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The Lattice of Varieties of Implication Semigroups

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Abstract

An implication semigroup is an algebra of type $(2,0)$ with a binary operation $\to$ and a 0-ary operation 0 satisfying the identities $(x \to y) \to z \approx x \to (y \to z)$, $(x \to y) \to z \approx \big[(z' \to x) \to (y \to z)'\big]'$ and $0'' \approx 0$ where $\mathbf{u}'$ means $\mathbf{u} \to 0$ for any term $\mathbf{u}$. We completely describe the lattice of varieties of implication semigroups. It turns out that this lattice is non-modular and consists of 16 elements.

Keywords Implication semigroup · Variety · Lattice of varieties

1 Introduction and summary

In the article [8], the second author introduced and examined a new type of algebras as a generalization of De Morgan algebras. These algebras are of type $(2,0)$ with a binary operation $\to$ and a 0-ary operation 0 satisfying the identities

$$(x \to y) \to z \approx \big[(z' \to x) \to (y \to z)'\big]' \quad \text{and} \quad 0'' \approx 0,$$

where $\mathbf{u}'$ means $\mathbf{u} \to 0$ for any term $\mathbf{u}$. Such algebras are called *implication zroupoids*. We refer an interested reader to [8] for detailed explanation of the background and motivations.

The class of all implication zroupoids is a variety denoted by **IZ**. It seems very natural to examine the lattice of its subvarieties. One of the important and interesting subvarieties of

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IZ is the class of all associative implication zroupoids, that is algebras from **IZ** satisfying the identity

$$(x \to y) \to z \approx x \to (y \to z).$$

It is natural to call such algebras *implication semigroups*. The class **IS** of all implication semigroups forms a subvariety in **IZ**. This subvariety was implicitly mentioned in [8, Lemma 8.21] and investigated more explicitly in the articles [3–5]. (Incidentally, we should mention here that implication zroupoids are referred to as "implicator groupoids" in [4].) But only the location of **IS** in the subvariety lattice of the variety **IZ** and "interaction" of **IS** with other varieties from this lattice were studied in those articles. The aim of this paper is to examine the lattice of subvarieties of the variety **IS**. Our main result gives a complete description of this lattice.

For convenience of our considerations, we turn to the notation generally accepted in the semigroup theory. As usual, we denote the binary operation by the absence of a symbol, rather than by $\to$. Since this operation is associative, we will, as a rule, omit brackets in terms. Besides that, the notation 0 for the 0-ary operation seems to be inappropriate in the framework of examination of implication semigroups, because it is associated with the operation of fixing the zero element in a semigroup with zero. For this reason, we will denote the 0-ary operation by the symbol ω which does not have any predefined a priori meaning. In this notation, implication semigroups are defined by the associative law $(xy)z \approx x(yz)$ and the following two identities:

$$xyz \approx z\omega xyz\omega^2, \tag{1.1}$$

$$\omega^3 \approx \omega. \tag{1.2}$$

To formulate the main result of the article, we need some notation. As usual, elements of the free implication semigroup over a countably infinite alphabet are called *words*, while elements of this alphabet are called *letters*. Words rather than letters are written in bold. We connect two sides of identities by the symbol $\approx$. We denote by **T** the trivial variety of implication semigroups. The variety of implication semigroups given (within **IS**) by the identity system Σ is denoted by var Σ. Let us fix notation for the following concrete varieties:

$$\begin{aligned}
\mathbf{B} &:= \operatorname{var}\{x \approx x^2\},\\
\mathbf{K} &:= \operatorname{var}\{xyz \approx x^2 \approx \omega,\ xy \approx yx\},\\
\mathbf{L} &:= \operatorname{var}\{xyz \approx x^2 \approx \omega\},\\
\mathbf{M} &:= \operatorname{var}\{xyz \approx \omega,\ xy \approx yx\},\\
\mathbf{N} &:= \operatorname{var}\{xyz \approx \omega\},\\
\mathbf{SL} &:= \operatorname{var}\{x \approx x^2,\ xy \approx yx\},\\
\mathbf{ZM} &:= \operatorname{var}\{xy \approx \omega\}.
\end{aligned}$$

The lattice of all varieties of implication semigroups is denoted by $\mathbb{IS}$.

