COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
39. SEMIGROUPS, SZEGED (HUNGARY), 1981.

REGULAR INVOLUTION SEMIGROUPS

K.S.S. NAMBOORIPAD — F.J.C.M. PASTIJN

The fundamental representation of regular involution semigroups is studied, and the different approaches which are suggested in [13], [14], [15], [17], [30] are presented in an integrated view. We show how some of the results by Foulis [9], [10], [11], Imaoka [22], [23], [24], [25] and Yamada [45] fit in the general framework which is outlined here.

1. PRELIMINARIES

We shall use the standard notation and terminology, as established in [4], [7], [20], [30], [39].

Let S be a semigroup. A transformation $^\star: S \to S,\ x \to x^\star$ will be called an *involution* on S if for all $x, y \in S$

(1.1) $\quad (xy)^\star = y^\star x^\star$

and

(1.2) $\quad (x^\star)^\star = x.$

The purpose of this note is to sketch a general scheme for describing the structure of regular semigroups with involution. This first section will introduce some definitions which lead to a preliminary classification, a list of "natural" examples, and a first investigation on the behaviour of congruences.

It follows at once that the involution * on S gives rise to a permutation on S which induces a permutation on the set $E(S)$ of idempotents of S. Note that * maps $\mathscr{L}$-classes onto $\mathscr{R}$-classes, $\mathscr{R}$-classes onto $\mathscr{L}$-classes, and $\mathscr{H}$-classes onto $\mathscr{H}$-classes. If an element is fixed by *, then its $\mathscr{H}$-class is left invariant (setwise). This is e.g. the case with the $\mathscr{H}$-classes which contain elements of the form $aa^\star$, $a \in S$. An idempotent of S which is fixed by the involution * will be called a *projection* of S. Evidently an $\mathscr{L}$-class or an $\mathscr{R}$-class of S can contain at most one projection.

Theorem 1.1. *Let S be a regular semigroup with involution *. Then the following statements and their duals are equivalent.*

(i) *Every $\mathscr{L}$-class of S contains a projection,*

(ii) *for every $x \in S$ we have $x^\star x \mathscr{L} x$,*

(iii) *for every $x \in S$, $x^\star$ is $\mathscr{L}$-related to some inverse of x.*

Proof. Obviously (iii) implies (ii). Let us suppose that (ii) holds, and let $x \in S$. From (ii) it follows that $xx^\star \mathscr{L} x^\star$, and hence also $xx^\star = (xx^\star)^\star \mathscr{R} (x^\star)^\star = x$ for all $x \in S$. Thus $x \mathscr{R} xx^\star \mathscr{L} x^\star$ and consequently also $x^\star \mathscr{R} x^\star x \mathscr{L} x$. It follows that the $\mathscr{H}$-classes $H_{xx^\star}$ and $H_{x^\star x}$ contain idempotents. Since $xx^\star$ and $x^\star x$ are fixed by the involution, we have that * induces an involution on the groups $H_{xx^\star}$ and $H_{x^\star x}$ respectively. Thus the identities of these groups are projections of S. We proved (i) and its dual.

Let us now assume that (i) holds, and let $x \in S$. Let e be the projection in the $\mathscr{L}$-class of $x^\star$. From $e \mathscr{L} x^\star$, it follows that $e = e^\star \mathscr{R} (x^\star)^\star = x$. Thus $x \mathscr{R} e \mathscr{L} x^\star$, and we conclude that (iii) holds. It is now easy to show that (ii) and (iii) are equivalent to their duals.

Corollary 1.2. *The equivalent statements of Theorem* 1.1 *are equivalent to*

(iv) *for every* $x \in S$, $x^\star$ *is* $\mathscr{H}$*-equivalent to some inverse of* x,

(v) *for every* $x \in S$ *there exists an element* $x^\dagger$ *which satisfies*

(1.3) $\quad x^\dagger x x^\dagger = x^\dagger, \quad x x^\dagger x = x, \quad (x x^\dagger)^\star = x x^\dagger, \quad (x^\dagger x)^\star = x^\dagger x.$

Proof. If $x \in S$, then the element $x^\dagger$ is the inverse of x which is $\mathscr{L}$-related to $x^\star$, $x^\dagger x$ is the projection which is $\mathscr{L}$-related to x, and $x x^\dagger$ is the projection which is $\mathscr{R}$-related to x.

A regular semigroup S with involution * which satisfies the equivalent conditions of Theorem 1.1 and Corollary 1.2 will be called a *$\star$-regular semigroup.* Our definition accords with the definition of $\star$-regularity which has been given in [8], [10]. As in [8] we may call the inverse $x^\dagger$ of x which is defined by (1.3) the MP- *(Moore–Penrose) inverse* of x (see also [11], [19]). We may consider the class of $\star$-regular semigroups as the variety of algebras of type $\langle 2, 1, 1 \rangle$ defined by the identities (1.1), (1.2), (1.3), together with the identity which guarantees the associativity of the binary operation. In $\star$-regular semigroups each $\mathscr{L}$-class and each $\mathscr{R}$-class contains exactly one projection. The elements which are fixed by * must belong to a maximal subgroup which contains a projection.

Let R be a regular ring, and let * be a transformation of R. Then R is called a *$\star$-regular ring* if (1.1), (1.2),

(1.4) $\quad (x + y)^\star = x^\star + y^\star$

and

(1.5) $\quad x x^\star = 0 \Rightarrow x = 0$

are satisfied [44]. One can easily show that the multiplicative reduct of a $\star$-regular ring yields a $\star$-regular semigroup ([44], Proposition 88). Further, every $\star$-regular semigroup which has a zero 0 satisfies (1.5). Thus $\star$-regular rings yield the first examples of $\star$-regular semigroups, and in fact they formed much of the motivation for introducing $\star$-regularity for semigroups in the way described above.

The following may be recorded for later use.

Theorem 1.3. *Let* S *be a* $\star$*-regular semigroup. Then for every* $x \in S$, *we have*

$$(1.6) \qquad (x^\star)^\dagger = (x^\dagger)^\star .$$

Proof. Clearly $(x^\star)^\dagger$ and $(x^\dagger)^\star$ are $\mathcal{H}$-related to x. Therefore it suffices to show that $(x^\dagger)^\star$ is an inverse of $x^\star$. And indeed,

$$x^\star (x^\dagger)^\star x^\star = (x x^\dagger x)^\star = x^\star ,$$

$$(x^\dagger)^\star x^\star (x^\dagger)^\star = (x^\dagger x x^\dagger)^\star = (x^\dagger)^\star .$$

A semigroup S with involution * will be called a *special* $\star$*-semigroup* if (1.1), (1.2) and

$$(1.7) \qquad x x^\star x = x, \quad x^\star x x^\star = x^\star$$

are satisfied. A special $\star$-semigroup is a $\star$-regular semigroup, where

$$(1.8) \qquad x^\dagger = x^\star .$$

That (1.7) or (1.8) is a strong restriction is suggested by the following.

