorem 3.5, the bimorphism $\theta = \boldsymbol{E}(\phi)$ is regular. Statement (3) implies that any ρ-class containing an idempotent is a subgroup and hence contains only one idempotent. It follows that θ is injective. By Theorem 2.41, it is surjective. Thus θ is a bijective regular bimorphism and so, by Corollary 3.25, θ is an isomorphism.

(4) ⇒ (1). Statement (4) clearly implies that no ρ-class contain more than one idempotent.

Let $e \in E(R_x)$. If $u \in \rho(e)$, then $(e, u) \in \rho$ and so, $(x, ux) = (ex, ux) \in \rho$. Hence $\rho(e)x \subseteq \rho(x)$. If $y \in \rho(x)$, and if $x' \in \mathscr{V}(x)$ with $xx' = e$, then $yx' \in \rho(e)$. Also, since $x \mathscr{H} y$, $y \mathscr{L} x'x$ and so, $y = yx'x \in \rho(e)x$. Thus $\rho(x) = \rho(e)x$. Dually $\rho(x) = x\rho(f)$ for any $f \in E(L_x)$. □

Idempotent separating congruences on semigroups that are not regular, may not satisfy condition (2), (3) or (4) above (see Example 4.10).

Recall that (see Subsection 3.3.3) a regular semigroup S is *fundamental* if there is no non-trivial idempotent separating congruence on S (see Proposition 3.47). By Proposition 3.46 and Proposition 3.46 $\mu(S) = \mathscr{H}_{(c)}$ is the maimum idempotent congruence on a regular semigroup S and the semigrup $S/\mu(S)$ is fundamental.

Let ρ be an idempotent separating congruence on the regular semigroup S. By Theorem 4.32(4), $\boldsymbol{E}(S)$ is isomorphic to $\boldsymbol{E}(S/\rho)$ and hence we may identify these biordered sets (idetifying $e \in \boldsymbol{E}(S)$ with $e\rho^{\#}$). Therefore the KN-system $\mathcal{A}_\rho$ (Equation (4.7b)) of ρ may be regarded as a function on $\boldsymbol{E}(S)$ taking values in the set of all subgroups of S. In this case, more is true: they are group-valued functors on the preorder $(\boldsymbol{E}, \omega)$.

Given a biordered set $\boldsymbol{E}$, let $\boldsymbol{E}_\omega$ denote the preorder $(\boldsymbol{E}, \omega)$ (see Subsection 1.3.1 for more details). Suppose that $F : \boldsymbol{E}_\omega \to \boldsymbol{C}$ be a functor to a category $\boldsymbol{C}$. For $e \in \boldsymbol{E}$, the ω-partial functor of F on $\omega(e)$ is the restriction

$$F_e = F \mid \omega(e) \tag{4.8a}$$

of F to the preorder on the biordered subset $\omega(e) \subseteq \boldsymbol{E}$.

Let ρ be an idempotent separating congruence on the regular semigroup S. For all $e \in \boldsymbol{E}$ and $f \leq e$, define

$$\boldsymbol{G}(e) = \rho(e) \quad \text{and} \quad u\boldsymbol{G}(f, e) = uf, \quad u \in \boldsymbol{G}(e) \tag{4.8b}$$

Since ρ is idempotent separating, $\boldsymbol{G}(e) = \rho(e)$ is a subgroup of H_e for all $e \in \boldsymbol{E}$. Also, if $f \ \omega \ e$ and $u \in \boldsymbol{G}(e)$, the fact that ρ is a congruence gives

$fu, uf \in \boldsymbol{G}(f)$ and $fu, uf \leq u$ in the natural partial order. Hence, $fu = uf$ by Proposition 4.7(c). It follows that

$$\mathfrak{a}(u, u^{-1}) = 1_{\omega(e)} \quad \text{for all} \quad u \in \boldsymbol{G}(e).$$

Also if $u, v \in \boldsymbol{G}(e)$, then we have

$$\begin{aligned}(u\boldsymbol{G}(f,e))(v\boldsymbol{G}(f,e)) &= (fu)(fv)\\ &= f(uv) = (uv)\boldsymbol{G}(e,f)\end{aligned}$$

so that $\boldsymbol{G}(f,e) : \boldsymbol{G}(e) \to \boldsymbol{G}(f)$ is a homomorphism such that

$$fu = u\boldsymbol{G}(e,f) \leq u.$$

Again, for $g \ \omega \ f \ \omega \ e$, we have $\boldsymbol{G}(f,e)\boldsymbol{G}(g,f) = \boldsymbol{G}(g,e)$; also $\boldsymbol{G}(e,e) = 1_{\omega(e)}$ for all $e \in \boldsymbol{E}$. Since, for each $f \ \omega \ e$, (f,e) is the unique morphism from f to e in the preorder $\boldsymbol{E}_\omega$, it follows that $\boldsymbol{G} : \boldsymbol{E}_\omega \to \mathbf{Grp}$ is a contravariant functor.

Let $x \in S$ and $x' \in \mathscr{V}(x)$. Recall from Lemma 2.67 that $\mathfrak{a}(x,x') : h \mapsto x'hx$ is an order isomorphism of $\omega(xx')$ onto $\omega(x'x)$ so that $\mathfrak{a}(x,x')$ is an isomorphism of the preorder on $\omega(xx')$ onto the preorder $\omega(x'x)$. Again, the fact that ρ is a congruence gives that the map $\mathfrak{c}^\rho_g : u \mapsto x'ux$ is an isomorphism of $\boldsymbol{G}(g)$ to $\boldsymbol{G}(x'gx) = \boldsymbol{G}(g\mathfrak{a}(x,x'))$ for all $g \ \omega \ xx'$. Moreover, if $h \ \omega \ g \ \omega \ e$, the following diagram commute:

$$\begin{array}{ccc}\boldsymbol{G}(g) & \xrightarrow{\ \mathfrak{c}^\rho_g\ } & \boldsymbol{G}\left(g\mathfrak{a}(x,x')\right)\\ {\scriptstyle \boldsymbol{G}(h,g)}\big\downarrow & & \big\downarrow{\scriptstyle \boldsymbol{G}\left(h\mathfrak{a}(x,x'),\, g\mathfrak{a}(x,x')\right)}\\ \boldsymbol{G}(h) & \xrightarrow[\ \mathfrak{c}^\rho_h\ ]{} & \boldsymbol{G}\left(h\mathfrak{a}(x,x')\right)\end{array} \tag{d.gkr}$$

For, if $u \in \boldsymbol{G}(g)$, we have

$$\begin{aligned}u\mathfrak{c}^\rho_g\boldsymbol{G}\left(h\mathfrak{a}(x,x'), g\mathfrak{a}(x,x')\right) &= (x'hx)(x'ux) = x'h(xx')ux = x'hux\\ &= (u\boldsymbol{G}(h,g))\mathfrak{c}^\rho_h\end{aligned}$$

It follows that to each $x \in S$ and $x' \in \mathscr{V}(x)$, there corresponds a transformation (see § Subsection 1.2.2)$\mathfrak{c}^\rho(x,x') : \boldsymbol{G}_{xx'} \to \boldsymbol{G}_{x'x}$ of the partial functor $\boldsymbol{G}_{xx'}$ of $\boldsymbol{G} = \boldsymbol{G}^\rho$ to $\boldsymbol{G}_{x'x}$ such that

$$\mathfrak{v}\mathfrak{c}^\rho(x,x') = \mathfrak{a}(x,x'), \tag{4.8c}$$

and the component of the natural transformation $\mathfrak{c}^\rho(x,x')$ at $g \ \omega \ xx'$ is given by

$$u\mathfrak{c}^\rho_g(x,x') = x'ux \quad \text{for all} \quad u \in \boldsymbol{G}(g). \tag{4.8d}$$

We have thus proved the direct part of the following theorem:

THEOREM 4.33. *Let ρ be an idempotent separating congruence on a regular semigroup S. Then Equation* (4.8b) *defines a contravariant functor $\boldsymbol{G} = \boldsymbol{G}^{\rho} : \boldsymbol{E}_{\omega} \to \mathsf{Grp}$ satisfying the following conditions:*

(Gkr1) For all $e \in \boldsymbol{E}$, $\boldsymbol{G}(e)$ is a subgroup of H_e such that

$$\boldsymbol{G}(e) \subseteq \{u \in H_e : \mathfrak{a}(u, u^{-1}) = 1_{\omega(e)}\}.$$

(Gkr2) For $g \;\omega\; e$, we have $u\boldsymbol{G}(g, e) \leq u$ for all $u \in \boldsymbol{G}(e)$.

(Gkr3) For $x \in S$ and $x' \in \mathscr{V}(x)$, there is a unique a transformation $\mathfrak{c}^{\rho}(x, x') : \boldsymbol{G}_{xx'} \to \boldsymbol{G}_{x'x}$ satisfying Equation (4.8c) *and Equation* (4.8d).

Conversely, if $\boldsymbol{G}$ is a contravariant group-valued functor on $\boldsymbol{E}_{\omega}(S)$ satisfying the conditions above, then

$$\rho = \{(x, y) : \textit{for some } e \in E(R_x)\ \exists\, u \in \boldsymbol{G}(e) \textit{ with } y = ux\} \tag{4.9}$$

is an idempotent separating congruence on S such that $\boldsymbol{G}^{\rho} = \boldsymbol{G}$.

Proof. Since we have already proved the direct part, it is sufficient to verify the converse. Suppose that $\boldsymbol{G}$ is a contravariant group-valued functor on $\boldsymbol{E}_{\omega}(S)$ satisfying conditions (Gkri), $i = 1, 2, 3$ and let ρ be defined by Equation (4.9). First, we note that $\rho \subseteq \mathscr{H}$. For, let $(x, y) \in \rho$. By the definition there is some $e \in E(R_x)$ and $u \in \boldsymbol{G}(e) \subseteq H_e$ such that $y = ux$. Now, Corollary 2.27, the translation $\rho_x : L(e) \to L(x)$ is an isomorphism of left ideals and so, by Theorem 2.25, ρ_x is a bijection of H_e onto H_x. Hence $y = ux \in H_x$. Moreover,

$$\begin{aligned} \rho &= \{(x, y) : \text{for all } e \in E(R_x)\ \exists\, u \in \boldsymbol{G}(e) \text{ with } y = ux\}; \\ &= \{(x, y) : \text{for all } f \in E(L_x)\ \exists\, v \in \boldsymbol{G}(f) \text{ with } y = xv\}. \end{aligned} \tag{4.9*}$$

Let σ be the relation defined by the first equation above and let $(x, y) \in \rho$. Then $y = ux$ for some $u \in \boldsymbol{G}(e)$ with $e \in E(R_x)$. For any $e' \in E(R_x)$, $e \in \mathscr{V}(e')$ and so by (Gkr3), there exist a transformation $\mathfrak{c}^{\rho}(e', e) : \boldsymbol{G}_e \to \boldsymbol{G}_{e'}$ whose component $\mathfrak{c}^{\rho}_e$ is the isomorphism $u \mapsto ue'$ of $\boldsymbol{G}(e)$ to $\boldsymbol{G}(e')$. Since $y = ux$, we have $ue' \in \boldsymbol{G}(e')$ and $(ue')x = u(e'x) = ux = y$. This implies that $\rho \subseteq \sigma$. The reverse inclusion clearly hold and so, $\rho = \sigma$. Again, let τ be the relation defined by the second equality in Equation (4.9*). If $f \in E(L_x)$ and if x' is the inverse of x in $L_e \cap R_f$, then $y = ux$ if and only if $y = x(x'ux)$. By axiom (Gkr 3), $u \mapsto x'ux$ is an isomorphism of the group $\boldsymbol{G}(xx') = \boldsymbol{G}(e)$ onto $\boldsymbol{G}(x'x) = \boldsymbol{G}(f)$ and so, $x'ux \in \boldsymbol{G}(f)$. Thus $\rho \subseteq \tau$. The reverse inclusion follows by duality. Therefore ρ satisfies Equation (4.9*).

Clearly ρ is reflexive. If $(x, y) \in \rho$, then from $y = ux$, $u \in \boldsymbol{G}(e)$, we have $x = u^{-1}y$. By (Gkr1), $u^{-1} \in \boldsymbol{G}(e)$ and so, $(y, x) \in \rho$ and so ρ is symmetric. Transitivity can be proved in a similar way. Thus ρ is an equivalence relation.

If $u \in \boldsymbol{G}(e)$ and $f \ \omega \ e$, then from (Gkr1), $f\mathfrak{a}(u, u^{-1}) = f$ and so $fu = uf$. Since $fu \ \mathscr{R} \ f \ \mathscr{L} \ uf$, it follows that $fu \in H_e$ and $fu \leq u$. Since by (Gkr2), $u\boldsymbol{G}(e, f) \leq u$, it follows by Proposition 4.7(c) that $fu = uf = u\boldsymbol{G}(e, f)$.

kernel!group – $\boldsymbol{G}^{\rho}$: kernel of ρ

Now suppose that $(x, y) \in \rho$ and $z \in S$. Let $x' \in \mathscr{V}x$. By Equation (4.9*), $y = ux$ for some $u \in \boldsymbol{G}(e)$ with $e = xx'$. Let $f = x'x$, $g \in E(R_z)$ and $h \in \mathscr{S}(f, g)$. Then $h' = xhx' \ \omega \ e$ and so $uh' = h'u = u\boldsymbol{G}(e, h') \leq u$ by axioms (Gkr1) and (Gkr2). By Theorem 3.7,

$$xz = (xh) * (hz), \quad yz = (yh) * (hz), \quad \text{and} \quad h' \in E(R_{xz}).$$

It follows from Corollary 2.27 that the translation $\rho_x : L(e) \to L(x) = L(f)$ is an isomorphism of left ideals and so, by Theorem 2.25, ρ_x is a bijection of $H_{h'}$ onto $(H_{h'})\rho_x = H_{h'x}$. Hence, since $uh' \ \mathscr{H} \ h'$, we have $h'x \ \mathscr{H} \ uh'x = h'ux = h'y$. Therefore

$$yh = uxh = uxhx'x = uh'x \ \mathscr{H} \ h'x = xh \quad \text{and so,} \quad xz \ \mathscr{H} \ yz.$$

Since $yz = uh'(xz)$ and $uh' \in \boldsymbol{G}(h')$, $(xz, yz) \in \rho$ by Equation (4.9).

By Equation (4.9*), the definition of ρ is selfdual. Hence dualizing the arguements above, we conclude that $(zx, zy) \in \rho$. Thus ρ is a congruence. It follows from Theorem 4.32(3) and axiom (Gkr1) that ρ is idempotent separating. By the definition of ρ, it is clear that $\boldsymbol{G}^{\rho}(e) = \rho(e) = \boldsymbol{G}(e)$ for all $e \in \boldsymbol{E}(S)$. Similarly for all $f \ \omega \ e$ and $u \in \boldsymbol{G}(e)$ we have

$$u\boldsymbol{G}^{\rho}(e, f) = fu = u\boldsymbol{G}(e, f).$$

Therefore $\boldsymbol{G}^{\rho} = \boldsymbol{G}$. □

The contravariant group-valued functor $\boldsymbol{G} : \boldsymbol{E}_{\omega}(S) \to \mathsf{Grp}$ satisfying the conditions (Gkr$i$), $i = 1, 2, 3$ will be called a *group kernel* on S. If ρ is an idempotent separating congruence on S, the group kernel $\boldsymbol{G}^{\rho}$ is called the *kernel* of ρ. Notice that the KN-system $\mathcal{A}_{\rho}$ Remark 4.4 of ρ is, in this case, a set of subgroups of S and is the image of the vertex-map of the functor $\boldsymbol{G}^{\rho}$. Consequently, the map $\mathfrak{v}\boldsymbol{G} : e \mapsto \boldsymbol{G}(e)$ completely determine the functor $\boldsymbol{G}$.

PROPOSITION 4.34. *Let $\mu = \mu(S)$ denote the largest idempotent separating congruence on the regular semigroup S and let $\boldsymbol{G}^{\mu}$ denote its kernel. Then for each $e \in \boldsymbol{E}(S)$,*

$$\boldsymbol{G}^{\mu}(e) = \{u \in H_e : \mathfrak{a}(u, u^{-1}) = 1_{\omega(e)}\}. \tag{4.10}$$

Proof. For each $e \in \boldsymbol{E} = \boldsymbol{E}(S)$, let C_e denote the set on the right of Equation (4.10). Then by Lemma 2.67,

$$C_e = \{u \in H_e : ug = gu \quad \text{for all} \quad g \ \omega \ e\}.$$

It is clear that, if $u, v \in C_e$, then $uv \in C_e$. If $u \in C_e$ and $g \ \omega \ e$, then from $ug = gu$, we have $gu^{-1} = u^{-1}g$; hence $u^{-1} \in C_e$. Thus C_e is a subgroup of H_e. Furthermore, if $f \ \omega \ e$, then the map

$$C_{(e,f)} : u \in C_e \mapsto fu$$

is a homomorphism of C_e to C_f. We proceed to show that

$$\boldsymbol{G} : e \mapsto C_e, \quad (f, e) \mapsto C_{(e,f)}$$

is a group kernel on S in the sense defined above (that is, satisfies conditions (Gkr$i$), $i = 1, 2, 3$).

By the remarks above, $\boldsymbol{G}$ satisfies (Gkr1). For $g \ \omega \ e$, and $u \in H_e$, $gu \ \mathscr{R} \ g$ and $ug \ \mathscr{L} \ g$. Hence $ug = gu$ implies that $ug \ \mathscr{L} \ g \ \mathscr{R} \ gu$ and so, $v = ug = gu \in H_g$. It is clear that v commutes with every $h \ \omega \ g$ and so, $v \in C_g$. Also $v = gu \leq u$. Hence $C_{(e,g)} : u \mapsto gu$ is a homomorphism of C_e to C_g that satisfies (Gkr2).

To prove (Gkr3), let $x \in S$ and $x' \in \mathscr{V}(x)$. If $e = xx'$ and $f = x'x$, by Lemma 2.67 $\mathfrak{a}(x, x') : h \mapsto x'hx$ is an order isomorphism from $\omega(e)$ onto $\omega(f)$. Now $v \in C_e$ if and only if $hv = vh$ for all $h \ \omega \ e$. This is true if and only if

$$(x'hx)(x'vx) = x'hvx = x'vhx = (x'vx)(x'hx) \quad \text{for all} h \ \omega \ e.$$

It follows that $v \in C_e$ if and only if $x'vx \in C_f$. Let $g \ \omega \ e$ and $u, v \in C_g$. Since

$$x'(uv)x = (x'ux)(x'vx) \quad \text{for all} u, v \in C_g$$

the map $\gamma_g : u \mapsto x'ux$ is an isomorphism of C_g onto $C_{x'gx}$. A routine verifoica-tion shows that for all $h \ \omega \ g$, the following diagram commutes.

$$\begin{array}{ccc} C_g & \xrightarrow{\gamma_g} & C_{(x'gx)} \\ {\scriptstyle C_{(g,h)}} \downarrow & & \downarrow {\scriptstyle C_{(x'gx, x'hx)}} \\ C_{(h)} & \xrightarrow[\gamma_h]{} & C_{(x'hx)} \end{array} \qquad \text{(d1.gkr)}$$

It follows that there is a transformation $\mathfrak{c}^\rho(x, x')$ of the partial functor : $\boldsymbol{G}_e$ of $\boldsymbol{G}$ to $\boldsymbol{G}_f$ such that

$$\mathfrak{v}\mathfrak{c}^\rho = \mathfrak{a}(x, x')$$

and the natural transformation $\mathfrak{c}^\rho$ is the map

$$g \in \omega(e) \mapsto \gamma_g.$$

Therefore (Gkr3) holds. Consequently, by Theorem 4.33, $\boldsymbol{G}$ is a group kernel and there is a unique idempotent separating congruence σ defined by Equation (4.9) with $\boldsymbol{G} = \boldsymbol{G}^\sigma$.

Since μ is an idempotent separating congruence, by axiom (Gkr1), $\mu(e) \subseteq C_e = \sigma(e)$. This implies, by Equation (4.32) that $\mu \subseteq \sigma$. Since μ is the largest idempotent separating congruence on S, we have $\mu = \sigma$ and so $\boldsymbol{G} = \boldsymbol{G}^\mu$. $\square$

Clearly, every group kernel $\boldsymbol{G}$ of S is a subfunctor of $\boldsymbol{G}^\mu$ in the sense defined by Equations (1.51) and (1.52) and so, $\boldsymbol{G}^\mu$ is the maximum group kernel on S and is closely related to the structure of S. The following theorem list a number of equivalent descriptions of the maximum idempotent separating congruence $\mu = \mu(S)$ on a regular semigroup S. The statement (2) below is due to Hall Hall [1973] and is a straightforward generalization of Howie's description Howie [1964] of $\mu(S)$ on an inverse semigroup S ([Clifford and Preston, 1967, see also]). Statements (3) and (4) are due to Nambooripad [1979] and Grillet [1974a] respectively and are related to the fundamental representations of regular semigroups. We shall come back to this later in this chapter.

THEOREM 4.35. *Let S be a regular semigroup. The following statements are equivalent for $(x, y) \in \mathscr{H}$.*

(1) $(x, y) \in \mu(S)$;

(2) for all $x' \in \mathscr{V}(x)$ and $g \; \omega \; e = xx'$, there exist a unique $y' \in \mathscr{V}(y)$ such that $x'gx = y'gy$.

(3) there exists $x' \in \mathscr{V}(x)$ and $y' \in \mathscr{V}(y)$ such that $\mathfrak{a}(x, x') = \mathfrak{a}(y, y')$;

(4) for each $z \in S$, $L_{zx} = L_{zy}$ and $R_{xz} = R_{yz}$;

Proof. (1) ⇒ (2). Let $x' \in \mathscr{V}(x)$ and $g \; \omega \; e = xx'$. (1) implies by Equation (4.9) that $y = ux$ where $ug = gu$ for all $g \; \omega \; e$. Now the translation $\lambda_{x'} : R(x) = R(e) \to R(x')$ is an isomorphism of right ideals. Since $u^{-1} \mathscr{H} e$, $x'u^{-1} \mathscr{H} x'$. Now $y' = x'u^{-1}$ is an inverse of $y = ux$ in $H_{x'}$ and

$$y'gy = x'u^{-1}(gu)x = x'u^{-1}(ug)x = x'(u^{-1}u)gx = x'gx$$

for all $g \; \omega \; e$. If $y'' \in \mathscr{V}(y)$ also satisfies this, taking $g = e = xx'$, we have $x'x = y''y$. Hence

$$x' = y''yx' = y''uxx' = y''ue = y''u \quad \text{and so,} \quad y'' = x'u^{-1} = y'.$$

This proves the uniqueness of $y' \in \mathscr{V}(y)$.

(2) ⇒ (3). This is an immediate consequence of the definition of the map $\mathfrak{a}(x, x')$ (see Equation (2.44)).

(3) ⇒ (4). Let $z \in S$. By (3), there exist $x' \in \mathscr{V}(x)$, $y' \in \mathscr{V}(y)$ such that $\mathfrak{a}(x, x') = \mathfrak{a}(y, y')$. Since $x \mathscr{H} y$, $\mathscr{S}(z, x) = \mathscr{S}(z, y)$ (see Proposition 3.12). If $h \in \mathscr{S}(z, x)$, $e = xx'$ and $f = x'x$, then $he\ \omega\ e$ and so, by (3), we have $x'hx = x'hex = y'hey = y'hy$. Now

$$(hx)(x'h)(hx) = h(xx')hhx = hx, \quad (x'h)(hx)(x'h) = x'hh(xx')h = x'h$$

and so $x'h$ is the inverse of hx in $L_h \cap R_{x'hx}$. Similarly, $y'h$ is the inverse of hy in $L_h \cap R_{y'hy}$. Therefore the equality $x'hx = y'hy$ implies that $hx \mathscr{H} hy$. Hence

$$zx = (zh) * (hx) \mathscr{H} (zh) * (hy) = zy$$

which gives $L_{zx} = L_{zy}$. Similarly, $R_{xz} = R_{yz}$.

(4) ⇒ (1). This is a consequence of the following proposition. □

PROPOSITION 4.36. *Let S be a regular semigroup. Then*

$$\mu_l(S) = \{(x, y) : L_{zx} = L_{zy} \quad \forall\, z \in S\} \tag{4.11a}$$

is the largest congruence on S contained in $\mathscr{L}$. Dually,

$$\mu_r(S) = \{(x, y) : R_{xz} = R_{yz} \quad \forall\, z \in S\} \tag{4.11b}$$

is the largest congruence on S contained in $\mathscr{R}$. Moreover, $\mu(S) = \mu_l(S) \cap \mu_r(S)$.

Proof. Clearly, $\mu_l = \mu_l(S)$ is an equivalence relation. Let $(x, y) \in \mu_l$ and $u \in S$. Then for any $z \in S$,

$$zx \mathscr{L} zy \Rightarrow z(xu) \mathscr{L} z(yu), \quad \text{and}$$
$$z(ux) = (zu)x \mathscr{L} (zu)y = z(uy)$$

and so $(xu, yu), (ux, uy) \in \mu_l$. So μ_l is a congruence. Also for $(x, y) \in \mu_l$, $x \mathscr{L} ey$ where $e \in E(R_x)$. Hence $x \in Sy$. Similarly $y \in Sx$ and so $x \mathscr{L} y$. Therefore $\mu_l \subseteq \mathscr{L}$. If ρ is any congruence contained in $\mathscr{L}$, then for any $(x, y) \in \rho$ and $z \in S$, $(xz, yz) \in \rho$ which implies that $L_{zx} = L_{zy}$. Thus $\rho \subseteq \mu_l$.

Dually $\mu_r = \mu_r(S)$ is the largest congruence contained in $\mathscr{R}$. Now, since $\mu = \mu(S) \subseteq \mathscr{H} \subseteq \mathscr{L}$, $\mu \subseteq \mu_l$ and similarly, $\mu \subseteq \mu_r$. Hence $\mu \subseteq \mu_l \cap \mu_r$. Since $\mu_l \cap \mu_r \subseteq \mathscr{L} \cap \mathscr{R} = \mathscr{H}$, by Theorem 4.32, $\mu_l \cap \mu_r$ is an idempotent separating congruence on S. Hence we conclude, by Proposition 3.47, that $\mu = \mu_l \cap \mu_r$ is a congruence contained in $\mathscr{H}$ and so $\mu \subseteq \mu(S)$. On the other hand, it is clear that $\mu(S) \subseteq \mu_l$ and $\mu(S) \subseteq \mu_r$ and so, $\mu(S) \subseteq \mu$. Hence $\mu = \mu(S)$. □

By Proposition 3.47 a congruence ρ on S is idempotent separating if and only if $\rho \subseteq \mu = \mu(S)$. Hence the set of all idempotent separating congruences on S is the order ideal $\mathfrak{L}(\mu)$ of the lattice $\mathfrak{L} = \mathfrak{L}_S$. Hence the lattice $\mathfrak{L}(\mu)$ of all idempotent separating congruences on S is a complete lattice. Further for $\rho \in \mathfrak{L}(\mu)$ and each $e \in \boldsymbol{E}(S)$, $\boldsymbol{G}^\rho(e) = \rho(e)$ is the kernel of the homomorphism $\rho^\#|H_e$ of H_e. Hence for every $e \in \boldsymbol{E}(S)$, $\boldsymbol{G}^\rho(e)$ is a normal subgroup of H_e. Hence $\boldsymbol{G}^\rho(e)$ is a member of the lattice $\mathcal{N}(H_e)$ of all normal subgroup of H_e (see Example 1.2). Therefore the map $\boldsymbol{G}^\rho : e \mapsto \boldsymbol{G}^\rho(e)$ is a member of the product lattice

$$\mathcal{N} = \prod_{e \in \boldsymbol{E}(S)} \mathcal{N}(H_e).$$

By Theorem 4.33, the map $\rho \mapsto \boldsymbol{G}^\rho$ is a bijection of the lattice $\mathfrak{L}(\mu)$ and the set all group kernels on S. By Theorem 4.32, the vertex map

$$\mathfrak{v}\boldsymbol{G} : e \mapsto \boldsymbol{G}(e) = \rho(e)$$

of a group kernel $\boldsymbol{G} = \boldsymbol{G}^\rho$ completely dertermine it. These functions are in $\mathcal{N}$. Hence there is a bijection

$$\mathfrak{v}\boldsymbol{G} : \rho \mapsto \mathfrak{v}\boldsymbol{G}^\rho$$

between idempotent separating congruences on S and functions in $\mathcal{N}$ that are vertex maps of group kernels. Now the order in the product lattice $\mathcal{N}$ is defined componentwise; that is if $\alpha, \beta \in \mathcal{N}$, then

$$\alpha \leq \beta \iff \alpha_e \subseteq \beta_e$$

for all $e \in \boldsymbol{E}(S)$. Also for $\rho, \sigma \in \mathfrak{L}(\mu)$

$$\begin{aligned} \rho \subseteq \sigma &\iff \boldsymbol{G}^\rho(e) \subseteq \boldsymbol{G}^\sigma(e) \\ &\iff \boldsymbol{G}^\rho \subseteq \boldsymbol{G}^\sigma. \end{aligned}$$

Hence $\mathfrak{v}\boldsymbol{G}$ is an order embedding of $\mathfrak{L}(\mu)$ into $\mathcal{N}$. Since the $\wedge$ operation is the intersection in both $\mathfrak{L}(\mu)$ and $\mathcal{N}(H_e)$ (for every $e \in \boldsymbol{E}(S)$), it is clear that $\mathfrak{v}\boldsymbol{G}$ preserves $\wedge$. To see that $\mathfrak{v}\boldsymbol{G}$ also preserves $\vee$, let $\Omega \subseteq \mathfrak{L}(\mu)$ and let $\sigma = \vee\Omega$. By Proposition 2.6,

$$\sigma = (\cup\Omega)^{(t)}.$$

Hence if $e \in \boldsymbol{E}(S)$ and $u \in \boldsymbol{G}^\sigma(e) = \sigma(e)$, then there exists $\rho_i \in \Omega$, $i = 1, 2, \ldots, r$ such that

$$\begin{aligned} u &\in (\rho_1 \vee \rho_2 \vee \cdots \vee \rho_r)(e) \\ &\in \rho_1(e) \cdot \ldots \rho_r(e) \end{aligned}$$

by Example 1.2

$$= u_1 u_2 \dots u_r$$

where $u_i \in \rho_i(e)$, $i = 1, 2, \dots, r$. This shows, by the definition of $\vee$ in $\mathcal{N}(H_e)$, that $u \in \vee\{\rho(e) : \rho \in \Omega\}$ where the right-hand side denotes the join in $\mathcal{N}(H_e)$. Hence

$$\sigma(e) = \vee\{\rho(e) : \rho \in \Omega\}$$

for all $e \in \boldsymbol{E}(S)$. Therefore

$$\boldsymbol{G}^{\vee\Omega} = \boldsymbol{G}^{\sigma} = \vee\left\{\boldsymbol{G}^{\rho} : \rho \in \Omega\right\}.$$

This shows that $\boldsymbol{G} : \mathfrak{L}(\mu) \to \mathcal{N}$ is a lattice embedding. Since each lattice $\mathcal{N}(H_e)$ is modular by Example 1.2, $\mathcal{N}$ is a product of modular lattices and so, since $\mathfrak{L}(\mu)$ is isomorphic to a sublattice of $\mathcal{N}$, $\mathfrak{L}(\mu)$ is modular (see § Subsection 1.1.3). We thus have

THEOREM 4.37. *Let S be a regular semigroup. Then the lattice (under inclusion) of all idempotent separating congruences on S is a complete modular sublattice of the lattice $\mathfrak{L}_S$ of all congruences on S with $\mathbf{1} = \mu(S)$.* □

Example 4.10: Let $M = X^*$. Then M is a semigroup with only one idempotent and so, any congruence on M is idempotent separating. Since any monoid is a homomorphic image of a free monoid, there are non-trivial congruences on M. But the $\mathscr{H}$ relation on M is the identity relation and so, no non-trivial congruence on M satisfies condition (2) of Theorem 4.32.

Example 4.11: Let $S = \mathscr{T}_X$ be the semigroup of all transformations of a set X. Then S is regular (see Examples 2.10 and 2.15). Suppose that $e \in \boldsymbol{E}(S)$. If $\alpha \in \mu(e)$, then $\alpha \mathscr{H} e$ and so $\operatorname{Im}\alpha = \operatorname{Im} e = Y$ and $\pi_\alpha = \pi_e = \pi$ (say). By Proposition 4.34, $g\alpha = \alpha g$ for all $g \;\omega\; e$. Now, $g \;\omega\; e$ if and only if $\operatorname{Im} g \subseteq Y$ and $\pi_g \supseteq \pi_e$. So, for any $x \in \operatorname{Im} g$, $xg\alpha = x\alpha = x\alpha g$ which implies that $x\alpha \in \operatorname{Im} g$ for all $x \in \operatorname{Im} g$. Now let $x \in Y$ and let c_x be the constant transformation with value x. Then c_x is an idempotent with $c_x \;\omega\; e$ and $\operatorname{Im} c_x = \{x\}$. It follows from the remarks above that $x\alpha = x$. This is true for all $x \in Y$ and so $\alpha = e$. Hence $\mu(e) = \{e\}$ and so μ is the identity congruence on S; that is, S is fundamental.

Example 4.12: Let $S = \mathscr{L}\mathscr{T}(V)$ be the semigroup of all linear transformations of a vector space V over a field $\Bbbk$. Then S is regular (see Examples 2.11 and 2.15). Suppose that $e \in \boldsymbol{E}(S)$. If $\alpha \in \mu(e)$, then $\alpha \mathscr{H} e$ and so $\operatorname{Im}\alpha = \operatorname{Im} e = U$ and $\boldsymbol{N}(\alpha) = \boldsymbol{N}(e) = N$ (say). By Proposition 4.34, $g\alpha = \alpha g$ for all $g \;\omega\; e$. Suppose that $e \neq 0$ and let $v \in U$, $v \neq 0$. Then for any $g \;\omega\; e$, $g \neq 0$ and $v \neq 0 \in \operatorname{Im} g$, $(v)g\alpha = (v)\alpha = (v)\alpha g$ and so, $v\alpha \in \operatorname{Im} g$ for all $v \in \operatorname{Im} g$. If $\dim U = 1$, so that $U = \langle v \rangle$ for some $v \in V$, there is $k \in \Bbbk^*$ with $v\alpha = kv$ since $v\alpha \in U$ and α is a linear isomorphism of U onto itself. Then for all $w \in U$, $w = k'v$ and so, $w\alpha = k'(v\alpha) = k'kv = k(k'v) = kw$. Thus $\alpha = ke$. Here $\Bbbk^*$ denote the set of all non-zero elements of $\Bbbk$. If $\dim U > 1$ we can see similarly that for any $v \neq 0 \in U$ there is $k_v \in \Bbbk^*$ such that $v\alpha = (k_v)v$. Hence if $v, w \in U$ are linearly independent, we have

$$(v + w)\alpha = v\alpha + w\alpha = (k_v)v + (k_w)w = k_{v+w}(v + w)$$

and the linear independence gives $k_v = k_{v+w} = k_w$. Thus there is $k \in \Bbbk^*$ such that $\alpha = ke$. It follows that for all $e\boldsymbol{E}(S)$ with $e \neq 0$, $\mu(e) = \Bbbk^* e$. By Equation (4.32), if $x \in S$ and $e \in E(R_x)$, we have

$$\mu(x) = \Bbbk^* ex = \Bbbk^* x.$$

congruence!primitive – semigroup!–, categorical at 0 ideal!categorical –

This completely determine the congruence μ on S. This equality has a nice geometric interpretation. Note that $S = \mathscr{L}\mathscr{T}(V)$ is a vector space over $\Bbbk$. Then the congruence class $\mu(x)$ can be identified with the projective point of x (or line joining x and 0 in S).

4.2.3 Primitive congruences on regular semigroups

In this subsection, to avoid repeating, by a primitive regular semigroup, we mean a primitive regular semigroup with zero. Note that by Theorem 4.24, primitive regular semigroup with out zero is completely simple. Therefore a primitive regular semigroup with out zero will be referred to explicitly as completely simple semigroup. Also, recall from § Subsection 2.1.1 that, given a semigroup S, we write $S = S^0$ to mean that the semigroup S has zero 0.

A congruence ρ on a semigroup S is called a *primitive congruence* if S/ρ is a primitive semigroup; ρ is a *completely simple congruence* if S/ρ is completely simple. Recall § Subsection 2.7.2 that a congruence ρ on a semigroup $S = S^0$ is 0-restricted if $\rho(0) = \{0\}$.

We say that a semigroup $S = S^0$ is *categorical at* 0 if S satisfy the condition

$$xyz = 0 \Rightarrow \text{either} \quad xy = 0 \quad \text{or} \quad yz = 0 \tag{4.12a}$$

for $x, y, z \in S$. An ideal I in S (not necessarily having 0) is called a *categorical ideal* if I satisfies the condition

$$xyz \in I \Rightarrow \text{either} \quad xy \in I \quad \text{or} \quad yz \in I \tag{4.12b}$$

for $x, y, z \in S$. It is clear that I is a categorical ideal in S if and only if the Rees quotient S/I is categorical at 0.

Recall that given a surjective homomorphism $f : S \to T$ there is a $\vee$-homomorphism $f^* : \mathfrak{L}_S \to \mathfrak{L}_T$ and a lattice isomorphism $f_* : \mathfrak{L}_T \to [\boldsymbol{\kappa} f, \mathbf{1}]$ (defined by Equation (2.19)) such that $f_* \circ f^* = 1_{\mathfrak{L}_T}$ (see Proposition 2.8). We use these in the following statement.