The main result of the article is the following

Theorem 1.1 *The lattice* $\mathbb{IS}$ *has the form shown in* Fig. 1.

In [8, Problem 5], the second author formulated the question of whether the lattice of all varieties of implication zroupoids is distributive. The following assertion immediately follows from Fig. 1 and provides the negative answer to this question.

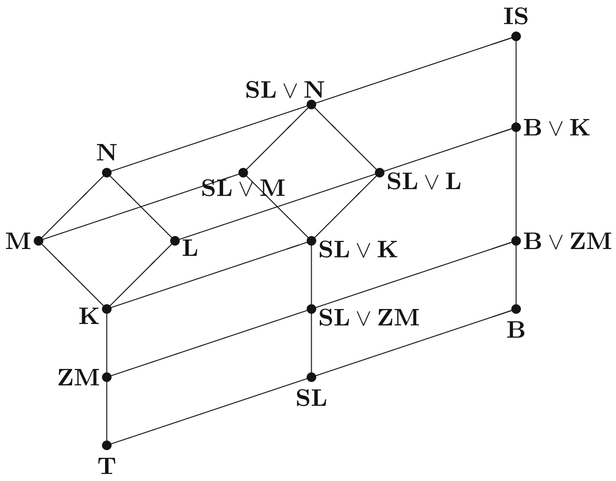

Fig. 1 The lattice $\mathbb{IS}$

Corollary 1.2 *The lattice $\mathbb{IS}$ is non-modular.*

This article consists of three sections. Section 2 is devoted to the proof of Theorem 1.1, while Section 3 contains several open problems.

2 Proof of the Main Result

To verify Theorem 1.1, we need a few auxiliary assertions. If $\mathbf{u}$ and $\mathbf{v}$ are words and ε is an identity then we will write $\mathbf{u} \stackrel{\varepsilon}{\approx} \mathbf{v}$ in the case when the identity $\mathbf{u} \approx \mathbf{v}$ follows from ε.

Lemma 2.1 *The variety* **IS** *satisfies the following identities:*

$$\omega^2 \approx \omega, \tag{2.1}$$

$$\omega x \approx x\omega, \tag{2.2}$$

$$xyz \approx xyz\omega. \tag{2.3}$$

Proof The following three chains of identities provide deductions of the identities (2.1)–(2.3) from the identities that hold in the variety **IS**:

$$\begin{aligned}
(2.1):\quad & \omega^2 \stackrel{(1.2)}{\approx} \omega^{10} = \omega^2\omega\omega\omega^2\omega^2\omega^2 \stackrel{(1.1)}{\approx} \omega\omega^2\omega^2 = \omega^5 \stackrel{(1.2)}{\approx} \omega,\\
(2.2):\quad & \omega x \stackrel{(2.1)}{\approx} \omega\omega x \stackrel{(1.1)}{\approx} x\omega\omega\omega x\omega^2 \stackrel{(2.1)}{\approx} (x\omega\omega\omega x\omega^2)\omega\\
& \stackrel{(2.1)}{\approx} (\omega\omega x)\omega \approx \omega\omega x\omega\omega\omega^2 \stackrel{(1.1)}{\approx} x\omega\omega \stackrel{(2.1)}{\approx} x\omega,\\
(2.3):\quad & xyz \stackrel{(1.1)}{\approx} z\omega xyz\omega^2 \stackrel{(2.1)}{\approx} (z\omega xyz\omega^2)\omega \stackrel{(1.1)}{\approx} (xyz)\omega.
\end{aligned}$$

Lemma is proved. □

An idempotent e of a semigroup S that commutes with every element in S is said to be a *central idempotent*. The identities (2.1) and (2.2) show that if S is an implication semigroup then the distinguished element ω of S is a central idempotent. This explains our interest in the following assertion which is a part of semigroup folklore. We provide its proof here for the sake of completeness.