Lemma 1.4. *Let S be a $\star$-regular semigroup which satisfies*

$$(1.9) \qquad x^\star x = x^\star y = y^\star x = y^\star y \Rightarrow x = y .$$

Then S is reductive. S is a special $\star$-semigroup if and only if S is an inverse semigroup where $x^\star = x^\dagger = x^{-1}$ for all $x \in S$.

Proof. Let us suppose that S is a $\star$-regular semigroup which satisfies (1.9). It was remarked in [8] that S must be reductive. We give a short proof. Let $a, b \in S$ such that $ta = tb$ for every $t \in S$. Then $a^\star a = a^\star b$ and $b^\star a = b^\star b$. Thus $a^\star b = (b^\star a)^\star = (b^\star b)^\star = b^\star a$, and we may put $a^\star a = a^\star b = b^\star a = b^\star b$. By (1.9) we have $a = b$, and so S is left reductive. From the fact that * is an involution we may immediately conclude that S is also right reductive.

Let us now suppose that S is a special $\star$-semigroup which satisfies (1.9). Let e be any idempotent of S. Then $e^\star$ is an idempotent which is an inverse of e, and $e, ee^\star, e^\star, e^\star e$ form an E-square. Clearly, $ee^\star$ and $e^\star e$ are projections. Putting $x = e^\star$ and $y = ee^\star$, we see that the antecedent of (1.9) is satisfied. Thus $e^\star = ee^\star$, and we see that the

E-square considered above reduces to one element. Thus every idempotent is a projection. Since in a $\star$-regular semigroup every $\mathscr{L}$-class and every $\mathscr{R}$-class contains exactly one projection, we may conclude that S is an inverse semigroup where $x^\star = x^\dagger = x^{-1}$ for all $x \in S$. The converse is obvious.

Corollary 1.5. *A $\star$-regular ring R has a reduct which is a special $\star$-semigroup if and only if R is an abelian regular ring where* $x^\star = x^\dagger$.

Proof. A $\star$-regular ring satisfies (1.9) (see e.g. [8], [11]).

Note that we use the terminology "abelian regular ring" in the sense of [12].

In [22], [23], [24], [25], [35], [40], [45], special $\star$-semigroups are called regular $\star$-semigroups. In order to avoid confusion with the situation for $\star$-regular rings, and also in view of Corollary 1.5, we do not want to adopt this terminology any longer, and we rather prefer the terminology of [8], [10]. The name SIR (Special Involution Regular) semigroup was used in [3] for what we here will call a special $\star$-semigroup which is completely simple.

We illustrate the foregoing with three major classes of examples.

Example 1.6. Let S be an orthodox semigroup, and let $\mathbf{S}$ be the set which consists of the pairs (x, x'), where $x' \in V(x)$, $x \in S$. On $\mathbf{S}$ we define a multiplication and an involution by

$$(x, x')(y, y') = (xy, y'x'),$$

$$(x, x')^\star = (x', x),$$

for all $(x, x'), (y, y') \in \mathbf{S}$. It follows from [41], [42] that the above product is well defined, and that * yields an involution. Obviously $\mathbf{S}$ becomes a special $\star$-semigroup which is orthodox. The projections are of the form (e, e), $e \in E(S)$.

One easily checks that $(e, e)\,\omega\,(f, f)$ in $E(\mathbf{S})$ if and only if $e\,\omega\,f$ in $E(S)$. Thus, the poset of projections of $\mathbf{S}$ is order-isomorphic to the poset $(E(S), \omega)$. This also leads to the surprising conclusion that bands, endowed with the natural partial order, constitute regular partially ordered sets (in the sense of [13]).

If S is a band, then $\mathbf{S}$ is a $\star$-regular band. $\star$-regular bands were studied extensively in [1], [2].

Example 1.7. A rectangular band $I \times \Lambda$ of inverse semigroups $S_{i\lambda}$, $(i, \lambda) \in I \times \Lambda$, is called elementary if $S_{i\lambda} S_{j\mu} = S_{i\mu}$ for all $(i, \lambda), (j, \mu) \in$ $\in I \times \Lambda$ [38]. We shall describe here how to introduce an involution on an elementary rectangular band of E-unitary inverse semigroups. The corresponding results for completely simple semigroups and E-unitary inverse semigroups will come as special cases.

Let G be a group which acts (from the left) on the partially ordered set $\mathscr{X}$ as a group of order automorphisms; let $\mathscr{Y}$ be a subsemilattice and an ideal of $\mathscr{X}$, such that $\mathscr{X} = G\mathscr{Y}$ and such that $a\mathscr{Y} \cap \mathscr{Y} \neq \square$ for every $a \in G$ (see also [27]). Let α_1 be an automorphism of order 2 of $\mathscr{X}$ which leaves $\mathscr{Y}$ (setwise) invariant, and let α_2 be an automorphism of order 2 of G, such that $(gA)^{\alpha_1} = g^{\alpha_2} A^{\alpha_1}$ for all $g \in G$ and $A \in \mathscr{Y}$. Let I be an index set, and let $P = (p_{ij})$ be an $I \times I$-matrix with entries in G, such that

(i) for each $(i, j) \in I \times I$, p_{ij} induces an automorphism on $\mathscr{Y}$,

(ii) for each $(i, j) \in I \times I$, $p_{ji} = (p_{ij}^{-1})^{\alpha_2}$.

Let $\mathscr{M} = \mathscr{M}(P(G, \mathscr{X}, \mathscr{Y}); I; P; \alpha_1, \alpha_2)$ be the set which consists of the elements $(A, g)_{ij}$, where $(A, g) \in \mathscr{Y} \times G$, $i, j \in I$, and $g^{-1}A \in \mathscr{Y}$. On $\mathscr{M}$ we define a multiplication and a $\star$-operation by

(1.10) $$(A, g)_{ij}(B, h)_{mn} = (A \wedge g p_{jm} B, g p_{jm} h)_{in}$$

and

(1.11) $$(A, g)^{\star}_{ij} = ((g^{-1})^{\alpha_2} A^{\alpha_1}, (g^{-1})^{\alpha_2})_{ji}.$$

Then $\mathscr{M}$ is an elementary rectangular band of E-unitary inverse semigroups, and $\star$ is an involution. Conversely, every elementary rectangular band of E-unitary inverse semigroups which has an involution, can be so constructed. The proof of this statement is routine from [36], [38]. We obtain the corresponding result for E-unitary inverse semigroups by putting $|I| = 1$ and $P = (1)$, and we obtain the corresponding result for completely simple semigroups by putting $\mathcal{X} = \mathcal{Y} = \{1\}$ in which case we may identify $P(G, \mathcal{X}, \mathcal{Y})$ with G.