THEOREM 4.38. *Let σ be a primitive congruence on the semigroup S. Then $I = \sigma(0)$ is a categorical ideal in S and $q_I^*(\sigma)$ is a 0-restricted primitive congruence on S/I, where $q_I : S \to S/I$ is the quotient homomorphism. Conversely if I is a categorical ideal in S and ρ is a 0-restricted primitive congruence on S/I, then $(q_I)_*(\rho)$ is a primitive congruence on S such that $(q_I)_*(\rho)(0) = I$.*

Proof. Let $\phi = \sigma^\# : S \to S/\sigma = T$ be the quotient homomorphism. If σ is a primitive congruence, then T is a primitive semigroup. Let $u \in S$ and

$a \in I = \sigma(0)$. Then $(ua)\phi = u\phi a\phi = u\phi 0 = 0$ since $a\phi = \sigma(0)$ is the zero of S/σ. Hence $ua \in \sigma(0) = I$. Similarly $au \in I$ and so I is an ideal. Suppose that $a, b, c \in S$ and $ab, bc \notin I$. Then $ab\phi = a\phi b\phi$ is a non-zero element of a primitive semigroup and so, by Theorem 4.24, $a\phi$ and $b\phi$ are elements of the non-zero $\mathscr{D}$-class of a completely 0-simple semigroup whose product is not zero. Hence by Theorem 2.66(1) and Equation (2.48a), the trace product $a\phi * b\phi$ exists. Similarly, trace product $b\phi * c\phi$ also exists and so the trace product $a\phi * b\phi c\phi = (abc)\phi$ exists and is not zero. Therefore $abc \notin I$. Hence I is a categorical ideal. Let $\rho = q_I^*(\sigma)$. Since the Rees congruence (§ Subsection 2.2.1) $\rho_I \subseteq \sigma$, by Proposition 2.8(c), S/σ and $(S/I)/\rho$ are isomorphic and so, ρ is a primitive congruence on S/I. To show that ρ is 0-restricted, let $u \in \rho(0)$ and let $a \in S$ with $aq_I = u$. Again by Proposition 2.8(c), $a\phi = 0$ since $u\sigma'^{\#} = 0$. Thus $a \in I$ and so $u = aq_I = 0$. Therefore ρ is 0-restricted.

Conversely, let I be a categorical ideal and ρ be a 0-restricted primitive congruence on S/I. If $\sigma = q_{I*}(\rho)$, then, it follows from Proposition 2.8(c) as above, that σ is a primitive congruence on S with $\sigma(0) = I$. □

The theorem above shows that primitive congruences on a semigroups are determined by categorical ideals in S and 0-restricted primitive congruences on semigroups that are categorical at 0. We proceed to study the later congruences on regular semigroups.

Recall from Proposition 2.7(a) that, given any relation ρ on a semigroup S, $\rho^{(c)}$ denotes the smallest congruence containing ρ (that is, the congruence generated by the relation ρ).

THEOREM 4.39. *Let $S = S^0$ be a regular semigroup which is categorical at* 0 *and let*

$$\beta(S) = \{(x, y) : \text{for some } z \in S - \{0\}, \quad z \leq x,\ z \leq y\} \cup \{(0, 0)\} \tag{4.13}$$

and let

$$\beta_0(S) = \beta(S)^{(c)}.$$

Then $\beta_0(S)$ is the finest 0*-restricted primitive congruence on S.*

Proof. For brevity, let $\beta = \beta(S)$ and $\beta_0 = \beta_0(S)$. We first show that β_0 is a 0-restricted primitive congruence on S. Let $\phi = \beta_0^{\#} : S \to S/\beta_0 = T$ be the quotient homomorphism. Suppose that $\bar{x} \leq \bar{y}$, $\bar{x} \neq 0$ in T. Then by Theorem 4.20, for each $y \in S$ with $y\phi = \bar{y}$ we can find $x \leq y$, $x \neq 0$ such that $x\phi = \bar{x}$. Then $(x, y) \in \beta$ and so, $\bar{x} = x\phi = y\phi = \bar{y}$. Hence T is a primitive semigroup. To show that β_0 is 0-restricted, we must show that, if $(u, 0) \in \beta_0$, then $u = 0$. Now, since β is reflexive and symmetric, $\beta_0 = \beta^{(c)}$ is the transitive closure of the smallest compatible relation

directed subset

$$\beta^c = \{(axb, ayb) : a, b \in S^1 \quad \text{and} \quad (x, y) \in \beta\}. \tag{4.14}$$

containing β (see the proof of Proposition 2.7). Hence $(u, 0) \in \beta_0$ implies there is a finite sequence $u_0 = u, u_1, \ldots, u_n = 0$ in S such that $(u_{i-1}, u_i) \in \beta^c$ for $i = 1, \ldots, n$. Hence, by induction, the desired conclusion will follow if we show that $(u, 0) \in \beta^c$ implies $u = 0$. By Equation (4.14), if $(u, 0) \in \beta^c$, then there exist $(x, y) \in \beta$ and $a, b \in S^1$ such that $u = axb$ and $0 = ayb$. If $(x, y) = (0, 0)$, then clearly, $u = 0$. Otherwise, there is $z \neq 0$ such that $z \leq x$ and $z \leq y$. Since S is categorical at 0, $ayb = 0$ implies either $ay = 0$ or $yb = 0$. Assume that $ay = 0$. Since $z \leq x, z \leq y$, by Proposition 4.3, there exists $f, g \in E(L_z)$ with $z = xf = yg$. Then $axf = az = ayg = 0$ and since S is categorical at 0 and $xf = z \neq 0$, we have $ax = 0$. Therefore $u = axb = 0$. If $yb = 0$, we can similarly show that $u = 0$. We have thus shown that β_0 is a 0-restricted primitive congruence on S.

Now let σ be any 0-restricted primitive congruence on S and let $(x, y) \in \beta$. If $(x, y) = (0, 0)$, clearly $(x, y) \in \sigma$. Otherwise there is $z \neq 0$ with $z \leq x$ and $z \leq y$. Let $\psi = \sigma^\# : S \to S/\sigma$ be the quotient homomorphism. Since σ is 0-restricted, $z\psi \neq 0, z\psi \leq x\psi$ and $z\psi \leq y\psi$. Since S/σ is primitive, this implies $x\psi = y\psi$ and so $(x, y) \in \sigma$. Hence $\beta \subseteq \sigma$. Since β_0 is the smallest congruence containing β, we have, $\beta_0 \subseteq \sigma$. □

Many authors have noted that the relation $\beta(S)$ is the finest 0-restricted primitive congruence on an inverse semigroup which is categorical at 0 (see for example, Hall [1968], McAlister [1968]). This is not true for arbitrary regular semigroups. We show below that under a mild restriction on the biordered set $\boldsymbol{E}(S)$ of a regular semigroup $S = S^0$ which is categorical at 0, $\beta(S)$ is a congruence and the classical result mentioned above follows as a consequence.

In what follows by a *directed subset* of a partially ordered set X we mean a subset Y of X with the property that for all $x, y \in Y$, there is $z \in Y$ with $z \leq x$ and $z \leq y$. A directed subset of a semigroup is a subset which is directed with respect to the natural partial order. Again, for brevity, we write β for $\beta(S)$ and β_0 for $\beta_0(S)$, if there is no ambiguity.

PROPOSITION 4.40. *For a regular semigroup $S = S^0$ which is categorical at 0, the following statements are equivalent.*

(a) For every $e \in \boldsymbol{E}(S) - \{0\}$, $\omega(e) - \{0\}$ is directed.

(b) β is an equivalence relation.

(c) $\beta = \beta_0$.

Proof. (a) $\Rightarrow$ (b): Clearly β is reflexive and symmetric. To prove transitivity assume that $(x,y),(y,z) \in \beta$. Then either $x = y = z = 0$ or none of them is 0. In the former case, clearly, $(x,z) \in \beta$. In the latter case, there exist $u_1, U_2 \in S - \{0\}$ with $u_1 \leq x$, $u_1 \leq y$, $u_2 \leq y$ and $u_2 \leq z$. Choose $f \in E(R_y)$. Then, by Proposition 4.3, there exist $e_i \in E(R_{u_i}) \cap \omega(f)$ such that $u_i = e_i y$. Since $e_i \mathscr{R} u_i \neq 0$, $e_i \neq 0$ for $i = 1,2$. By (a), there exists $g \in \omega(f) - \{0\}$ such that $g\ \omega\ e_i$, $i = 1,2$. Then $g \mathscr{R} gy \neq 0$ and $gy = ge_1y = gu_1 \leq u_1 \leq x$. Similarly, $gy \leq u_2 \leq z$. Hence by the definition of β, $(x,z) \in \beta$.

(b) $\Rightarrow$ (c): We must show that β is compatible. Let $x,y,c \in S$ with $x \leq y$ and $x \neq 0$. Choose $y' \in \mathscr{V}(y)$ and let $f = yy'$, $f' = y'y$. Then by Proposition 4.3(2), there is $e \in \omega(f)$ with $x \mathscr{R} e$ and $x = ey = ye'$ where $e' = y'ey\ \omega\ f'$. If $cy = 0$, then $cx = cfx = (cy)y'x = 0$. Conversely, if $cx = 0$, then $cye' = 0$. Since S is categorical at 0 and $ye' = x \neq 0$, $cy = 0$. Therefore when either cx or cy is zero, the other is zero and $(cx,cy) \in \beta$. Next assume that $cx \neq 0 \neq cy$. Let $g \in E(L_c)$, $h \in \mathscr{S}(g,f)$ and $k \in \mathscr{S}(g,e)$. Then, by (Theorem 3.7), $cy = (ch) * (hy)$ and it is easy to see that $h' = y'hy \in E(L_{cy}) \cap \omega(f')$. Similarly, $k' = y'ky \in E(L_{cx}) \cap \omega(f')$. By Equation (4.13), every non-zero idempotent in $\omega(f')$ is β-related to f'. Since $e', h', k' \in \omega(f') - \{0\}$, we have $e'\beta h'\beta k'$ by (b). It follows from Equation (4.13) that the set $F = \omega(e') \cap \omega(h') \cap \omega(k') - \{0\} \neq \emptyset$. Choose $t \in F$. Then $z = cyt \leq cy$ and $z = cyt = cye't = cxt \leq cx$ by Proposition 4.3(3). Since $z \mathscr{L} k \neq 0$, $z \neq 0$. Hence by Equation (4.13), $(cx,cy) \in \beta$.

Let $(u,v) \in \beta$ and $c \in S$. If $u = v = 0$ then clearly $(cu,cv) \in \beta$. Otherwise, there is $z \neq 0$ such that $z \leq u$ and $z \leq v$. Then by the above $(cz,cu),(cz,cv) \in \beta$. Since β is an equivalence relation we have $(cu,cv) \in \beta$. In a similar way, we can prove that $(uc,vc) \in \beta$. Hence β is a congruence and so $\beta = \beta_0$.

(c) $\Rightarrow$ (a): Let $e \in \boldsymbol{E}(S) - \{0\}$, $f,g \in \omega(e) - \{0\}$. Then we have $(f,e),(g,e) \in \beta$ and since β is a congruence $(f,g) \in \beta$. Then by Equation (4.13), there is $z \in S$ such that $z \neq 0$, $z \leq f$ and $z \leq g$. Then by Proposition 4.7(a), $z \in \boldsymbol{E}(S)$. This implies that $\omega(e) - \{0\}$ is directed. □

The fact that on an inverse semigroup $S = S^0$ which is categorical at 0, β is the finest 0-restricted primitive congruence is a consequence of the following more general result.

COROLLARY 4.41. *Let $S = S^0$ be a locally inverse semigroup which is categorical at* 0. *Then β is the finest* 0*-restricted primitive congruence on S.* □

Proof. Let $e \in \boldsymbol{E}(S) - \{0\}$, $f,g \in \omega(e) - \{0\}$. Since S is categorical at 0, $0 = fg = feg$ implies either $fe = f = 0$ or $eg = g = 0$. Hence $fg \neq 0$. Since $\omega(e)$ is a semilattice, $fg\ \omega\ f$ and $fg\ \omega\ g$. Hence $\omega(e)$ is directed and the result follows from the theorem above. □

Theorem 4.39 applies to regular semigroups $S = S^0$ which is categorical at 0. For regular semigroups not necessarily having 0, we have the following weaker form of this result.

THEOREM 4.42. *Let I be a categorical ideal in the regular semigroup S and let $q_I : S \to S/I$ denote the quotient map. Let*

$$\beta_I(S) = (q_I)_* (\beta_0(S/I)) \tag{4.15}$$

where $(q_I)_*$ is the lattice isomorphism of Equation (2.19) determined by q_I. Then $\beta_I = \beta_I(S)$ is the finest primitive congruence on S such that $\beta_I(0) = I$.

Proof. By Theorem 4.38, β_I is a primitive congruence on S such that $\beta_I(0) = I$. Let σ be any other primitive congruence on S with $\sigma(0) = I$. Then by Theorem 4.38, $\sigma' = (q_I)^*(\sigma)$ is a 0-restricted primitive congruence on $T = S/I$ and so $\beta_0(T) \subseteq \sigma'$. Hence, using Proposition 2.8(b), we have

$$\beta_I = (q_I)_* (\beta_0(T)) \subseteq (q_I)_* (\sigma')) = \sigma.$$ □

Finally, we apply Theorem 4.39 to obtain the finest completely simple congruence on a regular semigroup S. Notice that the congruence ρ below is trivial if S has 0.

THEOREM 4.43. *Let S be a regular semigroup with out 0. Let*

$$\rho = \rho(S) = \{(x, y) \in S \times S : z \leq x,\ z \leq y \quad \text{for some} \quad z \in S\}. \tag{4.16}$$

Then $\rho(S)^{(c)}$ is the finest congruence on S such that $S/\rho(S)^{(c)}$ is completely simple. $\rho(S) = \rho(S)^{(c)}$ if and only if every ω-ideal in $\mathbf{E}(S)$ is directed. In particular, for a locally inverse semigroup S with out 0, we have $\rho(S) = \rho(S)^{(c)}$.

Proof. Since S does not have 0, S^0 is a regular semigroup which is categorical at 0 and so $\beta_0(S^0)$ is the finest 0-restricted primitive congruence on S^0. Then $T = S^0/\beta_0(S^0)$ is a primitive regular semigroup whose non-zero elements T' form a subsemigroup. Now the natural partial order on T' is the restriction of the natural partial order of T to T' by Corollary 4.4. Since T is primitive, it follows that T' is a primitive semigroup with out 0 and hence, by Theorem 4.24, T' is completely simple. Now $\rho = \rho(S)$ is the restriction of the relation $\beta(S^0)$ defined by Equation (4.13) to S. Since $\beta_0(S^0)$ is 0-restricted, it is clear that $\rho' = \rho^{(c)}$ is the restriction of $\beta_0(S^0)$ to S and $T' = S/\rho'$. Hence ρ' is a congruence on S such that S/ρ' is completely simple. Moreover if σ is any congruence on S such that S/σ

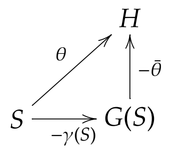
$\gamma(S)$:the universal group homomorphism on S
universal group homomorphism on S

is completely simple, then $\sigma' = \sigma \cup \{(0,0)\}$ is a 0-restricted primitive congruence on S^0 and hence $\beta_0(S^0) \subseteq \sigma'$. Hence $\rho' \subseteq \sigma$. The remaining statements readily follow from Proposition 4.40 and Corollary 4.41. □

If ρ is any group congruence on a regular semigroup S, then the identity ρ-class contain $\boldsymbol{E}(S)$. Therefore intersection of any set of group congruences on S is a group congruence. Therefore S has the finest group congruence $\boldsymbol{\kappa}\gamma(S) = \boldsymbol{\kappa}\gamma$ and let

$$\gamma(S) : S \to G(S) = S/\boldsymbol{\kappa}\gamma \tag{4.17}$$

denote the quotient homomorphism. The minimality of the congruence $\boldsymbol{\kappa}\gamma$ implies that the homomorphism $\gamma(S) = \gamma$ has the following universal property: Given any homomorphism $\theta : S \to H$ to a group H, there is a unique homomorphism $\bar{\theta} : G(S) \to H$ such that the following diagram commute:



(4.18)

This is an immediate consequence of the third isomorphism theorem (see Theorem 2.5). The homomorphism $\gamma(S)$ will be called the universal group homomorphism on S. Furthermore, since by Theorem 2.43, homomorphic image of an inverse semigroup is inverse and since a completely simple inverse semigroup is a group, the relation $\rho(S)$ on an inverse semigroup S is the finest group congruence on S. Thus, from the remarks above and properties of natural partial order on inverse semigroups (Theorem 4.24), we have:

Proposition 4.44. *Every regular semigroup S has the finest group congruence $\boldsymbol{\kappa}\gamma$. Let $\gamma(S) : S \to G(S)$ be the quotient homomorphism. Then $\gamma = \gamma(S)$ has the universal property that given any homomorphism $\theta : S \to H$ to a group H there is a unique homomorphism $\bar{\theta} : G(S) \to H$ making the diagram 4.18 commute. Further, if S is an inverse semigroup then*

$$\boldsymbol{\kappa}\gamma(S) = \rho(S) = \{(x, y) \in S \times S : ex = ey \quad \text{for some} \quad e \in \boldsymbol{E}(S)\} \tag{4.19}$$

where $\rho(S)$ is the relation defined by Equation (4.16). □

Example 4.13: We give an example to show that a regular semigroup S may not have the finest primitive congruence; in particular, the congruence β_0 is not the smallest. For let $E = \{e, f, g\}$ be the semilattice with $ef = g$ and $F = E^0 = \{e, f, g, 0\}$ be the semilattice obtained by adjoining a zero to E. Then $\beta = \beta_0$ is the relation with the partition $\{E, \{0\}\}$. Also $I = \{g, 0\}$ is a categorical ideal in F and the Rees congruence ρ_I is a primitive congruence on F (so that $\rho_I = \beta_I$). Clearly, in this case, β_0 and ρ_I are not comparable.

4.3 DECOMPOSITIONS OF SEMIGROUPS

decomposition!band – decomposition!semilattice – congruence!band – congruence!semilattice –

Decomposing a given semigroup S into semigroups of knowm type, say T, is very useful in getting an insight into the structure of the semigroup S. Often it is also an effective method of determining the structure of a semigroup S relative to the structure of semigroups of type T. For example, if S is completely simple, then by Theorem 2.65 S has a decomposition into groups and by Corollary 2.80, it structure is determined relative to groups. In this section we consider two such decompositions: the *band decomposition* and *semilattice decomposition* of semigroups.

4.3.1 Band and semilattice decompositions

Let S be a semigroup and let

$$\mathcal{B} = \{S_\alpha : \alpha \in B\} \quad \text{where} \quad S_\alpha \cap S_\beta = \emptyset \quad \text{if} \quad \alpha \neq \beta \tag{4.20}$$

be a decomposition of S into subsets S_α (see Equation (1.9b)). It is called a *band decomposition* if and only if $\mathcal{B}$ is the decomposition associated with a *band congruence* Equation (1.9b); that is, a congruence ρ such that $B = S/\rho$ is a band (idempotent semigroup). If this is the case it is clear that the partition class S_α is a subsemigtoup of S for each $\alpha \in B$. Similarly, the decomposition $\mathcal{Y}$ is a *semilattice decomposition* if it is the decomposition associated with a *semilattice congruence*. In general, we shall say that S is a band [semilattice] $\mathcal{B}$ of semigroups S_α if there is a congruence σ on S such that S/σ is isomorphic to $\mathcal{B}$ and for each $\alpha \in \mathcal{B}$, the σ-class

$$\alpha(\sigma^\#)^{-1} = \{x \in S : x\sigma^\# = \alpha\}$$

is isomorphic to S_α.

THEOREM 4.45. *The decomposition $\mathcal{B} = \{S_\alpha : \alpha \in B\}$ of the semigroup S is a band decomposition if and only if*

(A) S_α is a subsemigroup of S for each $\alpha \in B$;

(B) for $\alpha, \beta \in B$, there is a unique $\gamma \in B$ such that

$$S_\alpha S_\beta \subseteq S_\gamma.$$

Proof. Let $\mathcal{B}$ satisfy the given conditions (A) and (B). Since the subsets of S in $\mathcal{M}$ are pairwiase disjoint, the relation

$$\rho = \rho_{\mathcal{B}} = \{(x, y) : x, y \in S_\alpha;\ \alpha \in B\}$$

is an equivalence relation on S. Let $(x, y) \in \rho$ and $z \in S$. Then there exists $\alpha, \beta \in B$ such that $x, y \in S_\alpha$ and $z \in S_\beta$. By (B), there exists $\gamma \in B$ with $xz, yz \in S_\gamma$ which implies that $(xz, yz) \in \rho$. Similarly, $(zx, zy) \in \rho$. Hence ρ is a congruence on S. The condition (A) implies that every element in $B = S/\rho$ is an idempotent. Therefore B is a band and so $\mathcal{B}$ is a band decomposition. Conversely, if $\mathcal{B}$ is a band decomposition, and if $\rho = \rho_{\mathcal{B}}$ is the associated band congruence on S, then every element in $B = S/\rho$ is an idempotent and so (A) holds. Given $\alpha, \beta \in B$, let $\gamma = \alpha\beta$. Since $\phi = \rho^\# : S \to B$ is a homomorphism, for any $x \in S_\alpha$ and $y \in S_\beta$,

$$\alpha\beta = (x\phi)(y\phi) = (xy)\phi.$$

Since $xy \in S_{\alpha\beta} = S_\gamma$, we have

$$S_\alpha S_\beta \subseteq S_{\alpha\beta} = S_\gamma.$$

Hence (B) also holds. □

Notice that the condition (B) implies that for $\alpha, \beta \in B$, there is $\gamma, \delta \in B$ such that

$$S_\alpha S_\beta \subseteq S_\gamma \quad \text{and} \quad S_\beta S_\alpha \subseteq S_\delta.$$

It is clear that the band B will be commutative if and only if we always have $\gamma = \delta$. Since a semilattice is a commutative band, we have the following:

COROLLARY 4.46. *The decomposition $\mathcal{B}$ of S is a semilattice decomposition if and only if $\mathcal{B}$ satisfies condition (A) of the theorem above and the following:*

(C) for $\alpha, \beta \in B$, there is a unique $\gamma \in B$ such that

$$S_\alpha S_\beta \subseteq S_\gamma \quad \text{and} \quad S_\beta S_\alpha \subseteq S_\gamma.$$

It may be noted that a decomposition of a semigroup S into subsemigroups need not be a band decomposition (see Example 4.14 below). Also, any semigroup has at least one band decomposition since the universal congruence is trivially a band congruence. If $\{\rho_i\}$ is any set of band congruences on S, then $\sigma = \cap_i\{\rho_i\}$ is a band congruence. For, if $x \in S$, then $\sigma(x) = \cap_i\{\rho_i(x)\}$ is a subsemigroup of S since each $\rho_i(x)$ is a subsemigroup. It follows that every element of S/σ is an idempotent and so, σ is a band congruence. In particular every semigroup has a finest band decomposition. Similarly, given any set of semilattice congruences $\{\sigma_i\}$ on S, then $\sigma = \cap_i\{\sigma_i\}$ is a semilattice congruence on S. For if $x, y \in S$, then $xy, yx \in \sigma_i(xy)$ for every i since each σ_i is a semilattice congruence. Hence

$$xy, yx \in \cap_i\{\sigma_i(xy)\} = \sigma(xy)$$

and so, σ is a semilattice congruence. It follows that every semigroup has the finest semilattice decomposition. Thus we have:

THEOREM 4.47. *Every semigroup S has the finest band decomposition as well as the finest semilattice decomposition.* □

$a \mid b$: a divides b
semigroup!archimedean –
semigroup!completely regular –

The theorem above gives the existance of the finest band decomposition and the finest semilattice decomposition of a semigroup. Note that these may turn out to be trivial; thus for example, the finest semilattice decomposition of a simple semigroup is trivial. However, in particular cases, such decompositions turn out to be very useful— see Example 4.15 below. Next section discuss another important example.

Example 4.14: Let $S = \mathbb{Z}_2 \cup \{e, f\}$ where $\mathbb{Z}_2 = \{1, u\}$ is the group of order 2. Define multiplication in S so that 1 is the identity, e and f are $\mathscr{R}$-related idempotents and

$$eu = f, \; fu = e, \; ue = e \text{ and } uf = f.$$

Then S can be shown to be a semigroup in which every $\mathscr{H}$-class is a group and so S has a decomposition into groups. However,

$$\mathbb{Z}_2 \cdot H_e = \{e, f\}$$

and so does not satisfy the condition (B) of Theorem 4.45. Hence the decomposition of S into groups does not give a band of groups.

Example 4.15: Let S be a commutative semigroup. Given $a, b \in S$, we shall say that a *divide*s b, written $a \mid b$, if $ax = b$ for some $x \in S^1$. Define the relation η on S by:

$$x \; \eta \; y \iff \text{for some } m, n \geq 1, \quad a \mid b^m, \; b \mid a^n. \tag{4.21}$$

Clearly, η is a reflexive and transitive relation. If $a \mid b^m$ and $b \mid c^p$, then $ax = b^m$ and $by = c^p$ for $x, y \in S^1$ and so, $axy^m = (by)^m = c^{mp}$ and so, $a \mid c^{mp}$. It follows that η is an equivalence relation. Further, if $ax = b^m$, then for any $z \in S^1$, $(az)u = (bz)^m$ if $u = xz^{m-1}$. Hence if $(a, b) \in \eta$, then for all $z \in S^1$, $(az, bz) \in \eta$; thus η is a congruence on S. Evidently $a \; \eta \; a^2$ for any $a \in S$. Since S is commutative, this implies that η is a semilattice congruence on S. If ρ is any semilattice congruence on S, and if $a \mid b^n$, then we must have $b\rho^\# = (b^n)\rho^\# \leq a\rho^\#$. It follows that if $(a, b) \in \eta$, then $(a, b) \in \rho$. Therefore η is the smallest semilattice congruence on S; consequently, S/η is the maximum semilattice homomorphic image of S.

A commutative semigroup S is said to be *archimedean* if for any $a, b \in S$, there exists integers $m, n \geq 1$ such that $a \mid b^m$ and $b \mid a^n$; that is the congruence η on S is the universal congruence. Thus the congruence η on S gives a decomposition of S into maximal archimedeam subsemigroups. These subsemigroups are called *archimedean components* of S. Therefore any commutative semigroup S has a unique decomposition into archimedean components and this decomposition is the finest semilattice congruence on S.

4.3.2 Completely regular semigroups

A semigroup S is said to be *completely regular* if S is a union of groups. If S is completely regular, then each $x \in S$ is contained in a subgroup of S and so H_x is a group. Therefore in a completely regular semigroup every $\mathscr{H}$-class of S is a group and it is a *disjoint union* of groups. Thus a completely regular semigroup has a decomposition into groups. In particular, if e and f are $\mathscr{D}$-related idempotents in S, then both $L_e \cap R_f$ and $R_e \cap L_f$ contains idempotents. Hence it follows from the definition of solid biordered sets (see Subsection 3.4.2) that the biordered set of a completely regular semigroup S is solid.

Moreover, it is clear that every completely regular S semigroup is regular; in fact, every $x \in S$ has a unique group inverse x^* (see Equation (2.40)). The converse also holds; that is, a semigroup S is completely regular if every element $x \in S$ has a group inverse in the sense of Subsection 2.6.2. For convenience of later reference, we summerise the discussion as:

PROPOSITION 4.48. *The following statements are equivalent for a semigroup S.*

(a) *S is completely regular;*

(b) *every $\mathscr{H}$-class of S is a group;*

(c) *S is a disjoint union of groups;*

(d) *every $x \in S$ has a group inverse.*

In particular, when S is completely regular, $\mathbf{E}(S)$ is a solid biordered set. □

The equivalent conditions above are simple consequences of definition of completely regular semigroups and they do not yield any significant insight into the structure of these semigroups. The next theorem provide some illumination in this direction. Recall that $\mathbf{J} = \mathbf{J}_S$ (see Subsection 2.1.1) denote the partially ordered set of all principal ideals of S under inclusion.

THEOREM 4.49. *The following statements concerning a semigroup S are mutually equivalent.*

(a) *S is completely regular.*

(b) *S is completely semisimple and every $\mathscr{D}$-class of S is a subsemigroup of S.*

(c) *S is completely semisimple and the Green's relation $\mathscr{D}$ is a congruence.*

(d) *The partially ordered set $\mathbf{J}$ is a semilattice with respect to intersection and S is a semilattice $\mathbf{J}$ of completely simple semigroups.*

Proof. **(a)** $\iff$ **(b)**: Suppose that D is a $\mathscr{D}$-class of S and $a, b \in D$. Then $L_a \cap R_b$ is an $\mathscr{H}$-class in S. So if (a) holds, by Proposition 4.48(b), $L_a \cap R_b$ contains an idempotent. Therefore, by Equation (2.48a), the trace product $a * b = ab$ exists. This implies that $ab \in D$ and so, D is a subsemigroup of S. Suppose that $e, f \in E(D)$ and $e \;\omega\; f$. Then by (a), $L_e \cap R_f$ contains an idempotent g. Since $e \;\omega\; f$, $e \;\omega^r\; g$; also $e \;\mathscr{L}\; g$. Hence $e \;\omega\; g$ which implies $e = g$. Similarly, from

$e \mathscr{R} f$ and $e \,\omega\, f$, we have $e = f$. This proves by Lemma 2.86 that S satisfies the condition $\boldsymbol{M}_E^*$ and so, by Theorem 2.87 S is completely semisimple. Conversely if (b) holds, then by Theorem 2.87 each $\mathscr{D}$-class D is a regular semigroup in which every element is minimal with respect to the natural partial order and so D is a primitive regular semigroup with out 0. Hence, by Theorem 4.24, D is completely simple and so, S satisfies (a).

(a) $\iff$ (c): Suppose that S satisfies (a). By (b), S is completely semisimple and so, to prove (c), it is sufficient to show that $\mathscr{D}$ is a congruence. Let $a \mathscr{D} b$, $c \in S$ and $h \in \mathscr{S}(e, f)$ where $e \in E(L_c)$ and $f \in E(R_a)$. Then by Theorem 3.7, $h, ch, ha \in D_{ab}$, $ch \leq c$ and $ha \leq a$. Then by Proposition 4.12, there exists $b_1 \in S$ with $ha \mathscr{D} b_{\leq}b$. By (a), the $\mathscr{H}$-class $L_{ch} \cap R_{b_1}$ contains an idempotent, say k. If $g \in E(R_b)$, then, by Proposition 4.3(2), there exists $g' \in E(R_{b_1})$ such that $g' \,\omega\, g$ and $b_1 = g'b$. Hence $k \,\omega^r\, g$. Similarly, $k \,\omega^l\, e$. Therefore $k \in M(e, g)$ and so, it follows from Theorem 3.7 that $ckb \mathscr{D} k \mathscr{D} cb$. Since $ckb = (ckc')(cb)$ for any $c' \in \mathscr{V}(c)$, we have

$$J(ca) = J(ckb) \subseteq J(cb). \quad \text{Similarly,} \quad J(cb) \subseteq J(ca).$$

Hence $ca \mathscr{J} cb$. Since S is completely semisimple, $ca \mathscr{D} cb$ by Corollary 2.88. It can be shown, in a similar way, that $ac \mathscr{D} bc$. Therefore $\mathscr{D}$ is a congruence on S. Conversely if S satisfies (c) the fact that S is completely semisimple implies that the congruence $\mathscr{D}$ satisfies the condition

$$x \leq y \quad \text{and} \quad x \mathscr{D} y \Rightarrow x = y.$$

So, by Theorem 4.22, every $\mathscr{D}$-class of S is a completely simiple subsemigroup of S. Thus S satisfies (a).

(a) $\iff$ (d): Trivially (d) $\Rightarrow$ (a). Suppose that (a) holds. Then by (c) $\mathscr{D}$ is a congruence on S. Now let $x, y \in S$ and $h \in \mathscr{S}(e, f)$ where $e \in E(L_x)$ and $f \in E(R_y)$. By (a), there is an idempotent $k \in R_{xh} \cap L_{hy}$. Since $xh \leq x$ and $hy \leq y$, as in the last paragraph, we see that $k \in M(f', e')$ where $e' \in E(R_x)$ and $f' \in E(L_y)$. Then again by Theorem 3.7, we have

$$J(xy) = J(ykx) \subseteq J(yx). \quad \text{Similarly,} \quad J(yx) \subseteq J(xy).$$

Thus $xy \mathscr{J} yx$ and by (b), $xy \mathscr{D} yx$. Hence by Theorem 4.45, $\mathscr{D}$ is a semilattice congruence on S. Since S is completely semisimple, by Corollary 2.88 the map $\phi : D_x \to J(x)$ is a bijection of $S/\mathscr{D}$ onto $\mathbf{J}$ (see Subsection 2.6.1). To prove (d), it is sufficient to show that ϕ is an order isomorphism. If $D_x \leq D_y$ in $S/\mathscr{D}$, then $D_x = D_x D_y$ and since the map $a \mapsto D_a$ is a homomorphism, we have $D_x = D_{xy}$.

Z(S):center of S
semigroup!center of the –

Hence $J(x) = J(xy) \subseteq J(y)$. Conversely, if $J(x) \subseteq J(y)$, then $x = uyv$ for $u, v \in S^1$ and so,

$$D_x = D_u D_y D_v \leq D_y$$

in the semilattice $S/\mathscr{D}$. This proves (d) □

Recall that a semigroup S is a rectangular band if and only if it is a completely simple semigroup over the trivial group (see Example Subsection 2.1.3). Therefore as a corollary of the theorem above, we have:

COROLLARY 4.50. *A semigroup B is a band if and only if it is a semilattice of rectangular bands.*

Similarly, from the observation that a completely simple inverse semigroup is a group, we obtain:

COROLLARY 4.51. *A semigroup S is a semilattice of groups if and only if S is a completely regular, inverse semigroup.*

The structure of completely simple semigroups are known by Ress Theorem (see Corollary 2.80) relative to groups. By Theorem 4.49(d), a completely regular semigroup is a semilattice $\mathcal{Y}$ of completely simple semigroups S_α, $\alpha \in \mathcal{Y}$. This, therefore, enables us to obtain an isight into the structure of completely regular semigroups relative to groups and semilattices. However, given a semilattice $\mathcal{Y}$ and completely simple semigroups $\{S_\alpha : \alpha \in \mathcal{Y}\}$, it is possible to have more than one binary operation on the set $S = \cup\{S_\alpha : \alpha \in \mathcal{Y}\}$ that make S, a completely regular semigroup that indues the given semilattice decomposition on S. Thus Theorem 4.49 does not determine the structure of completely regular semigroups relative to groups and semilattices. Note that, by Corollary 4.50, a structure theorem for completely regular semigroups must yield, as a special case, a structure theorem for bands. However, most of the existing structure theorems valid for arbitrary completely regular semigroups are quite complicated and does not provide any more insight into their structure than can be obtained from the theorem above.

On the other hand, quite illuminating strutheorems for some subclasses of the class of completely regular semigroups exists. The classical theorem Clifford [1941] due to Clifford on the structure of semilattices of groups is an especially simple example of this type. We need the following lemma.

LEMMA 4.52. *Let S be a semilattice $\mathcal{Y}$ of groups G_α and let $E = \boldsymbol{E}(S)$. Then E is a semilattice isomorphic to $\mathcal{Y}$. Moreover, $\mu(S) = \mathscr{D}$ and $E \subseteq Z(S)$ where*

$$Z(S) = \{z \in S : zs = sz \text{ for all } s \in S\}$$

is the center *of S.*

Proof. Since S is a semilattice of groups, by Theorem 4.49 and Corollary 4.51, each $\mathscr{D}$-class of S is a completely simple inverse semigroup and hence a group. Therefore $\mathscr{D}$=$\mathscr{H}$ and so, by Theorem 4.49(c), $\mathscr{H}$ is a congruence on S. Hence, by Proposition 3.46, $\mu(S)$ =$\mathscr{H}$=$\mathscr{D}$.

Since S is an inverse semigroup, by Theorem 2.44, E is a commutative subsenigroup of S and so ϕ =$\mathscr{D}^{\#}$ is a homomorphism of E onto $\mathcal{Y}$; since $\mathscr{D}$ is idempotent separating, ϕ is an isomorphism of E onto $\mathcal{Y}$.

To show that $E \subseteq Z(S)$, let $g \in E$ and $a \in S$. Then $a \in H_f$for some $f \in E$. Since E is commutative, $e = gf = fg\ \omega\ f$ and since $\mu(S)$ =$\mathscr{H}$, by Equation (4.10), $ga = gfa = ea = ae = afg = ag$. Therefore $g \in Z(S)$. □

Recall Subsection 1.3.1 that any partially ordered set can be regarded as a category having utmost one morphism between any two vertices. In particular a semilattice $\mathcal{Y}$ is a category. If this is the case, the category $\mathcal{Y}^{\mathrm{op}}$ is also a semilattice; in fact, an upper semilattice if $\mathcal{Y}$ is a lover semilattice . Also, if $\theta : \mathcal{Y} \to \mathcal{Y}'$ is any semilattice homomorphism, then θ is a functor of the category (preorder) $\mathcal{Y}$ to $\mathcal{Y}'$ (see Example Subsection 2.1.3). We use these in the following statement.

THEOREM 4.53. *Let $\mathcal{Y}$ be a semilattice and let $\Phi : \mathcal{Y}^{\mathrm{op}} \to \mathsf{Grp}$ be a functor from $\mathcal{Y}^{\mathrm{op}}$ to the category Grp of groups. Let*

$$S = \{(\alpha, a) : \alpha \in \mathcal{Y}, \quad a \in \Phi(\alpha)\}. \tag{4.22a}$$

Define product in S by

$$(\alpha, a)(\beta, b) = (\alpha\beta, c) \quad where \quad c = (a\Phi(\alpha, \alpha\beta))\,(b\Phi(\beta, \alpha\beta))\,. \tag{4.22b}$$

This defines a single valued binary operation in S and S with this product is a semigroup $S(\Phi)$ which is a semilattice $\mathcal{Y}$ of groups $\Phi(\alpha)$.

Conversely, let S be a semilattice of groups and $E = \boldsymbol{E}(S)$. Then

$$\Phi_S(e) = H_e \quad \textit{for } e \in E, \textit{ and} \quad \Phi_S(e, f) : a \mapsto fa$$

for all $f \leq e$ and $a \in H_e$, defines a functor Φ_S of E^{op} to Grp such that the map

$$\xi_S : a \mapsto (e, a) \quad a \in H_e$$

is an isomorphism of the semigroup S onto $S(\Phi_S)$.