Lemma 2.2 *If e is a central idempotent of a semigroup S then S is a subdirect product of its ideal eS and the Rees quotient S/eS.*

Proof Clearly, eS is an ideal of S and the natural homomorphism $\eta\colon S \to S/eS$ has the property that $\eta(x) = \eta(y)$ if and only if either $x = y$ or $x, y \in eS$. On the other hand, the map $\varphi\colon S \to eS$ given by the rule $\varphi(x) = ex$ is a homomorphism of S onto eS and $ex = ey$ implies $x = y$ for $x, y \in eS$. Therefore, if $x, y \in S$ are such that $\eta(x) = \eta(y)$ and $\varphi(x) = \varphi(y)$, then $x = y$. We see that φ and η are surjective homomorphisms from S onto eS and S/eS respectively, and the intersection of kernels of these homomorphisms is the equality relation. Hence S is a subdirect product of eS and S/eS. □

Recall that a semigroup is called a *band* if it satisfies the identity $x^2 \approx x$. We call a variety of implication semigroups $\mathbf{V}$ a *monoid variety* if the identities $x\omega \approx \omega x \approx x$ hold in $\mathbf{V}$. Obviously, this means that every semigroup in $\mathbf{V}$ has an identity element and the operation ω fixes just this element in each semigroup from $\mathbf{V}$.

Lemma 2.3 *A variety of implication semigroups is a monoid variety if and only if it is a variety of bands.*

Proof Any monoid variety satisfies the identities $x \approx \omega^2 x \stackrel{(1.1)}{\approx} x\omega^3 x\omega^2 \approx x^2$, while any variety of bands satisfies the identities $\omega x \stackrel{(2.2)}{\approx} x\omega \approx x^3\omega \stackrel{(2.3)}{\approx} x^3 \approx x$. □

Lemma 2.4 *If $\mathbf{V}$ is an implication semigroup variety then $\mathbf{V} = (\mathbf{V} \wedge \mathbf{B}) \vee (\mathbf{V} \wedge \mathbf{N})$.*

Proof We can assume that the variety $\mathbf{V}$ is generated by an implication semigroup S. In view of Lemmas 2.1 and 2.2, the set ωS is an ideal of S and S is a subdirect product of ωS and the Rees quotient $S/\omega S$. Clearly, ωS is an implication semigroup with the distinguished element ω and $\omega x = x\omega = x$ for every $x \in \omega S$. Then $\omega S \in \mathbf{B}$ by Lemma 2.3. Note also that $S/\omega S$ is an implication semigroup with the distinguished element ωS and $xyz \stackrel{(2.3)}{=} xyz\omega \stackrel{(2.2)}{=} \omega xyz \in \omega S$ for every $x, y, z \in S$. This implies that $S/\omega S$ satisfies the identity $xyz \approx \omega$ and therefore, is contained in the variety $\mathbf{N}$. Thus, we have proved that S is a subdirect product of the implication semigroups $\omega S \in \mathbf{B}$ and $S/\omega S \in \mathbf{N}$. This implies the required conclusion. □

As usual, we denote by $L(\mathbf{X})$ the subvariety lattice of the variety $\mathbf{X}$.

Proof of Theorem 1.1 According to Lemma 2.3, $\mathbf{B}$ is a monoid variety. Therefore, it satisfies the identities $xyx \approx x\omega yx\omega^2 \stackrel{(1.1)}{\approx} \omega yx \approx yx$. The lattice of varieties of band monoids is completely described in [11]. In view of [11, Proposition 4.7], the lattice $L(\mathbf{B})$ is the 3-element chain $\mathbf{T} \subset \mathbf{SL} \subset \mathbf{B}$.