Though the involution semigroup $\mathcal{M}$ which is defined above is a regular semigroup with involution, it is not in general a $\star$-regular semigroup, even not if $\mathcal{M}$ reduces to an E-unitary inverse semigroup. One can check that $\mathcal{M}$ is $\star$-regular if and only if $\alpha_1 \mid \mathcal{Y}$ is the identity on $\mathcal{Y}$. This condition is obviously satisfied if $\mathcal{M}$ reduces to a completely simple semigroup. One sees that in this case for every $(A, g)_{ij} \in \mathcal{M}$, $(A, g)_{ij}^{\dagger} = $ $= (p_{jj}^{-1} g^{-1} A, p_{jj}^{-1} g^{-1} p_{ii}^{-1})_{ji}$.

The semigroup with involution which is defined above is a special $\star$-semigroup if and only if α_1 is the identity on $\mathcal{X}$ and α_2 is the identity on G. If $\mathcal{M}$ reduces to an inverse semigroup, we have $^{\star} = {}^{-1} = {}^{\dagger}$, and in case $\mathcal{M}$ reduces to a completely simple semigroup we obtain a result from [40].

Example 1.8. Let X be a set, and B_X the semigroup of all binary relations on the set X. For any relation $\alpha \in B_X$, let α^{-1} be the inverse relation: $(a, b) \in \alpha \Leftrightarrow (b, a) \in \alpha^{-1}$, for all $a, b \in X$. It is well known that B_X, endowed with the unary operation $^{-1}$ becomes an involution semigroup. Since $^{-1}$ leaves the greatest regular ideal M_X of B_X setwise invariant, we can state that M_X becomes a regular involution semigroup [18]. Yet, M_X is not a $\star$-regular semigroup. To see this, we take $X =$ $= \{0, 1\}$, and we consider the $\mathcal{D}$-class consisting of the four three-element binary relations: in this $\mathcal{D}$-class $^{-1}$ interchanges the two idempotents, and fixes the two non-idempotents. B_X has a smallest non-zero ideal which consists of the so-called rectangular binary relations [43], [46]. The involution $^{-1}$ turns this ideal into a special $\star$-semigroup: the projections are the rectangular binary relations which are of the form $A \times A$, for $A \subseteq X$.

Example 1.9. The following example is taken from the work of Foulis. We refer to [4], [9], [10], [11], [26] for more details and for more references to the relevant literature in this connection.

Let $(L, \wedge, \vee, 0, 1, {}^{\perp})$ be an orthocomplemented lattice, and let $\mathrm{Res}\,(L)$ be the semigroup of residuated mappings of L into L. If $f \in \mathrm{Res}\,(L)$, and if $\overset{+}{f}$ is the residual of f, then

$$^{\perp}\overset{+}{f}{}^{\perp} = f^{\star} : L \to L, \quad a \mapsto (a^{\perp}\overset{+}{f})^{\perp}$$

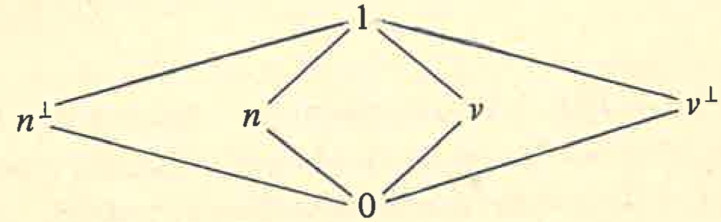
is a residuated mapping, and

$$\star : \mathrm{Res}\,(L) \to \mathrm{Res}\,(L), \quad f \to f^{\star}$$

is an involution of $\mathrm{Res}\,(L)$. If L is an orthocomplemented modular lattice, then the strongly range-closed residuated mappings form a subsemigroup $B(L)$ which is closed for the involution $\star$. The semigroup $B(L)$ is a strongly regular Baer semigroup, which is at the same time a $\star$-regular semigroup. Here the projections are the so-called Sasaki projections $L \to L$, $a \mapsto n \wedge (n^{\perp} \vee a)$, $n \in L$.

Remark that a $\star$-regular ring R yields a strongly regular Baer semigroup with involution $\star$, which is at the same time a $\star$-regular semigroup. Further, if L is the orthocomplemented modular lattice which is coordinatized by R, then there exists a canonical idempotent-separating (multiplicative) homomorphism of R into $B(L)$ which respects the involution (see also [26]). Corollary 1.5 states that if R yields a special $\star$-semigroup, then L must be a Boolean algebra, and in this case $B(L)$ will be an inverse semigroup.

It could be interesting to characterize the orthocomplemented modular lattices L which can be coordinatized by some strongly regular Baer $\star$-semigroup which is also a special $\star$-semigroup. The answer to this question will be given in Example 2.14. If L is the lattice which is depicted below, then the idempotent-generated part of $B(L)$ is a special $\star$-semigroup.



We conclude this section with some observations concerning congruences. A semigroup congruence ρ on an involution semigroup S will be called a $\star$-congruence if $x \rho y$ implies $x^\star \rho y^\star$ for all $x, y \in S$. A $\star$-homomorphism between involution semigroups will be defined accordingly (see also [24], [35]).

In the following, S will be a regular involution semigroup, and $\mathcal{L}^\star(S)$ the complete lattice of $\star$-congruences on S. If X is any set, then we denote the partition lattice on X by $\Pi(X)$. It was already observed in [35] that $\mathcal{L}^\star(S)$ is a complete sublattice of $\Pi(S)$. In particular, $\mathcal{L}^\star(S)$ is a complete sublattice of $\mathcal{L}(S)$, i.e. the complete lattice of all semigroup congruences on S. Let θ be the relation on $\mathcal{L}(S)$ which is defined by

(1.12) $$\theta = \{(\rho, \sigma) \in \mathcal{L}(S) \times \mathcal{L}(S) \colon \rho \mid E(S) = \sigma \mid E(S)\}.$$

Using [16], we can state that θ induces a congruence on $\mathcal{L}^\star(S)$ in such a way that $\theta^\natural$ induces a complete lattice homomorphism of $\mathcal{L}^\star(S)$ into the partition lattice $\Pi(E(S))$. Since the θ-classes form complete modular sublattices of $\mathcal{L}(S)$ [41], the $\theta \mid \mathcal{L}^\star(S)$-classes will form complete modular sublattices of $\mathcal{L}^\star(S)$.