Proof. Since $\mathcal{Y}$ is a semilattice, $\alpha\beta$ is a well-defined element of $\mathcal{Y}$ for all $\alpha, \beta \in \mathcal{Y}$. Since $\alpha\beta \leq \alpha$, $(\alpha, \alpha\beta)$ is a unique morphism of the calegory $\mathcal{Y}^{\mathrm{op}}$ from α to $\alpha\beta$. Since Φ is a functor of $\mathcal{Y}^{\mathrm{op}}$ to Grp, $\Phi(\alpha, \alpha\beta) : \Phi(\alpha) \to \Phi(\alpha\beta)$ is a homomorphism of groups. Hence for all $a \in \Phi(\alpha)$, $a\Phi(\alpha, \alpha\beta)$ is an element of $\Phi(\alpha\beta)$. Similarly $b\Phi(\beta, \alpha\beta)$ is an element of $\Phi(\alpha\beta)$. Since $\Phi(\alpha\beta)$ is a group, it follows that

$$c = (a\Phi(\alpha, \alpha\beta))\,(b\Phi(\beta, \alpha\beta))$$

is a unique element in $\Phi(\alpha\beta)$ for all $a \in \Phi(\alpha)$ and $b \in \Phi(\beta)$. Hence Equation (4.22b) gives a well-defined binary operation in the set S defined by Equation (4.22a). Let (α, a), (β, b) and (γ, c) be in S. Then

$$\begin{aligned} (\alpha\beta)\gamma = \alpha(\beta\gamma) \le \alpha\beta \le \alpha \qquad & \text{and since } \Phi \text{ is a functor,} \\ a\Phi(\alpha, \alpha\beta\gamma) = a\Phi(\alpha, \alpha\beta)\Phi(\alpha\beta, \alpha\beta\gamma) & \end{aligned}$$

Using Equation (4.22b) and results similar to those above, we obtain

$$((\alpha, a)(\beta, b))(\gamma, c) = (\alpha\beta\gamma, d)$$

where

$$\begin{aligned} d &= (((\alpha, a)(\beta, b))\,\Phi(\alpha\beta, \alpha\beta\gamma))\,((\gamma, c)\Phi(\gamma, \alpha\beta\gamma)) \\ &= ((a\Phi(\alpha, \alpha\beta))\,(b\Phi(\beta, \alpha\beta)))\,\Phi(\alpha\beta, \alpha\beta\gamma)\,(c\Phi(\gamma, \alpha\beta\gamma)) \\ &= ((a\Phi(\alpha, \alpha\beta)\Phi(\alpha\beta, \alpha\beta\gamma))\,(b\Phi(\beta, \alpha\beta)\Phi(\alpha\beta, \alpha\beta\gamma)))\,(c\Phi(\gamma, \alpha\beta\gamma)) \\ &= ((a\Phi(\alpha, \alpha\beta\gamma))\,(b\Phi(\beta, \alpha\beta\gamma)))\,(c\Phi(\gamma, \alpha\beta\gamma))\,. \end{aligned}$$

Similarly if

$$d' = (a\Phi(\alpha, \alpha\beta\gamma))\,((b\Phi(\beta, \alpha\beta\gamma))\,(c\Phi(\gamma, \alpha\beta\gamma)))$$

we have

$$(\alpha, a)\,((\beta, b)(\gamma, c)) = (\alpha\beta\gamma, d').$$

Since the binary operation in $\Phi(\alpha\beta\gamma)$ is associative, it follows that $d = d'$. Therefore the binary operation in S defined by Equation (4.22b) is associative and so, S is a semigroup. Also, it is clear from Equation (4.22b) that the projection $\pi : (\alpha, a) \mapsto \alpha$ is a homomorphism of the semigroup S onto the semilattice $\mathcal{Y}$ and for each $\alpha \in \mathcal{Y}$, $\alpha\pi^{-1} = \{\alpha\} \times \Phi(\alpha)$ which is isomorphic to the group $\Phi(\alpha)$. Therefore S is a semilattice $\mathcal{Y}$ of groups $\Phi(\alpha)$.

Conversely, let S be a semilattice of groups. Since S is an inverse semigroup (by Corollary 4.51), $E = \boldsymbol{E}(S)$ is a semilattice and by Lemma 4.52, E is contained in the center of S. Hence for $e, f \in E$ with $e \le f$, the map

$$\Phi_S(f, e) : a \mapsto ea \quad \text{for all} \quad a \in H_f$$

is a homomorphism of H_f into H_e. It is clear that

$$\Phi_S(e, e) = 1_{H_e} \quad \text{for all} \quad e \in E;\ \Phi_S(g, f) \circ \Phi_S(f, e) = \Phi_S(g, e) \quad \text{for all} \quad e \le f \le g.$$

Hence the assignments

$$R\Phi_S : e \mapsto H_e, \quad (f, e) \mapsto \Phi_S(f, e) \quad \text{for} \quad e \le f,$$

is a a functor $\Phi_S : E^{\mathrm{op}} \to \mathsf{Grp}$. Let $T = S(\Phi_S)$ be the semigroup constructed by Equations (4.22a) and (4.22b) above. By the first part T is a semilattice of groups. Let $(e, a), (f, b) \in T$. Then by Equation (4.22b) and the definition of Φ_S we get

$$(e, a)(f, b) = (ef, c)$$

where

$$\begin{aligned} c &= (a\Phi_S(e, ef))\,(b\Phi_S(f, ef)) && \text{by Equation (4.22b)} \\ &= (efa)(efb) = (ef)ab && ef \in E. \end{aligned}$$

Hence it follows that the $T = S(\Phi_S)$ is a semilattice of groups isomorphic to S. By Equation (4.22a), $T = \{(e, a) : e \in E\ a \in H_e\}$. Cearly, $a \mapsto (e, a)$ $(a \in H_e)$ is a bijection of S onto T which by the above, is an isomorphism of S onto T. □

***Remark* 4.5:** It is easy to see that the set of all contravariant group-valued functors on semilattices form a subcategory $\mathfrak{Y}\mathfrak{g} \subseteq [-, \mathsf{Grp}]$ of all group-valued small functors (see § Subsection 1.2.2). The theorem above shows that each functor $\phi \in \mathfrak{v}\mathfrak{Y}\mathfrak{g}$ determines a semilattice of groups $S(\phi)$. It can be shown that each morphism (transformation) $t : \phi_1 \to \phi_2$ determine a unique homomorphism $S(t) : S(\phi_1) \to S(\phi_2)$ and viceversa. In fact the assignments

$$S : \phi \mapsto S(\phi); \quad t :\mapsto S(t)$$

is a category equivalence S of the category of contravariant group valued functors on semilattices on to the category of semilattice of groups.

CHAPTER 5

Inverse semigroups

Recall, from Subsection 2.6.2, that an *inverse semigroup* is a regular semigroup such that every $x \in S$ has exactly one inverse. The study of this class of semigroups was started with the publication of the papers Vagner [1953a,b] Vagner. Later Preston [Preston, 1954a,b, see] also discovered this class of semigroups, as well as the now famous Vagner - Preston representation of an inverse semigroup, independently. Since then large number of important contributions have appeared about inverse semigroups and it has now become an important branch of both the theory of semigroups as well as the theory of groupoids. We do not propose to give a systematic eccount of the theory of inverse semigroups here; the reader may refer to, [for example, Munn, 1970, **?**] for such a treatment. However, given the fact that, most of the present day structure theory for arbitrary regular semigroups is a stright-forward generalization of the structure theory for inverse semigroups, a discussion of the later will provide a good model for the more general theory to be given in the next chapter.

In the first section we define the Schein's concept of an inductive groupoid [see Schein, 1966] and show that its category is equivalent to the category of inverse semigroups. Part of our motivation here is the fact that Schein's theory of inductive groupoids provide a simple introduction to the more general concept of inductive groupoids which will be considered in the next chapter. Inductive groupoids affords a neat separation of the global and local structure of an inverse semigroup. This is of considerable help in formulating results about of regular semigroups and proving them. The remaining part of the chapter discusses some illustration of the tenique of inductive groupoids.

Recall that Theorem 2.44 provides some equivalent characterizations of inverse semigroups. In particular, if $\phi : S \to T$ is any homomorphism, $\operatorname{Im} \phi = \phi(S)$ is an inverse subsemigroup of T. Since inclusion clearly provide a choice of subobjects for the category $\mathfrak{IS}$ of inverse semigroups, it follows that the category $\mathfrak{IS}$ has images.

$\boldsymbol{G}(S)$:Inductive groupoid of S

Throughout this chapter, unless otherwise explicitly specified, S will denote an inverse semigroup and $\boldsymbol{E} = \boldsymbol{E}(S)$, its semilattice (biordered set) of idempotents.

5.1 INDUCTIVE GROUPOIDS OF INVERSE SEMIGROUPS

Recall from Section 1.4 that a groupoid is a small category in which every morphism is an isomorphism. Here we shall discuss B. M. Schein's theory of inductive groupoids Schein [1966]. Schein's far reaching contribution showed that the theory of inverse semigroup is equivalent to the theory of ordered groupoids (see Subsection 1.4.2) that satisfyies the condition that the vertex set is a semilattice under the induced order.

Suppose that $S(*)$ is the trace of an inverse semigroup S (see Subsection 2.7.3). By Equation (2.48a) the trace product $x * y$ of $x, y \in S$ exists if and only if $L_x \cap R_y$ contains an idempotent. Since S is inverse, by Theorem 2.44, this is true if and only if $x^{-1}x = f_x = e_y = yy^{-1}$. Also, if $e \in \boldsymbol{E}$ and $x \in S$ the trace product $e * x$ exists if and only if $e = e_x$ and $x * e$ exists if and only if $e = f_x$. We can verify the following (see axioms for categories on page 9 of MacLane [1971]):

THEOREM 5.1. *Let S be an inverse semigroup. Then the trace $S(*)$ is a groupoid with objects (identities) $\boldsymbol{E}(S)$, morphism set S with composition as the trace product. Moreover, the natural partial order on S gives a partial order on this groupoid and*

$$\boldsymbol{G}(S) = (S(*), \leq)$$

is an ordered groupoid such that $\mathfrak{v}\boldsymbol{G}(S) = \boldsymbol{E}(S)$ is a semilattice.

Proof. Since S is inverse, as noted above, the trace product $x * y$ exists if and only if the right identity f_x of x is the same as the left identity e_y of y. Also, in this case, $x * y = xy \in R_x \cap L_y$ by Theorem 2.34. Suppose that the pairs (x, y) and (xy, z) are composable. Then $L_x \cap R_y$ contain an idempotent, say, e and $L_{xy} \cap R_z$ contain the idempotent g. Since

$$z \mathscr{R} g \mathscr{L} xy \mathscr{L} y \quad \text{and} \quad x \mathscr{L} e \mathscr{R} y \mathscr{R} yz$$

it follows that the pairs (y, z) and (x, yz) are composable. It follows that $S(*)$ is a category. Moreover, for any $x \in S$, we have

$$x * x^{-1} = xx^{-1} = e_x, \quad \text{and} \quad x^{-1} * x = f_x$$

which shows that every morphism in $S(*)$ is an isomorphism. Hence $S(*)$ is a groupoid.

groupoid!inductive –
functor!inductive –
groupoid!Schein's –

We now verify that $\boldsymbol{G}(S)$ satisfies the axioms of Definition 1.6. Let $x_i, y_i \in S$ with $y_i \leq x_i$ and suppose that (x_1, x_2) and (y_1, y_2) are composable. If $e \in L_{x_1} \cap R_{x_2}$ and $g \in L_{y_1} \cap R_{y_2}$ are idempotents, using Proposition 4.3 and the fact that S is inverse, we deduce that $g \leq e$ and $y_1 = x_1 g$, $y_2 = g x_2$. Hence $y_1 y_2 = x_1 g x_1^{-1} x_1 x_2$. Since

$$x_1 g x_1^{-1} \leq x_1 e x_1^{-1} = x_1 x_1^{-1} = e_{x_1 x_2}$$

it follows that $y_1 y_2 \leq x_1 x_2$. This proves axiom (1) of Definition 1.6. If $y \leq x$, then $e_y \leq e_x$ and $y = e_y x$. Hence $y^{-1} = x^{-1} e_y$ and so, axion (2) also hoplds. To verify axiom (3) of Definition 1.6, let $x|g = gx$ for all $x \in S$ and $g \leq e_x$. Since $gx \leq x$ and $e_{gx} = g$, axiom (3) is verified. Therefore $\boldsymbol{G}(S)$ is an ordered groupoid. □

We have noted that the set $V = \mathfrak{v}\mathcal{G}$ of vertices of an ordered groupoid $\mathcal{G}$ is an order ideal under the induced partial order on V. Schein [1966] defined an *inductive groupoid* as an ordered groupoid $\mathcal{G}$ in which the orderideal V of vertices of $\mathcal{G}$ is a semilattice. The theorem above says that the trace of an inverse semigroup S is an inductive groupoid $\boldsymbol{G}(S)$ in the sense above with respect to the natural partial order. An order-preserving functor $f : \mathcal{G} \to \mathcal{H}$ of inductive groupoids is called an *inductive functor*. Thus we have category $S\mathfrak{I}\mathfrak{G}$ whose oblects are Schein's inductive groupoids and morphisms are order preserving functors. Thus $S\mathfrak{I}\mathfrak{G}$ is a full subcategory of the category $\mathfrak{O}\mathfrak{G}$ of all ordered groupoids. We observe that, even though, the partially ordered set of morphisms of the inductive groupoid $\boldsymbol{G}(S)$ may not be a semilattice, the order strcture is closely related to semilattices. In fact, as observed above, its vertex set $\boldsymbol{E}$ is a semilattice and by Proposition 4.8, every principal order ideal is a semilattice.

A more general concept of inductive groupoids will be introduced in the next chapter. We shall see that these are essentially ordered groupoids whose vertex sets carry the structure of biordered sets. This will reduce to Schein's inductive groupoids when the vertex biordered set is a semilattice so that the former concept is a non-trivial generalization of Schein's inductive groupoid. To avoid ambiguity and for brevity we shall call Schein's inductive groupoid as *Schein's groupoid*.

We proceed to prove the basic result due to Schein [1966] ([see also **?**]) that every Schein's groupoid $\mathcal{G}$ arises as the Schein's groupoid $\boldsymbol{G}(S)$, defined in Theorem 5.1, for a suitable inverse semigrp S. Here, as for our definition of Schein's groupoid, we formulate the result in terms of ordered groupoids. In the following, except for those explicitly specified, we use notations of Subsection 1.4.2.

THEOREM 5.2. *Let $\mathcal{G}$ be a Schein's groupoid with the vertex-semilattice $\boldsymbol{E}$. Suppose that $*$ denote the composition in $\mathcal{G}$. For $x, y \in \mathcal{G}$ let*

$$xy = (x \centerdot g) * (g \centerdot y) \quad \textit{where} \quad g = f_x e_y \tag{5.1}$$

where $x \centerdot g$ $[g \centerdot y]$denote the corestriction [restriction] of x $[y]$ to g (see Definition 1.6 and Equation(1.62)*). This defines a binary operation on the set of morphisms of $\mathcal{G}$ making $\mathcal{G}$ an inverse semigroup $\boldsymbol{S}(\mathcal{G})$ such that*

$$\boldsymbol{G}(\boldsymbol{S}(\mathcal{G})) = \mathcal{G}. \quad \textit{Moreover, we also have} \quad \boldsymbol{S}(\boldsymbol{G}(S)) = S.$$

for any inverse semigroup S.

Proof. Since $g \leq f_x$, by Proposition 1.18, $x \centerdot g$ is a unique element og $\mathcal{G}$ such that $f_{x \centerdot g} = g$. Similarly, since $g \leq e_y$, by Definition 1.6, $g \centerdot y \leq y$ and $e_{g \centerdot y} = g$. Therefore $(x \centerdot g, g \centerdot y)$ is a composable pair in $\mathcal{G}$. Hence Equation (5.1) gives a well-defined binary operation in the set $\mathcal{G}$. We now show that the product defined by Equation (5.1) is associative. Consider $x, y, z \in \mathcal{G}$. Then

$$(xy)z = (((x \centerdot g) * (g \centerdot y)) \centerdot h') * (h' \centerdot z)$$

where $g = f_x e_y$ and $h' = f_{g \centerdot y} e_z$

$$= (x \centerdot h'') * ((g \centerdot y) \centerdot h') * (h' \centerdot z)$$

by Proposition 1.19 where

$$h'' = e_{(g \centerdot y) \centerdot h'}$$

Since $(g \centerdot y) \centerdot h' \leq g \centerdot y \leq y$ and $e_{(g \centerdot y) \centerdot h'} = h'$, by Proposition 1.18, the element $(g \centerdot y) \centerdot h'$ is the corestriction of y to h'. Hence $(g \centerdot y) \centerdot h' = y \centerdot h'$ and so

$$(xy)z = (x \centerdot h'') * (y \centerdot h') * (h' \centerdot z).$$

Similarly if $h = f_y e_z$, $g' = e_{y \centerdot h} f_x$, then $g' \centerdot (y \centerdot h) = g' \centerdot y$ and

$$x(yz) = (x \centerdot g') * (g' \centerdot y) * (g'' \centerdot z)$$

where $g'' = f_{g' \centerdot y}$. Now $y \centerdot h' \leq g \centerdot y \leq y$ implies $h' \leq f_y$. Since $h' \leq e_z$ by definition, we have $h' \leq h$. Hence $h'' = e_{y \centerdot h'} \leq e_{y \centerdot h}$. Also since $h' \leq f_{g \centerdot y}$, by Proposition 1.20, $h'' = e_{y \centerdot h'} \leq e_{g \centerdot y} = g \leq f_x$. Therefore $h'' \leq g'$. It can be shown dually that $g'' \leq h'$. This implies that $g' \leq h''$ by Proposition 1.20. Thus $g' = h''$ and similarly, $g'' = h'$. Hence $g' \centerdot y = y \centerdot h'$, and

$$\begin{aligned}(xy)z &= (x \centerdot g') * (y \centerdot h') * (h' \centerdot z)\\ &= (x \centerdot g') * (g' \centerdot y) * (h' \centerdot z)\\ &= x(yz).\end{aligned}$$

Therefore $\mathcal{G}$ is a semigroup $\boldsymbol{S}(\mathcal{G})$ with respect to the product defined by Equation (5.1). If (x, y) is a composable pair in $\mathcal{G}$ then $f_x = e_y = g$ (say) and by Equation (5.1),

$$xy = (x \mathbin{\centerdot} g) * (g \mathbin{\centerdot} y) = x * y.$$

Also $e_x x = e_x * x = x$ and $xx^1 = x * x^{-1} = e_x$. Hence $e_x \mathscr{R}\ x$ in $\boldsymbol{S}(\mathcal{G})$. Similarly $f_x \mathscr{L}\ x$ in $\boldsymbol{S}(\mathcal{G})$. If $e, f \in \mathfrak{v}\mathcal{G}$ it is easy to see that the product ef defined by Equation (5.1) coincides with their product in the semilattice $\mathfrak{v}\mathcal{G}$. Therefore $\boldsymbol{S}(\mathcal{G})$ is an inverse semigroup with $\boldsymbol{E} = \mathfrak{v}\mathcal{G}$ as the semilattice of idempotents. The arguement above also implies that (x, y) is composable in $\mathcal{G}$ if and only if the trace product of x and y exists in $\boldsymbol{S}(\mathcal{G})$. Furthermore, $y \leq x$ in the ordered groupoid $\mathcal{G}$ if and only if $e_y \leq e_x$ and $y = e_y \mathbin{\centerdot} x$. By Equation (5.1), this is true if and only if $y = e_y x$. Therefore $y \leq x$ in $\mathcal{G}$ if and only if $y \leq x$ with respect to the natural partial order on $\boldsymbol{S}(\mathcal{G})$. It follows from Theorem 5.1 that

$$\boldsymbol{G}\left(\boldsymbol{S}(\mathcal{G})\right) = \mathcal{G}.$$

Let S be an inverse semogroup and let $\mathcal{G} = \boldsymbol{G}(S)$ be the inductive groupoid of Theorem 5.1. For $x, y \in \mathcal{G}$, let $x \cdot y$ denote the product defined by Equation (5.1). If $g = f_x e_y$, by Equation (5.1), we have

$$x \cdot y = (x \mathbin{\centerdot} g) * (g \mathbin{\centerdot} y)$$

By the definition of restriction and corestriction in $\mathcal{G}$, $x \mathbin{\centerdot} g = xg$ and $g \mathbin{\centerdot} y = gy$. Since $*$ on the right of the equation above denote trace product in S, we yhave

$$x \cdot y = (xg) * (gy) = (xg)(gy) = xgy = xf_x e_y y = xy.$$

Therefore $S = \boldsymbol{S}\left(\boldsymbol{G}(S)\right)$. □

The constructions of the Schein's groupoid $\boldsymbol{G}(S)$ from an inverse semigroup S (cf. Theorem 5.1) and the inverse semigroup $\boldsymbol{S}(\mathcal{G})$ from the Schein's groupoid $\mathcal{G}$ (cf. Theorem 5.2) are functorial in the sense that $S \mapsto \boldsymbol{G}(S)$ and $\mathcal{G} \mapsto \boldsymbol{S}(\mathcal{G})$ are object maps of functors $\boldsymbol{G} : \mathfrak{IS} \to S\mathfrak{IG}$ from the category of inverse semigroups to the category of Schein's groupoids and $\boldsymbol{S} : S\mathfrak{IG} \to \mathfrak{IS}$ from the category of Schein's groupoids to the category of inverse semigroups. For let $\phi : S \to S'$ be a homomorphism of inverse semigroups. It is clear that ϕ preserves trace products and natural partial order and so, it is an order preserving functor of $\boldsymbol{G}(S)$. Thus ϕ determine a unique inductive functor of $\boldsymbol{G}(S)$ to $\boldsymbol{G}(S')$ which we denote by $\boldsymbol{G}(\phi)$. Notice that, set-theoretically, ϕ and $\boldsymbol{G}(\phi)$ denote the same map of the set S to the set S'. The functorial property of this assignment is obvious. Similarly, if $\gamma : \mathcal{G} \to \mathcal{H}$ is an inductive fumctor, then

$$\gamma(x * y) = \gamma(x) * \gamma(y)$$

for every composable pair (x, y) of morphisms in $\mathcal{G}$. Further, for $x \in \mathcal{G}$ and $g \le e_x$, $h \le f_x$, we have

$$\gamma(g \centerdot x) = \gamma(g) \centerdot \gamma(x), \quad \gamma(x \centerdot h) = \gamma(x) \centerdot \gamma(h).$$

Hence for any $x, y \in \mathcal{G}$, by Equation (5.1),

$$\begin{aligned} \gamma(xy) &= \gamma(x \centerdot g) * \gamma(g \centerdot y) \\ &= (\gamma(x) \centerdot \gamma(g)) * (\gamma(g) \centerdot \gamma(y)) \\ &= \gamma(x)\gamma(y). \end{aligned}$$

Thus γ induces a unique homomorphism $\boldsymbol{S}(\gamma) : \boldsymbol{S}(\mathcal{G}) \to \boldsymbol{S}(\mathcal{H})$. The assignment $\boldsymbol{S}$ is also functorial. We have thus proved the following.

THEOREM 5.3. *For every homomorphism $\phi : S \to S'$ of inverse semigroups there is a unique inductive functor $\boldsymbol{G}(\phi) : \boldsymbol{G}(S) \to \boldsymbol{G}(S')$ such that the assignments*

$$S \mapsto \boldsymbol{G}(S) \quad \text{and} \quad \phi \mapsto \boldsymbol{G}(\phi)$$

is a functor $\boldsymbol{G} : \mathfrak{IS} \to \mathcal{S}\mathfrak{IG}$. Similarly, each inductive functor $\gamma : \mathcal{G} \to \mathcal{H}$ determines a unique homomorphism $\boldsymbol{S}(\gamma) : \boldsymbol{S}(\mathcal{G}) \to \boldsymbol{S}(\mathcal{H})$ such that the assignments

$$\mathcal{G} \mapsto \boldsymbol{S}(\mathcal{G}) \quad \text{and} \quad \gamma \mapsto \boldsymbol{S}(\gamma)$$

is a functor $\boldsymbol{S} : \mathcal{S}\mathfrak{IG} \to \mathfrak{IS}$. Furthermore, the functors $\boldsymbol{G}$ and $\boldsymbol{S}$ are mutually inverse.

The theorems above shows that *inverse semigroups* and *Schein's groupoids* are equivalent mathematical structures. Schein's groupoid $\boldsymbol{G}(S)$ of an inverse semogroup S afford the separation of the structure of S into the lacal structure of S represented by the trace groupoid $S(*)$ and the global structure of S represented by the natural partial order on S.

We illustrate the use of the inductive groupoid technique below by applying to some important constructions.

5.2 FUNDAMENTAL INVERSE SEMIGROUPS

Many examples of ordered groupoids given in Subsection 1.4.2 (see Example 1.24) are inductive groupoids and hence represent inverse semigroups. Thus $\boldsymbol{I}_X$ is an ordered groupoid with respect to groupoid composition and "the usual inclusion" (see Example 1.21). Identities in $\boldsymbol{I}_X$ are identity maps on subsets of X and so $\mathfrak{v}\boldsymbol{I}_X$ may be identified with the set of all subsets of X which is a semilattice with respect to intersection. Hence by Theorem 5.2, $\boldsymbol{I}_X$ is an inverse semigroup with the semilattice of idempotents as the set of all subsets of X under intersection. The binary operation in $\boldsymbol{I}_X$ is defined by Equation (5.1):

$$\alpha\beta = (\alpha \centerdot g) * (g \centerdot \beta) \quad \text{where} \quad g = f_\alpha e_\beta \quad \text{and} \quad \alpha, \beta \in \boldsymbol{I}_X.$$

The identity g denotes $1_{\operatorname{cod}\alpha \cap \operatorname{dom}\beta}$. So, $\alpha \centerdot g$ denote the range restriction of α to $\operatorname{dom} g = \operatorname{cod}\alpha \cap \operatorname{dom}\beta$. Similarly $g \centerdot \beta$ denote the domain restriction of β to $\operatorname{dom} g$. It follows that, in this case $\alpha\beta$ is the "usual composition" of partial transformations.

Similarly, if X is a partially ordered set, and if I is any set of order-ideals in X such that intersection of any two order ideals in I is an order ideal in I, then, the set I^* of all order-isomorpisms of ideals in I is an ordered subgroupoid $\boldsymbol{OI}_I$ of $\boldsymbol{I}_X$. Also the set $\mathfrak{v}\boldsymbol{OI}_I$ is orderisomorphic with I which is a semilattice under intersection. Hence $\boldsymbol{OI}_I$ is an inductive groupoid. Therefore by Teorem 5.2, $\mathsf{T}(I) = \boldsymbol{S}(\boldsymbol{OI}_I)$ is an inverse semigroup in which the semilattice of idempotents $\boldsymbol{E}(\mathsf{T}(I))$ is isomorphic to I. As above it follows from Equation (5.1) that the binary product in $\mathsf{T}(I)$ is the composition of partial isomorphisms of semilattices.

In particular if E is a semilattice, then

$$E(ef) = E(e) \cap E(f) \quad \text{for all} \quad e \in E$$

where $E(e) = \{g \in E : g \leq e\}$ denote the orderideal of E generated by $e \in E$. Hence the set of all principal order ideals $\{E(e) : e \in E\}$ is closed with respect to intersection. By the remarks above, the ordered groupoid $\mathsf{T}^*(E)$ of all isomorphisms of principal ideals of E is an inductive groupoid in which the semilattice (under intersection)of identities is $\{E(e) : e \in E\}$. Since $e \mapsto E(e)$ is a semilattice isomorphism of E onto $\{E(e) : e \in E\}$ we shall identify $\mathfrak{v}\mathsf{T}^*(E)$ with E. Therefore, by 5.2,

$$\mathsf{T}(E) = \boldsymbol{S}(\mathsf{T}^*(E))$$

is an inverse semigroup with $\boldsymbol{E}(\mathsf{T}^*(E)) = E$. $\mathsf{T}^*(E)$ is called the *Munn semigroup* of the semilattice E. The following result is a particular case of 6.28 in Chapter 6 and is equivalent to Munn's theorem on fundamental inverse semigroups [Munn, 1970, see].

THEOREM 5.4. *Let G be an inductive groupid with $\mathfrak{v}G = E$. For $x \in G$ and $e \in \omega(e_x)$ let*

$$e\mathfrak{a}(x) = f_{e \centerdot x}. \tag{5.2}$$

Then we have the following:

(1) The map $\mathfrak{a}(x) : \omega(e_x) \to \omega(f_{x:})$ is an ω-isomorphism.

$T^*(G)$:Fundamental image of inductive groupoid G
$T^*(\phi)$:Fundamental image of the inductive functor ϕ
$\mathcal{A}_\rho$:The kernel normal system of ρ

(2) *There is an inductive functor $\mathfrak{a}_G : G \to T^*_E$ with $\mathfrak{v}\mathfrak{a}_G = 1_e$ and whose morphism map is $x \mapsto \mathfrak{a}(x)$.*

(3) *If G is a $\mathfrak{v}$-full inductive subgroupoid of T^*_E then $\mathfrak{a}_G$ is the inclusion of G in T^*_E. In particular, $\mathfrak{a}_{T^*_E} = 1_{T^*_E}$.*

(4) *Let $T^*(G) = \operatorname{Im}\mathfrak{a}_G$. If $\phi : G \to G'$ is an inductive functor which is a $\mathfrak{v}$-surjection, then*

$$T^*(\phi)\,(\mathfrak{a}_G(x)) = \mathfrak{a}_{G'}\left(\phi(x)\right) \tag{5.3}$$

defines an inductive functor $T^*(\phi) : T^*(G) \to T^*(G')$. Furthermore, if ϕ and ϕ' are inductive $\mathfrak{v}$-surjections for which $\phi\phi'$ exists, then

$$T^*(\phi\phi') = T^*(\phi)T^*(\phi').$$

(5) *If ϕ is a $\mathfrak{v}$-isomorphism, then $T^*(\phi)$ is an injection. In particular, if $\mathfrak{v}\phi = 1_E$, then $T^*(\phi)$ is the inclusion $T^*(G) \subseteq T^*(G')$.*

THEOREM 5.5. *Let E be a semilattice. Then the set $\mathsf{T}(E)$ of all isomorphisms of principal ideals of E is an inverse subsemigroup of $\boldsymbol{I}_E$. Furthermore $\mathsf{T}(E)$ is a fundamental inverse semigroup with semilattice of idempotents isomorphic to E. If S is any fundamental inverse semigroup with semilattice of idempotents isomorphic to E then S is isomorphic to a full subsemigroup of $\mathsf{T}(E)$.*

5.3 CONGRUENCES ON INVERSE SEMIGROUPS

If ρ is any congruence on an inverse semigroup S, by Corollary 4.30(b), any idempotent congruence class of ρ is an inverse subsemigroup of S and by Theorem 4.31, the set

$$\mathcal{A}_\rho = \{\mathcal{A}(e) : e \in E = \boldsymbol{E}(S)\}$$

of these inverse subsemigroups forms a kernel normal system of S (see Equation (4.7a)). In the case of inverse semigroups, it is posible to characterize the kernel normal systems abstractly (see Clifford and Preston [1967], § 7.4). Here we provide a characterization in terms of Schein's groupoid $G(S)$.

To simplify statement of the desired result we meed the following simple consequence of the fact that E is a commutative subsemigroup of the inverse semigroup S (see Theorem 2.44).

LEMMA 5.6. *Let S be an inverse semigroup. For each $a \in S$ and $e \in \boldsymbol{E}$ let*

$$e\mathfrak{C}(a) = a^{-1}ea$$

Then $\mathfrak{C}(a) : e \mapsto e\mathfrak{C}(a)$ is an endomorphism of $\boldsymbol{E}$ and $\mathfrak{C} : a \mapsto \mathfrak{C}(a)$ is a representation of S in the semigroup End $\boldsymbol{E}$ *of enomorphisms of $\boldsymbol{E}$.*

THEOREM 5.7. *Let ρ be a congruence on the inverse semigroup S. Then the set*

$$\mathcal{A}_\rho = \{\rho(e) : e \in E\}$$

satisfies the following:

(K1) $e \in \rho(e)$.

(K2) If, for $e, f \in E$, $\rho(e) \cap \rho(f) \neq \emptyset$, *then* $\rho(e) = \rho(f)$.

(K3) For each $a \in S$ *and* $e \in E$, $a^{-1}(\rho(e))a \subseteq \rho(e\mathfrak{C}(a))$.

(K4) If $a, ab, bb^{-1} \in \rho(e)$ *then* $b \in \rho(e)$.

Converasely, if $\mathcal{A} = \{\mathcal{A}(e) : e \in \boldsymbol{E}(S)\}$ *is any family of inverse subsemigroup of* S *satisfying the conditions above, then the relation*

$$\rho_{\mathcal{A}} = \{(a, b) \in S \times S : aa^{-1}, bb^{-1}, ab^{-1} \in \mathcal{A}(e) \quad \textit{for some} \quad e \in E\} \tag{5.4}$$

is a congruence on S whose kernel normal system is $\mathcal{A}$. Moreover the correspondances

$$\rho \mapsto \mathcal{A}_\rho \quad \textit{and} \quad \mathcal{A} \mapsto \rho_{\mathcal{A}}$$

are mutually inverse bijections of the set of all congruences on S with the set of all kernal normal systems on S.

To simplify the proof, we shall prove some preliminary lemmas. In the following, we assume that $\mathcal{A} = \{\mathcal{A}(e) : e \in \boldsymbol{E}(S)\}$ is a set of inverse subsemigroups of S satisfying conditions $(K1) \ldots (K4)$.

LEMMA 5.8. *Suppose that* $ab^{-1} \in \mathcal{A}(e)$ *for* $a, b \in S$ *and* $e \in E$. *Then* $\mathcal{A}(e\mathfrak{C}(a)) = \mathcal{A}(e\mathfrak{C}(b))$.

Proof. Let $f = e\mathfrak{C}(a) = a^{-1}ea$, $g = e\mathfrak{C}(b)$ and $u = ab^{-1}$. Since $\mathcal{A}(e)$ is an inverse subsemigroup, $u^{-1} = ba^{-1} \in \mathcal{A}(e)$ and so,

$$(ab^{-1})(ba^{-1}) = uu^{-1},\ (ba^{-1})(ab^{-1}) = u^{-1}u \in \mathcal{A}(e).$$

Therefore

$$(a^{-1}a)(b^{-1}b) = (a^{-1}a)(b^{-1}b)(a^{-1}a) = a^{-1}\left((ab^{-1})(ba^{-1})\right)a \in \mathcal{A}(f);$$

and

$$(b^{-1}b)(a^{-1}a) = (b^{-1}b)(a^{-1}a)(b^{-1}b) = b^{-1}\left((ba^{-1})(ab^{-1})\right)b \in \mathcal{A}(g)$$

Since E is commutative, we have

$$(a^{-1}a)(b^{-1}b) \in \mathcal{A}(f) \cap \mathcal{A}(g).$$

By (K2), it follows that $\mathcal{A}(f) = \mathcal{A}(g)$. □

LEMMA 5.9. *If* $aa^{-1}, bb^{-1}, ab^{-1} \in \mathcal{A}(e)$ *then* $a^{-1}a, b^{-1}b, a^{-1}b \in \mathcal{A}(e\mathfrak{C}(a))$. *In particular,* $(a,b) \in \rho_{\mathcal{A}}$ *implies* $(a^{-1}, b^{-1}) \in \rho_{\mathcal{A}}$.

Proof. Let $f = c\mathfrak{C}(a)$ and $g = e\mathfrak{C}(b)$. The given conditions imply by Lemma 5.8 that $\mathcal{A}(f) = \mathcal{A}(g)$. Hence, by (K3),

$$a^{-1}a = (aa^{-1})\mathfrak{C}(a), b^{-1}b = (bb^{-1})\mathfrak{C}(b) \in \mathcal{A}(f).$$

To show that $v = a^{-1}b \in \mathcal{A}(f)$, we observe that

$$uv = b^{-1}(ba^{-1})b \in b^{-1}\mathcal{A}(e)b \subseteq \mathcal{A}(g) = \mathcal{A}(f),$$

where $u = b^{-1}b$ and

$$vv^{-1} = a^{-1}(bb^{-1})a \in a^{-1}\mathcal{A}(e)a \subseteq \mathcal{A}(f).$$

Therefore, $u, uv, vv^{-1} \in \mathcal{A}(f)$ and so, $v \in \mathcal{A}(f)$ by the condition (K4). The last statement now follows from the definition of the relation $\rho_{\mathcal{A}}$. □

LEMMA 5.10. *If* $aa^{-1} \in \mathcal{A}(e)$ *then* $\mathcal{A}(e\mathfrak{C}(aa^{-1})) = \mathcal{A}(e)$.

Proof. By Lemma 5.6, we have

$$\mathfrak{C}(aa^{-1}) = \mathfrak{C}(a)\mathfrak{C}(a^{-1}).$$

Hence

$$(aa^{-1})\mathcal{A}(e)(aa^{-1}) \subseteq \mathcal{A}(e\mathfrak{C}(aa^{-1})).$$

But since $\mathcal{A}(e)$ is an inverse semigroup containing aa^{-1}, $(aa^{-1})\mathcal{A}(e)(aa^{-1}) \subseteq \mathcal{A}(e)$. Hence, by (K2), we have $\mathcal{A}(e) = \mathcal{A}(e\mathfrak{C}(aa^{-1}))$. □

Proof of Theorem 5.7. Suppose that $\mathcal{A} = \mathcal{A}_\rho$. Conditions (K1) and (K2) are consequences of the fact that $\rho(e)$ is the congruence class containing $e \in E$. If $a \in S$, $e \in E$ and $u \in \rho(e)$, then

$$a^{-1}ua \;\rho\; a^{-1}ea = e\mathfrak{C}(a) \quad \text{which implies} \quad a^{-1}ua \in \rho(e\mathfrak{C}(a)).$$

Therefore (K3) holds. Let $a, ab, bb^{-1} \in \rho(e)$. Then

$$bb^{-1} \;\rho\; a \Rightarrow b = bb^{-1}b \;\rho\; ab \;\rho\; e.$$

Hence $\mathcal{A}_\rho$ satisfies (K4).

Conversely let $\mathcal{A} = \{\mathcal{A}(e) : e \in E(S)\}$ be a collection of inverse subsemigroupps satisfying the conditions (K1) ... (K4) and let $\rho = \rho_{\mathcal{A}}$ be the relation defined by Equation (5.4).