The variety $\mathbf{N}$ satisfies the identities $\omega x \stackrel{(2.2)}{\approx} x\omega \stackrel{(2.1)}{\approx} x\omega^2 \approx \omega$. Hence every semigroup from $\mathbf{N}$ contains the zero element and the operation ω fixes just this element in each semigroup from $\mathbf{N}$. This means that $\mathbf{N}$ is nothing but the variety of all 3-nilpotent semigroups. The subvariety lattice of this variety has the form shown in Fig. 1. This claim can be easily verified directly and is a part of semigroup folklore. It is known at least from the beginning of 1970's (see [7], for instance).

Recall that commutative bands are called *semilattices*. We fix notation for the following semigroups:

$$\begin{aligned}
A &:= \{0, 1\} \text{ — the 2-element semilattice,}\\
B &:= \langle e, f, 1 \mid ef = f^2 = f, fe = e^2 = e\rangle = \{e, f, 1\},\\
K &:= \langle a, b, 0 \mid ab = ba, a^2 = b^2 = 0\rangle = \{a, b, ab, 0\}\\
L &:= \langle a, b, 0 \mid ba = a^2 = b^2 = 0\rangle = \{a, b, ab, 0\},\\
M &:= \langle a, b, 0 \mid ab = ba, a^2 = ab^2 = b^3 = 0\rangle = \{a, b, b^2, ab, 0\},\\
Z &:= \langle a, 0 \mid a^2 = 0\rangle = \{a, 0\}
\end{aligned}$$

where 0 and 1 have the usual sense in semigroup context (the zero element of a semigroup and the identity one, respectively). All these semigroups can be considered as implication semigroups. Indeed, it is easy to see that putting $\omega = 1$ in A, B and $\omega = 0$ in K, L, M, Z, we achieve the fulfillment of the identities (1.1) and (1.2). The variety generated by an implication semigroup S is denoted by var S. It is well known and easily verified that $\mathbf{B} = \operatorname{var} B$, $\mathbf{K} = \operatorname{var} K$, $\mathbf{L} = \operatorname{var} L$, $\mathbf{M} = \operatorname{var} M$, $\mathbf{SL} = \operatorname{var} A$ and $\mathbf{ZM} = \operatorname{var} Z$.

Now we are going to prove that the lattice $L(\mathbf{SL} \vee \mathbf{N})$ has the form shown in Fig. 1. Clearly, the implication semigroups A, L and M satisfy the identity $xy\omega \approx yx\omega$. So, this identity holds in $\mathbf{SL} \vee \mathbf{N}$. Since it is false in B, we have that $(\mathbf{SL} \vee \mathbf{N}) \wedge \mathbf{B} = \mathbf{SL}$. This fact and Lemma 2.4 imply that $\mathbf{V} = (\mathbf{V} \wedge \mathbf{SL}) \vee (\mathbf{V} \wedge \mathbf{N})$ for every subvariety $\mathbf{V}$ of $\mathbf{SL} \vee \mathbf{N}$. Then $\mathbf{SL} \vee \mathbf{N}$ has at most 12 subvarieties, namely, the ones shown in Fig. 1. We need to verify that these subvarieties are different from each other. For a class $\mathbf{X}$ of implication semigroups, let $\overline{\mathbf{X}}$ stand for the class of all semigroup reducts of implication semigroups in $\mathbf{X}$. Since $\omega \approx x^3$ in $\mathbf{N}$, we see that $\overline{\mathbf{V}}$ is a subvariety of $\overline{\mathbf{N}}$ whenever $\mathbf{V}$ is a subvariety of $\mathbf{N}$. Now let $\mathbf{V}$ and $\mathbf{W}$ be two different subvarieties of $\mathbf{N}$. Then the semigroup varieties $\overline{\mathbf{V}}$ and $\overline{\mathbf{W}}$ are different as well. It is well known that the semigroup variety $\overline{\mathbf{SL}}$ of all semilattices constitutes a neutral element of the lattice of all semigroup varieties (it is proved explicitly in [10, Proposition 4.1]), whence $\overline{\mathbf{V}} \vee \overline{\mathbf{SL}} \neq \overline{\mathbf{W}} \vee \overline{\mathbf{SL}}$. Any semigroup identity that differentiates $\overline{\mathbf{V}} \vee \overline{\mathbf{SL}}$ from $\overline{\mathbf{W}} \vee \overline{\mathbf{SL}}$ will also differentiate the implication semigroup varieties $\mathbf{V} \vee \mathbf{SL}$ and $\mathbf{W} \vee \mathbf{SL}$.