The semigroup congruence κ^c which is generated by a relation κ will be a $\star$-congruence if

(1.13) $$(x, y) \in \kappa \Rightarrow (x^\star, y^\star) \in \kappa^c$$

is satisfied. If S is an involution semigroup, then it follows from [17] and the foregoing that the greatest idempotent-separating congruence μ_S of S is also a $\star$-congruence. Let ρ be a $\star$-congruence, and let $\rho_{\min}$ [$\rho_{\max}$] be the least [greatest] element in the θ-class of ρ. Putting $\kappa = \rho \mid E(S)$, we have that $\rho_{\min} = \kappa^c$, and that (1.13) is satisfied. Thus $\rho_{\min}$ is a $\star$-congruence. Further, $\rho_{\max}$ is the congruence which is induced by the canonical $\star$-homomorphism $\rho^\natural \mu^\natural_{S/\rho}$, and so $\rho_{\max}$ is also a $\star$-congruence. We conclude that, whenever a θ-class contains a $\star$-congruence, then its least and its greatest elements are also $\star$-congruences. If S is a special $\star$-semigroup, then the behaviour of the $\star$-congruences is even nicer. We refer to [35] for more details concerning this case. One can also directly verify that the least group congruence (generated by $E(S) \times E(S)$) and the least completely simple congruence (generated by the natural partial order on $E(S)$) are $\star$-congruences. The least quasi-orthodox congruence, the least completely regular congruence, the least inverse congruence, the least band congruence, the least semilattice congruence, . . . , are all $\star$-congruences: it is routine to show this from the foregoing, and from the results in [21].

Theorem 1.10. *Let S be a $\star$-regular semigroup. The semigroup congruence which is generated by*

$$\kappa = \{(x^\star, x^\dagger) \mid x \in S\}$$

is a $\star$-congruence. This $\star$-congruence is the least among all $\star$-congruences ρ on S for which the quotient is a special $\star$-semigroup.

Proof. From Theorem 1.3 it follows that the congruence κ^c satisfies (1.13). Hence κ^c is a $\star$-congruence. Now $\star$-congruences on $\star$-regular semigroups are always compatible with the †-operation; thus S/κ^c is a special $\star$-semigroup since it satisfies (1.8). Further, every $\star$-congruence ρ on S which yields a special $\star$-semigroup S/ρ filters through κ^c. Hence the statement of our theorem holds.

2. BIORDERED SETS OF REGULAR INVOLUTION SEMIGROUPS

Let us first characterize the biordered set of regular involution semigroups. If $^*: S \to S$ is an involution of the regular semigroup S, then * gives rise to an isomorphism of S onto S^{op} (S^{op} is the left-right dual of S). Therefore the restriction * of * to $E(S)$ is an isomorphism of the biordered set $E(S)$ onto the biordered set $E(S^{op})$. It is clear that * is a mapping of $E(S)$ onto itself which satisfies the following conditions:

(i) $e^{\star\star} = e$ for all $e \in E(S)$,

(ii) $(ef)^\star = f^\star e^\star$ for all $e, f \in E(S)$ for which the basic product ef exists in the biordered set.

In particular, for all $e, f \in E(S)$, we have

(2.1) $\quad e\, \omega^r f \Leftrightarrow e^\star\, \omega^l f^\star,$

(2.2) $\qquad S(e,f)^{\star} = S(f^{\star}, e^{\star})$.

Let E be a biordered set, and let ${}^{\star}\colon E \to E$ be a mapping of E onto itself which satisfies the above conditions (i) and (ii). Then * will be called an involution on E.

Theorem 2.1. *Let S be a regular semigroup with an involution *. Let * be the restriction of * to $E(S)$. Then * is an involution of the biordered set $E(S)$.*

Conversely, if E is a biordered set with an involution *, then there exists an involution * of $T(E)$ which extends the involution * of $E = E(T(E))$.

Proof. It is sufficient to prove the converse. Let us consider the set $T^{\star}(E)$ of ω-isomorphisms of the biordered set E, and let $\alpha\colon \omega(e_\alpha) \to \omega(f_\alpha)$ be any element of $T^{\star}(E)$. We define ${}^{\star}(\alpha^{-1})^{\star}$ to be the ω-isomorphism

(2.3) $\qquad {}^{\star}(\alpha^{-1})^{\star}\colon \omega(f_\alpha) \to \omega(e_\alpha), \quad g \mapsto ((g^{\star})\alpha^{-1})^{\star}$.

The mapping

$$\phi\colon T^{\star}(E) \to T^{\star}(E), \qquad \alpha \mapsto {}^{\star}(\alpha^{-1})^{\star} = \phi\alpha$$

is obviously a permutation of order 2 of the set $T^{\star}(E)$. Let $e, f \in E$ and let $e \,\mathcal{R}\, f$. Then $e^{\star} \,\mathcal{L}\, f^{\star}$, and

$$\phi(\tau^r(e,f)) = {}^{\star}\tau^r(f,e)^{\star} = \tau^l(f^{\star}, e^{\star}).$$

Let us now suppose that α and β are p-related in $T^{\star}(E)$, i.e.

(2.4) $\qquad e_\alpha \,\mathcal{R}\, e_\beta, \quad f_\alpha \,\mathcal{L}\, f_\beta, \quad \text{and} \quad \alpha\tau^l(f_\alpha, f_\beta) = \tau^r(e_\alpha, e_\beta)\beta$.