ρ is an equivalence relation: If $a \in S$, then by the definition of ρ (Equation (5.4)) and (K1), $(a, a) \in \rho$ and so, ρ is reflexive. To prove symmetry, let $(a, b) \in \rho$. Then by Equation (5.4), $aa^{-1}, bb^{-1}, ab^{-1} \in \mathcal{A}(e)$ for some $e \in \boldsymbol{E}(S)$ and so, $\mathcal{A}(aa^{-1}) = \mathcal{A}(bb^{-1}) = \mathcal{A}(e)$ by (K2). Since $\mathcal{A}(e)$ is an inverse subsemigroup of S, we have

$$aa^{-1}, bb^{-1}, ba^{-1} = (ab^{-1})^{-1} \in \mathcal{A}(e).$$

Thus $(b, a) \in \rho$ and hence ρ is symmetric.

Transitivity: Let $(a, b), (b.c) \in \rho$. Then, by Equation (5.4) and (K2), we have

$$aa^{-1},\ bb^{-1},\ cc^{-1},\ ab^{-1},\ bc^{-1} \in \mathcal{A}(e).$$

Then if $f = e\mathfrak{C}(a)$, $g = e\mathfrak{C}(b)$ and $h = e\mathfrak{C}(c)$, by Lemma 5.8,

$$\mathcal{A}(f) = \mathcal{A}(g) = \mathcal{A}(h).$$

Hence to prove transitivity, it is sufficient to show that $ac^{-1} \in \mathcal{A}(e)$. Let $u = ab^{-1}$ and $v = ca^{-1}$. Since $bc^{-1} \in \mathcal{A}(e)$, by Lemma 5.9, $b^{-1}c \in \mathcal{A}(e\mathfrak{C}(a)) = \mathcal{A}(f)$. Therefore

$$uv = (ab^{-1})(ca^{-1}) = a(b^{-1}c)a^{-1} \in a\mathcal{A}(f)a^{-1}$$

By (K3) and Lemma 5.10,

$$a\mathcal{A}(f)a^{-1} \subseteq \mathcal{A}(f\mathfrak{C}(a^{-1})) = \mathcal{A}(e).$$

Similarly, by (K3),

$$vv^{-1} = (ca^{-1})(ac^{-1}) = c(a^{-1}a)c^{-1}.$$

Since

$$a^{-1}a = (aa^{-1})\mathfrak{C}(a) \in \mathcal{A}(f) = \mathcal{A}(h),$$

by Lemma 5.10, we have

$$vv^{-1} = c(a^{-1}a)c^{-1} \in \mathcal{A}(h\mathfrak{C}(c^{-1})) = \mathcal{A}(e)$$

Thus we have shown that $u, uv, vv^{-1} \in \mathcal{A}(e)$ and so, $v \in \mathcal{A}(e)$ by the condition (K4). Therefore $v^{-1} = ac^{-1} \in \mathcal{A}(e)$ which proves that ρ is transitive.

ρ is a congruence: Consider $(a, b) \in \rho$ and $c \in S$. Then we shall show that

$$(1)\ (ca, cb) \in \rho, \quad (2)\ (ac, bc) \in \rho.$$

For, since $aa^{-1}, bb^{-1}, ab^{-1} \in \mathcal{A}(e)$, by (K3), we have

$$\begin{aligned}(ca)(ca)^{-1} &= c(aa^{-1})c^{-1} \in \mathcal{A}(e\mathfrak{C}(c^{-1}));\\ (cb)(cb)^{-1} &= c(bb^{-1})c^{-1} \in \mathcal{A}(e\mathfrak{C}(c^{-1}));\\ (ca)(cb)^{-1} &= c(ab^{-1})c^{-1} \in \mathcal{A}(e\mathfrak{C}(c^{-1})).\end{aligned}$$

By Equation (5.4), this gives (1). Now the hypothesis gives, by Lemma 5.9 that,

$$(a^{-1}, b^{-1}) \in \rho. \quad \text{Hence} \quad \big((ac)^{-1}, (bc)^{-1}\big) = (c^{-1}a^{-1}, c^{-1}b^{-1}) \in \rho$$

by the proof above. Again, using Lemma 5.9, we conclude that $(ac, bc) \in \rho$. Thus ρ is a congruence.

The kernel of ρ is $\mathcal{A}$: Suppose that $e \in E$. If $u \in \rho(e)$ then $(e, u) \in \rho$. By Equation (5.4), $e, uu^{-1}, eu^{-1} \in \mathcal{A}(e)$. Hence by (K4), $u \in \mathcal{A}(e)$. Conversely, let $u \in \mathcal{A}(e)$. Then $\mathcal{A}(e)$ is an inverse subsemigroup of S which contain u. By (K1), $e \in \mathcal{A}(e)$. Hence $e, uu^{-1}, eu^{-1} \in \mathcal{A}(e)$ and so $u \in \rho(e)$ by Equation (5.4). Therefore $\rho(e) = \mathcal{A}(e)$ for all $e \in E$ so that $\mathcal{A} = \mathcal{A}_\rho$.

Finally, consider the maps

$$\theta : \rho \mapsto \mathcal{A}_\rho$$

from the set of all congruences on S to the set of kernel normal systems of S and

$$\phi : \mathcal{A} \mapsto \rho_{\mathcal{A}}$$

from the set of all kernjel normal systems to the set of all congruences on S. The proof above shows that

$$\mathcal{A}_{\rho_{\mathcal{A}}} = \mathcal{A} \quad \text{that is} \quad \phi \circ \theta(\mathcal{A}) = \mathcal{A}$$

and hence $\phi \circ \theta$ is identity on the set of all kernel normal systems of S. Now if ρ is any congruence on S and if $\rho' = \rho_{\mathcal{A}_\rho}$, then ρ and ρ' are congruences having the same congruence classes containing idempotents. Therefore $\rho = \rho'$ by Theorem 4.31. This gives that $\theta \circ \phi$ is identity on the set of all congruences on S. □

The theorem above gives a direct characterization of kernal normal systems independent of the congruence it determines on S. This allowas us to study congruences in terms kernel normal systems The description of congruences by their kernal normal systems can be simplified condiserably in the case of idempotent separarting ongruences (see Subsection 4.2.2). A characterization of idempotent separating kernels is provided by Theorem 4.33 which holds for all regular semigroups. Some further notational simplifications are possible in the case of inverse semigroups. In this case, the biordered set $\boldsymbol{E}(S)$ is completely determined by the partial order ω of the semilattice and so the group kernel $\boldsymbol{G}$ on S (cf. Theorem 4.33) are contravariant group-valued functors on E satisfying axioms (Gkr1), (Gkr2) and (Gkr3). The uniquiness of the inverse implies that the transformation $\mathfrak{c}^{\rho}(x, x')$ of axiom (Gkr3) depends only on $x \in S$. As observed in Subsection 4.2.2, any group kernel $\boldsymbol{G}$ on S is a subfunctor of $\boldsymbol{G}^{\mu}$, the group kernel associated with the maximum idempotent separating congruence $\mu(S)$. Also $\boldsymbol{G}^{\mu}$ is closely related to the structure of S.

unitary!left – unitary!right – unitary!E-unitary

5.4 CONJUGATE EXTENSIONS

Composition of transformations $\sigma : F \to G$ and $\tau : G \to H$ can be defined as the transformation $\sigma \circ \tau$ with

$$\mathfrak{v}(\sigma \circ \tau) = (\mathfrak{v}\sigma) \circ (\mathfrak{v}\tau) \tag{5.5a}$$

and for any $c \in \mathfrak{v}\,\mathrm{dom}\,F$,

$$(\sigma \circ \tau)_c = \sigma_c \circ \tau_{\bar{c}} \tag{5.5b}$$

where $\bar{c} = \mathfrak{v}\sigma(c)$. It is easy to vereify that $\sigma \circ \tau$ is a transformation from F to H.

5.5 *e*-UNITARY INVERSE SEMIGROUPS

A subset U of a semigroup S is *left [right] unitary* if $u \in U$ and $ux \in U$ $[xu \in U]$ for $x \in S$ together implies $x \in U$. U is [two-sided] *unitary* if U is both left and right unitary. We say that the semigroup S is [left, right, twosided] *E-unitary* if $\boldsymbol{E}(S)$ is a [left, right, twosided] unitary subset of S.

LEMMA 5.11. *Let S be a regular semigroup which is left [right or two-sided] E-unitary. Then S satisfies the following condition:*

(EU) *If $x \in S$ and $g \leq x$ for $g \in E$ then $x \in E$.*

For an inverse semigroup S, the consdition above is also sufficient for S to be E-unitary.

Proof. If S is left E-unitary. If $g \leq x$ for $g \in E$ and $x \in S$, then by Corollary 4.3, $g = ex$ for some $e \in E$. Since S is left unitary, we have $x \in E$. So, S satisfies (EU). Similarly S satisfies (EU) if S is rignt unitary or unitary.

Now suppose that S is inverse and satisfies (EU). Let $x \in S$ and $e, ex \in E$. Then $ex \leq x$ by Corollary 4.3 and so $x \in E$ by (EU). Therefore S is left unitary. Similarly, S is right unitary. Consequently S is unitary. □

A useful characterization of an E-unitary inverse semigroups S is is in terms of the universal group homomorphism $\gamma(S)$ on S (see Proposition 4.44).

PROPOSITION 5.12. *For an inverse semigroup S the following statements are equivalent.*

(a) S is E-unitary;

(b) E is a congruence class of a congruence on S;

(c) For each $x \in S$, the universal group homomorphism $\gamma(S)$ is injective on L_x $[R_x]$.

Proof. (a) ⇒ (b): Notice that E is contained in a single congruence class C of $\rho = \boldsymbol{\kappa}\gamma(S)$ where $\gamma(S)$ is the universal group homomorphism. Hence $x\rho e$ for any $x \in C$ and $e \in E$. By Equation (4.19), $gx = ge$ for some $g \in E$. Then $gx \in E$ and $gx \leq x$. Hence it follows from (a) that $x \in E$. Therefore $C = E$.
(b) ⇒ (c): Suppose that $x \mathscr{L} y$ and $x\rho y$. Then

$$f_x = x^{-1}x = y^{-1}y = f_y \quad \text{and} \quad e_x = xx^{-1} \mathscr{L} yx^{-1}.$$

Since $yx^{-1}\rho xx^{-1} = e_x$ and $e_x \mathscr{L} yx^{-1} \in E$, by (b), $e_x = yx^{-1}$. Therefore

$$y = yy^{-1}y = yx^{-1}x = xx^{-1}x = x.$$

This shows that $\gamma(S)$ is injective on every $\mathscr{L}$-class. Similarly $\gamma(S)$ is injective on every $\mathscr{R}$-class.
(c) ⇒ (a): Let $x \in S$, $e \in E$ and $ex \in E$. If $g = ex$, then $g \leq x$ and so $x\gamma(S)g$ by Equation (4.19). Also, $gf_x = g$ and so, $g \leq f_x$ which implies again by Equation (4.19) that $g\gamma(S)f_x$. Therefore $x\gamma(S)f_x$ and since $x \mathscr{L} f_x$, $x = f_x \in E$ by (c). Similarly it can be shown that if $xe \in E$, then $x \in E$. Thus S is E-unitary. □

CHAPTER 6

Inductive groupoids

In this chapter we discuss one approach to the structure theory of regular semigroups using inductive groupoids defined in Section 6.1. We refer the reader to the introduction of Chapter 3 for a discussion of development of structure theory of regular semigroups.

In Chapter 5 we discusses inductive groupoids of inverse semigroups due to Schein [1966]. Notice that inverse semigroups may be classified in terms of biordered sets as those regular semigroups whose biordered sets are semilattices (see Chapter 3). This is the starting point of our discussion of inductive groupoids. We can see that inductive groupoids of regular semigroups is a far-reaching generalization of Schein's theory. We show that the category of inductive groupoids is naturally equivalent to the category of regular semigroups. Consequently, one can replace regular semigroups by their inductive groupoids or vice-versa. The inherent symmetry of the groupoids could be exploited to simplify formulation as well as proof of results. In particular, this technique enable one to formulate and prove many results for general regular semigroups that are available for inverse semigroups.

In section 3, we apply the theory of inductive groupoids to discuss the fundamental regular semigroups. This leads to a generalization of Munn's theorem for fundamental inverse semigroups. In section 3, is devoted to regular semibands. We determine all regular semibands generated by a given regular biordered set. We also obtain an alternate constructions of the free semiband $\boldsymbol{B}_0(E)$ and fundamental semiband $\boldsymbol{B}_\tau(E)$ generated by a regular biordered set E in terms of their inductive groupoids. The last section discuss some special classes of semigroups and their inductive groupoids.

In this chapter S will denote a regular semigroup and E will be a regular biordered set unless otherwise made explicit.

6.1 DEFINITION AND BASIC PROPERTIES

We noted in Chapter 5 (see Theorem 5.1) that, given an inverse semigroup S, $\boldsymbol{G}(S)$, the algebra on the set S with trace product (see)and the natural partial order, is an ordered groupoid (see Theorem 6.28). However, for a regular semigroup S the partial algebra $S(*)$ (see (**??**) and (**??**)) is not, in general, a groupoid. To overcome this problem, we consider the relation

$$\boldsymbol{G}(S) = \{(x, x') :\in S,\ x' \in \mathscr{V}(x)\{(x, x') :\in S,\ x' \in \mathscr{V}(x)\}.\}. \tag{6.1}$$

Several authors, among them [Schein, 1966], considered this relation. Schein observed that $\boldsymbol{G}(S)$ is a semigroup under the multiplication

$$(x, x')(y, y') = (xy, y'x')$$

if the semigroup is orthodox (see Theorem 2.43). However, this clearly does not work for arbitrary regular semigroups.

On the other hand, when S is inverse, the relation defined above can be identified with $S(*)$ by identifying $(x, x^{-1}) \leftrightarrow x$ which is an isomorphism of groupoids. Nambooripad [1979] showed that it is possible to extend this definition of $\boldsymbol{G}(S)$ for inverse semigroups to arbitrary regular semigroups. Our aim in this section is to present the definition of inductive groupoids, its morphisms and certain basic properties. We begin with some auxiliary definitions and results needed for the definition of inductive groupoids.

6.1.1 The groupoid of E-chains

Let E be a (regular) biordered set. By Equation (3.1), the relations $\mathscr{L}_E = \mathscr{L}$ and $\mathscr{R}_E = \mathscr{R}$ are equivalence relations on E and hence represents simplecial groupoids with vertex set E (see Example 1.20). Observe that in $\mathscr{L}$ the composite $(e, f)(g, h)$ exist if and only if $f = g$; in particular (e, e) is the unique left identity of (e, f) and (f, f) is the unique right identity so that we may identify the set of vertexes of $\mathscr{L}$ with E. Similar observations are valid for $\mathscr{R}$ also (see Example 1.20).

LEMMA 6.1. *Let E be a biordered set. For $(e, f), (g, h) \in \mathscr{L}$ define*

$$(e, f) \le (g, h) \iff e\ \omega\ g, \quad \text{and} \quad h = fg.$$

This defines a partial order on $\mathscr{L}$ and $\mathscr{L}$ is an ordered groupoid. If $\theta : E \to E'$ is a bimorphism, then

$$\mathscr{L}_\theta : (e, f) \mapsto (e\theta, f\theta)$$

is an order-preserving functor $\mathscr{L}_\theta : \mathscr{L}_E \to \mathscr{L}_{E'}$. Furthermore, the assignments

$$\mathscr{L} : E \mapsto \mathscr{L}_E \quad \text{and} \quad \theta \mapsto \mathscr{L}_\theta$$

is a functor $\mathscr{L} : \mathfrak{RB} \to \mathfrak{OG}$.

graph
free category
path

Proof. First consider $\mathscr{L}$. Since ω is a partial order and since fg is the basic product, it is clear that the relation $\leq$ is reflexive and antisymmetric. If $(k,l) \leq (g,h) \leq (e,f)$, then $k\omega g\omega e$ and $l = hk = (fg)k = f(gk) = fk$. Hence $(k,l) \leq (e,f)$ and so $\leq$ is a partial order on $\mathscr{L}$.

We now verify axioms (OGi), $i = 1,2,3$. Suppose that $(e,f),(f,g),(e',f')$ and (f',g') are morphisms in $\mathscr{L}$ such that $(e',f') \leq (e,f)$ and $(f',g') \leq (f,g)$. Then

$$g' = gf' = g(fe') = (gf)e' = ge'.$$

which implies that $(e',f')(f',g') = (e',g') \leq (e,g) = (e,f)(f,g)$. Hence $\mathscr{L}$ satisfies axiom (OG1) of Definition 1.6. If $(g,h) \leq (e,f)$ then it is clear from the definition of $\leq$ above and axiom (B2) for biordered sets that $h\ \omega\ f$. Also $eh = e(fg) = eg = g$. Hence $(h,g) \leq (f,e)$ and so, axiom (OG2) holds. If we set restriction in $\mathscr{L}$ as

$$g \centerdot (e,f) = (e,f) \mid g = (g,fg), \tag{6.2}$$

then $g \centerdot (e,f)$ is a unique morphism in $\mathscr{L}$ such that $g \centerdot (e,f) \leq (e,f)$ and the left identity of $g \centerdot (e,f)$ is (g,g). Hence axiom (OG3) also holds. Thus $\mathscr{L}_E$ is an ordered groupoid. If $\theta : E \to E'$ is a bimorphism, it is clear that the assignment $(e,f) \mapsto (e\theta, f\theta)$ is functor. Also, since θ is a bimorphism, we have

$$(g \centerdot (e,f))\theta = (g\theta,(fg)\theta) = (g\theta,(f\theta)(g\theta)) = (g\theta) \centerdot (e\theta, f\theta).$$

Therefore, the functor $\mathscr{L}_\theta : \mathscr{L}_E \to \mathscr{L}_{E'}$ is order preserving. Finally, it is routine to check that the given assignment is a functor. $\square$

It is clear that the dual of the above lemma also holds. Thus for each biordered set E, the relation $\leq$ defined by the equation dual to Equation (6.2) is a partial order on $\mathscr{R}_E = \mathscr{R}$ and $\mathscr{R}$ is an ordered groupid with respect to $\leq$. Further, for each bimorphism $\theta : E \to E'$, the map

$$\mathscr{R}_\theta : (e,f) \mapsto (e\theta, f\theta)$$

an order preserving functor $\mathscr{R}_\theta : \mathscr{R}_E \to \mathscr{R}_{E'}$ such that the assignments

$$\mathscr{R} : E \mapsto \mathscr{R}_E \quad \text{and} \quad \theta \mapsto \mathscr{R}_\theta$$

is a functor $\mathscr{R} : \mathfrak{RB} \to \mathfrak{OG}$.

In the following discussion, we follow MacLane [1971] for concepts such as a *graph*, *free category* generated by a graph, etc. Suppose that $G = (E, \mathscr{L}_E \cup \mathscr{R}_E)$ be the graph with vertex set $\mathfrak{v}G = E$ and edge set $EG = \mathscr{L}_E \cup \mathscr{R}_E$ [see MacLane, 1971, Page 10]. Notice that any edge in G may be represented uniquely as a pair $(e,f) \in \mathscr{L}_E \cup \mathscr{R}_E$ since $\mathscr{L}_E$ and $\mathscr{R}_E$ are simplecial groupoids. We say that two edges (e,f) and (g,h) in G are composable if and only if $f = g$. A *path* in G

vertex!inessential
category!congruence on –

is a finite sequence $s = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ of edges in which adjacent edges $\alpha_i,\ \alpha_{i+1}$ are composable; that is $\operatorname{cod}\alpha_i = \operatorname{dom}\alpha_{i+1}$ for $i = 1, 2, \ldots, n$.

Let $F = F_E$ be the free category generated by G [see MacLane, 1971, Page 50]; that is F is the category with $\mathfrak{v}F = E$ and for $e, f \in E$, the home-set $F(e, f)$ is the set of all paths in G from e to f. Since edges in G are represented as pairs of vertexes, a path in G from e to f can be represented as a finite sequences of vertexes

$$s = (e = e_0, e_1, \ldots, e_n = f) = (e, e_1)(e_1, e_2)\ldots(e_{n-1}, f)$$

where $(e_{i-1}, e_i) \in \mathscr{L}_E \cup \mathscr{R}_E$ for all $i = 1, 2 \ldots, n$. Here the vertexes e_i, $i = 0, 1, \ldots, n$ will be called the vertexes of the path s. We shall say that a vertex e_i of s is *inessential* if both edges (e_{i-1}, e_i) and (e_i, e_{i+1}) belongs to $\mathscr{L}_E$ or both belong to $\mathscr{R}_E$. If e_i is inessential, the sequence

$$s' = (e = e_0, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n = f) \quad \text{where} \quad 0 < i < n,$$

is also a path in G from e to f. We shall write $s \leftrightarrow s'$ to mean that the path s' is obtained by removing from s or introducing into s an inessential vertex. This clearly defines a symmetric relation on the morphism set of F. Let σ denote the transitive closure of this relation. The symmetry of $\leftrightarrow$ implies that σ is an equivalence relation (see the discussion of equivalence relations in Subsection 1.1.1). Then by Equation (1.8a) we have

$$s\ \sigma\ s' \iff \begin{cases} s = s' & \text{or} \\ \exists s_i \in F & \text{such that } s_0 = s,\ s_n = s' \\ & \text{and} \quad s_{i-1} \leftrightarrow s_i, \quad 0 < i < n. \end{cases} \tag{6.3}$$

Notice that when s and s' are related in this way, then $s \in F(e, f)$ if and only if $s' \in F(e, f)$. It follows that the restriction of σ to $F(e, f)$ is an equivalence relation for every $e, f \in F$. Moreover, for $u, v \in F$ if the product usv exists in F, then

$$s\ \sigma\ s' \Rightarrow usv\ \sigma\ us'v.$$

Consequently σ is a *congruence* on the category F in the sense of [MacLane, 1971, Page 52]. It is easy to see that F/σ is the morphism set of a category for which composition is defined by

$$\sigma(s)\sigma(s') = \sigma(ss') \tag{6.4}$$

for all $s, s' \in F$ such that ss' exists in F. Now identities in F are trivial paths of the form $s(e, e)$ so that identities in F/σ are $\sigma(s(e, e))$, $e \in E$. Consequently, we have a small category $\mathfrak{C}(E)$ in which morphisms set is F/σ and $\mathfrak{v}\mathfrak{C}(E) = E$. Since σ is a congruence, the quotient map $\sigma^{\#}$ preserves composition and hence there is a functor from F to $\mathfrak{C}(E)$, also denoted by $\sigma^{\#}$ with $\mathfrak{v}\sigma^{\#} = 1_E$. Now if $s = (e, e_1) \ldots (e_{n-1}, f)$ is a path in $F(e, f)$, it is clear that $s^* = (f, e_{n-1}) \ldots (e_1, e)$ is a path in $F(f, e)$ such that $ss^* \; \sigma \; s(e, e)$ and $s^*s \; \sigma \; s(f, f)$. Hence, by Equation (6.4), $\sigma(s)\sigma(s^*) = \sigma(s(e, e))$ and $\sigma(s^*)\sigma(s) = \sigma(s(f, f))$. Therefore $\sigma(s^*) = (\sigma(s))^{-1}$ in $\mathfrak{C}(E)$ and so, $\mathfrak{C}(E)$ is a groupoid. We have thus proved the following.

E-chain
groupoid of E-chains

LEMMA 6.2. *F/σ is the morphism set of a (small) category $\mathfrak{C}(E)$ such that $\mathfrak{v}\mathfrak{C}(E) = \mathfrak{v}F = E$. The composition in $\mathfrak{C}(E)$ is defined by*

$$\sigma(s)\sigma(s') = \sigma(ss')$$

for all $s, s' \in F$ such that the composite ss' exists in F. Also there is a functor $\sigma^{\#} : F \to \mathfrak{C}(E)$ which sends each $s \in F$ to $\sigma(s)$ and $\mathfrak{v}\sigma^{\#} = 1_E$. Moreover $\mathfrak{C}(E)$ is a groupoid. □

For any $s = s(e_0, e_1, \ldots, e_n) \in F$, we write

$$\sigma(s(e_0, e_1, \ldots, e_n)) = c(e_0, e_1, \ldots, e_n).$$

$\sigma(s)$ is called an *E-chain* in E and the groupoid $\mathfrak{C}(E)$ is called the *groupoid of E-chains* of the biordered set E. Since we have identified vertexes and identities (see Subsection 1.3.1), each $e \in E$ will also stand for the corresponding identity $c(e, e)$. In particular, for any $c \in \mathfrak{C}(E)$, $e_c = e_0$ will stand for the domain of $c = c(e_0, \ldots, e_n)$ in $\mathfrak{C}(E)$ as well as the left identity; similarly, $f_c = e_n$ will denote the co-domain as well as the right identity of c. Recall also that for each $(e, f) \in \mathscr{L}$, $\tau(e, f)$ (see Corollary 3.16) is an ω-isomorphism of $\omega(e)$ onto $\omega(f)$ such that the assignments of Equation (3.14) is a functor of $\mathscr{L}$ to the the ordered groupoid T_E^* of ω-isomorphisms of E. Dually the assignments of Equation (3.14*) gives a functor of $\mathscr{R}$ into T_E^*.

We now show that we can define *restriction* in $\mathfrak{C}(E)$ which makes it an ordered groupoid. Equations (6.5a) and (6.5b) below define operations that are more general than necessary for the present purpose. However, they will be needed in the sequel for discussing inductive groupoids and associated semigroups.

LEMMA 6.3. *Let $c = c(e_0, e_1, \ldots, e_n) \in \mathfrak{C}(E)$. If $h \; \omega^r \; e_0$ then*

$$\begin{aligned} h \,{\scriptstyle\blacksquare}\, c = c(h, h_0, h_1, \ldots, h_n) \quad &\text{where} \quad h_0 = he_0 \\ \text{and for} \quad i = 1, 2, \ldots, n, \quad &h_i = h_{i-1}\tau(e_{i-1}, e_i). \end{aligned} \tag{6.5a}$$

is a well-defined E-chain. Dually, if $k \; \omega^l \; f$, then

$$\begin{aligned} c \,{\scriptstyle\blacksquare}\, k = c(k_0, k_1, \ldots, k_n, k) \quad &\text{where} \quad k_n = e_n k \\ \text{and for} \quad i = 0, 1, \ldots, n-1, \quad &k_i = k_{i+1}\tau(e_{i+1}, e_i). \end{aligned} \tag{6.5b}$$

$\mathfrak{C}(E)$:*The groupoid of E-chains*

is a well-defined E-chain. Now define

$$c \le c' \iff e_c\ \omega\ e_{c'} \quad \text{and} \quad c = e_c \centerdot c'. \tag{6.5c}$$

Then $\le$ is a partial order on $\mathfrak{C}(E)$ such that $\mathfrak{C}(E)$ is an ordered groupoid.

Proof. By axiom (B2), $h\ \mathscr{R}\ he_0 = h_0\ \omega\ e_0$. Since for $i = 1, \dots, n$, if $(e_{i-1}, e_i) \in \mathscr{L}$ then by Corollary 3.16, $(h_{i-1}, h_i) \in \mathscr{L}$ and dually, if $(e_{i-1}, e_i) \in \mathscr{L}$ then $(h_{i-1}, h_i) \in \mathscr{L}$. Hence $h \centerdot c$ is an E-sequence. To show that $h \centerdot c$ is a well-defined E-chain we must show that when $c(s) = c(s')$, $h \centerdot c(s) = h \centerdot c(s')$. Suppose that e_i is inessential in c so that $e_{i-1}\ \mathscr{R}\ e_i\ \mathscr{R}\ e_{i+1}$ (or $e_{i-1}\ \mathscr{L}\ e_i\ \mathscr{L}\ e_{i+1}$). Then by Equation (3.14) (or Equation (3.14*)),

$$h_{i+1} = h_i\tau(e_i, e_{i+1}) = h_{i-1}\tau(e_{i-1}, e_i)\tau(e_i, e_{i+1}) = h_{i-1}\tau(e_{i-1}, e_{i+1}).$$

It follows that h_i is inessential in $h.c$. Consequently if $s \leftrightarrow s'$, then $h.c(s) = h.c(s')$. By finite induction we conclude that $h \centerdot c(s) = h \centerdot c(s')$ if $s\sigma s'$. It follows that the E-chain $h \centerdot c$ is well-defined. Dually, for $k\ \omega^l\ e_n$, $c \centerdot k$ is a well-defined E-chain.

The relation $\le$ defined by Equation (6.5c) is clearly reflexive and antisymmetric. Suppose that $g\ \omega\ h\ \omega\ e_c$ where $c = c(e_0, e_1, \dots, e_n)$. Then by Equation (6.5a), $h \centerdot c$ has the form $c(h = h_0, h_1, \dots, h_n)$ and $h \centerdot c \le c$. Let $g \centerdot c = c(g = g_0, \dots, g_n)$ and $g \centerdot (h \centerdot c) = c(g'_0, g'_1, \dots, g'_n)$. If $e_{i-1}\ \mathscr{R}\ e_i$ then

$$g'_i = g'_{i-1}h_i = g'_{i-1}(h_{i-1}e_i) = (g'_{i-1}h_{i-1})e_i = g'_{i-1}e +_i .$$

If $e_{i-1}\ \mathscr{L}\ e_i$, we similarly have $g'_i = e_i g'_{i-1}$. Since $g'_0 = g_0 = g$, it follows by induction from the above that $g'_i = g_i$ for all $i = 1, 2, \dots, n$. Therefore

$$g \centerdot c = g \centerdot (h \centerdot c) \quad \text{for all} \quad c \in \mathfrak{C}(E) \quad \text{and} \quad g\ \omega\ h\ \omega\ e_c. \tag{6.6}$$

This in particular shows that $\le$ is transitive and so, a partial order on $\mathfrak{C}(E)$.

Suppose that $c_i \le d_i$, $i = 1, 2$ and assume that products c_1c_2 and d_1d_2 exists in $\mathfrak{C}(E)$. Then $f_{c_1} = h = e_{c_2}$ and $f_{d_1} = h' = e_{d_2}$. Since $c_2 \le d_2$, by definition, $h = e_{c_2}\omega f_{d_2} = h'$. Also since $c_1 \le d_1$, we gave $g = e_{c_1}\ \omega\ e_{d_1} = g'$. Let

$$g \centerdot d_1 = c(g = g_0, \dots, g_n = h), \quad h \centerdot d_2 = c(h = h_0, \dots, h_m) \quad \text{and}$$
$$g \centerdot d_1d_2 = c(g = k_0, \dots, k_n, k_{n+1}, \dots, k_{n+m}).$$

Then by Equation (6.5a), we have

$$k_i = \begin{cases} g\tau(e_0, e_1)\tau(e_1, e_2)\dots\tau(e_{i-1}, e_i) = g_i & \text{if} \quad 1 \le i \le n; \\ h\tau(e_n, f_1)\dots\tau(f_{i-n-1}, f_{i-n}) = h_{i-n} & \text{if} \quad n < i \le m+n. \end{cases}$$

Therefore

$$g \centerdot d_1d_2 = (g \centerdot d_1)(h \centerdot d_2) \tag{6.7}$$

It follows that $\mathfrak{C}(E)$ satisfies axiom (OG1). Again let $g \ \omega \ e_c$, $g \centerdot c = c(g = g_0, \dots g_n = h)$ and if $h \centerdot c^{-1} = c(h = h_0, \dots, h_n)$ where $c = c(e_0, \dots, e_n)$, then by Equation (6.5a), we have

$$\begin{aligned} h_i &= h\tau(e_n, e_{n-1}) \dots \tau(e_{n-i+1}, e_{n-i}) \\ &= g\tau(e_0, e_1) \dots \tau(e_{n-1}, e_n)\tau(e_n, e_{n-1}) \dots \tau(e_{n-i+1}, e_{n-i}) \\ &= g\tau(e_0, e_1) \dots \tau(e_{n-i-1}, e_{n-i}) = g_{n-i}. \end{aligned}$$

It follows that

$$h \centerdot c^{-1} = (g \centerdot c)^{-1}. \tag{6.8}$$

By Equation (6.5c) axiom (OG2) holds. Axiom (OG3) also follows if we define restriction of $c \in \mathfrak{C}(E)$ to $g \ \omega \ e_c$ as $g \centerdot c$. Therefore $\mathfrak{C}(E)$ is an ordered groupoid with respect to the partial order defined by Equation (6.5c). □

Notice that if $g \ \omega \ e_c$ then the left restriction $g \centerdot c$ is the restriction in the ordered groupoid $\mathfrak{C}(E)$ (see 1.6, axiom (OG3)) and so, there is no ambiguity in the notation defined in the lemma above. Similarly, if $h \ \omega \ f_c$ then $c \centerdot h$ is the co-restriction or the co-domain restriction in $\mathfrak{C}(E)$ (see Subsection 1.4.2).

The groupoid of E-chains can be characterized as a push out in the category $\mathfrak{OG}$ (see Example 1.6).

PROPOSITION 6.4. *Let E be a biordered set and let*

$$\jmath_r : 1_E \subseteq \mathscr{R}_E \quad \text{and} \quad \jmath_l : 1_E \subseteq \mathscr{L}_E.$$

Then there exists order preserving functors $L_E : \mathscr{L}_E \to \mathfrak{C}(E)$ and $R_E : \mathscr{R}_E \to \mathfrak{C}(E)$ such that the following diagram is a push-out in $\mathfrak{OG}$.

$$\begin{array}{ccc} 1_E & \xrightarrow{\jmath_r} & \mathscr{R}_E \\ {\scriptstyle \jmath_l}\downarrow & & \downarrow{\scriptstyle R_E} \\ \mathscr{L}_E & \xrightarrow[L_E]{} & \mathfrak{C}(E) \end{array} \tag{6.9}$$

Consequently $\mathfrak{C}(E) = \mathscr{L}_E \coprod_{1_E} \mathscr{R}_E$.

Proof. Since $\mathfrak{v}\mathscr{L}_E = E = \mathfrak{v}\mathscr{R}_E$, and 1_E is trivially an ordered groupoid with $\mathfrak{v}1_E = E$, the inclusions $\jmath_r$ and $\jmath_l$ are order preserving functors with $\mathfrak{v}\jmath_r = 1_E = vrt\jmath_l$. Let R_E be defined by the assignments

$$R_E : (e, f) \in \mathscr{R}_E \mapsto c(e, f), \quad e \mapsto e.$$

By Equation (6.2) and Equation (6.5c), $R_E : \mathscr{R}_E \to \mathfrak{C}(E)$ is an order preserving functor with $\mathfrak{v}R_E = 1_E$. Dually the assignments

$$L_E : (e, f) \in \mathscr{L}_E \mapsto c(e, f), \quad e \mapsto e.$$

is an order preserving functor $L_E : \mathscr{L}_E \to \mathfrak{C}(E)$ with $\mathfrak{v}L_E = 1_E$. It is clear from the definitions that the given diagram commutes.

To prove that the diagram above is a pushout, consider order preserving functors $F_r : \mathscr{R}_E \to G$ and $F_l : \mathscr{L}_E \to G$ such that

$$\jmath_r \circ F_r = \jmath_l \circ F_l \quad \text{or equivalently,} \quad \mathfrak{v}F_r = \mathfrak{v}F_l.$$

Define $F : \mathfrak{C}(E) \to G$ by

$$\mathfrak{v}F = \mathfrak{v}F_r = \mathfrak{v}F_l, \quad \text{and} \quad F(c) = F_1(e_0, e_1)F_2(e_1, e_2)\dots F_n(e_{n-1}, e_n)$$

for all $c = c(e_0, e_1, \dots, e_n) \in \mathfrak{C}(E)$, where

$$F_i(e_{i-1}, e_i) = \begin{cases} F_r(e_{i-1}, e_i) & \text{if} \quad e_{i-1} \mathscr{R}\, e_i; \\ F_l(e_{i-1}, e_i) & \text{if} \quad e_{i-1} \mathscr{L}\, e_i. \end{cases}$$

Since $\mathfrak{v}F_r = \mathfrak{v}F_l$, the compositions in the expression for $F(c)$ exists in G. If e_i is inessential, $F_i(e_{i-1}, e_i)$ and $F_{i+1}(e_i, e_{i+1})$ are images of composable morphisms in $\mathscr{R}_E$ or $\mathscr{L}_E$ and hence

$$F_i(e_{i-1}, e_i)F_{i+1}(e_i, e_{i+1}) = F_{i+1}(e_{i-1}, e_{i+1}).$$

Consequently $F(c)$ is well-defined. It is clear that

$$R_E \circ F = F_r \quad \text{and} \quad L_E \circ F = F_r.$$

These equations also shows that F is the unique functor satisfying these equations.