Further, we are going to prove that the lattice $L(\mathbf{B} \vee \mathbf{ZM})$ has the form shown in Fig. 1. First of all, we note that the identity $xy \approx xy\omega$ holds in B and Z but fails in K. Therefore, $(\mathbf{B} \vee \mathbf{ZM}) \wedge \mathbf{N} = \mathbf{ZM}$. This fact and Lemma 2.4 imply that $\mathbf{V} = (\mathbf{V} \wedge \mathbf{B}) \vee (\mathbf{V} \wedge \mathbf{ZM})$ for every subvariety $\mathbf{V}$ of $\mathbf{B} \vee \mathbf{ZM}$. Then $\mathbf{B} \vee \mathbf{ZM}$ has at most 6 subvarieties, namely, the ones shown in Fig. 1. We need to verify that these subvarieties are different from each other. In view of the observations made in the first, the second and the fourth paragraphs of the proof of Theorem 1.1, it remains to show that $\mathbf{SL} \vee \mathbf{ZM} \subset \mathbf{B} \vee \mathbf{ZM}$. This follows from the fact that the identity $xy \approx yx\omega$ holds in $\mathbf{SL} \vee \mathbf{ZM}$ but fails in $\mathbf{B}$.

Lemma 2.4 with $\mathbf{V} = \mathbf{IS}$ implies that $\mathbf{IS} = \mathbf{B} \vee \mathbf{N}$. Since $\mathbf{B}$ has exactly 3 subvarieties and $\mathbf{N}$ has exactly 6 ones, we have that $\mathbf{IS}$ has at most 18 subvarieties. Now we aim to show that $\mathbf{B} \vee \mathbf{K} = \mathbf{B} \vee \mathbf{L}$ and $\mathbf{B} \vee \mathbf{M} = \mathbf{B} \vee \mathbf{N}$. The subset $I = \{(e, 0), (f, 0), (1, 0)\}$ of the direct product $B \times K$ forms an ideal of $B \times K$. The Rees quotient $(B \times K)/I$ is a 3-nilpotent implication semigroup that satisfies the identity $x^2 \approx \omega$ but violates the commutative law. Indeed, $(e, a)(f, b) = (f, ab) \neq (e, ab) = (f, b)(e, a)$. We see that $(B \times K)/I$ lies in $\mathbf{L}$ but does not lie in $\mathbf{K}$. Note that $\mathbf{K}$ is the only maximal subvariety of $\mathbf{L}$. Whence $(B \times K)/I$ generates the variety $\mathbf{L}$. Since $(B \times K)/I \in \mathbf{B} \vee \mathbf{K}$, we have that $\mathbf{L} \subseteq \mathbf{B} \vee \mathbf{K}$. We conclude that $\mathbf{B} \vee \mathbf{L} \subseteq \mathbf{B} \vee \mathbf{K}$, and the converse inclusion is clear. Thus $\mathbf{B} \vee \mathbf{K} = \mathbf{B} \vee \mathbf{L}$. Further,