We then have

$$\begin{aligned}
(2.4) \Leftrightarrow e_\alpha \,\mathcal{R}\, e_\beta, \quad f_\alpha \,\mathcal{L}\, f_\beta \quad &\text{and} \quad ({}^{\star}\tau^l(f_\beta, f_\alpha)^{\star})({}^{\star}(\alpha^{-1})^{\star}) = \\
&= ({}^{\star}(\beta^{-1})^{\star})({}^{\star}\tau^r(e_\beta, e_\alpha)^{\star}) \Leftrightarrow \\
\Leftrightarrow e_\alpha^{\star} \,\mathcal{L}\, e_\beta^{\star}, \quad f_\alpha^{\star} \,\mathcal{R}\, f_\beta^{\star} \quad &\text{and} \quad \tau^r(f_\beta^{\star}, f_\alpha^{\star})(\phi\alpha) = \\
&= (\phi\beta)\tau^l(e_\beta^{\star}, e_\alpha^{\star}) \Leftrightarrow
\end{aligned}$$

$$\Leftrightarrow e_{\phi\alpha} \mathcal{R} e_{\phi\beta}, \quad f_{\phi\alpha} \mathcal{L} f_{\phi\beta} \quad \text{and} \quad \tau^r(e_{\phi\beta}, e_{\phi\alpha})(\phi\alpha) = $$
$$= (\phi\beta)\tau^l(f_{\phi\beta}, f_{\phi\alpha}) \Leftrightarrow$$
$$\Leftrightarrow \phi\alpha\, p\, \phi\beta.$$

We see that $\alpha p \beta$ in $T^\star(E)$ if and only if $\phi\alpha\, p\, \phi\beta$ in $T^\star(E)$. For any $\alpha \in T^\star(E)$, $\bar{\alpha}$ denotes the p-class of α. The above reasoning shows that

(2.5) $\quad {}^\star\colon T(E) \to T(E), \quad \bar{\alpha} \mapsto \bar{\alpha}^\star = \overline{\phi\alpha}$

is a permutation of order 2 of $T(E)$.

Let $\alpha, \beta \in T(E)$, and let $h \in S(f_\alpha, e_\beta)$. We then have ([30], Theorem 4.12)

$$(\bar{\alpha}\bar{\beta})^\star = \overline{\alpha\tau^l(f_\alpha h, h)\tau^r(h, he_\beta)\beta}^\star =$$
$$= \overline{\phi(\alpha\tau^l(f_\alpha h, h)\tau^r(h, he_\beta)\beta)} =$$
$$= \overline{{}^\star\beta^{-1}\tau^r(he_\beta, h)\tau^l(h, f_\alpha h)\alpha^{-1\star}} =$$
$$= \overline{({}^\star\beta^{-1\star})({}^\star\tau^r(he_\beta, h)^\star)({}^\star\tau^l(h, f_\alpha h)^\star)({}^\star\alpha^{-1\star})} =$$
$$= \overline{(\phi\beta)\tau^l((e_\beta^\star)(h^\star), h^\star)\tau^r(h^\star, (h^\star)(f_\alpha^\star))(\phi\alpha)} =$$
$$= \overline{(\phi\beta)\tau^l(f_{\phi\beta}(h^\star), h^\star)\tau^r(h^\star, (h^\star)e_{\phi\alpha})(\phi\alpha)} =$$
$$= \overline{\phi\beta}\,\overline{\phi\alpha} = \bar{\beta}^\star\bar{\alpha}^\star$$

since $h^\star \in S(f_\alpha, e_\beta)^\star = S(e_\beta^\star, f_\alpha^\star) = S(f_{\phi\beta}, e_{\phi\alpha})$. Thus * is an involution of $T(E)$. For any $e \in E$, ϵ_e denotes the identity transformation on $\omega(e)$. Then $\bar{\epsilon}^\star = \overline{\phi\epsilon_e} = \bar{\epsilon}_{e^\star}$. If we identify E with $E(T(E))$, by the canonical biorder isomorphism $E \to E(T(E))$, $e \mapsto \bar{\epsilon}_e$, then the above shows that * extends the involution * of $E = E(T(E))$.

Let S be a regular semigroup. If $x \in S$, and if x' is an inverse of x in S, then $\theta_{x,x'}\colon \omega(xx') \to \omega(x'x)$, $g \mapsto x'gx$ belongs to $T^\star(E(S))$, and $\overline{\theta_{x,x'}}$ is independent of the choice of x' in $V(x)$, i.e. the set of inverses of x. The mapping $\theta\colon S \to T(E(S))$, $x \mapsto \overline{\theta_{x,x'}}$ is an idempotent separating homomorphism of S onto a full regular subsemigroup of $T(E(S))$, and θ is called the canonical homomorphism of S onto $T(E(S))$ [30].

Theorem 2.2. *Let S be a regular semigroup with involution $\star$, and let $\star$ be the involution of $E = E(S)$ which is induced by $\star$. Let $\star$ be the involution of $T(E)$ which is defined by* (2.3) *and* (2.4). *Then the canonical homomorphism $\theta: S \to T(E)$ is a $\star$-homomorphism, i.e. $\overline{\theta_{x,x'}}^{\star} = \overline{\theta_{x^\star,x'^\star}}$ for all $x \in S$, $x' \in V(x)$.*

If in particular S is a full regular subsemigroup of $T(E)$, then the involution $\star$ on S must be the restriction to S of the involution $\star$ on $T(E)$.

Proof. Let $x \in S$ and $x' \in V(x)$. Then

$$\overline{\theta_{x,x'}}^{\star} = \overline{\phi(\theta_{x,x'})} = \overline{{}^{\star}\theta_{x',x}{}^{\star}}.$$

Here

$${}^{\star}\theta_{x',x}{}^{\star}: \omega((x'x)^{\star}) \to \omega((xx')^{\star}), \quad h \mapsto (x(h^{\star})x')^{\star}.$$

Since $(x'x)^{\star} = x^{\star}x'^{\star}$, we see that $\theta_{x^\star,x'^\star}$ and ${}^{\star}\theta_{x',x}{}^{\star}$ have the same domain. Further, if $h \in \omega((x'x)^{\star}) = \omega(x^{\star}x'^{\star})$, then

$$h({}^{\star}\theta_{x',x}{}^{\star}) = (x(h^{\star})x')^{\star} = x'^{\star}hx^{\star} = h\theta_{x^\star,x'^\star}$$

and so ${}^{\star}\theta_{x',x}{}^{\star} = \theta_{x^\star,x'^\star}$. Consequently $\overline{\theta_{x,x'}}^{\star} = \overline{\theta_{x^\star,x'^\star}}$. Thus θ is a $\star$-homomorphism.

If S is a full regular subsemigroup of $T(E)$, then the canonical homomorphism $\theta: S \to T(E)$ is the inclusion mapping and so the last assertion in the statement of the theorem is clear.

Corollary 2.3. *Let $E, \star$ and $\star$ be as in Theorem* 2.1. *Then $\star$ is the unique involution on $T(E)$ which extends the involution $\star$ of $E = E(T(E))$.*

Remark that from Theorem 2.2 it again follows that the greatest idempotent separating congruence on the regular involution semigroup S must be compatible with the involution. Indeed, the canonical homomorphism θ of S into $T(E(S))$ induces the greatest idempotent separating congruence on S [30].