Let $c = c(e_0, e_1, \dots, e_n) \in \mathfrak{C}(E)$ and $h \;\omega\; e_0$. Then by Lemma 6.3 $h \centerdot c = c(h_0, h_1, \dots, h_n) \leq c$. Also, since $h_i \;\omega\; e_i$ for all i, by the dual of Lemma 6.1, $(h_{i-1}, h_i) = h_{i-1} \centerdot (e_{i-1}, e_i) \leq (e_{i-1}, e_i)$ in $\mathscr{R}_E$ if $(e_{i-1}, e_i) \in \mathscr{R}$ and similarly for $\mathscr{L}$. Then

$$F_i(h_{i-1}, h_i) = F_i(h_{i-1} \centerdot (e_{i-1}, e_i)) = F_i(h_{i-1}) \centerdot F_i(e_{i-1}, e_i) = F(h_{i-1}) \centerdot F(e_{i-1}, e_i)$$

since both F_r and F_l are order preserving and since $\mathfrak{v}F_r = \mathfrak{v}F_l = \mathfrak{v}F$. Therefore

$$\begin{aligned} F(h \centerdot c) &= (F(h_0) \centerdot F(e_0, e_1)) \dots (F(h_{n-1}) \centerdot F(e_{n-1}, e_n)) \\ &= F(h_0) \centerdot (F(e_0, e_1) \dots F(e_{n-1}, e_n)) && \text{by Proposition 1.19(2)} \\ &= F(h) \centerdot F(c). \end{aligned}$$

Thus F is order preserving. ☐

The proposition above constructs the ordered groupoid $\mathfrak{C}(E)$ for every biordered set E. Let $\theta : E \to E'$ be a bimorphism. Then by the dual of Lemma 6.1, $\mathscr{R}_\theta : \mathscr{R}_E \to \mathscr{R}_{E'}$ is an order preserving functor. Hence by the above $\mathscr{R}_\theta \circ R_{E'}$ is an order preserving functor of $\mathscr{R}_E$ to $\mathfrak{C}(E')$. Dually, $\mathscr{L}_\theta \circ L_{E'} : \mathscr{L}_E \to \mathfrak{C}(E')$ is an order preserving functor. Also since $\mathfrak{v}(\mathscr{R}_\theta \circ R_{E'}) = \mathfrak{v}(\mathscr{L}_\theta \circ L_{E'}) = \theta$, we have

$$\jmath_r \circ \mathscr{R}_\theta \circ R_{E'} = \jmath_l \circ \mathscr{L}_\theta \circ L_{E'}.$$

Since the diagram 6.9 is pushout, there is a unique order preserving functor $\mathfrak{C}(\theta) : \mathfrak{C}(E) \to \mathfrak{C}(E')$ such that

$$\mathscr{R}_\theta \circ R_{E'} = R_E \circ \mathfrak{C}(\theta) \quad \text{and} \quad \mathscr{L}_\theta \circ L_{E'} = L_E \circ \mathfrak{C}(\theta).$$

These equations imply that

$$\mathfrak{C}(\theta)(c) = c(e_0\theta, e_1\theta, \dots, e_n\theta) \tag{6.10}$$

for all $c = c(e_0, \dots, e_n) \in \mathfrak{C}(E)$.

PROPOSITION 6.5. *The assignments*

$$\mathfrak{C} : E \mapsto \mathfrak{C}(E) \quad \textit{and} \quad \theta \mapsto \mathfrak{C}(\theta) \tag{6.11}$$

is functor $\mathfrak{C} : \mathfrak{RB} \to \mathfrak{OG}$.

Proof. The vertex map of $\mathfrak{C}$ is well-defined by Proposition 6.4 and for each $\theta : E \to E'$, $\mathfrak{C}(\theta)$, defined above, is a unique order preserving functor of $\mathfrak{C}(E)$ to $\mathfrak{C}(E')$. It is clear from Equation (6.10) that $\mathfrak{C}(1_E) = 1_{\mathfrak{C}(E)}$. If $\theta : E \to E'$ and $\theta' : E' \to E''$ are bimorphisms, using Equation (6.10), we have

$$\begin{aligned}\mathfrak{C}(\theta \circ \theta')(c) &= c\left((e_0)\theta\theta', \dots, (e_n)\theta\theta'\right)\\ &= \mathfrak{C}(\theta')\left(c(e_0\theta, \dots, e_n\theta)\right)\\ &= \mathfrak{C}(\theta')\left(\mathfrak{C}(\theta)(c)\right) = \mathfrak{C}(\theta) \circ \mathfrak{C}(\theta')(c)\end{aligned}$$

for all $c \in \mathfrak{C}(E)$. Hence

$$\mathfrak{C}(\theta\theta') = \mathfrak{C}(\theta) \circ \mathfrak{C}(\theta')$$

Therefore $\mathfrak{C} : \mathfrak{RB} \to \mathfrak{OG}$ is a functor. □

6.1.2 Definition and basic properties of inductive groupoids

The ordered groupoid $\boldsymbol{G}(S)$ is the inductive groupoid of S when S is an inverse semigroup so that we can reconstruct S from $\boldsymbol{G}(S)$ (see Theorem 5.2). In general the local structure of the regular semigroup S is represented by a

ϵ-commutative
groupoid!inductive –
ε_G:Evaluation of the inductive groupoid G
inductive groupid!biordered set of –
inductive groupoid!evaluation ot the –

suitably constructed ordered groupoid $\boldsymbol{G}(S)$ (see Subsection 6.2.1) while the the global structure of S is not adequately reflected in it. In particular, the relation between the the biordered set $\boldsymbol{E}(S) = E$ and $\boldsymbol{G}(S)$ is not strong enough to be able to recover the biordered set of S from $\boldsymbol{G}(S)$. We therefore add a new layer of structure to the ordered groupoid $\boldsymbol{G}(S)$ by defining an *evaluation* of the groupoid $\mathfrak{C}(E)$ in $\boldsymbol{G}(S)$.

Recall that an E-square in a biordered set E is a 2×2-matrix $A = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ (see Section 3.2) where $e \mathscr{R} f \mathscr{L} h \mathscr{R} g \mathscr{L} e$. Moreover if $g, h \in \omega^r(e)$ and $g \mathscr{L} h$ or if g, h and e satisfy the dual conditions, then we have an E-squares $\begin{pmatrix} g & ge \\ h & he \end{pmatrix}$ and $\begin{pmatrix} g & h \\ ge & he \end{pmatrix}$ respectively; these are called singular E-squares (see Section 3.2).

Recall also that a $\mathfrak{v}$-isomorphism of ordered groupoids is an order preserving functor that induces an order isomorphism of the set of vertexes (see Subsection 1.4.2). Let E be a biordered set and $\epsilon : \mathfrak{C}(E) \to G$ be a $\mathfrak{v}$-isomorphism of $\mathfrak{C}(E)$ to an ordered groupoid G. We say that an E-square $A = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ is *ϵ-commutative* if the following equality holds in G:

$$\epsilon(e, f)\epsilon(f, h) = \epsilon(e, g)\epsilon(g, h).$$

Here, for brevity, we have written $\epsilon(e, f)$, $(e, f) \in \mathscr{L}_E \cup \mathscr{R}_E$ for $\epsilon(c(e, f))$. We shall use such simplifications whenever it is convenient.

DEFINITION 6.1. Let E be a biordered set and $\varepsilon_G : \mathfrak{C}(E) \to G$ be a $\mathfrak{v}$-isomorphism of $\mathfrak{C}(E)$ to an ordered groupoid G. We say that the pair (G, ε_G) is an *inductive groupoid* if the following axioms hold:

(IG1) Let $x \in G$ and $e_i, f_i \in E$ such that $\varepsilon_G(e_i) \leq e_x$ and $\varepsilon_G(f_i) = f_{\varepsilon_G(e_i) \centerdot x}$ for $i = 1, 2$.

(a) If $e_1 \; \omega^r \; e_2$ then $f_1 \; \omega^r \; f_2$ and

$$\varepsilon_G(e_1, e_1e_2)\,(\varepsilon_G(e_1e_2) \centerdot x) = (\varepsilon_G(e_1) \centerdot x)\,\varepsilon_G(f_1, f_1f_2).$$

(b) If $e_1 \; \omega^l \; e_2$ then $f_1 \; \omega^l \; f_2$ and

$$\varepsilon_G(e_1, e_2e_1)\,(\varepsilon_G(e_2e_1) \centerdot x) = (\varepsilon_G(e_1) \centerdot x)\,\varepsilon_G(f_1, f_2f_1).$$

(IG2) All singular E-squares in E are ε_G-commutative.

E is called the *biordered set* of the inductive groupoid (G, ε_G) and ε_G is called the *evaluation* of $\mathfrak{C}(E)$ in G (or the evaluation of (G, ε_G)).

To simplify the notation we shall avoid the pair notation for inductive groupoids if no ambiguity is likely. We write G, G', etc. for inductive groupoids with biordered sets E, E', etc. and evaluations ε, ε' etc. Since $\varepsilon_G = \varepsilon$ is a $\mathfrak{v}$-isomorphism, it naturally induces a biorder structure on $\mathfrak{v}G$ which makes it a biordered set isomorphic to E. We shall identify $\mathfrak{v}G$ with E by ε and consider $\mathfrak{v}G$ itself as the biordered set of G; moreover, $\mathfrak{v}\varepsilon = 1_E$. From now on, we shall follow these conventions (if no ambiguity is likely).

inductive functor
$\mathfrak{IG}$:The category of inductive groupoids
inductive subgroupoid
$\mathfrak{v}$ – full inductive subgroupoid
inductive groupoid!isomorphism of –

DEFINITION 6.2. Let G and G' be inductive groupoids. An *inductive functor* $\phi : G \to G'$ is an order preserving functor such that

$$\begin{array}{ccc} \mathfrak{C}(\mathfrak{v}G) & \xrightarrow{\varepsilon_G} & G \\ {\scriptstyle \mathfrak{C}(\mathfrak{v}\phi)}\downarrow & & \downarrow{\scriptstyle \phi} \\ \mathfrak{C}(\mathfrak{v}G') & \xrightarrow[\varepsilon_{G'}]{} & G' \end{array} \tag{6.12}$$

$\mathfrak{v}\phi : \mathfrak{v}G \to \mathfrak{v}G'$ is a regular bimorphism making the diagram 6.12 is commutative.

It is clear that for every inductive groupoid G, the identity $1_G : G \to G$ is an inductive functor. Further, if $\phi : G \to G'$ and $\sigma : G' \to G''$ are inductive functors, then an easy verification with the diagram 6.12 above shows that $\phi \circ \psi : G \to G''$ is inductive. It follows that inductive groupoids with inductive functors as morphisms form a category $\mathfrak{IG}$. An inductive groupoid G' is an *inductive subgroupoid* of an inductive groupoid G if G' is an ordered subgroupoid of G and the inclusion $G' \subseteq G$ is inductive; that is G' is a subobject of G in $\mathfrak{IG}$. Also G' is a $\mathfrak{v}$ *– full inductive subgroupoid* of G if $\mathfrak{v}G' = \mathfrak{v}G$. An inductive functor $\phi : G \to G'$ is an *isomorphism* of inductive groupoids if ϕ is an isomorphism of ordered groupoids and $\mathfrak{v}\phi$ is a biorder isomorphism. It is easy to see from Diagram 6.12 that, in this case, $\phi^{-1} : G' \to G$ is also an inductive functor and hence an isomorphism in $\mathfrak{IG}$. Our aim in this chapter is to prove that the category $\mathfrak{IG}$ is naturally equivalent to the category $\mathfrak{RS}$ of regular semigroups.

Remark* 6.1*: Clearly there exists a forgetful functor $\mathbf{U}_\varepsilon : \mathfrak{IG} \to \mathfrak{OG}$ (that forgets evaluation) to the category of ordered groupoids. Again the assignments

$$\mathfrak{v} : G \mapsto \mathfrak{v}G, \quad \text{and} \quad \phi \mapsto \mathfrak{v}\phi$$

is a functor $\mathfrak{v} : \mathfrak{IG} \to \mathfrak{RB}$. Therefore, if $\mathfrak{C} : \mathfrak{RB} \to \mathfrak{OG}$ is the functor defined in Proposition 6.5, $\mathfrak{v} \circ \mathfrak{C} : \mathfrak{IG} \to \mathfrak{OG}$ is a functor. The diagram 6.12 shows that the evaluations are components of a natural transformation $\varepsilon : \mathbf{U}_\varepsilon \xrightarrow{n} \mathfrak{v} \circ \mathfrak{C}$.

The following facts about inductive groupoids are immediate consequences of the definitions.

PROPOSITION 6.6. *For an inductive groupoid G we have the following.*

h . x:Left restriction
x . h:Right restriction
restriction!left –
restriction!right –

(1) *An inductive groupoid G' is an inductive subgroupoid of G if and only if G' is an ordered subgroupoid of G such that*

$$\varepsilon_{G'} = \varepsilon_G \mid \mathfrak{C}(\mathfrak{v}G').$$

(2) $\operatorname{Im}\varepsilon_G$ *is an inductive subgroupoid of G with respect to the evaluation ε_G. Furthermore, a $\mathfrak{v}$-full ordered subgroupoid G' of G (so that $\mathfrak{v}G' = \mathfrak{v}G$) is an inductive subgroupoid if and only if*

$$\operatorname{Im}\varepsilon_G \subseteq G'.$$

(3) *The lattice of all $\mathfrak{v}$-full inductive subgroupoid of G is a complete lattice with* $\operatorname{Im}\varepsilon_G$ *as the* 0*-element.* □

Let G be an inductive groupoid, $x \in G$, $h\ \omega^r\ e_x$ and $g\ \omega^l\ f_x$. As in the ordered groupoid $\mathfrak{C}(E)$ (see Lemma 6.3), we define the morphisms $h \centerdot x$ and $x \centerdot g$ in G as follows:

$$h \centerdot x = \varepsilon(h, he_x)(he_x \centerdot x) = \varepsilon(h, he_x)(x|he_x) \tag{6.13}$$

where $he_x \centerdot x = x|he_x$ denote the restriction of x to $he_x\ \omega\ e_x$ in the ordered group of G; and

$$x \centerdot g = (x \centerdot f_x g)\varepsilon(f_x g, g) \tag{6.13*}$$

where $x \centerdot f_x g$ denote the co-restriction of x to $f_x g\ \omega\ f_x$.

For $h\ \omega^r\ e_x$ [$h\ \omega^l\ f_x$] the morphism $h \centerdot x$ [$x \centerdot h$] defined by Equation (6.13) [by Equation (6.13)*] is called the *left restriction*[the *right restriction*] of x to h. Clearly, if $h\ \omega\ e_x$ then the left restriction $h \centerdot x$ is the usual or the domain restriction of x to h and if $g\ \omega\ f_x$ then $x \centerdot g$ is the co-restriction or the co-domain restriction of x to g (see Subsection 1.4.2).

PROPOSITION 6.7. *Let $\phi : G \to G'$ be an inductive functor and $\mathfrak{v}\phi = \theta$.*

(1) *Let $x \in G$. If $h\ \omega^r\ e_x$ then*

$$\phi(h \centerdot x) = h\theta \centerdot \phi(x)$$

and if $g\ \omega^l\ f_x$, then

$$\phi(x \centerdot g) = \phi(x) \centerdot (g\theta).$$

(2) $\operatorname{Im}\phi = H$ *is an inductive subgroupoid of G'.*

(3) *If ϕ is $\mathfrak{v}$-bijective, then it is a $\mathfrak{v}$-isomorphism agnd if ϕ is a bijection, then ϕ is an isomorphism.*

Proof. **(1)** Since ϕ is an order preserving functor such that $\mathfrak{v}\phi = \theta : E \to E'$ is a regular bimorphism, by Equation (6.13),

$$\begin{aligned}
\phi(h \centerdot x) &= \phi(\varepsilon(h, he_x))\left((he_x)\theta \centerdot \phi(x)\right), \\
&= \varepsilon'(\mathfrak{C}(E)(\theta)(h, he_x))\left(h\theta e_x\theta \centerdot \phi(x)\right) && \text{since } \phi \text{ is inductive,} \\
&= \varepsilon'(h\theta, h\theta e_{x\phi})\left(h\theta e_{x\phi} \centerdot \phi(x)\right) && \text{since } e_x\theta = e_x\phi = e_{x\phi} \\
&= h\theta \centerdot \phi(x).
\end{aligned}$$

The remaining part of (1) follows by duality.

(2) Let $x', y' \in H$ and that $x'y'$ exists in G' so that $f_{x'} = e_{y'}$. Let $x, y \in G$ with $\phi(x) = x'$, $\phi(y) = y'$ and $h \in \mathscr{S}(f_x, e_y)$. Since θ is a regular bimorphism, we have

$$h\theta \in \mathscr{S}(f_x\theta, e_y\theta) = \mathscr{S}(f_{\phi(x)}, e_{\phi(y)}) = \mathscr{S}(f_{x'}, e_{y'})$$

and so, $h\theta = f_{x'} = e_{y'}$. Therefore by Equation (6.13), its dual and (1), we have

$$\begin{aligned}
\phi\left((x \centerdot h)(h \centerdot y)\right) &= \left(\phi(x \centerdot h)\right)((h \centerdot y)) \\
&= \left(\phi(x) \centerdot h\theta\right)\left(h\theta \centerdot \phi(y)\right) \\
&= (x' \centerdot f_{x'})(e_{y'} \centerdot y') = x'y'.
\end{aligned}$$

Therefore $x'y' \in H$. Since $u^{-1} \in H$ for all $u \in H$, H is a subgroupoid of G'. Let $x' \in H$ and let $h\ \omega\ e_{x'}$ where $h \in E_1 = \mathfrak{v}H$. Let $x \in G$ with $\phi(x) = x'$ so that $e_x\theta = e_{x'}$. Since θ is a regular bimorphism $E_1 = \operatorname{Im}\theta$ is a regular biordered subset of $E' = \mathfrak{v}G'$. It follows from Proposition 3.24 that there is $g \in E$ with $g\ \omega\ e_x$ and $g\theta = h$. Since ϕ is order preserving, we have $h \centerdot x' = g\theta \centerdot \phi(x) = \phi(g \centerdot x)$ and so, $h \centerdot x' \in H$. Therefore H is an ordered subgroupoid of G'.
Since $\mathfrak{v}H = E_1$, to prove that H is inductive, by Proposition 6.6, it is sufficient to prove that $\varepsilon_1 = \varepsilon' \mid \mathfrak{C}(E_1)$ maps $\mathfrak{C}(E_1)$ into H. If $c \in \mathfrak{C}(E_1)$ we have by 6.12,

$$\varepsilon\left(\mathfrak{C}(\theta)(c)\right) = \phi(\varepsilon(c)).$$

Hence we must show that every E-chain $c' \in \mathfrak{C}(E_1)$ there is $c \in \mathfrak{C}(E)$ such that $\mathfrak{C}(\theta)(c) = c'$. Assume inductively that every E-chain in E_1 with n vertexes satisfy this and let $c' = c(e'_0, e'_1, \ldots, e'_n)$ be an E-chain with $n+1$ vertexes. Then by hypothesis there is a chain

$$c = c(e_1, \ldots, e_n) \in \mathfrak{C}(E) \quad \text{with} \quad \mathfrak{C}(\theta)(c) = c(e'_1, \ldots, e_{n'})$$

so that $e'_i = e_i\theta$ for $i = 1, 2, \ldots, n$. Let $e'_0\ \mathscr{R}\ e'_1$. Then, by Proposition 3.24, there exists $h\ \omega^r\ e_1$ such that $h\theta = e'_0$. Then, by Equation (6.5a), $h \centerdot c = c(h = h_0, h_1, \ldots, h_n)$ is the left restriction of c to h and by Equation (6.10),

$$\mathfrak{C}(\theta)(h \centerdot c) = c(h'_0, h'_1, \ldots, h'_n) \quad \text{where} \quad h'_i = h_i\theta, \quad i = 0, 1, \ldots, n.$$

Now by the choice of $h = h_0$,

$$h_0\theta = h_{0'} = e'_0, \quad h'_1 = (he_1)\theta = e'_0 e'_1 = e'_1$$

and for each $i = 2, \dots, n$, by Equation (6.5a)

$$h'_i = h_i\theta = (h_{i-1}\tau(e_{i-1}, e_i))\,\theta$$
$$= (h_{i-1}\theta)\tau(e_{i-1}\theta, e_i\theta) = h'_{i-1}\tau(e'_{i-1}, e'_i).$$

Inductively, $h'_i = e'_i$ for all $i = 0, 1, \dots, n$. Therefore $\mathfrak{C}(\theta)(h \centerdot c) = c'$. If $e'_0 \mathscr{L} e'_1$, again by Proposition 3.24, there is $k\ \omega^l\ e_1$ such that $k\theta = e'_0$. Then $(e_1 k)\theta = e'_1 e{+}'_0 = e'_1$ and, as before, we can show that

$$\mathfrak{C}(\theta)\,(c(k, e_1k)(e_1k \centerdot c)) = c'.$$

(3) If ϕ is a $\mathfrak{v}$-bijection by Corollary 3.25, $\mathfrak{v}\phi = \theta$ is a biorder isomorphism. If $\phi : G \to G'$ is a bijection, it is clearly an isomorphism of groupoids. By the above, $\mathfrak{v}\phi = \theta$ is a biorder isomorphism. Let $x' \le y'$ for $x', y' \in G'$. Then $\phi(x) = x'$ and $\phi(y) = y'$ for $x, y \in G$. Since $x' \le y'$, $e_x\theta = e_{x'}\ \omega\ e_{y'} = e_y\theta$. Therefore $e_x\ \omega\ e_y$. Also we have

$$\phi(x) = e_x\theta \centerdot \phi(y) = \phi(e_x \centerdot y)$$

and so, $x = e_x \centerdot y$. Therefore $x \le y$ and hence ϕ is an order isomorphism. Therefore, by definition, ϕ is an isomorphism of inductive groupoids. □

6.2 THE INDUCTIVE GROUPOID OF A REGULAR SEMIGROUP

We proceed to show that we can associate a unique inductive groupoid with every regular semigroups. This is similar to the situation for inverse semigroups even though the relation between a regular semigroup S and its inductive groupoid $\boldsymbol{G}(S)$ considerably subtler.

We begin by constructing the ordered group of $\boldsymbol{G}(S)$.

6.2.1 The ordered groupoid $\boldsymbol{G}(S)$

LEMMA 6.8. *Let* $\boldsymbol{G}(S)\{(x, x') :\in S,\ x' \in \mathscr{V}(x)\}$ *(see Equation* (6.1)*). For* $(x, x'), (y, y') \in \boldsymbol{G}(S)$ *define*

$$(x, x')(y, y') = (x * y, y' * x') \quad \text{if} \quad x'x = yy'. \tag{6.14}$$

Then $\boldsymbol{G}(S)$ *is a groupoid with respect to the composition defined above. For* $(x, x') \in \boldsymbol{G}(S)$, $e_{(x,x')} = (xx', xx')$ *is the left identity,* $f_{(x,x')} = (x'x, x'x)$ *is the right identity and* (x', x) *is the inverse.*

Proof. First observe that when the condition $x'x = yy'$ is satisfied, the trace products $x * y$ and $y' * x'$ exist and $(x, x')(y, y') \in \boldsymbol{G}(S)$ by Theorem 3.7. Suppose that (u, u') is a left identity of (x, x'). Then we must have

$$u'u = xx', \quad ux = x \quad \text{and} \quad x'u' = x'.$$

These give
$$u = uu'u = uxx' = xx', \quad \text{and} \quad u' = xx'u' = xx'.$$

Hence (xx', xx') is the unique left identity of (x, x'). Similarly $(x'x, x'x)$ is the unique right identity of (x, x'). Associativity of the composition defined by Equation (6.14) is a consequence of the associativity of trace products. Hence $\boldsymbol{G}(S)$ is a groupoid in which the inverse of the morphism (x, x') is (x', x). □

In what follows, we denote by $\boldsymbol{G}(S)$ the groupoid in which morphisms are pairs (x, x') with $x \in S$ and $x' \in \mathscr{V}(x)$ and with composition defined by Equation (6.14). It is clear that the map $g \mapsto (g, g)$ is a bijection of $\boldsymbol{E}(S)$ onto $\mathfrak{v}\boldsymbol{G}(S)$. Therefore, in the following, we shall regard $\boldsymbol{G}(S)$ as a groupoid with $\mathfrak{v}\boldsymbol{G}(S) = \boldsymbol{E}(S)$.

LEMMA 6.9. *Let $\boldsymbol{G}(S)$ be the groupoid defined in Lemma 6.8. Then*

$$(x, x') \leq (y, y') \quad \text{if} \quad x = (xx')y, \; x' = y'(xx') \quad \text{and} \quad xx' \omega yy'. \tag{6.15}$$

defines a partial order on $\boldsymbol{G}(S)$ with respect to which $\boldsymbol{G}(S)$ is an ordered groupoid such that $\mathfrak{v}\boldsymbol{G}(S)$ is order isomorphic with $(\boldsymbol{E}(S), \omega)$.

Proof. The relation $\leq$ defined by Equation (6.15) is clearly reflexive and anti-symmetric. If $(x, x') \leq (y, y') \leq (z, z')$ then $xx' \omega zz'$ and

$$x = (xx')y = (xx')(yy')z = (xx')z \quad \text{and} \quad x' = z'(xx').$$

Therefore $(x, x') \leq (z, z')$. Thus $\leq$ is a partial order on $\boldsymbol{G}(S)$. From Equation (6.14) we see that the partial order induced by this order $\leq$ on $\mathfrak{v}\boldsymbol{G}(S) = \boldsymbol{E}(S)$ coincide with ω.

Let $(x, x') \leq (y, y')$. Then we have

$$x'x = y'(xx')(xx')y = y'(xx')y \;\omega\; y'y.$$

Therefore
$$x = (xx')y = (yy')(xx')y = y(y'(xx')y) = y(x'x)$$
and
$$x' = (x'x)y'.$$

Consequently, $(x, x')^{-1} = (x', x) \leq (y', y) = (y, y')^{-1}$. Hence axiom (OG2) of Definition 1.6 hold. Axiom (OG3) hold if we define restriction in $\boldsymbol{G}(S)$ as follows.

$$e \centerdot (x, x') = (x, x')|e = (ex, x'e) \quad \text{for all} \quad (x, x') \in \boldsymbol{G}(S) \quad \text{and} \quad e \;\omega\; xx'. \tag{6.15*}$$

Now let $(u, u') \leq (x, x')$ and $(v, v') \leq (y, y')$. If products $(u, u')(v, v')$ and $(x, x')(y, y')$ exists in $\boldsymbol{G}(S)$, then

$$(uv)(v'u') = u(vv')u' = u(u'u)u' = uu'$$
$$\omega\ xx' = (xy)(y'x');$$
$$(uv)(v'u')xy = (uu')xy = uy = u(vv')y = uv;$$

and similarly, $$(y'x')(uv)(v'u') = v'u'.$$

Therefore $$(u, u')(v, v') = (uv, v'u') \leq (xy, y'x') = (x, x')(y, y').$$

Therefore axiom (OG1) also holds. □

6.2.2 The inductive groupoid

We now define an evaluation (see Definition 6.1) of $\mathfrak{C}(\boldsymbol{E}(S))$ in $\boldsymbol{G}(S)$ so that $\boldsymbol{G}(S)$ becomes an inductive groupoid.

LEMMA 6.10. *There is an order preserving functor $\varepsilon_S : \mathfrak{C}(E) \to \boldsymbol{G}(S)$ with $\mathfrak{v}\varepsilon_S = 1_E$ and morphism map defined as follows: For each $c = c(e_0, e_1, \dots, e_n) \in \mathfrak{C}(E)$*

$$\varepsilon_S(c) = (w_c, w_{c^{-1}}) \tag{6.16}$$

where $w_c = e_0 e_1 \dots e_{n-1} e_n$.

Proof. First notice that $w_{c^{-1}} \in \mathscr{V}(w_c)$. For either $e_{i-1} \mathscr{R} e_i$ so that $e_{i-1}e_i = e_i$ or $e_{i-1} \mathscr{L} e_i$ so that $e_{i-1}e_i = e_{i-1}$. Hence

$$w_c w_{c^{-1}} = e_0 e_1 \dots e_n e_n e_{n-1} \dots e_0 = e_0 \qquad \text{and}$$
$$w_{c^{-1}} w_c = e_n e_{n-1} \dots e_0 e_0 e_1 \dots e_n = e_n.$$

It follows that $w_c w_{c^{-1}} w_c = w_c$ and $w_{c^{-1}} w_c w_{c^{-1}} = w_{c^{-1}}$. This proves that $(w_c, w_{c^{-1}}) \in \boldsymbol{G}(S)$. Also,

$$w_c \mathscr{R} e_0 = e_c \mathscr{L} w_{c^{-1}} \mathscr{R} e_n = f_c \mathscr{L} w_c.$$

Suppose that $c, d \in \mathfrak{C}(E)$ and that cd exists. Then $f_c = e_d$ and so, $w_{c^{-1}} w_c = f_c = e_d = w_d w_{d^{-1}}$. Therefore, by Equation (6.14) the composite $(w_c, w_{c^{-1}})(w_d, w_{d^{-1}})$ exists in $\boldsymbol{G}(S)$. Moreover $w_{cd} = w_c w_d$ and $w_{(cd)^{-1}} = w_{d^{-1}} w_{c^{-1}}$. Therefore

$$\begin{aligned} \varepsilon_S(s)\varepsilon_S(d) &= (w_c, w_{c^{-1}})(w_d, w_{d^{-1}}) \\ &= (w_c w_d, w_{d^{-1}} w_{c^{-1}}) = (w_{cd}, w_{(cd)^{-1}}) \\ &= \varepsilon_S(sd) \end{aligned}$$

by the definition of ε_S. Since $\varepsilon_S(c(e, e)) = (e, e)$, $\varepsilon_S : \mathfrak{C}(E) \to \boldsymbol{G}(S)$ is a functor such that $\mathfrak{v}\varepsilon_S = 1_E$. Let $c = c(e_0, e_1, \dots, e_n) \in \mathfrak{C}(E)$ and $h\ \omega\ e_0$. If

$$h \centerdot c = c(h, h_1, \dots, h_n) \quad \text{then} \quad h_i = e_i h_{i-1} e_i \quad \text{for all} \quad i = 1, \dots, n.$$

This gives

$$\begin{aligned} hh_1h_2\dots h_n &= h(e_1e_0he_0e_1)\dots(e_ne_{n-1}\dots e_0he_0e_1\dots h_n) \\ &= he_0e_1\dots e_n \quad \text{and similarly,} \\ h_nh_{n-1}\dots h_1h &= e_ne_{n-1}\dots e_0h. \end{aligned}$$

Therefore

$$\begin{aligned} \varepsilon_S(h \centerdot c) &= (hh_1h_2\dots h_n, h_nh_{n-1}\dots h_1h) \\ &= (he_0e_1\dots e_n, e_ne_{n-1}\dots e_0h) \\ &= h \centerdot (e_0e_1\dots e_n, e_ne_{n-1}\dots e_0) \\ &= h \centerdot \varepsilon_S(c). \end{aligned}$$

Thus $\varepsilon_S : \mathfrak{C}(E) \to \boldsymbol{G}(S)$ is an order preserving functor. □

THEOREM 6.11. *Let S be a regular semigroup and let* $E = \boldsymbol{E}(S)$.

(a) Let $\boldsymbol{G}(S)$ *be the ordered groupoid of S and* ε_S *be the order preserving* $\mathfrak{v}$*-isomorphism defined in the lemma above. Then* $(\boldsymbol{G}(S), \varepsilon_S)$ *is an inductive groupoid.*

(b) Let $\phi : S \to S'$ *be a homomorphism of regular semigroups. Then there exists an inductive functor* $\boldsymbol{G}(\phi) : \boldsymbol{G}(S) \to \boldsymbol{G}(S')$ *such that*

$$\begin{aligned} \mathfrak{v}\boldsymbol{G}(\phi) &= \boldsymbol{E}(\phi) \quad \text{and} \\ \boldsymbol{G}(\phi)(x, x') &= (x\phi, x'\phi) \quad \text{for all} \quad (x, x') \in \boldsymbol{G}(S). \end{aligned} \tag{6.17}$$

(c) The assignments

$$\boldsymbol{G} : S \mapsto \boldsymbol{G}(S) \quad \text{and} \quad \phi \mapsto \boldsymbol{G}(\phi)$$

is a functor $\boldsymbol{G} : \mathfrak{RS} \to \mathfrak{IG}$

Proof. **(a)** By Lemmas 6.8 and 6.9, $\boldsymbol{G}(S)$ is an ordered groupoid. Lemma 6.10 constructs an evaluation $\varepsilon = \varepsilon_S : \mathfrak{C}(E) \to \boldsymbol{G}(S)$ (see Equation (6.16)) where $E = \boldsymbol{E}(S)$. So, to show that $\boldsymbol{G}(S)$ is an inductive groupoid, it is sufficient to verify axioms (IG1) and (IG2). So, let $(x, x') \in \boldsymbol{G}(S)$, $e_1, e_2 \in \omega(e_{(x,x')})$. If $f_i = f_{e_i \centerdot (x,x')}$, then by Equation (6.15*)

$$f_i = f_{(e_ix, x'e_i)} = (x'e_i)(e_ix) = x'(e_i)x$$

so that $f_1 \ \omega^r \ f_2$ if and only if $e_1 \ \omega^r \ e_2$. Similarly $f_1 \ \omega^l \ f_2$ if and only if $e_1 \ \omega^l \ e_2$. Also, if $e_1 \ \omega^r \ e_2$, then

$$\begin{aligned} \varepsilon(e_1, e_1e_2)\,(e_1e_2 \centerdot (x, x')) &= (e_1e_2, e_1)(e_1e_2x, x'e_1e_2) \\ &= (e_1e_2x, x'e_1); \end{aligned}$$

and

$$\begin{aligned}(e_1 \centerdot (x,x'))\,\varepsilon(f_1, f_1f_2) &= (e_1x, x'e_1)(f_1f_2, f_1) = (e_1xf_1f_2, f_1x'e_1)\\ &= (e_1(xx')e_1(xx')e_2x, x'e_1(xx')e_1)\\ &= (e_1e_2x, x'e_1).\end{aligned}$$

This proves axiom (IG1)(a). (IG1)(b) is proved dually. Now suppose that $\left(\begin{smallmatrix} g & ge\\ h & he\end{smallmatrix}\right)$ be a column singular E-square so that $g, h \in \omega^r(e)$ and $g \mathscr{L} h$. Then

$$\begin{aligned}\varepsilon(g,h)\varepsilon(h,he) &= (g,h)(he,h) = (ge,h)\\ \varepsilon(g,ge)\varepsilon(ge,he) &= (ge,g)(ge,he) = (ge,h).\end{aligned}$$

Dually all row singular E-squares also commute.

(b) Equation (6.17) shows that $\boldsymbol{G}(\phi)$ maps $\boldsymbol{G}(S)$ to $\boldsymbol{G}(S')$. If $(x,x')(y,y')$ exists in $\boldsymbol{G}(S)$, then $x'x = yy'$ and so

$$(x')\phi(x)\phi = (x'x)\phi = (yy')\phi = (y)\phi(y')\phi.$$

Therefore the product

$$\big((x,x')\boldsymbol{G}(\phi)\big)\big((y,y')\boldsymbol{G}(\phi)\big) = (x\phi, x'\phi)(y\phi, y'\phi)$$

exists in $\boldsymbol{G}(S')$ and

$$\begin{aligned}\big((x,x')\boldsymbol{G}(\phi)\big)\big((y,y')\boldsymbol{G}(\phi)\big) &= (x\phi y\phi, y'\phi x'\phi) = \big((xy)\phi, (y'x')\phi\big)\\ &= ((x,x')(y,y'))\,\boldsymbol{G}(\phi)\end{aligned}$$

Thus $\boldsymbol{G}(\phi) : \boldsymbol{G}(S) \to \boldsymbol{G}(S')$ is a functor. Also, for any $g\omega e_{(x,x')}$, $(x,x') \in \boldsymbol{G}(S)$,

$$\begin{aligned}(g \centerdot (x,x'))\,\boldsymbol{G}(\phi) &= (gx, x'g)\boldsymbol{G}(\phi)\\ &= \big((gx)\phi, (x'g)\phi\big)\\ &= \big(g\theta)(x\phi), (x'\phi)(g\theta)\big) \qquad \text{where} \quad \theta = \boldsymbol{E}(\phi)\\ &= (g\theta) \centerdot (x,x')\boldsymbol{G}(\phi).\end{aligned}$$

This shows that $\boldsymbol{G}(\phi)$ is order preserving. If $c = c(e_0, e_1, \ldots, e_n) \in \mathfrak{C}(E)$ then

$$w_c\phi = w_{c\theta}.$$

Therefore

$$\begin{aligned}(\varepsilon_S(c))\,\boldsymbol{G}(\phi) &= (w_c, w_{c^{-1}})\boldsymbol{G}(\phi)\\ &= (w_c\phi, (w_{c^{-1}})\phi) = (w_{c\theta}, w_{(c\theta)^{-1}})\\ &= \varepsilon_{S'}(c\theta).\end{aligned}$$

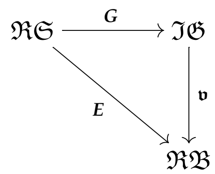
This proves that $\boldsymbol{G}(\phi) : \boldsymbol{G}(S) \to \boldsymbol{G}(S')$ is an inductive functor.

(c) By (a) and (b), given assignments are single valued. Suppose that $\phi : S \to S'$ and $\psi : S' \to S''$ be homomorphisms of regular semigroups. Then for any $(x, x') \in \boldsymbol{G}(S)$ we have

$$(x, x')\boldsymbol{G}(\phi\psi) = (x\phi\psi, x'\phi\psi) = \big((x, x')\boldsymbol{G}(\phi)\big)\,\boldsymbol{G}(\psi).$$

Hence $\boldsymbol{G}(\phi\psi) = \boldsymbol{G}(\phi) \circ \boldsymbol{G}(\psi)$ and so $\boldsymbol{G}$ is a functor as desired. □

By the convention established above, we have $\boldsymbol{E}(S) = \mathfrak{v}\boldsymbol{G}(S)$ for every regular semigroup S and $\boldsymbol{E}(\phi) = \mathfrak{v}\boldsymbol{G}(\phi)$ for all homomorphism $\phi : S \to S'$ of regular semigroups. Thus the following diagram of categories and functors commute:



(6.18)

6.2.3 Exercise

Exercise 6.1: Determine the inductive groupoid $\boldsymbol{G}(S)$ in the fallowing cases.

1. $S = \mathscr{T}_X$ where X is a set.
2. $S = \mathscr{L}\mathscr{T}(V)$ where V is a vector space over a field $\Bbbk$.

Exercise 6.2: Let G be an ordered groupoid. Show that it is possible to have more than one biorder structure on $E = \mathfrak{v}G$ which makes G an inductive groupoid.

Exercise 6.3: Let S be an orthodox semigroup (see **??**). Show that $\boldsymbol{G}(S)$ be comes an orthodox semigroup $\tilde{S}$ if we extend the composition in $\boldsymbol{G}(S)$ by:

$$(x, x')(y, y') = (xy, y'x').$$

Find the biordered set $\boldsymbol{E}(\tilde{S})$. Can you characterize all those orthodox semigroups that arise as $\tilde{S}$ for some orthodox semigroup S ?