$\mathbf{B} \vee \mathbf{M} \supseteq \mathbf{B} \vee \mathbf{K} = \mathbf{B} \vee \mathbf{L}$. Therefore, $\mathbf{L} \subseteq \mathbf{B} \vee \mathbf{M}$, whence $\mathbf{N} = \mathbf{M} \vee \mathbf{L} \subseteq \mathbf{B} \vee \mathbf{M}$. We get that $\mathbf{B} \vee \mathbf{N} \subseteq \mathbf{B} \vee \mathbf{M}$. The converse inclusion is clear, whence $\mathbf{B} \vee \mathbf{M} = \mathbf{B} \vee \mathbf{N}$. Thus, we have proved that **IS** has at most 16 subvarieties, namely, the ones shown in Fig. 1. We need to verify that these subvarieties are different from each other. In view of what is said in the fourth and the fifth paragraphs of the proof of Theorem 1.1, it remains to show that $\mathbf{B} \vee \mathbf{K} \subset \mathbf{IS}$. This follows from the above-mentioned equalities $\mathbf{IS} = \mathbf{B} \vee \mathbf{N} = \mathbf{B} \vee \mathbf{M}$ and the fact that the identity $x\omega \approx x^2$ holds in K and B but fails in M. □

3 Open Problems

We denote by $\mathbb{IZ}$ the lattice of all varieties of implication zroupoids. Theorem 1.1 shows that the lattice $\mathbb{IZ}$ is non-modular but the following problem still remains open.

Problem 3.1 Determine whether the lattice $\mathbb{IZ}$ satisfies any non-trivial lattice identity.

Recall that a lattice $\langle L; \vee, \wedge\rangle$ with the least element 0 is called 0-*distribuive* if it satisfies the implication

$$\forall x, y, z \in L: \quad x \wedge z = y \wedge z = 0 \longrightarrow (x \vee y) \wedge z = 0.$$

Lattices of varieties of all classical types of algebras (groups, semigroups, rings, lattices etc.) are well-known to be 0-distributive. The following question seems to be interesting.

Problem 3.2 Determine whether the lattice $\mathbb{IZ}$ is 0-distributive.

This problem is closely related to knowing the set of all atoms of the lattice $\mathbb{IZ}$. This set is known but not yet published. Indeed, it is well known that any non-trivial variety of algebras contains a simple algebra, i.e. algebra without congruences except the trivial and the universal ones (see [1, Theorem 10.13], for instance). The complete list of simple implication zroupoids is provided by [2, Theorem 5.8]. The variety generated by one of these algebras contains either **ZM** or **SL** or the variety **BA** of all Boolean algebras. On the other hand, it is easy to see that these three varieties are atoms of $\mathbb{IZ}$. Combining these observations, we have the following

Remark 3.3 The varieties **ZM**, **SL** and **BA** are the only atoms of the lattice $\mathbb{IZ}$.

Returning to Problem 3.2, it is easy to see that this problem is equivalent to the following claim: if **A** is an atom of the lattice $\mathbb{IZ}$ and **X**, **Y** are varieties of implication zroupoids with $\mathbf{X}, \mathbf{Y} \not\supseteq \mathbf{A}$ then $\mathbf{X} \vee \mathbf{Y} \not\supseteq \mathbf{A}$. We have a proof of this fact in the case when **A** is one of the varieties **SL** or **BA**. But the case when $\mathbf{A} = \mathbf{ZM}$ still remains open.

An element x of a lattice L is called *neutral* if, for any $y, z \in L$, the elements x, y and z generate a distributive sublattice of L. Neutral elements play an important role in the lattice theory. If a is a neutral element of a lattice L then L is a subdirect product of the principal ideal and the principal filter of L generated by a (see [6, proof of Theorem 254]). So, the knowledge of the set of neutral elements of a lattice gives significant and important information about the structure of this lattice. Figure 1 shows that the varieties **SL** and **ZM** are neutral elements of the lattice $\mathbb{IS}$. The following problem seems to be very interesting.

Problem 3.4 Determine whether **SL**, **ZM** and **BA** are neutral elements of the lattice $\mathbb{IZ}$.

Note that the varieties of all semilattices and of all semigroups with zero multiplication considered as simply semigroup varieties are neutral elements of the lattice of all semigroup varieties (see [10, Proposition 4.1] or Theorem 3.4 in the survey [9]).

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