Theorem 2.4. *Let S be a regular semigroup with involution *. Then S is $\star$-regular if and only if the involution induced on $E(S)$ satisfies the following condition:*

(A) *for all $e \in E(S)$, there exist $h, k \in E(S)$, such that*

$$\begin{pmatrix} e & h \\ k & e^\star \end{pmatrix} \tag{2.6}$$

is an E-square.

Proof. If S is a $\star$-regular semigroup, and if $e \in E(S)$, then it is easy to show from Corollary 1.2 that $h = ee^\dagger$ and $k = e^\dagger e$ are projections such that (2.6) is an E-square.

Conversely, if for $e \in E(S)$ there exist $h, k \in E(S)$ such that (2.6) is an E-square, then k must be a projection, and k must be in the $\mathcal{L}$-class of e. Thus, if the regular involution semigroup S satisfies the above mentioned condition, then it immediately follows from Theorem 1.1 that S is $\star$-regular.

Combining Theorems 2.1 and 2.4 we see that a biordered set E is the biordered set of some $\star$-regular semigroup if and only if E admits an involution such that the condition (A) is satisfied. In what follows we shall call such a biordered set a *$\star$-regular biordered set.* We shall also use the notation $P(E)$ to denote the set of projections of E, that is the set of elements of the biordered set E which are fixed by the involution.

$\star$-regular biordered sets have the following important property.

Proposition 2.5. *Let E be a $\star$-regular biordered set. Then for all $e, f \in P(E)$, there exists a unique $h \in S(e,f)$ such that eh and hf are projections.*

Proof. In view of Theorem 2.4, we may assume that E is the biordered set of some $\star$-regular semigroup S. Let $e, f \in P(E)$, and put $h = (ef)^\dagger$. Then $(ef)(ef)^\dagger$ and $(ef)^\dagger(ef)$ are projections. It follows from $ef(ef)^\dagger \, \omega^r \, e$ that $ef(ef)^\dagger \, \omega \, e$, and from $(ef)^\dagger (ef) \, \omega^l f$ that $(ef)^\dagger (ef) \, \omega f$. Hence $f(ef)^\dagger e = (ef)^\dagger$, and so $h = (ef)^\dagger \in S(e,f)$ ([30], Theorem 1.1).

Let k be any element of $S(e,f)$ such that ek and kf are projections. Then $(ef)^\dagger (ef)$ and kf are both projections in the $\mathscr{L}$-class of ef, and thus $(ef)^\dagger (ef) = kf$. Analogously $(ef)(ef)^\dagger = ek$. We conclude that $(ef)^\dagger \ \mathscr{H} k$, thus $(ef)^\dagger = k$.

Given the $\star$-regular biordered set E, and $e, f \in P(E)$, we denote the element $h \in S(e,f)$ which satisfies the condition of the above proposition by $h = (ef)^\dagger$. The proof given shows that h is in fact equal to $(ef)^\dagger$ whenever E appears as the biordered set of a $\star$-regular semigroup. Also, since * and † commute (see (1.6)), we also have

$$(2.7) \qquad (fe)^\dagger = ((ef)^\dagger)^\star \in S(f,e)$$

for all $e, f \in P(E)$.

We now turn to the characterization of regular involution semigroups that are special $\star$-semigroups in terms of their regular partial bands. We refer to [25], Section 2, for another approach to the converse of the following theorem.

Theorem 2.6. *Let S be a special $\star$-semigroup. Then the regular partial band of S satisfies the following condition:*

(B) *for all* $e \in E(S)$

$$\begin{pmatrix} e & ee^\star \\ e^\star e & e^\star \end{pmatrix}$$

is a 2×2 rectangular subband of $E(S)$.

Conversely, if the regular partial band of a regular involution semigroup S satisfies (B), *then*

$$(2.8) \qquad T = \{x \in S \mid xx^\star \in E(R_x)\}$$

is the greatest full subsemigroup of S which is a special $\star$-semigroup.

Proof. If S is a special $\star$-semigroup, and if $e \in E(S)$, then $e^\star = e^\dagger$ is an idempotent inverse of e, and so (B) is satisfied.

Let us conversely suppose that S is a regular involution semigroup whose regular partial band $E(S)$ satisfies (B). In particular S satisfies condition (A) of Theorem 2.4, and so S is $\star$-regular. Thus, if $e, f \in \in P(E(S))$, then by Proposition 2.5, $(ef)^\dagger$ is an idempotent, and so $((ef)^\dagger)^\star$ is an idempotent inverse of $(ef)^\dagger$ by the condition (B). Now ef is also an inverse of $(ef)^\dagger$, and we know that ef must be $\mathscr{H}$-equivalent to $((ef)^\dagger)^\star$. Thus $ef = ((ef)^\dagger)^\star$, and so ef is an idempotent. From this it follows that $fe = (ef)^\star = ((ef)^\dagger)^{\star\star} = (ef)^\dagger$ is also an idempotent.

It is clear that $E(S) \subseteq T$. If $x \in T$, then from (2.8) is follows that $xx^\star \in P(E(S))$, thus $xx^\star = xx^\dagger$. Also, since $x^\star \mathscr{H} x^\dagger$, we must have $x^\star = x^\dagger$. Hence $x^\star x = x^\dagger x \in E(R_{x^\star})$, thus $x^\star = x^\dagger \in T$. Let $x, y \in T$. We have that $x^\star x$ and $yy^\star$ are projections, and that $x^\dagger = x^\star$ and $y^\dagger = y^\star$ belong to T. Therefore $(x^\star x)(yy^\star)$ is an idempotent, so that

$$(xy)(y^\star x^\star)(xy) = x(yy^\star)(x^\star x)y = x(x^\star x)(yy^\star)(x^\star x)(yy^\star)y =$$
$$= x((x^\star x)(yy^\star))^2 y = x(x^\star x)(yy^\star)y = xy$$

and by duality, $(y^\star x^\star)(xy)(y^\star x^\star) = y^\star x^\star$. We conclude that $(xy)^\star = = y^\star x^\star$ is an inverse of xy. Therefore $(xy)(xy)^\star \in E(R_{xy})$, and we may conclude that $xy \in T$. Thus T is a full subsemigroup of S. Further, since for every $x \in T$, we have $x^\star = x^\dagger \in T$, T is a full regular subsemigroup of S which must be $\star$-special. One readily verifies that every full regular subsemigroup of S which is $\star$-special must be contained in T.