6.3 STRUTURE OF REGULAR SEMIGROUPS

In Section 6.2 we have associated an inductive groupoid with every regular semigroups. Here we shall show that we an construct a regular semigroup $\boldsymbol{S}(G)$ from an inductive groupoid G and a hmomorphism $\boldsymbol{S}(\phi)$ from an inductive functor ϕ such that these assignments gives a functor $\boldsymbol{S} : \mathfrak{IG} \to \mathfrak{RS}$. Moreover, $\boldsymbol{S}$ is the adjoint inverse of the funcctor $\boldsymbol{G}$ of Theorem 6.11 so that $\boldsymbol{G}$ is an equivalence of the category $\mathfrak{RS}$ of regular semigroups with the category $\mathfrak{IG}$ of inductive groupoids.

6.3.1 The regular semigroup of an indutive groupoid

In the following G denotes an indutive groupoid with vertex biordered set E and evaluation ε. By convension established for cateories in Section 1.2, Chapter 1, G itself denote the set of morphisms of the groupoid G (see also Section 1.3).

On G define the relation $\boldsymbol{p}$ as follows:

$$x \boldsymbol{p} y \iff e_x \mathscr{R} e_y,\ f_x \mathscr{L} f_y \text{ and } x\varepsilon(f_x, f_y) = \varepsilon(e_x, e_y)y. \tag{6.19}$$

In view of the conditions $e_x \mathscr{R} e_y$ and $f_x \mathscr{L} f_y$ the last equality is equivalent to

$$x \centerdot f_y = e_x \centerdot y. \tag{6.19*}$$

It is easy to see that the relatin $\boldsymbol{p}$ is reflexive and symmetric. Also, if $x \boldsymbol{p} y \boldsymbol{p} z$, it follows from Equation (6.13) and Lemma 6.14 that

$$x \centerdot f_z = (x \centerdot f_y) \centerdot f_z = e_x \centerdot (y \centerdot f_z) = e_x \centerdot (e_y \centerdot z) = e_x \centerdot z.$$

Hence $x \boldsymbol{p} z$. Thus $\boldsymbol{p}$ is an equivalence relation. It is clear that no two dustinct morphisms in a home-set $G(e, f)$ can be $\boldsymbol{p}$ equivalent. In particular, no two identities are $\boldsymbol{p}$ equivalent.

LEMMA 6.12. *The relation $\boldsymbol{p}$ on (the morphism set) of an inductive groupoid G defined by Equation* (6.19) *is an equivalence relation such that $x, y \in G(e, f)$ and $x \boldsymbol{p} y$ implies $x = y$. In particular, no two identites are $\boldsymbol{p}$-equivalent.*

Next theorem gives the basic cconstruction of a regular semigroup from inductive groupoids.

THEOREM 6.13. *Let G be an inductive groupoid and let $\boldsymbol{S}(G) = G/\boldsymbol{p}$. For each $x \in G$, let $\bar{x}$ denote the $\boldsymbol{p}$-lass containing x. For $x, y \in G$ and $h \in \mathscr{S}(f_x, e_y)$ let*

$$\bar{x}\bar{y} = \overline{(x \circ y)_h}. \tag{6.20}$$

This defines a binary operation on $\boldsymbol{S}(G)$ and $\boldsymbol{S}(G)$ is a regular semigroup with respet to this operation. Furthermore, the map $\chi_G : e \mapsto \bar{e}$ is a biorder isomrphism of $\mathfrak{v}G = E$ onto $\boldsymbol{E}(\boldsymbol{S}(G))$.

We shall divide the proof into a number of preliminary lemmas. Recall for all $x \in G$, the map $\mathfrak{a}(x) : \omega(e_x) \to \omega(f_x)$ defined by Equation (6.27) is an ω-isomorphism and that the map $\mathfrak{a}_G : x \mapsto \mathfrak{a}(x)$ is a $\mathfrak{v}$-isomorphism of G to T^*_E (see Theorem 6.28).

LEMMA 6.14. *Let $x \in G$ and suppose that $h\ \omega^r\ e_x$ and $k\ \omega^l\ f_x$ such that $f_{h \centerdot x} = f_x k$. Then*

$$(h \centerdot x) \centerdot k = h \centerdot (x \centerdot k)$$

where $h \centerdot x$ and $x \centerdot k$ are left and right restrictions of x defined in Equation (6.13) *and its dual.*

Proof. It is clear from Equation (6.13) and its dual that the codomain of $h \centerdot x$ and $he_x \centerdot x$ are the same. Hence

$$f_{h\centerdot x} = f_{he_x\centerdot x} = (he_x)\mathfrak{a}(x) = f_x k$$

by Equation (6.27). Similarly

$$e_{x\centerdot k} = e_{x\centerdot f_x k} = f_{f_x k\centerdot (x)^{-1}} = (f_x k)\mathfrak{a}(x^{-1}) = he_x.$$

Therefore $f_{h\centerdot x} \mathscr{L}\ k$ and $e_{x\centerdot k} \mathscr{R}\ h$. Thus the expressions $(h \centerdot x) \centerdot k$ and $h \centerdot (x \centerdot k)$ are defined by Equations 6.13 and its dual. Again it follows from these that

$$\begin{aligned}(h \centerdot x) \centerdot k &= ((h \centerdot x) \centerdot (f_{h\centerdot x}k))\, \varepsilon(f_{h\centerdot x}k, k)\\ &= ((h \centerdot x) \centerdot f_{h\centerdot x})\, \varepsilon(f_{h\centerdot x}, k)\\ &= (h \centerdot x)\varepsilon(f_x k, k)\\ &= \varepsilon(h, he_x)\,(he_x \centerdot x)\, \varepsilon(f_x k, k).\end{aligned}$$

Similarly
$$h \centerdot (x \centerdot k) = \varepsilon(h, he_x)\,(x \centerdot f_x k)\, \varepsilon(f_x k, k).$$

Now $he_x \centerdot x \leq x$, $x \centerdot f_x k \leq x$. Also, by the given condition, the codomains of $he \centerdot x$ is $f_x k$ so that

$$\mathrm{cod}(he_x \centerdot x) = f_x k = \mathrm{cod}(x \centerdot f_x k).$$

Hence by the dual of axiom (OG3), we have $he_x \centerdot x = x \centerdot f_x k$. This proves the lemma. □

If h and k satisfy the conditions of Lemma 6.3 then the common value of the expressions $(h \centerdot x) \centerdot k$ and $h \centerdot (x \centerdot k)$ will be denoted by $h \centerdot x \centerdot k$.

LEMMA 6.15. *Let $x \in G$ and $g, h \in E$. If $g\ \omega^r\ h\ \omega^r\ e_x$ and $ge_x\ \omega\ he_x$, then $g \centerdot (h \centerdot x) = g \centerdot x$. If $g\ \omega^l\ h\ \omega^l\ f_x$ and $f_x g\ \omega\ f_x h$, then $(x \centerdot h) \centerdot g = x \centerdot g$*

Proof. By Equation (6.13) and Proposition 1.19(2), we have

$$g \centerdot (h \centerdot x) = \varepsilon(g, gh)\,(gh \centerdot \varepsilon(h, he_x))\,(k \centerdot x)$$

where $k = f_{gh\centerdot\varepsilon(h,he_x)}$. Since ε is order preserving, and $h\ \mathscr{R}\ he_x$ by Equation (6.2), we have

$$gh \centerdot \varepsilon(h, he_x) = \varepsilon(gh \centerdot (h, he_x)) = \varepsilon(gh, (gh)(he_x))$$

so that

$$k = (gh)(he_x) = g(he_x) = (ge_x)(he_x) = ge_x.$$

$(x \circ y)_h$:Defined by Eqrefeq:18ind

Since $g \mathscr{R} gh \mathscr{R} ge_x$, we have

$$g \centerdot (h \centerdot x) = \varepsilon(g, gh)\varepsilon(gh, ge_x)(ge_x \centerdot x) = \varepsilon(g, ge_x)(ge_x \centerdot x) = g \centerdot x.$$

The second statement follows by duality. □

LEMMA 6.16. *For $x, y \in G$ and $h \in \mathscr{S}(f_x, e_y)$, define*

$$(x \circ y)_h = (x \centerdot h)(h \centerdot y). \tag{6.21}$$

Then we have

$$k \centerdot (x \circ y)_h = (k \centerdot x \centerdot g)(g \centerdot y) \qquad \text{if } g \in M(f_x, h),\ k\ \omega^r\ e_{x\centerdot h} \text{ and } f_{k\centerdot x} = f_x g$$

and $$(x \circ y)_h \centerdot k = (x \centerdot g)(g \centerdot x \centerdot k) \qquad \text{if } g \in M(h, e_y),\ k\ \omega^l\ f_{h\centerdot y} \text{ amd } e_{y\centerdot k} = ge_y.$$

Proof. By Equations (6.21) and the dual of (6.13), we have $e_{x\centerdot f_x h} = e_{x\centerdot h} = e_{(x\circ y)_h}$. Then by Proposition 1.19, we have

$$k \centerdot (x \circ y)_h = \varepsilon(k, kh_1)\,(kh_1 \centerdot (x \centerdot h))\,(g_1 \centerdot (h \centerdot y))\,.$$

where $h_1 = e_{x\centerdot f_x h}$ and $g_1 = f_{kh_1\centerdot(x\centerdot h)}$. By Proposition 1.19, $h_1 \centerdot x = x \centerdot f_x h$ and so, by Equation (6.27) $f_x h = (h_1)\mathfrak{a}(x)$. Again, by Theorem 6.28 $\mathfrak{a}_G : G \to T^*_E$ is an inductive functor and so, we have

$$\begin{aligned}
g_1 &= (kh_1)\mathfrak{a}(x \centerdot h) && \text{by Equation (6.27)}\\
&= (kh_1)\mathfrak{a}(x \centerdot f_x h)\mathfrak{a}(\varepsilon(f_x h, h)) && \text{by Equation (6.13)}^*\\
&= (kh_1)\mathfrak{a}(h_1 \centerdot x)\tau(f_x h, h) && \text{since diagram 6.30 commutes}\\
&= (kh_1)\mathfrak{a}(x)\tau(f_x h, h) &&\\
&= (k)\mathfrak{a}(x)(h_1)\mathfrak{a}(x)\tau(f_x h, h) &&\\
&= h(f_x g)(f_x h) && \text{by the given conditions}\\
&= h(f_x(gh)) = gh. &&
\end{aligned}$$

By Proposition 1.19(1) $kh_1 \centerdot (x \centerdot h) = (x \centerdot h) \centerdot g_1$. Since $g_1\ \omega\ h\ \omega^l\ f_x$ and $g_1\ \omega\ h\ \omega^r\ e_y$, by Lemma 6.14

$$(x \centerdot h) \centerdot g_1 = x \centerdot g_1 \quad \text{and} \quad g_1 \centerdot (h \centerdot y) = g_1 \centerdot y.$$

Therefore $$\begin{aligned} k \centerdot (x \circ y)_h &= \varepsilon(k, kh_1)(x \centerdot g_1)(g_1 \centerdot y)\\ &= (k \centerdot x \centerdot g_1)(g_1 \centerdot y)\end{aligned}$$

using Equation (6.13).Again, since $g \mathscr{R} g_1$, by Equation (6.13) we have

$$g \centerdot y = \varepsilon(g, g_1)(g_1 \centerdot y)$$

and

$$\begin{aligned} k \centerdot x \centerdot g &= (k \centerdot x \centerdot f_x g)\varepsilon(f_x g, g) \\ &= (k \centerdot x \centerdot f_x g_1)\varepsilon(f_x g_1, f_x g)\varepsilon(f_x g, g) \\ &= (k \centerdot x \centerdot f_x g_1)\varepsilon(f_x g_1, g_1)\varepsilon(g_1, g) \qquad \text{by axiom (IG2)} \\ &= (k \centerdot x \centerdot g_1)\varepsilon(g_1, g). \end{aligned}$$

Hence

$$\begin{aligned} k \centerdot (x \circ y)_h &= (k \centerdot x \centerdot g)\varepsilon(g, g_1)\varepsilon(g_1, g)(g \centerdot y) \\ &= (k \centerdot x \centerdot g)(g \centerdot y). \end{aligned}$$

The second statement follows by dualiity. □

LEMMA 6.17. *Let $x, y, z \in G$, $h_1 \in \mathscr{S}(f_x, e_y)$ and $h_2 \in \mathscr{S}(f_y, e_z)$. Write $h'_1 = f_{h_1 \centerdot y}$ and $h'_2 = e_{y \centerdot h_2}$. Then there exist $h \in \mathscr{S}(f_x, h'_2)$ and $h' \in \mathscr{S}(h'_1, e_z)$ such that*

$$((x \circ y)_{h_1} \circ z)_{h'} = (x \circ (y \circ z)_{h_2})_h.$$

Proof. Since $h'_1 = f_{h_1 \centerdot y} = f_{h_1 e_y \centerdot y}$, and $h'_2 = e_{y \centerdot f_y h_2}$ by Equation (6.27),

$$h'_1 = (h_1 e_y)\mathfrak{a}(y) \quad \text{and} \quad h'_2 = (f_y h_2)\mathfrak{a}(y^{-1})$$

By Corollary 3.23 there is $h \in \mathscr{S}(h_1, h'_2) \subseteq \mathscr{S}(f_x, h'_2)$ and $h' \in \mathscr{S}(h'_1, h_2) \subseteq \mathscr{S}(h'_1, e_z)$ such that $(h e_y)\mathfrak{a}(y) = f_y h'$. Then we have $f_{h \centerdot y} = f_y h'$. Since $h' \in M(f_y, h_2)$, by Lemma 6.17,

$$h \centerdot (y \circ z)_{h_2} = (h \centerdot y \centerdot h')(h' \centerdot z).$$

Therefore

$$(x \circ (y \circ z)_{h_2})_h = (x \centerdot h)(h \centerdot y \centerdot h')(h' \centerdot z).$$

Since h_1, h'_1, h and h' satisfy the dual hypothesis, we obtain by dual arguements that

$$((x \circ y)_{h_1} \circ z)_{h'} = (x \centerdot h)(h \centerdot y \centerdot h')(h' \centerdot z).$$

This proves the lemma. □

LEMMA 6.18. *Let $x \boldsymbol{p} y$ in G. Tnen*

$$h \centerdot x \,\boldsymbol{p}\, h \centerdot y \qquad \textit{for all } h \,\omega^r\, e_x$$

and dually,

$$x \centerdot g \,\boldsymbol{p}\, y \centerdot g \qquad \textit{for all } g \,\omega^l\, f_x.$$

Proof. By Equation (6.13) $h \centerdot x = \varepsilon(h, he_x)(he_x \centerdot x)$. Let $h_1 = f_{he_x \centerdot x}$. Then by Proposition 1.19,

$$\begin{aligned} he_x \centerdot \big(x\varepsilon(f_x, f_y)\big) &= (he_x \centerdot x)\big(h_1 \centerdot \varepsilon(f_x, f_y)\big) = (he_x \centerdot x)\varepsilon(h_1, f_y h_1); \\ he_x \centerdot \big(\varepsilon(e_x, e_y)y\big) &= \varepsilon(he_x, he_y)(he_y \centerdot y). \end{aligned}$$

Since $x\varepsilon(f_x, f_y) = \varepsilon(e_x, e_y)y$, we have

$$\begin{aligned}(h \centerdot x)\varepsilon(h_1, f_y h_1) &= \varepsilon(h, he_x)(he_x \centerdot x)\varepsilon(h_1, f_y h_1)\\ &= \varepsilon(h, he_x)\varepsilon(he_x, he_y)(he_y \centerdot y)\\ &= \varepsilon(h, he_y)(he_y \centerdot y) = h \centerdot y.\end{aligned}$$

Clearly $h_1 \mathscr{L} f_y h_1$ and so, $h \centerdot x \; \boldsymbol{p} \; h \centerdot y$. The second statement is the dual of the first. □

Lemma 6.19. *Let $x \; \boldsymbol{p} \; x'$, $y \; \boldsymbol{p} \; y'$ in G and $h \in \mathscr{S}(f_x, e_y)$ in E. Then $(x \circ y)_h \; \boldsymbol{p} \; (x' \circ y')_h$.*

Proof. Given conditions imply that $f_x \mathscr{L} f_{x'}$ and $e_x \mathscr{R} e_y$. Hence by Proposition 3.12, $\mathscr{S}(f_x, e_y) = \mathscr{S}(f_{x'}, e_{y'})$. Hence the expression $(x' \circ y')_h$ is defined by Equation (6.21). By Lemma 6.18, $x \centerdot h \; \boldsymbol{p} \; x' \centerdot h$. Since the codomains of there are the same, by Equation (6.19), we have

$$x' \centerdot h = \varepsilon(h_1', h_1)(x \centerdot h)$$

where h_1 and h_1' are domains of $x \centerdot h$ and $x' \centerdot h$ respectively. Dually, if h_2 and h_2' are codomains of $h \centerdot y$ and $h \centerdot y'$ respectively,

$$h \centerdot y' = (h \centerdot y)\varepsilon(h_2, h_2').$$

Therefore

$$\begin{aligned}(x' \circ y')_h &= (x' \centerdot h)(h \centerdot y')\\ &= \varepsilon(h_1', h_1)(x \centerdot h)(h \centerdot y)\varepsilon(h_2, h_2')\\ &= \varepsilon(h_1', h_1)(x \circ y)_h\varepsilon(h_2, h_2').\end{aligned}$$

Since $h_1' \mathscr{R} h_1$ and $h_2' \mathscr{L} h_2$, the lemma follows from Equation (6.19). □

Lemma 6.20. *Let $x, y \in G$ and $h, h' \in \mathscr{S}(f_x, e_y)$. Then $(x \circ y)_h = (x \circ y)_{h'}$.*

Proof. Let $h_1 = e_{x \centerdot h}$ and $h_2 = f_{h \centerdot y}$. Then by Equation (6.27) $h_1 = (f_x h)\mathfrak{a}(x^{-1})$ and $h_2 = (he_y)\mathfrak{a}(y)$. Similarly let $h_1' = e_{x \centerdot h'} = (f_x h')\mathfrak{a}(x^{-1})$ and $h_2' = (h' e_y)\mathfrak{a}(y)$.

First suppose that $h \mathscr{R} h'$. Then $he_y = h' e_y$ and so, $h_1 = h_1'$. Moreover, by Lemma 6.15,

$$\begin{aligned}h' \centerdot y &= h' \centerdot h \centerdot y\\ &= \varepsilon(h', h)\varepsilon(h, he_y)(he_y \centerdot y) = \varepsilon(h', h)(h \centerdot y).\end{aligned}$$

Now, by Proposition 1.19, $h_1 \centerdot x = x \centerdot f_x h$ and $h'_1 \centerdot x = x \centerdot f_x h'$ and so,

$$\begin{aligned} x \centerdot h' &= (x \centerdot f_x h')\varepsilon(f_x h', h') \\ &= (h'_1 \centerdot x)\varepsilon(f_x h', h') \\ &= \varepsilon(h'_1, h_1)(h_1 \centerdot x)\varepsilon(f_x h, f_x h')\varepsilon(f_x h', h') && \text{by axiom (IG1)} \\ &= \varepsilon(h'_1, h_1)(x \centerdot f_x h)\varepsilon(f_x h, h)\varepsilon(h, h') && \text{by axiom (IG2)} \\ &= \varepsilon(h'_1, h_1)(x \centerdot h)\varepsilon(h, h'). \end{aligned}$$

Therefore

$$\begin{aligned} (x \circ y)_{h'} &= (x \centerdot h')(h' \centerdot y) \\ &= \varepsilon(h'_1, h_1)(x \centerdot h)\varepsilon(h, h')\varepsilon(h', h)(h \centerdot y) \\ &= \varepsilon(h'_1, h_1)(x \centerdot h)(h \centerdot y) \\ &= \varepsilon(h'_1, h_1)(x \circ y)_h. \end{aligned}$$

Since $h'_1 \mathscr{R} h_1$ and $h'_2 = h_2$, it follows that $(x \circ y)_h \ \boldsymbol{p} \ (x \circ y)_{h'}$ in this case. In the case when $h \mathscr{L} h'$, the same conclusion follows dually. If $h, h' \in \mathscr{S}(f'_x e_y)$ are arbitrary, by Corollary 3.21, there is $h_1 \in \mathscr{S}(f_x, e_y)$ auch that $h \mathscr{R} h_1 \mathscr{L} h'$. Consequently the desired equality holds in all cases. ☐

Proof of Theorem 6.13. Lemmas 6.19 and 6.20, show that Equation (6.20) defines asingle valued binary operation on $\boldsymbol{S}(G)$. Let $x, y, z \in G$, $h_1 \in \mathscr{S}(f_x, e_y)$ and $h_2 \in \mathscr{S}(f_y, e_z)$. Then by Lemma 6.17 there is $h \in \mathscr{S}(f'_x e_{(y \circ z)_{h_2}})$ and $h' \in \mathscr{S}(f_{(x \circ y)_{h_1}}, e_z)$ such that

$$((x \circ y)_{h_1} \circ z)_{h'} = (x \circ (y \circ z)_{h_2})_h.$$

Now by Equation (6.20),

$$\begin{aligned} (\bar{x}\bar{y})\bar{z} &= \overline{((x \circ y)_{h_1} \circ z)_{h'}}; \\ \bar{x}(\bar{y}\bar{z}) &= \overline{(x \circ (y \circ z)_{h_2})_h}. \end{aligned}$$

It follows that $S = \boldsymbol{S}(G)$ is a semigroup. If the product xy exists in G, then $f_x = e_y$ and so, $\mathscr{S}(f_x, e_y) = \{f_x\}$. Therefore $\bar{x}\bar{y} = \overline{(x \circ y)_{f_x}} = \overline{xy}$. In particular, taking $y = x^{-1}$ we have

$$\bar{x}\overline{x^{-1}}\bar{x} = \overline{xx^{-1}x} = \bar{x} \quad \text{and similarly} \quad \overline{x^{-1}}\bar{x}\overline{x^{-1}} = \overline{x^{-1}}.$$

Therefore S is regular.

We now verify that $\chi : \mathfrak{v}G = E \to \boldsymbol{E}(S)$ is a biorder isomorphism. By Lemma 6.12 each $\boldsymbol{p}$-class of G ontain utmost one identity and so the map χ is injective. If $h \ \omega^r \ e$ by Proposition 3.9, $h \in \mathscr{S}(h, e)$ and $he \in \mathscr{S}(e, h)$. Then

$$\varepsilon(h, he) \ \boldsymbol{p} \ he \quad \text{and} \quad \varepsilon(he, h) \ \boldsymbol{p} \ h.$$

Hence by Equation (6.20)

$$\begin{aligned}\bar{e}\bar{h} &= \overline{(e \circ h)_{he}} = \overline{(e \centerdot he)(he \centerdot h)}\\ &= \overline{\varepsilon(he, h)} = \bar{h};\\ \bar{h}\bar{e} &= \overline{(h \circ e)_h \varepsilon(h, he)} = \overline{he}.\end{aligned}$$

If $h \; \omega^l \; e$, dually we have

$$\bar{e}\bar{h} = \overline{eh} \quad \text{and} \quad \bar{h}\bar{e} = \bar{h}$$

It follows that χ is a bimorphism. If $h \in \mathscr{S}(e, f)$, then

$$\begin{aligned}\bar{e}\bar{f} &= \overline{(e \centerdot h)(h \centerdot f)}\\ &= \overline{(\varepsilon(eh, h))(\varepsilon(h, hf))}\\ &= \overline{ehhf} && \text{since } \varepsilon(eh, h) \; \boldsymbol{p} \; eh \text{ and } \varepsilon(h, hf) \; \boldsymbol{p} \; hf\\ &= (\bar{e}\bar{h})(\bar{h}\bar{f}) = \bar{e}\bar{h}\bar{f} && \text{by the abve}\end{aligned}$$

By Proposition 3.4 $\bar{h} \in \mathscr{S}(\bar{e}, \bar{f})$ in $\boldsymbol{E}(S)$. Therefore χ is an injecctive regular bimorphism. Finally we show that $\chi : E \to \boldsymbol{E}(S)$ is surjective. Suppose that $x \in G$ such that $\bar{x} \in \boldsymbol{E}(S)$. If $h \in \mathscr{S}(f_x, e_x)$ then $(x \circ x)_h \; \boldsymbol{p} \; x$. Hence

$$e_x \mathscr{R} e_{(x \circ x)_h} = e_{x \centerdot h} = (f_x h)\mathfrak{a}(x^{-1}).$$

Therefore $(e_x)\mathfrak{a}(x)f_x \; \mathscr{R} \; f_x h \; \omega \; f_x$ which implies that $f_x = f_x h$. Thus $f_x \; \mathscr{L} \; h$. Dually $e_x \; \mathscr{R} \; h$. Since χ is a bimorphism, $\bar{e}_x \; \mathscr{R} \; \bar{h} \; \mathscr{L} \; \bar{f}_x$ in S. Further $\bar{x}\overline{x^{-1}} = \bar{e}_x$, $\bar{e}_x \bar{x} = \bar{x}$ and so, $\bar{e}_x \; \mathscr{R} \; \bar{x}$ in S. Similarly $\bar{f}_x \; \mathscr{L} \; \bar{x}$. Consequently, the hypothesis that $\bar{x}$ is an idempotent implies that $\bar{x}$ and $\bar{h}$ are $\mathscr{H}$-equivalent idempotents in the semigrup S. Therefore $\bar{x} = \bar{h}$ and so χ is surjective. By Corollary 3.25 χ is a biorder isomorphism. □

We proceed to show that the construction of Theorem 6.13 can be extended to a functor $\boldsymbol{S} : \mathfrak{IG} \to \mathfrak{RS}$. The following theorem constructs the morphism map of the desired functor $\boldsymbol{S}$.

THEOREM 6.21. *For an inductive functor $\phi : G \to G'$, define*

$$(\bar{x})\boldsymbol{S}(\phi) = \overline{\phi(x)} \quad \textit{for all} \quad \bar{x} \in \boldsymbol{S}(G). \tag{6.22}$$

Then $\boldsymbol{S}(\phi) : \boldsymbol{S}(G) \to \boldsymbol{S}(G')$ is a homomorphism of the semigroup $\boldsymbol{S}(G)$ to the semigroup $\boldsymbol{S}(G')$ such that the following diagram commute:

$$\begin{array}{ccc} \mathfrak{v}G & \xrightarrow{\chi_G} & \boldsymbol{E}(\boldsymbol{S}(G)) \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle \boldsymbol{E}(\boldsymbol{S}(\phi))} \\ \mathfrak{v}G' & \xrightarrow[\chi_{G'}]{} & \boldsymbol{E}(\boldsymbol{S}(G')) \end{array} \tag{6.23}$$

where $\chi_G : \mathfrak{v}G \to \boldsymbol{E}(\boldsymbol{S}(G))$ *and* $\chi_{G'} : \mathfrak{v}G' \to \boldsymbol{E}(\boldsymbol{S}(G'))$ *are biorder isomorpisms of Theorem 6.13 and* $\theta = \mathfrak{v}\phi$.

Moreover $\boldsymbol{S}(\phi)$ *is injective [surjective] if and only if* ϕ *has the corresponding property.*

Proof. We first show that $\boldsymbol{S}(\phi) : \boldsymbol{S}(G) \to \boldsymbol{S}(G')$ is a single valued mapping. To this end, assume that $x, y \in G$ and $x\ \boldsymbol{p}\ y$. Then by Equation (6.19) $x\varepsilon(f_x, f_y) = \varepsilon(e_x, e_y)y$ where $e_x \mathscr{R} e_y$ and $f_x \mathscr{L} f_y$. Now $\mathfrak{v}\phi = \theta : E \to E'$ is a regular bimorphism and so, $e_x\theta = e_{x\phi} \mathscr{R} e_y\theta$ and $f_x\theta \mathscr{L} f_y\theta$. Since ϕ is inductive,

$$\begin{aligned}(\phi(x))\varepsilon(f_x\theta, f_y\theta) &= (\phi(x))\phi(\varepsilon(f_x, f_y)) = \phi\left(x\varepsilon(f_x, f_y)\right)\\ &= \phi\left(\varepsilon(e_x, e_y)y\right) = \varepsilon(e_x\theta, e_y\theta)(\phi(y))\end{aligned}$$

Therefore $\phi(x)\ \boldsymbol{p}\ \phi(y)$. Hence $\boldsymbol{S}(\phi)$ is single valued. Again, let $x, y \in G$ and $h \in \mathscr{S}(f_x, e_y)$. Since θ is a regular bimorphism $h\theta \in \mathscr{S}(f_x\theta, y_y\theta)$. By Proposition 6.7(1) $\phi(x \centerdot h) = \phi(x) \centerdot h\theta$ and $\phi(h \centerdot y) = h\theta \centerdot \phi(y)$. Hence

$$\phi((x \circ y)_h) = (\phi(x) \circ \phi(y))_{h\theta}.$$

Therefore

$$\begin{aligned}(\bar{x}\bar{y})\boldsymbol{S}(\phi) &= \overline{\phi\left((x \circ y)_h\right)}\\ &= \overline{(\phi(x) \circ \phi(y))_{h\theta}}\\ &= \overline{\phi(x)}\ \overline{\phi(y)} = (\bar{x})\boldsymbol{S}(\phi)(\bar{(}y))\boldsymbol{S}(\phi).\end{aligned}$$

It follws that $\boldsymbol{S}(\phi)$ is a homomrphism. The definition of $\boldsymbol{S}(\phi)$ immediately imply commutativity of 6.23.

Suppose that $\phi : G \to G'$ is injective and that $\bar{x}\boldsymbol{S}(\phi) = \bar{y}\boldsymbol{S}(\phi)$. Then, by the definition of $\boldsymbol{S}(\phi)$, $\phi(x)\ \boldsymbol{p}\ \phi(y)$ and so $e_x\theta \mathscr{R} e_y\theta$ and $f_x\theta \mathscr{L} f_y\theta$ where $\theta = \mathfrak{v}\phi$. Sincce θ is a regular injective bimorphism, it is an isomorphism onto $E\theta$ by Proposition 3.24 and Corollary 3.25. Therefore $e_x \mathscr{R} e_y$ and $f_x \mathscr{L} f_y$. If $z = \varepsilon(e_x, e_y)y\varepsilon(f_y, f_x)$ then $z\ \boldsymbol{p}\ y$ and so, $\phi(z)\ \boldsymbol{p}\ \phi(y)\ \boldsymbol{p}\ \phi(x)$. Since $e_{\phi(x)} = e_{\phi(z)}$ and $f_{\phi(z)} = f_{\phi(x)}$, we have $\phi(x) = \phi(z)$. Since ϕ is ono-to-one, $x = z$. Therefore $x\ \boldsymbol{p}\ y$ and so $\bar{x} = \bar{y}$. Conversely suppose that $\boldsymbol{S}(\phi)$ is one-to-one and let $\phi(x) = \phi(y)$. Then

$$(\bar{x})\boldsymbol{S}(\phi) = \overline{\phi(x)} = \overline{\phi(y)} = (\bar{y})\boldsymbol{S}(\phi)$$

which implies that $\bar{x} = \bar{y}$. Also $\boldsymbol{E}(\boldsymbol{S}(\phi))$ is injective and so, by **??**, $\theta = \mathfrak{v}\phi$ is injective. Since $e_{\phi(x)} = e_x\theta = e_{\phi(y)} = e_y\theta$, we have $e_x = e_y$. Similarly $f_x = f_y$. Since $\bar{x} = \bar{y}$, it follows from Lemma 6.12 that $x = y$.

If ϕ is surjective, it is clear from Equation (6.22) that $\boldsymbol{S}(\phi)$ is surjective. So assume that $\boldsymbol{S}(\phi)$ is surjective. By Proposition 6.7, $G_1 = \operatorname{Im}\phi$ is an inductive subgroupoid of G'. Since $\boldsymbol{S}(\phi)$ is surjective, by Theorem 3.5 $\boldsymbol{E}(\boldsymbol{S}(\phi))$ is surjective and hence by 6.23, $\theta = \mathfrak{v}\phi$ is surjective. Hence if ε' denotes the evaluation of G', $\operatorname{Im}\varepsilon' \subseteq G_1$. Let $x' \in G'$. Then $\bar{x'} \in \boldsymbol{S}(G')$ and since $\boldsymbol{S}(\phi)$ is surjective, there exists $x \in G$ with $(\bar{x})\boldsymbol{S}(\phi) = \bar{x'}$. By the definition of $\boldsymbol{S}(\phi)$, $\phi(x)\ \boldsymbol{p}\ x'$. Therefore

$$x' = \varepsilon'(e_{x'}, e_{\phi(x)})\phi(x)\varepsilon'(f_{\phi(x)}, f_{x'}).$$

Since $\varepsilon'(e_{x'}, e_{\phi(x)}), \varepsilon'(f_{\phi(x)}, f_{x'}) \in \operatorname{Im}\varepsilon' \subseteq G_1$ and $\phi(x) \in G_1$ it follows that $x' \in G_1$. Hence $G_1 = G'$. This completes the proof. □

Equation (6.22) shows that $\boldsymbol{S}(1_G) = 1_{\boldsymbol{S}(G)}$. Moreover, if $\phi : G \to G'$ and $\psi : G' \to G''$ are composable inductive functors, then for all $x \in G$

$$\begin{aligned}(\bar{x})\boldsymbol{S}(\phi\psi) &= \overline{\psi(\phi(x))}\\ &= \Big((\bar{x})\boldsymbol{S}(\phi)\Big)\boldsymbol{S}(\psi).\end{aligned}$$

by Equation (6.22). Thus we have the following:

Theorem 6.22. *For each inductive groupoid, let $\boldsymbol{S}(G)$ denote the regular semigroup constructed in Theorem 6.13 and for each inductive functor $\phi : G \to G'$, let $\boldsymbol{S}(\phi) : \boldsymbol{S}(G) \to \boldsymbol{S}(G')$ be the homomrphism of Theorem 6.21. Then the assignments*

$$\boldsymbol{S} : G \mapsto \boldsymbol{S}(G), \qquad \phi \mapsto \boldsymbol{S}(\phi)$$

is a functor $\boldsymbol{S} : \mathfrak{IG} \to \mathfrak{RS}$. □

Notice that the diagram **??** shows that the map

$$\chi : G \mapsto \chi_G \quad \text{is a natural isomorphism} \quad \chi : \mathfrak{v} \xrightarrow{n} \boldsymbol{S} \circ \boldsymbol{E}.$$

As for the functor $\boldsymbol{G}$ (see Theorem 6.11), here also it may be convenient to identify $\mathfrak{v}G = \boldsymbol{E}(\boldsymbol{S}(G))$ for all inductive groupoid G by identifying $e \in \mathfrak{v}G$ with $\bar{e} = e\chi$. It follows from Equation (6.22) that this identification forces the identification $\mathfrak{v}\phi = \boldsymbol{E}(\boldsymbol{S}(\phi))$ for all inductive functor ϕ. Consequently the following diagram commute:

$$\begin{array}{ccc} \mathfrak{IG} & \xrightarrow{S} & \mathfrak{RS} \\ & \searrow^{\mathfrak{v}} & \downarrow{\scriptstyle E} \\ & & \mathfrak{RB} \end{array} \tag{6.24}$$

***Remark* 6.2:** Given any inductive groupoid G, Theorem 6.13 constructs a regular semigroup $S(G)$ with $\mathfrak{v}G$ is isomorphic to $\boldsymbol{E}(S)$. Given any biordered set E, by Proposition 6.27 the set of all ω-isomorphisms of E is an inductive groupoid T_E^* with $\mathfrak{v}T_E^*$ is isomorphic to E. Therefore, by Theorem 6.13, $S(T_E^*) = T(E)$ is a regular semigroup with biordered set isomorphic to E. This gives an alternate proof of the fact that any regular biordered set is the biordered set of a regular semigroup (see also Theorem 3.42).

6.3.2 The equivalence of $\mathfrak{IG}$ and $\mathfrak{RS}$

Suppose that G is an inductive groupoid and $x, y \in G$. If xy exists in G, clearly $y^{-1}x^{-1}$ also exists. It is immediate from Equation (6.20) that

$$\bar{x}\bar{y} = \overline{xy}, \quad \text{and} \quad \overline{y^{-1}}\,\overline{x^{-1}} = \overline{y^{-1}x^{-1}}.$$

In particular, $\bar{x}\overline{x^{-1}} = \bar{e_x}$, $\bar{x}\bar{f_x} = \bar{x} = \bar{e_x}\bar{x}$. Hence

$$\overline{e_x} \mathscr{R} \bar{x} \mathscr{L} \overline{f_x}$$

and $\overline{x^{-1}}$ is an inverse of $\bar{x}$ in $\boldsymbol{S}(G)$. When xy exists in G, the above equalities show that trace products $\bar{x} * \bar{y}$ and $\overline{y^{-1}} * \overline{x^{-1}}$ exists in $\boldsymbol{S}(G)$ (see Equation (2.48a)). Conversely, if $\bar{x}, \bar{y} \in \boldsymbol{S}(G)$ and if trace products $\bar{x} * \bar{y}$ and $\overline{y^{-1}} * \overline{x^{-1}}$ exist then there exist $g, h \in \boldsymbol{E}(\boldsymbol{S}(G))$ such that

$$\bar{x} \mathscr{L} g \mathscr{R} \bar{y} \quad \text{and} \quad \overline{y^{-1}} \mathscr{L} h \mathscr{R} \overline{x^{-1}}$$

Then $f_x \mathscr{L} g \mathscr{R} e_y$ and $f_y \mathscr{L} h \mathscr{R} e_x$. Hence if

$$u = \varepsilon(h, e_x)x\varepsilon(f_x, g) \quad \text{and} \quad v = \varepsilon(g, e_y)y\varepsilon(f_y, h)$$

then by Equation (6.19) $x \boldsymbol{p} u$, $y \boldsymbol{p} v$ and uv exists in G. To prove uniqueness, assume that $u \boldsymbol{p} u'$, $v \boldsymbol{p} v'$ and that products uv and $u'v'$ exists in G. Then $e_v = f_u \mathscr{L} f_{u'} = e_{v'}$. Since $v \boldsymbol{p} v'$, $e_v \mathscr{R} e_{v'}$ which implies that $f_u = e_v = e_{v'} = f_{u'}$. Similarly we have $f_v = e_u = e_{u'} = f_{v'}$. Therefore $u, u' \in G(e_u, f_u)$ and $v, v' \in G(e_v, f_v)$. By Lemma 6.12, $u = u'$ and $v = v'$.