Remark that in the statement of the above theorem, the semigroup T may also be characterized by

(2.9) $\qquad T = \{x \in S \mid x^\dagger = x^\star\}.$

Recall that an E-square

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

in a biordered set E is a 2×2 rectangular band in some regular semigroup S with $E(S) = E$ if and only if this E-square is τ-commutative [5], [30]. If this is the case, then this E-square forms a rectangular band in $T(E)$. We thus have the following characterization of biordered sets of special $\star$-semigroups.

Corollary 2.7. *A biordered set E is the biordered set of a special $\star$-semigroup if and only if E admits an involution * satisfying the following condition:*

(B′) *if $e \in E$, then there exist $h, k \in E$ such that*

$$\begin{pmatrix} e & h \\ k & e^\star \end{pmatrix}$$

is a τ-commutative E-square in E.

We shall see in Example 2.14 that not every $\star$-regular biordered set needs to be the biordered set of some special $\star$-semigroup.

A bimorphism $\alpha\colon E \to E'$ of involution biordered sets is called a *$\star$-bimorphism* if for all $e \in E$, $(e\alpha)^\star = e^\star\alpha$. Similarly a biorder congruence is called a *$\star$-congruence* if the associated canonical bimorphism is a $\star$-bimorphism. A $\star$-bimorphism which is a biorder isomorphism is called a *$\star$-isomorphism.*

Lemma 2.8. *Let E be a $\star$-regular biordered set. Then for $e \in E$, $\omega(e)$ is closed under * if and only if $e \in P(E)$.*

Proof. Let $\omega(e)$ be closed under * for $e \in E$. Then $e^\star \in \omega(e)$. From condition (A) follows that $e = e^\star$, and thus $e \in P(E)$. Let us conversely suppose that $e \in P(E)$, and let $f \in \omega(e)$. From condition (A) follows that there exist $g, j \in E$ such that

$$\begin{pmatrix} f & g \\ j & f^\star \end{pmatrix}$$

is an E-square. Clearly $g, j \in P(E)$. But then $g\,\omega^r\,e$ and $j\,\omega^l\,e$ imply $g, j \in \omega(e)$. Consequently $f^\star \in \omega(e)$, and we see that $\omega(e)$ is closed under *.

Theorem 2.9. *Let E be a special $\star$-biordered set. The set of all $\alpha \in T(E)$ for which α: $\omega(e_\alpha) \to \omega(f_\alpha)$, $e_\alpha, f_\alpha \in P(E)$, is a $\star$-isomorphism, forms the greatest full regular subsemigroup $\bar{T}(E)$ of $T(E)$ which is a special $\star$-semigroup.*

Proof. Let α: $\omega(e_\alpha) \to \omega(f_\alpha)$ be any element of $T^\star(E)$, and let h, k be the projections of E, with $h \mathcal{R} e_\alpha$ and $k \mathcal{L} f_\alpha$. Then

$$\beta = \tau^r(h, e_\alpha)\alpha\tau^l(f_\alpha, k)\colon \ \omega(h) \to \omega(k)$$

belongs to $T^\star(E)$ and $\alpha\, p\, \beta$ in $T^\star(E)$. Thus every p-class of $T^\star(E)$ contains a (unique) $\star$-isomorphism which maps a principal ω-ideal which is generated by a projection onto a principal ω-ideal generated by a projection.

Let $\bar{\alpha}$ be any element of the greatest full regular subsemigroup of $T(E)$ which is a special $\star$-semigroup. We may suppose that α: $\omega(e_\alpha) \to \omega(f_\alpha)$ is an element of $T^\star(E)$ such that $e_\alpha, f_\alpha \in P(E)$. We denote the involution on E by *. From Theorem 2.6 we know that $\bar{\alpha}^\star$ must be equal to $\bar{\alpha}^\dagger$ in $T(E)$. Thus $\bar{\alpha}\bar{\alpha}^\star$ must be the projection $\bar{\epsilon}_{e_\alpha}$ of $T(E)$. From Theorem 2.1, we know that $\bar{\alpha}^\star = \overline{{}^\star\alpha^{-1\star}}$. One verifies that ${}^\star\alpha^{-1\star}$: $\omega(f_\alpha) \to \omega(e_\alpha)$, since $e_\alpha, f_\alpha \in P(E)$. Thus $\alpha^\star\alpha^{-1\star}$: $\omega(e_\alpha) \to \omega(e_\alpha)$. Since $\alpha^\star\alpha^{-1\star}\, p\, \epsilon_{e_\alpha}$, we must have $\alpha^\star\alpha^{-1\star} = \epsilon_{e_\alpha}$ and thus ${}^\star\alpha^{-1\star} = \alpha^{-1}$. Consequently $\alpha^\star = {}^\star\alpha$, and we may conclude that α is a $\star$-isomorphism. Thus $\bar{T}(E)$ contains the greatest full regular subsemigroup of $T(E)$ which is a special $\star$-semigroup.

Let α be any element of $T(E)$, where α: $\omega(e_\alpha) \to \omega(f_\alpha)$, $e_\alpha, f_\alpha \in P(E)$, is a $\star$-isomorphism. If $e \in \omega(e_\alpha)$, then $e\alpha = (e^{\star\star})\alpha = (e^\star\alpha)^\star$. Hence $\alpha = {}^\star\alpha^\star$ and thus also $\alpha^{-1} = {}^\star\alpha^{-1\star}$. By Theorem 2.1, we have that $\bar{\alpha}^\star = \overline{{}^\star\alpha^{-1\star}} = \overline{\alpha^{-1}}$, and we see that $\bar{\alpha}^\star$ is an inverse of α. In other words, $\bar{\alpha}^\star = \bar{\alpha}^\dagger$, and $\bar{\alpha}$ belongs to the greatest full regular subsemigroup of $T(E)$ which is a special $\star$-semigroup. This completes the proof of the theorem.

We refer to [22], [23], [45] for other constructions of fundamental special $\star$-semigroups.

Example 2.10. It is clear that if B is a band which is $\star$-regular, then B must be a special $\star$-band: obviously all E-squares in B are τ-commutative. The orthodox $\star$-regular semigroup $T(B)$ then contains a greatest full regular subsemigroup $\bar{T}(B)$ which is a special $\star$-semigroup. The following shows that $\bar{T}(B)$ does not coincide with $T(B)$ in general. We suppose that B is the $\star$-regular band which consists of the E-square

$$\begin{pmatrix} e & h \\ k & e^\star \end{pmatrix}$$

with an identity 1 adjoined. The projections are h, k and 1. The permutation

$$\alpha = \begin{pmatrix} 1 & e & h & k & e^\star \\ 1 & h & e & e^\star & k \end{pmatrix}$$

of B is an ω-isomorphism, but not a $\star$-isomorphism. Thus, $\bar{\alpha}$ belongs to $T(B)$ but not to $\bar{T}(B)$.