For convenience of later reference, summarize the discussion above as:

LEMMA 6.23. *For $x, y \in G$ if the product xy exists in G then the trace products $\overline{x} * \overline{y}$ and $\overline{y^{-1}} * \overline{x^{-1}}$ exists in $\boldsymbol{S}(G)$. If this is the case, we have*

$$\overline{x} * \overline{y} = \overline{xy}, \quad \text{and} \quad \overline{y^{-1}} * \overline{x^{-1}} = \overline{y^{-1}x^{-1}}.$$

Conversely, if the trace products $\overline{x} * \overline{y}$ and $\overline{y^{-1}} * \overline{x^{-1}}$ exists in $\boldsymbol{S}(G)$, then there exists unique $u, v \in G$ such that $\bar{u} = \bar{x}$, $\bar{v} = \bar{y}$ and uv exists in G. In particular,

$$\overline{x}\overline{x^{-1}} = e_x, \quad \overline{x^{-1}}\overline{x} = f_x,$$
$$\bar{x}f_x = \bar{x} = e_x\bar{x}$$

for all $x \in G$. Consequently $\overline{x^{-1}} \in \mathscr{V}(\bar{x})$ for all $\bar{x} \in \boldsymbol{S}(G)$. □

Theorems 6.11 and 6.22 constructs functors $\boldsymbol{G} : \mathfrak{RS} \to \mathfrak{IG}$ and $\boldsymbol{S} : \mathfrak{IG} \to \mathfrak{RS}$ respectively. These constructions shows, in particular that, we can construct an inductive groupoid from a regular semigroup and conversely a regular semigroup can be constructed from any inductive groupoid. We proceed to show that every inductive groupoid is isomorphic to an inductive groupid of the from $\boldsymbol{G}(S)$ constructed from a regular semigroup S and every regular semigroup S is, upto isomorphism, a regular semigroup of the form $\boldsymbol{S}(G)$ constructed from an inductive groupoid G. Thus the mathematical structures *inductive groupoids* and *regular semigroups* are structurely equivalent (not equal). Here we prove this by showing that the functor $\boldsymbol{S}$ is the adjoint inverse of the functor $\boldsymbol{G}$ (see Subsection 1.2.4). It may be noted that our functorial approch gives a result considerably stronger than the structural equivalence of inductive groupoids and regular semigroups; in fact our result also includes the equivalence of inductive funtors and homomrphisms of regular semigroups. We shall illustrate some of the consequences of these equivalences later in this section.

We divide the proof of the equivalence of categories $\mathfrak{IG}$ and $\mathfrak{RG}$ into the following two propositions.

Proposition 6.24. *For any indutive groupoid G, there is an inductive isomorphism $\nu_G : G \to \boldsymbol{G}(\boldsymbol{S}(G))$ defined as follows:*

$$\nu_G(x) = (\overline{x}, \overline{x^{-1}}) \qquad \textit{for all morphism } x \in G$$

and

$$\mathfrak{v}\nu_G = \chi.$$

Furthermore

$$\nu : 1_{\mathfrak{IG}} \xrightarrow{n} \boldsymbol{S} \circ \boldsymbol{G}; \quad G \mapsto \nu_G$$

is a natural isomorphism.

Proof. For brevity, let us write $S = \boldsymbol{S}(G)$. We first observe that the morphism map of the functor ν_G given in the statement is single valued. Indeed, if $x \; \boldsymbol{p} \; y$ then it follows from Equation (6.19) that $y^{-'1} \; \boldsymbol{p} \; x^{-1}$. By Lemma 6.23, $\overline{x^{-1}} \in \mathscr{V}(\bar{x})$ and so, by Equation (6.1), $(\overline{x}, \overline{x^{-1}})$ is a morphism in $\boldsymbol{G}(S)$. Hence ν_G is a well defined map of the morphism set of G to the morphism set of $\boldsymbol{G}(\boldsymbol{S}(G))$. If xy exists in G, by Lemma 6.23,

$$\overline{x^{-1}}\overline{x} = f_x = e_y = \overline{y}\overline{y^{-1}}$$

and so, $\nu_G(x)\nu_G(y)$ exists in $\boldsymbol{G}(S)$ by Equation (6.14). Moreover,

$$\begin{aligned} \nu_G(x)\nu_G(y) &= (\overline{x}, \overline{x^{-1}})(\overline{y}, \overline{y^{-1}}) \\ &= (\overline{x} * \overline{y}, \overline{y^{-1}} * \overline{x^{-1}}) && \text{by Equation (6.14)} \\ &= (\overline{xy}, \overline{y^{-1}x^{-1}}) = (\overline{xy}, \overline{(xy)^{-1}}) && \text{by Lemma 6.23} \end{aligned}$$

Therefore $\qquad \nu_G(x)\nu_G(y) = \nu_G(xy).$

If $e \in G$ is an identity, (that is $e \in \mathfrak{v}G$) we have $e^{-1} = e$ and so,

$$\nu_G(e) = (\bar{e}, \bar{e}).$$

This shows that ν_G preserves identities and hence $\nu_G : G \to \boldsymbol{G}(S)$ is a functor. Moreover by Theorem 6.13 the map $\chi : e \mapsto \bar{e}$ is a biorder isomorphism and the map $e \mapsto (\bar{e}, \bar{e})$ induced by ν_G on the set of identities of G is a biorder isomorphism of $\mathfrak{v}G$ onto $\mathfrak{v}\boldsymbol{G}(S)$. In view of the identification $\mathfrak{v}\boldsymbol{G}(S) = \boldsymbol{E}(S)$ we may choose $\mathfrak{v}\nu_G = \chi$.

To show that ν_g is order preserving, consider $x \in G$, and $g\ \omega\ e_x$. Then by Proposition 3.9, $g \in \mathscr{S}(g, e_x) \cap \mathscr{S}(e_x, g)$. Hence by Equation (6.21)

$$(g \circ x)_h = g \centerdot x \quad \text{and} \quad (x^{-1} \circ g)_g = x^{-1} \centerdot g = (g \centerdot x)^{-1}.$$

Therefore by Equation (6.20)

$$\bar{g}\bar{x} = \overline{g \centerdot x} \quad \text{and} \quad \overline{x^{-1}}\bar{g} = \overline{(g \centerdot x)^{-1}}$$

Consequently, for all $x \in G$ and $g\ \omega\ e_x$,

$$\begin{aligned} \nu_G(g \centerdot x) &= (\overline{g \centerdot x}, \overline{(g \centerdot x)^{-1}}) \\ &= (\overline{g}\overline{x}, \overline{x^{-1}}\overline{g}) \\ &= \overline{g} \centerdot (\overline{x}, \overline{x^{-1}}) && \text{by Equation (6.15*)} \\ &= \overline{g} \centerdot \nu_G(x). \end{aligned}$$

We next verify that ν_G is inductive. Thus we must show that the diagram 6.12 commutes when we take $\nu_G = \phi$. Since $\mathfrak{v}\nu_G = \chi = \mathfrak{v}\mathfrak{C}(E)(\phi)$, the diagram of vertex maps in 6.12 commutes. To show that the diagram commutes also for morphism maps, it is sufficient to verify the commutativity for generating chains of $\mathfrak{C}(E)(E) = \mathfrak{C}(E)(\mathfrak{v}G)$; that is chains of the type $c(e, f)$ with either $e \mathscr{R} f$ or $e \mathscr{L} f$. Let $e \mathscr{R} f$. Then

$$\begin{aligned} \nu_G(\varepsilon(e, f)) &= \left(\overline{\varepsilon(e, f)}, \overline{\varepsilon(f, e)}\right) \\ &= (\overline{f}, \overline{e}) && \text{since } \varepsilon(e, f)\ \boldsymbol{p}\ f \text{ and } \varepsilon(f, e)\ \boldsymbol{p}\ e \\ &= (f\chi, e\chi) && \text{since } \mathfrak{v}\nu_G = \chi \\ &= \varepsilon_S(c(e\chi, f\chi)) && \text{by Equation (6.16)} \\ &= \varepsilon_S(\mathfrak{C}(E)(c(e, f))). \end{aligned}$$

This proves the ommutativity in the case when the chain is $c(e, f)$ with $e \mathscr{R} f$. The proof for the case $e \mathscr{L} f$ is dual.

This completes the proof that $\nu_G : G \to \boldsymbol{G}(\boldsymbol{S}(G))$ is an inductive homomorphism. To prove that ν_G is injective, assume that $\nu_G(x) = \nu_G(y)$ where $x, y \in G$. Then $x\ \boldsymbol{p}\ y$ and $x^{-1}\ \boldsymbol{p}\ y^{-1}$ so that $e_x \mathscr{R} e_y$ and $e_x = f_{x^{-1}} \mathscr{L} f_{y^{-1}} = e_y$ which gives $e_x = e_y$. Similarly $f_x = f_y$. Hence x and y are $\boldsymbol{p}$-related morphism in the same home-set of G and so, by Lemma 6.12 $x = y$. To prove that ν_G is surjective, let

$(u, u') \in \boldsymbol{G}(S)$ where $S = G/\,\boldsymbol{p}$ (see Theorem 6.13). Since the map $x \mapsto \bar{x}$ is surjective from the morphism set of G onto S, there is $x \in G$ with $u = \bar{x}$. Let $\bar{e} = uu'$ and $\bar{f} = u'u$. Tnen by Lemma 6.23 $e_x \mathscr{R} e$ and $f_x \mathscr{L} f$. So if $y = \varepsilon(e, e_x)x\varepsilon(f_x, f)$ then $\bar{y} = u$ and $y^{-1} = \varepsilon(f, f_x)x^{-1}\varepsilon(e_x, e)$. Again, by Lemma 6.23, it follows that $\overline{y^{-1}}$ is an inverse of $\bar{y} = u$ in the $\mathscr{H}$-class $R_{\bar{f}} \cap L_{\bar{e}}$. Thus $\overline{y^{-1}}$ and u' are $\mathscr{H}$-equivalent inverses of u and so, $u' = \overline{y^{-1}}$. Consequently $\nu_G(y) = (u, u')$. This proves, by Proposition 6.7 that ν_G is an inductive isomorphism.

Finally we show that $\nu : G \mapsto \nu_G$ is a natural isomorphism. Thus we must show that the following diagram commutes for all inductive functors $\phi : G \to G'$:

$$\begin{array}{ccc} G & \xrightarrow{\ \nu_G\ } & \boldsymbol{S} \circ \boldsymbol{G}(G) \\ \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \boldsymbol{S}\circ\boldsymbol{G}(\phi)} \\ G' & \xrightarrow[\ \nu_{G'}\ ]{} & \boldsymbol{S} \circ \boldsymbol{G}(G') \end{array} \tag{6.25}$$

Let $x \in G$. Then

$$\begin{aligned}
\Big(\nu_G \circ (\boldsymbol{S} \circ \boldsymbol{G})(\phi)\Big)(x) &= (\boldsymbol{S} \circ \boldsymbol{G}(\phi))(\overline{x}, \overline{x^{-1}}) \\
&= \Big(\boldsymbol{G}(\boldsymbol{S}(\phi))\Big)(\overline{x}, \overline{x^{-1}}) \\
&= \Big((\overline{x})\boldsymbol{S}(\phi), (\overline{x^{-1}})\boldsymbol{S}(\phi)\Big) && \text{by Equation (6.17)} \\
&= \Big(\overline{x\phi}, \overline{(x\phi)^{-1}}\Big) && \text{by Equation (6.22)} \\
&= \Big(\phi \circ \nu_{G'}\Big)(x).
\end{aligned}$$

This complete the proof of the proposition. □

PROPOSITION 6.25. *For each regular semigroup S, define the mapping $\eta_S : S \to (\boldsymbol{G} \circ \boldsymbol{S})(S)$ by*

$$x\eta_S = \overline{(x, x')}$$

for all $x \in G$ and $x' \in \mathscr{V}(x)$. Then η_S is an isomorphism of regular semigroups and

$$\eta : S \mapsto \eta_S : 1_{\mathfrak{RS}} \xrightarrow{n} \boldsymbol{G} \circ \boldsymbol{S}$$

is a natural isomorphism.

Proof. We first show that η_S is single valued; that is, for any $x', x'' \in \mathscr{V}(x)$, we have $(x, x')\ \boldsymbol{p}\ (x, x'')$. It follows from Proposition 2.40 that $e = xx' \mathscr{R} e' = xx''$ and $f = x'x \mathscr{L} x''x = f'$. Hence $e_{(x,x')} = e \mathscr{R} e' = e_{(x,x'')}$ and $f_{(x,x')} = f \mathscr{L} f' = f_{(x,x'')}$ in $\boldsymbol{G}(S)$. Also, by Equations (6.14) and (6.16), we have

$$\varepsilon_S(e', e)(x, x')\varepsilon_S(f, f') = (e, e')(x, x')(f, f') = (x, f'x'e') = (x, x'')$$

using Proposition 2.40. Therefore $(x,x')\ \boldsymbol{p}\ (x,x'')$ in $\boldsymbol{G}(S)$ and so η_S is single valued. If $\eta_S(x)=\eta_S(y)$, then $(x,x')\ \boldsymbol{p}\ (y,y')$ in $\boldsymbol{G}(S)$ and so $xx'\ \mathscr{R}\ yy'$, $x'x\ \mathscr{L}\ y'y$. Hene by Proposition 2.40, there is $y''\in\mathscr{V}(y)$ such that $y''\ \mathscr{H}\ x'$. Then $(x,x')\ \boldsymbol{p}\ (y,y')\ \boldsymbol{p}\ (y,y'')$. Since $y''\ \mathscr{H}\ x'$ it is easy to see that (x,x') and (y,y'') are morphisms in the home-set $\boldsymbol{G}(S)(xx',x'x)$ and hence $(x,x')=(y,y'')$ in $\boldsymbol{G}(S)$. Therefore $x=y$. This shows that η_S is injective. If $u\in\boldsymbol{S}(\boldsymbol{G}(S))$. By Theorem 6.13, there is $(x,x')\in\boldsymbol{G}(S)$ such that $u=\overline{(x,x')}=\eta_S(x)$ and so η_S is surjective. Hence η_S is an isomorphism. Suppose that $x,y\in S$. Let $x'\in\mathscr{V}(x)$, $y'\in\mathscr{V}(y)$ and $h\in\mathscr{S}(x'x,yy')$. Then by the definition of restriction and evaluation in $\boldsymbol{G}(S)$ (see Equations (6.15*) and (6.16)) we have

$$\begin{aligned} h \centerdot (y,y') &= \varepsilon_S(h,he_y)\Big((he_y)\centerdot(y,y')\Big) && \text{by Equation (6.21)}\\ &= (he_y,h)(he_yy,y'he_y = (hy,y'h). \end{aligned}$$

Dually $\qquad (x,x')\centerdot h = (xh,hx')$.

Hence
$$\begin{aligned} (x\eta_S)(y\eta_S) &= \overline{(x,x')}\ \overline{(y,y')}\\ &= \overline{((x,x')\circ(y,y'))_h} && \text{by Equation (6.20)}\\ &= \overline{((x,x')\centerdot h)\,(h\centerdot(y,y'))} && \text{by Equation (6.21)}\\ &= \overline{(xh,hx')(hy,y'h)}\\ &= \overline{(xy,y'hx')} = \eta_S(xy) && \text{by Theorem 3.7.} \end{aligned}$$

Therefore η_S is an isomorphism.

Finally, to prove that the map $S\mapsto\eta_S$ is a natural isomorphism we must prove that the following diagram commutex for all homomorphism $\phi: S\to S'$ of regular semigroups:

$$\begin{array}{ccc} S & \xrightarrow{\ \eta_S\ } & (\boldsymbol{G}\circ\boldsymbol{S})(S)\\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle(\boldsymbol{G}\circ\boldsymbol{S})(\phi)}\\ S' & \xrightarrow[\ \eta_{S'}\ ]{} & (\boldsymbol{G}\circ\boldsymbol{S})(S') \end{array} \tag{6.26}$$

Suppose that $x\in S$. Then

$$\begin{aligned} (x)\Big(\eta_S\circ(\boldsymbol{G}\circ\boldsymbol{S})(\phi)\Big) &= (\overline{(x,x')})(\boldsymbol{G}\circ\boldsymbol{S})(\phi) && \text{for some } x'\in\mathscr{V}(x)\\ &= (\overline{(x,x')})\Big(\boldsymbol{S}\Big(\boldsymbol{G}(\phi)\Big)\Big)\\ &= \overline{\boldsymbol{G}(\phi)(x,x')} && \text{by Equation (6.22)}\\ &= \overline{(x\phi,x'\phi)} && \text{by Equation (6.17)}\\ &= (x\phi)\eta_S = (x)(\phi\circ\eta_S). \quad\square \end{aligned}$$

Recall from Subsection 1.2.4 that an equivalence $< F, G, \eta, \nu >: C \stackrel{\cdot}{\rightleftharpoons} \mathcal{D}$ of categories consist of a pair of functors $F : C \to \mathcal{D}$ and $G : \mathcal{D} \to C$ and natural isomorphsms $\eta : 1_C \xrightarrow{n} F \circ G$ and $\nu : 1_{\mathcal{D}} \xrightarrow{n} G \circ F$. The Propositions 6.24 and 6.25 proves the following.

THEOREM 6.26. *Let* $\boldsymbol{G}$, $\boldsymbol{S}$, η *and* ν *be as above. Then*

$$< \boldsymbol{G}, \boldsymbol{S}; \eta, \nu >: \mathfrak{RS} \stackrel{\cdot}{\rightleftharpoons} \mathfrak{IG}$$

is an equivalence of the categories $\mathfrak{RS}$ *and* $\mathfrak{IG}$. □

***Remark* 6.3:** As already noted, the equivalence proved above enables one to replace regular semigrups by its inductive groupoids and vice versa according to the contest. Since functors $\boldsymbol{G}$ and $\boldsymbol{S}$ are equivalences they preserves all concepts defined categorically. However, since the categories we are concerned with are set-based and concrete, often we need preservation of properties such as injectiveness whose definition is set-theoretic rather than categoric. In the present context many such properties are also preserved. Thus if $\phi : G \to G'$ is an inductive functor, Theorem 6.21 shows that ϕ is injective or surjective if and only if $\boldsymbol{S}(\phi)$ has the corresponding property. The reverse implication is also true. Thus if $\sigma : S \to S'$ is a homomorphism of regular semigroup, $\boldsymbol{G}(\sigma)$ is injective or surjective according as σ is injective or surjective. For by diagram 6.26, σ is injective or surjective if and only if $\boldsymbol{S}(\boldsymbol{G}(\sigma))$ is injective or surjective. By Theorem 6.21 this is true if and only if $\boldsymbol{G}(\sigma)$ is injective or surjective. This in particular, enables us to define the concept of a congruence on an inductive groupoids (treating inductive grouoids as partial algebras).

6.3.3 Exercise

Exercise 6.4: Prove the following preservation properties of the functors $\boldsymbol{G}$ and $\boldsymbol{S}$: Given any homomorphism $\psi : S \to S'$ of regular semigroups, the morphism map of the inductive functor $\boldsymbol{G}(\psi) : \boldsymbol{G}(S) \to \boldsymbol{G}(S')$ is injetive [surjective] if and only if ψ has the corresponding property.

6.4 THE FUNDAMENTAL REPRESENTATION

Recall that a semigroup S is fundamental if $\mathscr{H}_{(c)} = 1_S$; that is, the only congruence contained in the relation $\mathscr{H}$ is the identity on S (see Proposition 3.46). Suppose that S is a regular semifroup. By Proposition 3.47 S is fundamental if and only if the only idempotent separating congruence on S is 1_S. Therefore S is fundamental if and only if every idempotent separating homomorphism $h : S \to S'$ (that is homomorphism $h : S \to S'$ such that $\boldsymbol{\kappa}\phi h$ is idempotent separating) is injective.

Let E denote a regular biordered set. Recall that an ω-isomorphism is a biorder isomorphisms of ω-ideals (see Subsection 3.2.1). By Proposition 3.17, the set T^*_E of all ω-isomorphisms of E is an ordered groupoid. Moreover, we have the commutative diagram 3.15 in the category $\mathfrak{OG}$ of ordered groupoids. By Proposition 6.4 the diagram 6.9 is a push-out in the category $\mathfrak{OG}$. Hence there exists a unique order preserving functor $\tau = \tau_E : \mathfrak{C}(E) \to T^*_E$ such that

$$R_E \circ \tau = \tau_R \quad \text{and} \quad L_E \circ \tau = \tau_L.$$

Furthermore, we have:

PROPOSITION 6.27. *The groupoid T^*_E of all ω-isomorphisms of the biordered set E is an inductive groupoid with evaluation $\tau = \tau_E$.*

Proof. Since $\mathfrak{v}R_E = \mathfrak{v}\tau_R = 1_E$ by definition, $\mathfrak{v}\tau = 1_E$. Hence to show that (T^*_E, τ) is an inductive groupoid, it is sufficient to verify axioms (IG1) and (IG2) of Definition 6.1. To verify axiom (IG1)(a), let $\alpha \in T^*_E$ and $e_1, e_2 \in \omega(e_\alpha)$ with $e_1 \; \omega^r \; e_2$. By Equation (3.11) we have $f_i = (e_i)\alpha = f_{e_i \cdot \alpha}$, $i = 1, 2$. Since α is a biorder isomorphism, $f_1 \; \omega^r \; f_2$ and for $g \; \omega \; e_1$

$$\begin{aligned} g\tau(e_1, e_1e_2)(e_1e_2 \cdot \alpha) &= (g(e_1e_2))\,\alpha \\ &= (g\alpha)f_1f_2 = g(\alpha|\omega(e_1))\tau(f_1, f_1f_2) \\ &= g(e_1 \cdot \alpha)\tau(f_1, f_1f_2). \end{aligned}$$

This proves that (IG1)(a) holds. (IG1)(b) is proved dually. Thus T^*_E satisfies axiom (IG1). Axiom (IG2) holds by Proposition 3.18. Therefore T^*_E is an inductive groupoid with evaluation τ_E. □

Recall that $\boldsymbol{OI}_X$ is an ordered groupid of all isomorphisms of order ideals of a partially ordered set X with $\mathfrak{v}\boldsymbol{OI}_X$ as the partially ordered set of all order ideals of X under inclusion (see Example 1.24). Now there is an order isomorphism of the set of principal order ideals of X with X so that the set $\boldsymbol{OI}^*_X$ of all isomorphisms of principal order-ideals is an ordered subgroupoid of $\boldsymbol{OI}_X$ whose vertex set can be identifies with X. In particular, if E is a regular biordered set T^*_E is an ordered subgroupoid of $\boldsymbol{OI}^*_E$. If G is any inductive groupid with $\mathfrak{v}G = E$, and $x \in G$, by Proposition 1.20 there is an order isomorphism $\mathfrak{a}(x) : \omega(e_x) \to \omega(f_x)$ and the map $x \mapsto \mathfrak{a}(x)$ is an order preserving $\mathfrak{v}$-isomorphism of G into $\boldsymbol{OI}_E$. The next theorem shows that $\mathfrak{a}_G$ is an important representation of G in T^*_E.

THEOREM 6.28. *Let G be an inductive groupid with $\mathfrak{v}G = E$. For $x \in G$ and $e \in \omega(e_x)$ let*

$$e\mathfrak{a}(x) = f_{e \cdot x}. \tag{6.27}$$

Then we have the following:

$T^*(G)$:Fundamental image of inductive groupoid G
$T^*(\phi)$:Fundamental image of the inductive functor ϕ

(1) *The map $\mathfrak{a}(x) : \omega(e_x) \to \omega(f_{x:})$ is an ω-isomorphism.*

(2) *There is an inductive functor $\mathfrak{a}_G : G \to T^*_E$ with $\mathfrak{v}\mathfrak{a}_G = 1_e$ and whose morphism map is $x \mapsto \mathfrak{a}(x)$.*

(3) *If G is a $\mathfrak{v}$-full inductive subgroupoid of T^*_E then $\mathfrak{a}_G$ is the inclusion of G in T^*_E. In particular, $\mathfrak{a}_{T^*_E} = 1_{T^*_E}$.*

(4) *Let $T^*(G) = \operatorname{Im}\mathfrak{a}_G$. If $\phi : G \to G'$ is an inductive functor which is a $\mathfrak{v}$-surjection, then*

$$T^*(\phi)\,(\mathfrak{a}_G(x)) = \mathfrak{a}_{G'}\big(\phi(x)\big) \tag{6.28}$$

defines an inductive functor $T^*(\phi) : T^*(G) \to T^*(G')$ such that the following diagram commutes:

$$\begin{array}{ccc} T^*(G) & \xrightarrow{T^*(\phi)} & T^*(G') \\ \uparrow{\scriptstyle \mathfrak{a}^\circ_G} & & \uparrow{\scriptstyle \mathfrak{a}^\circ_{G'}} \\ G & \xrightarrow[\phi]{} & G' \end{array} \tag{6.29}$$

Here $\mathfrak{a}^\circ_G$ [$\mathfrak{a}^\circ_{G'}$] denote the epimorphic compunent of $\mathfrak{a}_G$ [$\mathfrak{a}_{G'}$]. Furthermore, if ϕ and ϕ' are inductive $\mathfrak{v}$-surjections for which $\phi\phi'$ exists, then

$$T^*(\phi\phi') = T^*(\phi)T^*(\phi').$$

(5) *If ϕ is a $\mathfrak{v}$-isomorphism, then $T^*(\phi)$ is an injection. In particular, if $\mathfrak{v}\phi = 1_E$, then $T^*(\phi)$ is the inclusion $T^*(G) \subseteq T^*(G')$.*

Proof. **(1)** By Proposition 1.20(2), the map $\mathfrak{a}(x)$ is an order isomorphism of $\omega(e_x)$ onto $\omega(f_x)$. Let $e_1, e_2 \in \omega(e_x)$ and $e_1\ \omega^r\ e_2$. Let $f_1 = e_1\mathfrak{a}(x)$ and $f_2 = \mathfrak{a}(x)$. Then by (IG1)(a),

$$f_1 = f_{e_1\centerdot x}\ \omega^r\ f_{e_2\centerdot x} = f_2 \quad \text{and}$$
$$(e_1e_2)\mathfrak{a}(x) = f_{e_1e_2\centerdot x} = f_1f_2.$$

Similarly, by axiom (IG1)(b), the map $\mathfrak{a}(x)$ preserves ω^l and the associated basic product. Therefore $\mathfrak{a}(x)$ is a bimorphism. Similarly $\mathfrak{a}(x^{-1})$ is a bijective bimorphism of $\omega(f_x)$ onto $\omega(e_x)$. Since $\mathfrak{a}(x^{-1}) = (\mathfrak{a}(x))^{-1}$, $\mathfrak{a}(x)$ is a biorder isomorphism of $\omega(e_x)$ onto f_x.

(2) By Proposition 1.20(3), $\mathfrak{a} : G \to \boldsymbol{OI}_E$ is a $\mathfrak{v}$-isomorphism. In view of (1), the map $x \mapsto \mathfrak{a}(x)$ takes values in T^*_E. Hence the given assignments gives an order preserving $\mathfrak{v}$-isomorphism $\mathfrak{a}_G : G \to T^*_E$ with $\mathfrak{v}\mathfrak{a}_G = 1_E$. We now show that the following diagram commutes.

$$\begin{array}{ccc} \mathfrak{C}(E) & \xrightarrow{\varepsilon_G]} & G \\ & \searrow^{\tau_E} & \downarrow{\scriptstyle \mathfrak{a}_G} \\ & & T^*_E \end{array} \tag{6.30}$$

Let $c \in \mathfrak{C}(E)$ and $e \in \omega(e_c)$. By Equation (6.5a), $f_{e\centerdot c} = e\tau E(c)$ and since the evaluation $\varepsilon = \varepsilon_G$ is order preserving, we have

$$\begin{aligned}(\varepsilon \circ \mathfrak{a}_G)(c) &= \mathfrak{a}_G\,(\varepsilon(c))\\ &= f_{e\centerdot\varepsilon(c)} = f_{\varepsilon(e\centerdot c)} = f_{e\centerdot c} = e\tau_E(c).\end{aligned}$$

Therefore

$$\varepsilon \circ \mathfrak{a}_G = \tau_E$$

which proves that 6.30 commutes. Sine $\mathfrak{v}\mathfrak{a}_G = 1_E$, it follows that $\mathfrak{a}_G$ is inductive.

(3) Suppose that G be a $\mathfrak{v}$-full subgroupoid of T^*_E. Then for $\alpha \in G$ and $e \in \omega(e_\alpha)$, $e \centerdot \alpha = \alpha|\omega(e)$. Therefore, for all $\alpha \in G$ and $e \in \omega(e_\alpha)$, we have $e\mathfrak{a}_G(\alpha) = f_{e\centerdot\alpha} = e\alpha$ and so, $\mathfrak{a}_G(\alpha) = \alpha$.

(4) By Proposition 6.7(2), $T^*(G)$ is a an inductive subgroupoid of T^*_E where $E = \mathfrak{v}G$ which is $\mathfrak{v}$-full since $\mathfrak{v}\mathfrak{a}_G = 1_E$. Similarly $T^*(G')$ is a $\mathfrak{v}$-full inductive subgroupoid of $T^*_{E'}$ where $E' = \mathfrak{v}G$. Since $\phi : G \to G'$ is a $\mathfrak{v}$-surjection, $\mathfrak{v}\phi = \theta : E \to E'$ is a surjective (regular) bimorphism. We now show that $T^*(G)$ is well defined by Equation (6.28). Assume that $\mathfrak{a}_G(x) = \mathfrak{a}_G(y)$. Then by Theorem 6.28(1), x and y are in the same home-set of G so that $e_x = e_y$ and $f_x = f_y$. Moreover, for all $e \in \omega(e_x)$,

$$\begin{aligned}(e\theta)\mathfrak{a}_{G'}\big(\phi(x)\big) &= f_{e\theta\centerdot\phi(x)} = f_{\phi(e\centerdot x)} = (f_{e\centerdot x})\,\theta\\ &= (e\mathfrak{a}_G(x))\,\theta = (e\mathfrak{a}_G(y))\,\theta\\ &= (e\theta)\mathfrak{a}_{G'}\big(\phi(y)\big).\end{aligned}$$

Since θ is surjective, $\omega(e)\theta = \omega(e\theta)$ and it follows that $\mathfrak{a}_{G'}\big(\phi(x)\big) = \mathfrak{a}_{G'}\big(\phi(y)\big)$. Since $\mathfrak{a}_G$, $\mathfrak{a}_{G'}$ and ϕ are order preserving functors, it is immediate from Equation (6.28) that $T^*(\phi)$ is an order preserving functor. We also have the following commutative diagram:

$$\begin{array}{ccccc} \mathfrak{C}(E) & \xrightarrow{\varepsilon_G} & G & \xrightarrow{\mathfrak{a}_G} & T^*(G) \\ \downarrow{\scriptstyle \mathfrak{C}(\theta)} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle T^*(\phi)} \\ \mathfrak{C}(E') & \xrightarrow[\varepsilon_{G'}]{} & G' & \xrightarrow[\mathfrak{a}_{G'}]{} & T^*(G') \end{array} \tag{6.31}$$

inductive groupoid!fundamental –

The first square commutes since ϕ is inductive and the second square commutes by Equation (6.28). By 6.30,

$$\varepsilon_G \circ \mathfrak{a}_G = \tau_E \quad \text{and} \quad \varepsilon_{G'} \circ \mathfrak{a}_{G'} = \tau_{E'}.$$

Therefore $T^*(\phi)$ is inductive. Suppose that $\phi : G \to H$ and $\phi' : H \to K$ be $\mathfrak{v}$-surjections in $\mathfrak{IG}$. Then for any $x \in G$, we have

$$\begin{aligned} T^*(\phi\phi')\,(\mathfrak{a}_G(x)) &= \mathfrak{a}_K\left(\phi\phi'(x)\right) = \mathfrak{a}_K\left(\phi'\left(\phi(x)\right)\right) \\ &= T^*(\phi')\left(\mathfrak{a}_H\left(\phi(x)\right)\right) = T^*(\phi')\left(T^*(\phi)(\mathfrak{a}_G(x))\right) \\ &= \left(T^*(\phi)T^*(\phi')\right)(\mathfrak{a}_G(x)). \end{aligned}$$

Therefore $T^*(\phi\phi') = T^*(\phi)T^*(\phi')$.

(5) Assume that $\phi : G \to G'$ is a $\mathfrak{v}$-isomorphism so that $\mathfrak{v}\phi = \theta : E \to E'$ is a biorder isomorphism. Let $\mathfrak{a}_G(x), \mathfrak{a}_G(y) \in T^*(G)$ and $T^*(\phi)\,(\mathfrak{a}_G(x)) = T^*(\phi)\,(\mathfrak{a}_G(y))$. Then $\mathfrak{a}_{G'}\left(\phi(x)\right) = \mathfrak{a}_{G'}\left(\phi(x)\right)$ and so, $e_{\phi(x)} = (e_x)\theta = e_{\phi(y)} = (e_y)\theta$ and $f_{\phi(x)} = (f_x)\theta = f_{\phi(y)} = (f_y)\theta$. Since θ is an isomorphism, we have $e_x = e_y$ and $f_x = f_y$. Now, for any $e \in \omega(e_x)$

$$\begin{aligned} T^*(\phi)\,(\mathfrak{a}_G(e \,.\, x)) &= \mathfrak{a}_{G'}\left(\phi(e \,.\, x)\right) \\ &= (e\theta) \,.\, \mathfrak{a}_{G'}\left(\phi(x)\right) = (e\theta) \,.\, \mathfrak{a}_{G'}\left(\phi(y)\right) \\ &= \mathfrak{a}_{G'}\left(\phi(e \,.\, y)\right) = T^*(\phi)\,(\mathfrak{a}_G(e \,.\, y)). \end{aligned}$$

Hence $e\mathfrak{a}_G(x) = f_{e.x} = f_{e.y} = e\mathfrak{a}_G(y)$ for all $e \in \omega(e_x)$. Therefore $\mathfrak{a}_G(x) = \mathfrak{a}_G(y)$. Thus $T^*(\phi)$ is injective. If $\mathfrak{v}\phi = 1_E$, then for all $x \in G$, $e_x = e_{\phi(x)}$ and $f_x = f_{\phi(x)}$. It follows that $\mathfrak{a}_G(x) = \mathfrak{a}_{G'}(\phi(x))$ for all $x \in G$. Therefore $T^*(\phi) : T^*(G) \subseteq T^*(G')$. ☐

We shall say that an inductive groupoid G is *fundamental* if any $\mathfrak{v}$-isomorphism $\phi : G \to G'$ is injective. That is, G is fundamental if, for any inductive functor $\phi : G \to G'$, the morphism map of ϕ is injective whenever $\mathfrak{v}\phi : \mathfrak{v}G \to \mathfrak{v}G'$ is an isomorphism.

If G is fundamental, $\mathfrak{a}_G$ is injective and as in the proof of (5) above, we see that $G = T^*(G)$ in this case. By (3) above, any $\mathfrak{v}$-full inductive subgroupoid of T^*_E is fundamental. Hence fundamental inductive subgroupoids G with $\mathfrak{v}G = E$ are precisely $\mathfrak{v}$-full inductive subgroupoids of T^*_E.

***Remark* 6.4:** Notice that T^* defined in (4) above is not a functor on $\mathfrak{IG}$. For, it is easy to construct an example to show that the morphism $T^*(\phi)$ is not well-defined by Equation (6.28). However, $T^* : \mathfrak{IG}' \to \mathfrak{IG}'$ is a functor if $\mathfrak{IG}'$ is the category with inductive groupoids as objects and $\mathfrak{v}$-surjections as morphisms. In particular, if $\mathfrak{IG}_E$ is the inverse fiber of the functor $\mathfrak{v} : \mathfrak{IG} \to \mathfrak{RB}$ at E (that is, $\mathfrak{IG}_E$ is the category with objects as inductive groupoids G with $\mathfrak{v}G = E$ and with morphisms ϕ with $\mathfrak{v}\phi = 1_E$) then T^* is a functor of $\mathfrak{IG}_E$ to the preorder under inclusions of all $\mathfrak{v}$-full inductive subgroupoids of T^*_E.

extension
extensive families
ω-subgroupoid
generator

6.4.1 Exercise

Example 6.1: Suppose that G is an inductive groupoid in the sense of Schein (see Theorem 5.2). Prove that G is an inductive groupid according to Definition 6.1.

Example 6.2: Let G be a groupoid. Prove that there is a partial order on G which make it a Schein's groupoid if and only if either $\mathfrak{v}G$ is infinite or there is a component of G which is a group [see Schein, 1966]. However, on any groupoid G is the groupoid of an inductive groupoid; that is, a partial order can be defined on G making it an ordered groupoid, a biorder structure on $\mathfrak{v}G = E$ and an evaluation of $\mathfrak{C}(E)$ in G making making G an inductive groupoid [see Nambooripad, 1979, Page 55].

Example 6.3: Prove that there are ordered groupoids that does not arise as the ordered groupoid of an inductive groupoid. Also it may be possible to define more than one inductive groupoid structure on a given ordered groupoid.

Example 6.4: Prove that a groupoid with evaluation $G = (G, \varepsilon)$ satisfies the condition (IG1*) of Remark 6.5 if and only if it satisfies the axiom (IG1) of Definition 6.1.

6.5 EXTENSIONS

In general by an *extension* of an inducctive groupoid G, we mean a pair (G', ϕ) where G' is an inductive groupoid and $\phi : G \to G'$ is an embedding of the inductive groupoid G into G'; identifying G with the subgroupoid $\operatorname{Im}\phi$ we can regard the extension as the inductive groupoid G' containing G as an inductive subgroupoid.

In this section we study several classes of extensions of a regular semigroup S using the concepts of *extensive families*. Study of these extensions were originally done by [Pastijn and Petrich, 1986]. The version of these results presented here is due to [**?**] which illustrate the use of inductive groupoids in such constructions.

6.5.1 $\mathfrak{v}$ – full extensions of inductive groupoids

We continue to use the notation G for an arbitrary inductive groupoid with $E = \mathfrak{v}G$ and evaluation ε. An inductive groupoid G' is a $\mathfrak{v}$ *– full extension* of G if G is a $\mathfrak{v}$ – full inductive subgroupoid of G'. In this section, unless otherwise stated, by an extension, we shall mean a $\mathfrak{v}$ – full extension.

We begin by discussing some local properties of inductive groupoids. Call an inductive subgroupoid H of G to be an *ω-subgroupoid* if $\mathfrak{v}H = \omega(e)$ for $e \in E$; we write $H = H(e)$ and e is called the *generator* of H. Let $\sigma : H(e_\sigma) \to H(f_\sigma)$ and $\tau : H(e_\tau) \to H(f_\tau)$ be inductive isomorphisms of ω-subgroupoids of G. If the groupoid composite (that is, $H(f_\sigma) = H(e_\tau)$; see Example 1.21) of σ, τ exists, then it is clear that $\sigma\tau : H(e_\sigma) \to H(f_\tau)$ is an inductive isomorphism. Also $1_{H(e)} : H(e) \to H(e)$ is an inductive isomorphism. Therefore it is clear that there

$\mathfrak{G}G$:The groupoid of ω-subgroupoids of G

is a groupoid $\mathfrak{G}G$ of inductive isomorphisms of ω-subgroupoids which is a subgroupoid of the groupoid $\boldsymbol{I}_G$ of all partial bijections of G. Again if $\sigma \in \mathfrak{G}G$ then $\mathfrak{v}\sigma : \omega(e_\sigma) \to \omega(f_\sigma)$ is an ω-isomorphism of $E = \mathfrak{v}G$. If $H(g) \subseteq H(e_\sigma)$, then

$$\tau = (\sigma|H(g))^\circ$$

is an inductive isomorphism of $H(g)$ onto $\mathrm{Im}(\sigma|H(g)) = H(h)$ where $h = (g)|\sigma$. It follows that we can define a partial order on $\mathfrak{G}G$ as follows:

$$\sigma \leq \tau \iff G\langle e_\sigma\rangle \subseteq G\langle f_\tau\rangle \quad \text{and} \quad \sigma = (\tau|G\langle e_\sigma\rangle)^\circ \tag{6.32}$$

It is easy to verify that $\leq$ is the restriction of the partial order on $\boldsymbol{I}_G$ to $\mathfrak{G}G$ and hence the inclusion $\mathfrak{G}G \subseteq \boldsymbol{I}_G$ is an order-embedding of $\mathfrak{G}G$ into $\boldsymbol{I}_G$. Conequently we have:

LEMMA 6.29. *There is a groupoid $\mathfrak{G}G$ in which $\mathfrak{v}\mathfrak{G}G$ is the set of all ω-subgroupoids of G and morphisms are inductive isomorphisms. Moreover $\mathfrak{G}G$ is an ordered groupoid with respect to the relation $\leq$ defined by Equation* (6.32). □

Recall that for any $u \in G$, $\mathfrak{a}(u) : \omega(e_u) \to \omega(f_u)$ is an ω-isomorphism (see Theorem 6.28). The following proposition is due to **?**.

PROPOSITION 6.30. *For $e \in E = \mathfrak{v}G$ suppose that*

$$\mathcal{N}(e) = \{v \in G : e_v, f_v \in \omega(e)\}. \tag{6.33a}$$

Then $\mathcal{N}(e)$ is the morphism set of an inductive subgrupoid of G with

$$\mathfrak{v}\mathcal{N}(e) = \omega(e)$$

and evaluation $$\varepsilon_{\mathcal{N}(e)} = \varepsilon_G||\mathfrak{C}(\omega(e)).$$

For $u \in G$ define $\mathcal{N}(u) : \mathcal{N}(e_u) \to \mathcal{N}(f_u)$ by

$$(x)\mathcal{N}(u) = (u^{-1} \centerdot e_x)x(f_x \centerdot u) \quad \textit{for all} \quad x \in \mathcal{N}(e_u). \tag{6.33b}$$

Then $\mathcal{N}(u) : \mathcal{N}(e_u) \to \mathcal{N}(f)$ is an inductive isomorphism with

$$\mathfrak{v}\mathcal{N}(u) = \mathfrak{a}_G(u).$$

Proof. If $u, v \in \mathcal{N}(e)$ and uv exists, then $e_{uv} = e_u \ \omega \ e$ and $f_{uv} = f_v \ \omega \ e$ and so, $uv \in \mathcal{N}(e)$. Also $e_{u^{-1}} = f_u \ \omega \ e$ and $f_{u^{-1}} = e_u \ \omega \ e$ so that $u^{-1} \in G\langle e\rangle$. Hence $\mathcal{N}(e)$ is a subgroupoid of G. Further, if $g \ \omega \ e_u$ with $u \in \mathcal{N}(e)$ then $f_{e \centerdot u} \ \omega \ f_u \ \omega \ e$. Hence $e \centerdot u \in \mathcal{N}(e)$. It follows that $\mathcal{N}(e)$ is an ordered subgroupoid of G with $\mathfrak{v}\mathcal{N}(e) = \omega(e)$. Since $\mathfrak{C}(\omega(e))$ is an ordered subgroupoid of $\mathfrak{C}(E)$ and since $\varepsilon(c) \in \mathcal{N}(e)$ for all $c \in \mathfrak{C}(\omega(e))$, it is clear that $\mathcal{N}(e)$ is an inductive groupoid with respect to the evluation $\varepsilon' = \varepsilon|\mathfrak{C}(\omega(e))$.

By Proposition 1.18, $v^{-1} \centerdot e_u$ is the unique morphism with $v^{-1} \centerdot e_u \leq v^{-1}$ and $f_{v^{-1}\centerdot e_u} = e_u$. Hence $v^{-1} \centerdot e_u = (e_u \centerdot v)^{-1}$ and $e_{v^{-1}\centerdot e_u} \ \omega \ e_{v^{-1}} = f_v$. Since $f_{\mathcal{N}(v)} \ \omega \ f_v$ it follows that $\mathcal{N}(v)(u) \in \mathcal{N}(e)$ for all $u \in \mathcal{N}(e)$. If $w = (u)\mathcal{N}(v)$, an easy computation using the definition of $\mathcal{N}(v)$ shows that $(u)\mathcal{N}(v) = w$ if and only if $u = (w)\mathcal{N}(v^{-1})$ and so, $\mathcal{N}(v)$ is a bijection with $\mathcal{N}(v)^{-1} = \mathcal{N}(v^{-1})$. If $u, w \in \mathcal{N}(e)$ and if uw exists, then

$$
\begin{aligned}
(u)\mathcal{N}(v)(w)\mathcal{N}(v) &= \Big((v^{-1} \centerdot e_u)u(f_u \centerdot v)\Big)\Big((v^{-1} \centerdot e_w)w(f_w \centerdot v)\Big) \\
&= \Big((e_u \centerdot v)^{-1}u(f_u \centerdot v)\Big)\Big((e_w \centerdot v)^{-1}w(f_w \centerdot v)\Big) \\
&= \Big((e_u \centerdot v)^{-1}uf_uw(f_w \centerdot v)\Big) && \text{since } f_u = e_w \\
&= \Big((e_{uw} \centerdot v)^{-1}(uw)(f_{uw} \centerdot v)\Big) && \text{since } e_u = e_{uw} \text{ and } f_w = f_{uw} \\
&= (uw)\mathcal{N}(v).
\end{aligned}
$$

Also, if $g \in \mathcal{N}(e)$ is an identity then $g \ \omega \ e_v$. Hence

$$(g)\mathcal{N}(v) = (g \centerdot v)^{-1}g(g \centerdot v) = f_{g\centerdot v} = g\mathfrak{a}_G(v)$$

by Theorem 6.28 and so

$$\mathfrak{v}\mathcal{N}(v) = \mathfrak{a}_G(v)$$

Hence $\mathcal{N}(v)$ preserves composition and identities. Thus $\mathcal{N}(v)$ is a functor. Since $\mathcal{N}(v)$ is a bijection, it is an isomorphism of groupoids. To show that $\mathcal{N}(v)$ is order preserving, let $u, u' \in \mathcal{N}(e)[e_v]$ and $u \leq u'$. Then we have $e_u \ \omega \ e_{u'}$ and $f_u \ \omega \ f_{u'}$. Therefore $e_u \centerdot v \leq e_{u'} \centerdot v$ and $f_u \centerdot v \leq f_{u'} \centerdot v$ by axiom (oG3). Also $(e_u \centerdot v)^{-1} \leq (e_{u'} \centerdot v)^{-1}$ by axiom (OG2). Hence by axiom (OG1),

$$
\begin{aligned}
(u)\mathcal{N}(v) &= (e_u \centerdot v)^{-1}(u)(f_u \centerdot v) \\
&\leq (e_{u'} \centerdot v)^{-1}(u')(f_{u'} \centerdot v) = (u')\mathcal{N}(v)
\end{aligned}
$$

Thus $\mathcal{N}(v)$ is order preserving. Moreover, since $\mathfrak{v}\mathcal{N}(v) = \mathfrak{a}_G(v)$, by Theorem 6.28, $\mathfrak{v}\mathcal{N}(v)$ is an ω-isomorphism and hence a biorder isomorphism of $\mathfrak{v}\mathcal{N}(e_u)$ onto $\mathfrak{v}\mathcal{N}(f_u)$. Now cconsider $e_1, e_2 \in \omega(e_v) = \mathfrak{v}\mathcal{N}(e_u)$. If $e_1 \mathscr{R} e_2$, and if $f_1 = (e_i)\mathfrak{a}(v)$, $i = 1, 2$ then $f_1 \mathscr{R} f_2$ and

$$
\begin{aligned}
\varepsilon'(e_1, e_2)(e_2 \centerdot v) &= \varepsilon(e_1, e_2)(e_2 \centerdot v) && \text{where } \varepsilon' = \varepsilon|\mathcal{N}(e_v) \\
&= (e_1 \centerdot v)\varepsilon(f_1, f_2) && \text{by axiom (IG1)} \\
&= (e_1 \centerdot v)\varepsilon''(f_1, f_2). && \text{where } \varepsilon'' = \varepsilon|\mathcal{N}(f_v)
\end{aligned}
$$

Similarly, if $e_1 \mathscr{L} e_2$ then $f_1 \mathscr{L} f_2$ and

$$\varepsilon'(e_1, e_2)(e_2 \centerdot v) = (e_1 \centerdot v)\varepsilon''(f_1, f_2).$$

groupoid!– with evaluation
local subgroupoid
local isomorphism
local inner isomorphism

Consequently, for any $c \in \mathfrak{C}(\omega(e_v))$

$$\varepsilon'(c)(e_2 \centerdot v) = (e_1 \centerdot v)\varepsilon''(c\mathfrak{a}(v)).$$

It is now clear that the diagram 6.12 commutes for $\mathcal{N}(v)$ and so, by Definition 6.2, $\mathcal{N}(v)$ is an inductive isomorphism. □

***Remark* 6.5:** Notice that, the proof that $\mathcal{N}(v)$ is an order preserving functor of the ordered groupoid $\mathcal{N}(e_v)$ to $\mathcal{N}(f_v)$ does not use evaluation in any way and so the result is true for all ordered groupoids. However, the proof that $\mathcal{N}(v)$ is inductive uses evaluation. In fact the statement that $\mathcal{N}(v)$ is inductive for all $v \in G$ is equivalent to (IG1). For, call a pair (G, E_C) a *groupoid with evaluation* where G is an ordered group such that $\mathfrak{v}G = E$ is a biordered set and $\varepsilon : \mathfrak{C}(E) \to G$ is an evaluation; that is, an order preserving $\mathfrak{v}$-isomorphism. As for inductive groupoids, we shall abbreviate the notation of the groupoid with evaluation to G and denote the corresponding evaluation by ε_G and $\mathfrak{v}G = E$. A morphism $\phi : G \to G'$ is an order preserving functor $\phi : G \to G'$ of ordered groupoids such that $\theta = \mathfrak{v}\phi : E \to E'$ is a (regular) bimorphism making the diagram 6.12 commutative. The morphism ϕ is an isomorphism if ϕ is an isomorphism of ordered groupoids and $\theta = \mathfrak{v}\phi$ is a biorder isomorphism. This defines a category $\mathfrak{GE}$ which can be identified as a subcategory of the morphism category of the category $\mathfrak{OG}$ of ordered groupoids or the functor ccategory $[\mathbf{2}, \mathfrak{OG}]$ where $\mathbf{2}$ denote the category $\cdot \to \cdot$ with two objects and one morphism [see MacLane, 1971, Page 40]. For $e \in E = \mathfrak{v}G$, if we define subgropoid $\mathcal{N}(e)$ as above, it is a groupoid with evaluation $\varepsilon_{\mathcal{N}(e)} = \varepsilon_G|\mathcal{N}(e)$. It is not difficult to prove that a groupoid with evaluatin is an inductive grupoid if and only if it satisfies axiom (IG2) and the following:

(IG1)* For each $u \in G$, $\mathcal{N}(u) : \mathcal{N}(e_u) \to \mathcal{N}(f_u)$ is a local isomorphism.

(see also examples at the end of this section.)

For each $e \in E$ the inductive subgroupoid $\mathcal{N}(e) \subseteq G$ will be called a *local subgroupoid* of G at e. An iductive isomorphism $\sigma : \mathcal{N}(e_\sigma) \to \mathcal{N}(f_\sigma)$ is called a *local isomorphism* of G. The Proposition 6.30 above shows that, for $v \in G$, $\mathcal{N}(v) : \mathcal{N}(e_v) \to \mathcal{N}(f_v)$ is a local isomorphism of G. $\mathcal{N}(v)$ is called a *local inner isomorphism* of G.

The following is a useful consequence of Proposition 6.30

PROPOSITION 6.31. *Let G be an inductive groupoid with $\mathfrak{v}G = E$. Then the assignments*

$$\mathcal{N} : e \mapsto \mathcal{N}(e) = \mathcal{N}(e) \quad \textit{and} \quad v \mapsto \mathcal{N}(v)$$

is an order preserving $\mathfrak{v}$-isomorphism $\mathcal{N} : G \to \mathfrak{G}G$.

Proof. By Lemma 6.29, $\mathfrak{G}G$ is an ordered groupoid and by Proposition 6.30, $\mathcal{N} : v \mapsto \mathcal{N}(v)$ is a map of G into $\mathfrak{G}G$. Let $v, w \in G$ such that vw exists. Then for any $x \in \mathcal{N}(e_v)$,

extensive family morphism

$$\begin{aligned}(x)\mathcal{N}(v)\mathcal{N}(w) &= (h \centerdot w)^{-1}(e_x \centerdot v)^{-1}x(f_x \centerdot v)(k \centerdot w) \quad \text{where } h = (e_x)\mathfrak{a}(v) \text{ and } k = (f_x)\mathfrak{a}(v)\\ &= ((e_x \centerdot v)(h \centerdot w))^{-1}\, x(f_x \centerdot v)(k \centerdot w)\\ &= (e_x \centerdot vw)x(f_x \centerdot vw) = (x)\mathcal{N}(vw) \quad \text{by Proposition 1.19(2).}\end{aligned}$$

Also, taking $u = e$ in Equation (6.33b), we have $x\mathcal{N}(e) = x$ for al $x \in \mathcal{N}(e)$. Therefore $\mathcal{N} : G \to \mathfrak{G}G$ is a functor. If $v \leq w$ then $\mathcal{N}(e_v) \subseteq \mathcal{N}(e_w)$ and for any $x \in \mathcal{N}(e_v)$ we have

$$\begin{aligned}x\mathcal{N}(v) &= (e_x \centerdot v)^{-1}x(f_x \centerdot v) = (e_x \centerdot (e_v \centerdot w))^{-1}\, x\,(f_x \centerdot (f_v \centerdot w))\\ &= (e_x \centerdot w)^{-1}x(f_x \centerdot w) = (x)\mathcal{N}(w).\end{aligned}$$

Therefore $\mathcal{N}(v) \leq \mathcal{N}(w)$ by Lemma 6.29 and so $\mathcal{N}$ is order preserving. Since $e\ \omega\ f$ if and only if $\mathcal{N}(e) \subseteq \mathcal{N}(f)$, the map $e \mapsto \mathcal{N}(e)$ is an order embedding of $E = \mathfrak{v}G$ onto $\mathfrak{v}\mathfrak{G}G$ (see Lemma 6.29). This completes the proof. □

Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ to a category $\mathcal{D}$ with subobjects is a subfunctor of $H : \mathcal{C} \to \mathcal{D}$ if $F(c) \subseteq H(c)$ for all $c \in \mathfrak{v}\mathcal{C}$ and the map

$$j_F^H : c \mapsto j_{F(c)}^{H(c)}$$

is a natural transformation of F to H (see Equation (1.52)); we write $F \subseteq H$.

Definition 6.3. Let G be an inductive groupoid. An *extensive family* of G is an order preserving $\mathfrak{v}$-embedding $\mathcal{F} : G \to \mathfrak{G}G$ such that

(a) $\mathfrak{v}\,(\mathcal{F}(e)) = \omega(e)$ for all $e \in E$; and

(b) $\mathcal{F} \subseteq \mathcal{N}$.

An indutive isomorphism $\sigma : \mathcal{F}(e_\sigma) \to \mathcal{F}(f_\sigma)$ is called an $\mathcal{F}$*-morphism* if for all $g\ \omega\ e_\sigma$, the restriction of σ to $\mathcal{F}(g)$ in $\mathfrak{G}G$ is an isomorphism of $\mathcal{F}(g)$ to $\mathcal{F}(h)$ where $h = (g)\mathfrak{v}\sigma$.

Notice that $\mathcal{N}$ is, in particular, an extensive family and every local morphism of G is an $\mathcal{N}$-morphism.

Theorem 6.32. *Let $\mathcal{F}$ be an extensive family for the inductive groupoid G. Then there exists an inductive groupoid $\mathsf{A}_{\mathcal{F}}(G) = \mathsf{A}_{\mathcal{F}}$ such that $\mathfrak{v}\mathsf{A}_{\mathcal{F}} = \{\mathcal{F}(e) : e \in E\}$ and morphisms are $\mathcal{F}$- morphisms. Furthermore, with respect to this inductive structure on $\mathsf{A}_{\mathcal{F}}$, the functor $\mathcal{F} : G \to \mathsf{A}_{\mathcal{F}}$ is an inductive $\mathfrak{v}$-isomorphism.*

Proof. Let $\theta = \mathfrak{v}\mathcal{F}$ and $\tilde{E} = \{\mathcal{F}(e) : e \in E\}$. The given condition implies that the map $\theta : e \mapsto \tilde{e} = \mathcal{F}(e)$ is an order isomorphism of (E, ω) onto $\tilde{E}$. For $e, f \in E$ define

$$\tilde{e}\tilde{f} = \widetilde{ef}. \tag{6.34a}$$

This defines a partial binary operation on $\tilde{E}$ which makes it a biordered set such that $\theta : E \to \tilde{E}$ is a biorder isomorphism. Moreover, since $\mathcal{F}$ is a $\mathfrak{v}$-embedding, the original partial order on $\tilde{E}$ becomes the relation ω of the biordered set $\tilde{E}$.

Let σ and τ be $\mathcal{F}$-morphisms and assume that $\sigma\tau$ exists in $\mathfrak{G}G$. Then for any $\mathcal{F}(g) \subseteq \operatorname{dom}\sigma = \mathcal{F}(e_\sigma)$, by Definition 6.3 above

$$\sigma' = \sigma|\mathcal{F}(g) : \mathcal{F}(g) \to \mathcal{F}(h) \quad \text{where} \quad h = g\mathfrak{v}\sigma$$

is a morphism in $\mathfrak{G}G$. Then $h\ \omega\ f_\sigma = e_\tau$ and so, again by the definition above

$$\tau' = \tau|\mathcal{F}(h) : \mathcal{F}(h) \to \mathcal{F}(k) \quad \text{where} \quad k = h\mathfrak{v}\tau$$

is a morphism in $\mathfrak{G}G$. By Proposition 1.19, we have $\sigma\tau|\mathcal{F}(g) = \sigma'\tau'$ which is a morphism in $\mathfrak{G}G$ from $\mathcal{F}(g)$ to $\mathcal{F}(k)$. Hence $\sigma\tau$ is an $\mathcal{F}$- morphism. Similarly, since

$$\sigma^{-1}|\mathcal{F}(h) = (\sigma|\mathcal{F}(g))^{-1} \quad \text{where} \quad g\ \omega\ e_\sigma, \quad h = g\mathfrak{v}\sigma$$

it follows that σ is an $\mathcal{F}$-morphism if and only if σ^{-1} is an $\mathcal{F}$-morphism. Therefore there exists a groupoid $\mathsf{A}_{\mathcal{F}}$ in which morphisms are $\mathcal{F}$-morphisms and $\mathfrak{v}\mathsf{A}_{\mathcal{F}} = \tilde{E}$. Again if σ is an $\mathcal{F}$-morphism so is $\sigma|\mathcal{F}(g)$ for all $g\ \omega\ e_\sigma$. Therefore $\mathsf{A}_{\mathcal{F}}$ is an ordered subgroupoid of $\mathfrak{G}G$ with

$$\tilde{g} \centerdot \sigma = \sigma|\mathcal{F}(g) \quad \text{for all} \quad \sigma \in \mathsf{A}_{\mathcal{F}} \quad g\ \omega\ e_\sigma. \tag{6.34b}$$

Since $\theta : E \to \tilde{E}$ is an isomorphism, every E-chain in $\mathfrak{C}(\tilde{E})$ has the form $\tilde{c} = \mathfrak{C}(\theta)(c)$ for a unique $c \in \mathfrak{C}(E)$. Define

$$\tilde{\varepsilon}(\tilde{c}) = \mathcal{F}(\varepsilon(c)) \quad \text{for all} \quad c \in \mathfrak{C}(E). \tag{6.34c}$$

Taking $c = c(g, g) = g$ in the above we see that

$$\tilde{\varepsilon}(\tilde{g}) = \mathcal{F}(g) = \tilde{g} \quad \text{for all} \quad g \in E.$$

Thus $\tilde{\varepsilon} : \mathfrak{C}(E)(\tilde{E}) \to \mathfrak{G}G$ is a functor satisfying the condition

$$\mathfrak{C}(\theta) \circ \tilde{\varepsilon} = \varepsilon \circ \mathcal{F}. \tag{6.34c*}$$

Since ε, $\mathfrak{C}(\theta)$ and $\mathcal{F}$ are order-preserving $\mathfrak{v}$-isomorphisms, so is $\tilde{\varepsilon}$. We now show that $(\mathsf{A}_{\mathcal{F}}, \tilde{\varepsilon})$ satisfies axioms (IG1) and (IG2). Accordingly assume that $\sigma \in \mathsf{A}_{\mathcal{F}}$ and $\tilde{e_i}\ \omega\ \tilde{e_{si}}$, $i = 1, 2$. Then there exists unique $e_i\ \omega\ e_\sigma$, $i = 1, 2$, such that $\tilde{e_i} = (e_i)\theta$. Let $f_i = (e_i)\alpha$ where $\alpha = \mathfrak{v}\sigma$ is an ω-isomorphism. Then $\tilde{f_1}\ \omega^r\ \tilde{f_2}$ if and only if $\tilde{e_1}\ \omega^r\ \tilde{e_2}$ in $\tilde{E}$. To verify (IG1)(a), suppose that $e_1\ \omega^r\ e_2$ so that $\tilde{e_1}\ \mathscr{R}\ \widetilde{e_2e_2}$. Since α is an ω-isomorphism we have $f_1\ \omega^r\ f_2$ and $\tilde{f_1}\ \mathscr{R}\ \widetilde{f_1f_2}$. Then by Definition 6.3 $\mathcal{F}(e_1) \subseteq \mathcal{N}(e_1)$ and

$$\mathcal{F}(\varepsilon(e_1, e_ie_2)) = \mathcal{N}(\varepsilon(e_1, e_ie_2))|\mathcal{F}(e_1).$$

Therefore, for any $u \in \mathcal{F}(e_1)$, $e_u, f_u \in \omega(e_1)$ and

$$\begin{aligned}(u)\tilde{\varepsilon}(\tilde{e_1}, \widetilde{e_1e_2}) &= (u)\mathcal{F}\,(\varepsilon(e_1, e_1e_2)) && \text{by Equation (6.34c)}\\ &= (e_u \centerdot \varepsilon(e_1, e_1e_2))^{-1}\, u\,(f_u \centerdot \varepsilon(e_1, e_1e_2)) && \text{by Equation (6.33b)}\\ &= (\varepsilon(e_ue_2, e_u))\, u\,(\varepsilon(f_u, f_ue_2))\end{aligned}$$

using Equation (6.5a) and the fact that ε is order preserving. Therefore, since σ is an inductive functor, again using Equations (6.34b) and (6.34c) we have

$$\begin{aligned}(u)\tilde{\varepsilon}(\tilde{e_1}, \widetilde{e_1e_2})\,(\widetilde{e_1e_2} \centerdot \sigma) &= ((\varepsilon(e_ue_2, e_u))\, u\,(\varepsilon(f_u, f_ue_2))\,\sigma\\ &= (\varepsilon(e_{u\sigma}f_2, e_{u\sigma}))\,(u\sigma)\,(\varepsilon(f_{u\sigma}, f_{u\sigma}f_2)\\ &= (u\sigma)\mathcal{F}\,(\varepsilon(f_1, f_1f_2))\\ &= (u)\,(\tilde{e_1} \centerdot \sigma)\left(\tilde{\varepsilon}(\tilde{f_1}, \widetilde{f_1f_2})\right).\end{aligned}$$

Since this equality holds for all $u \in \mathcal{F}$, axiom (IG1)(a) is proved. Proof for axiom (IG1)(b) is dual. Hence $\mathsf{A}_{\mathcal{F}}$ satisfies axiom (IG1). To prove (IG2), let $A = \begin{pmatrix} \tilde{g} & \widetilde{ge} \\ \tilde{h} & \widetilde{he} \end{pmatrix}$ be a column-singular E-square in $\tilde{E}$ (so that $g, h \in \omega^r\,(e)$ and $g \mathscr{L} h$). Then $A' = \begin{pmatrix} g & ge \\ h & he \end{pmatrix}$ is a column-singular matrixx in E so that A' is ε-commutative in G. Since $\mathcal{F}$ is a functor, it follows that

$$\begin{aligned}\tilde{\varepsilon}(\tilde{g}, \widetilde{ge})\tilde{\varepsilon}(\widetilde{ge}, \widetilde{he}) &= \mathcal{F}\,(\varepsilon(g, ge))\mathcal{F}\,(\varepsilon(ge, he)) && \text{by (6.34c}^*)\\ &= \mathcal{F}\,(\varepsilon(g, ge)\varepsilon(ge, he)) && \text{since } \mathcal{F} \text{ is a functor}\\ &= \mathcal{F}\,(\varepsilon(g, H)\varepsilon(h, he)) && \text{by (IG2) for } G\\ &= \tilde{\varepsilon}(\tilde{g}, \tilde{h})\tilde{\varepsilon}(\tilde{h}, \widetilde{he}) && \text{again by (6.34c}^*).\end{aligned}$$

Hence A is $\tilde{\varepsilon}$-commutative in $\mathsf{A}_{\mathcal{F}}$. The proof of $\tilde{\varepsilon}$-commutativity of row-singular E-squares in $\mathsf{A}_{\mathcal{F}}$ is similar. Therefore axiom (IG2) also holds in $\mathsf{A}_{\mathcal{F}}$. Thus $\mathsf{A}_{\mathcal{F}}$ is an inductive groupoid.

Since G and $\mathsf{A}_{\mathcal{F}}$ are inductive groupoids, and $\mathcal{F}$ is an order preserving functor, Equation (6.34c*) shows that $\mathcal{F}$ is an inductive functor of G to $\mathsf{A}_{\mathcal{F}}$. □

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List of Notations

1_X: identity function on X, 2
1_a:, 12
1_C:, 14
$< E, D_E, * >$:partial algebra on E with domain D_E, 154
$< F, G, \eta, \sigma >: \mathcal{C} \overset{\cdot}{\rightharpoonup} \mathcal{D}$:adjunction from $\mathcal{C}$ to $\mathcal{D}$, 24
$< F, G; \eta, \nu >: \mathcal{C} \overset{\cdot}{\rightleftharpoons} \mathcal{D}$:A category equivalence, 24
$D(*)$: trace of the $\mathscr{D}$-class D, 120
D_X:domain of the partial binary operartion on X, 11
$D_{\mathbb{E}}$:domain of the partial operation on $\mathbb{E}$, 154
E:biordered set, 153
$F \cong G$:F is naturally equivalent to G, 15
F^{-1}:inverse of F, 14
F_e:ω-partial functor of F on $\omega(e)$, 236
$K(S)$: kernel of the semigroup S, 107
L_a, R_a, H_a, D_a, J_a: equivalence class of Green's relations, 85
$M(e, f)$:quasiordered set $(\omega^l(e) \cap \omega^r(f), \leq)$, 158
$M_{\text{row-mon}}$:semigroup of row-monomial matrices, 137
$R \circ R'$: composite of R and R', 1
S/I: Rees quotient semigroup of S by ideal I, 64
S/ρ: quotient semigroup, 62
$S \times_U T$:fibered product of S and T over U, 72
S^0: semigroup obtained by adjoining 0, 51
S^1: monoid obtained by adjoining 1 to S, 51
S^1x :cyclic left S-set generated by x, 81
S^{op}: Left-right dual of S, 50
S_λ: symmetric group of degree α, 104
S_l : left regular S-set, 82
S_r : right regular S-set, 82
T^*:dual of T, 154
$T^*(E)$:ordered groupoid of ω-isomorphisms of E, 170
T^{op}: dual of statement T, 50

X^n: Cartesian product of n copies of X, 49
Y:Yoneda equivalence, 20
$Z(S)$:center of S, 254
$[\mathcal{C}, \mathcal{D}]$:category of functors, 16
$\boldsymbol{B}_0(E)$: Free semigroup generated by E, 187
$\mathbb{C}$: complex numbers, 49
Cat:The category of small categories, 13
$\boldsymbol{\kappa}\theta$:the biorder congruence of the bimorphism θ, 178
$\Delta : \mathcal{D} \to [\mathcal{C}, \mathcal{D}]$, 21
Δ_d:constant functor with value d, 21
Grp:category of groups, 12
$\mathbf{J}_S,\ \Lambda_S,\ \mathbf{I}_S$: partially ordered set of principal ideals, 52
$\mathcal{A}_\rho$: Kernel normal system of ρ, 234
$\mathcal{A}_\rho$:The kernel normal system of ρ, 264
$\mathbb{M}\mathcal{C}$:subcategory of monomorphisms, 25
$\mathbb{N}$: natural numbers, 49
$\mathbb{N}$:system of natural numbers, 209
$\mathbb{N}^*,\ \mathbb{Q}^*,\ \mathbb{R}^*,\ \mathbb{C}^*$: set of non-zero numbers, 49
$\Omega(S)$:translational hull of S, 132
$\mathbb{Q}$: rational numbers, 49
$\mathbb{R}$: real numbers, 49
$\boldsymbol{B}_X$: set of relations on X, 3
Ab:category of abelian groups, 12
$\mathfrak{g}(H)$: Schützenberger group of H, 103
$\bigoplus_{\omega\in\Omega} \mathsf{M}_\omega$:direct sum of representations M_ω, 141
$\boldsymbol{E}(S)$: the biordered set of S, 52
$\boldsymbol{E}(S)$: the biordered set of idempotents of S, 97
$\boldsymbol{E}(S)$:set of idempotents of S, 110
$\mathcal{C}(-,-)$:the hom-functor, 17
$\mathcal{C}(-,c)$:contravariant hom-functor, 14
$\mathcal{C}(a,b)$:set of morphisms from a to b, 10
$\mathcal{C}(c,-)$:covariant hom-functor, 14
$\mathcal{C}(c,f)$:function $g \mapsto gf$, 14
$\mathcal{C}(f,c)$:function from $\mathcal{C}(c'',c)$ to $\mathcal{C}(c',c)$, 14
$\mathcal{C} \times \mathcal{D}$:, 16
$\mathcal{C}^* = [\mathcal{C}, \mathsf{Set}]$:, 19
$\mathcal{C}^{\mathrm{op}}$:the opposite category of $\mathcal{C}$, 12
$\cdot, +, *, \circ$: symbols for binary operations, 50
cod f: codomain of f, 2
cod f:codomain of f, 10

$\coprod_{i\in I} S_i$:free product of $\{S_i\}_{i\in I}$, 73
$\langle A; \{w_i = w_i', i \in I\}\rangle$:semigroup presented with generators A and relations R, 76
λ^D : anti-representation of S by partial transformations on D, 134
λ_a:partial left translation by a, 135
λ:representation by partial left translations, 135
$\varinjlim F$:direct limit of F, 22
dom R: domain of R, 1
dom f:domain of f, 2, 10
M_D^*:The dual Schützenberger representation of with respect to D, 140
$\eta(a)$:component of the natural transformation η, 15
$\eta : d \xrightarrow{n} F$:cone to the base F from vertex d, 21
E_C:evaluation functor, 20
$\langle E\varphi\rangle$:The fundamental semiband of E, 186
X^+: Free semigroup on X, 74
X^*: Free monoid on X, 74
φ_E: Fundamental embedding of the biordered set E, 186
$\gamma(S)$:the universal group homomorphism on S, 248
$\boldsymbol{G}^\rho$: kernel of ρ, 239
$\mathsf{Gl}_\alpha(\Bbbk)$: general linear group of degree α, 104
$\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$, $\mathscr{D}$, $\mathscr{J}$: Green's relations, 85
H_C, H^C:contra, co-variant representations, 19
$\boldsymbol{G}(S)$:Inductive groupoid of S, 260
$\mathfrak{i}\mathfrak{E}_U$:The category of ideal extensions of U, 147
$\mathfrak{I}_S$: lattice of ideals, 52
Im R: image of R, 1
j_c^d:inclusion of c in d, 29
$\varprojlim F$:inverse limit of F, 22
ϱ^D :Representation of S by partial transformations on the $\mathscr{D}$-class D, 133
ϱ^D :representation of S by partial transformations on the $\mathscr{D}$-class D, 134
ϱ_a:partial right translation by a, 135
ϱ:representation by partial right translations, 135
$\mathbb{L}(S)$:l-category of S, 84
$\boldsymbol{D}\lambda_a$:isodomain of λ_a, 135
$\leq_S$: The natural partial order on S, 215
$\leq_\sigma$:quotient of $\leq$ by σ, 222
$\leq_l$, $\leq_r$, $\leq_j$: quasi-orders induced by principal ideals, 85
$\mathfrak{L}\mathfrak{I}_S$: lattice of left ideals, 52
λ_S : left regular representation, 82
${}_S\mathsf{Set}$:category of left S-sets, 81
$\mathbb{E}C$:, 26

$\mathfrak{Y}_C$:Yoneda functor, 20
$\boldsymbol{M}_E^*$: minimum condition on idempotents in a $\mathscr{J}$-calss, 127
$\boldsymbol{M}_L^*$: minimum condition on $\mathscr{L}$-classes in a $\mathscr{J}$-class, 127
$\boldsymbol{M}_R^*$: minimum condition on $\mathscr{R}$-classes in a $\mathscr{J}$-class, 127
Nat(S, T):natural transformations from S to T, 16
ω^r:left quasiorder, 153
ω^r:right quasiorder, 153
$\mathcal{F}_a(S)$: Principal factor of S at a, 125
$\prod_{i\in I} S_i$:Direct product of the family $\{S_i\}$ of semigroups, 69
$\mathfrak{RB}$:subcategory of $\tilde{\mathfrak{B}}$ with objects as regular biordered sets, 159
$\mathbb{R}(S)$:r-category of S, 84
$\boldsymbol{D}\varrho_a$:isodomain of ρ_a, 135
$\boldsymbol{\pi}_S$: representation of S by bitranslations, 132
ρ^n:composite of n copies of ρ, 4
$\mathfrak{RI}_S$: lattice of right ideals, 52
$\mathcal{M}^0(G; I, \Lambda; P)$: Rees $I \times \Lambda$-matrix semigroup over G^0, 59
$\boldsymbol{\rho}_S$:right regular representation of S, 82
$\mathbf{Set}_S$:category of right S-sets, 81
$\mathscr{S}(e, f)$: Sandwich set, 158
$\mathscr{S}_1(e, f)$:see definition on 172, 159
$\mathscr{S}_2(e, f)$:see definition on 172, 159
Set:category of sets, 12
$\mathfrak{S}$: category of semigroups, 54
$\sigma : F \xrightarrow{n} d$:cone from base F to vertex d, 21
$\sigma : cone\boldsymbol{C} \xrightarrow{n} d$, 21
$\simeq$: equivalence relation $\leq \cap \leq^{-1}$, 158
$\mathscr{V}(a)$: The set of inverses of a, 94
M_D:The Schützenberger representation of with respect to D, 140
$\boldsymbol{\varpi}$:partial symmetric representation of on S, 135
$\boldsymbol{\varpi}^D$:partial symmetric representation of S on D, 134
$\tau^l(e)$:left translation, 153
$\tau^r(e)$:right translation, 153
τ_L:order preserving functor from $\mathscr{L}$ to T_E^*, 171
τ_R:order preserving functor from $\mathscr{R}$ to T_E^*, 171
$\tilde{\mathfrak{B}}$:subcategory of $\mathfrak{B}$ with morphisms as regular bimorphisms, 159
χ_E: Universal isomorphism of E, 187
$\vee\Lambda'$:meet of Λ':, 7
$\mathfrak{v}F$:The vertex map of F, 13
$\mathfrak{v}C$:vertex class of C, 10
$\mathfrak{v}$:functor from **Cat** to **Set**, 14

$\wedge\Lambda'$:, 6
$a \mid b$: a divides b, 251
a^{-1}: unique inverse of a, 100
$c \subseteq d$:c is a subobject of d, 29
$e \centerdot x$ or x, ϵ:restriction of x to e40
e_g and f_g:unique left and right identity of $g \in \mathcal{C}$, 12
$f \leq g$:, 25
$f \sim g$:equivalent monomorphisms, 25
f°: homomorphism onto the image of f, 53
f°:epimorphic component of f, 33
xRy: x and y are R-related, 1
xS^1 :cyclic right S-set generated by x, 81
$x * y$: trace product of x and y, 120
$x \mapsto x^*$: involution, 53
$x \centerdot f$:co-restriction of x to f, 41

$\mathcal{E}_X$:, 7

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S-set:S-subset, 81
S-set:right –, 80
$\mathcal{D}$-valued functors, 16
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ω-partial functor, 236
j_f:inclusion of f, 33
n-airy operation, 49

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