However, for pseudo-semilattices we have the following (see [29], [32], [33], [34] for definitions).

Theorem 2.11. *Let E be a $\star$-regular pseudo-semilattice. Then E is a special $\star$-biordered set, and $T(E)$ is a special $\star$-semigroup.*

Proof. Let g be any projection of the $\star$-regular pseudo-semilattice E, and let $e \in \omega(g)$. Then there exist projections h and k such that

$$\begin{pmatrix} e & h \\ k & e^\star \end{pmatrix}$$

is an E-square in E. From $h\,\omega^r\,g$ and $k\,\omega^l\,g$ now follows that $h, k \in \in \omega(g)$. Yet, $\omega(g)$ is a semilattice, and so the above E-square reduces to one single element. Thus, if g is a projection, then $\omega(g)$ is contained in the set of projections of E.

Let us now suppose that e is any element of E. Then

$$\begin{pmatrix} e & h \\ k & e^\star \end{pmatrix}$$

is an E-square in E for some projections h, k of E, since E is $\star$-regular. Let j be any element of $\omega(h)$. The above argument demonstrates that j must be a projection. Thus,

$$j\tau(h, e)\tau(e, k) = k(je) \in \omega(k)$$

and

$$j\tau(h, e)\tau(e, k) = (k(je))^* = (je)^* k = (e^* j)k = j\tau(h, e^*)\tau(e^*, k)$$

since $k(je)$ is then a projection. Hence the above considered E-square is τ-commutative, and E must be a special $\star$-biordered set.

Let α be any element of $T^\star(E)$. Then there exists a unique element $\beta \in T^\star(E)$ such that $e_\beta, f_\beta \in P(E)$ and $\alpha\, p\, \beta$. Since we know that $\omega(e_\beta)$ and $\omega(f_\beta)$ are subsets of $P(E)$, the ω-isomorphism β must be a $\star$-isomorphism. Thus $\bar{\alpha} = \bar{\beta}$ belongs to the greatest full regular subsemigroup of $T(E)$ which is a special $\star$-semigroup. We conclude that $T(E)$ itself must be a special $\star$-semigroup.

Example 2.12. The pseudo-semilattice which is determined by an elementary rectangular band of inverse semigroups must be of the form $E = (L; M_{i\lambda}; \phi_{i\lambda}, \psi_{i\lambda}; I, \Lambda)$ [38]. Let us suppose that $I = \Lambda$. We can always take $M_{i_0 i_0} = L$ and $\phi_{i_0 i_0} = \psi_{i_0 i_0} = \iota_L$ for some fixed $i_0 \in I$. Let us put

$$(2.10) \qquad \theta_{ij} = \psi_{i_0 i}^{-1} \phi_{i_0 i} \phi_{ji}^{-1} \psi_{ji} \psi_{ji_0}^{-1} \phi_{ji_0}$$

for all $i, j \in I$. Then θ_{ij}, $i, j \in I$, is an automorphism of L. Let $^\circ$ be an automorphism of order 2 of L, such that

$$(2.11) \qquad {}^\circ\theta_{ij}{}^\circ = \theta_{ij}^{-1}$$

for all $i, j \in I$. Then the mapping

$$(2.12) \qquad {}^\star : e_{ij} \to e_{ij}^\star = (e_{ij} \psi_{ij} \psi_{ii_0}^{-1} \phi_{ii_0})^\circ \psi_{i_0 i}^{-1} \phi_{i_0 i} \phi_{ji}^{-1}$$

is an involution on $E = (L; M_{ij}; \phi_{ij}, \psi_{ij}; I, I)$. Conversely, if the pseudo-semilattice of an elementary rectangular band of inverse semigroups admits an involution, then this pseudo-semilattice and the involution on it must be constructed in this way.

Let $E = (L; M_{ij}; \phi_{ij}, \psi_{ij}; I, I)$, let $M_{i_0 i_0} = L$ for some fixed i_0 in I, let θ_{ij} be defined by (2.10) for all $i, j \in I$, and let $^\circ$ be an automorphism of order 2 of L such that (2.11) is satisfied. Let * be the involution on E which is defined by (2.12). Let $T(L)$ be the Munn semigroup of L, and let $(\theta_{ij})_{i,j\in I} = Q$ be the $I \times I$-matrix which has θ_{ij} on the (i,j)-position. $\mathfrak{M} = \mathcal{M}(T(L); I; Q)$ is the set which consists of the elements α_{ij}, with $\alpha \in T(L)$, $i, j \in I$. The multiplication on $\mathfrak{M}$ is defined as follows. For all $\alpha_{ij}, \beta_{mn} \in \mathfrak{M}$

$$\alpha_{ij}\beta_{mn} = (\alpha\theta_{jm}\beta)_{in}.$$

One can show that $\mathfrak{M}$ is isomorphic to $T(E)$ [38]. Remark that $\mathfrak{M}$ is an elementary rectangular band of fundamental inverse semigroups which are all isomorphic with $T(L)$. On $\mathfrak{M}$ one defines an involution * by

$$(2.13) \qquad \alpha^*_{ij} = ({}^\circ(\alpha^{-1})^\circ)_{ji}.$$

The idempotents of $\mathfrak{M}$ are of the form $(\theta_{ji}^{-1}\iota_e)_{ij}$, where ι_e is the identity transformation on the principal ideal of L which is generated by e. One shows that

$$(2.14) \qquad E \to E(\mathfrak{M}), \quad e\psi_{i_0 j}^{-1}\phi_{i_0 j}\phi_{ij}^{-1} \to (\theta_{ji}^{-1}\iota_e)_{ij}, \quad e \in L,$$

is a biorder isomorphism. Further, if we identify E with $E(\mathfrak{M})$ by the biorder isomorphism (2.14), then the given pseudo-semilattice E with involution * is exactly the involution biordered set which is determined by the regular semigroup $\mathfrak{M}$ with involution *.

The above constructed pseudo-semilattice E is $\star$-regular if and only if the involution $^\circ$ on L, and the automorphisms θ_{ii}, $i \in I$, of L are equal to the identity transformation of L. Then the set of projections is given by $P(E) = \bigcup_{i\in I} M_{ii}$. Remark that in this case $\mathfrak{M}$ ($\cong T(E)$) becomes a special $\star$-semigroup: for all $\alpha_{ij} \in \mathfrak{M}$, we then have $\alpha^*_{ij} = (\alpha^{-1})_{ji} = \alpha^\dagger_{ij}$. This illustrates Theorem 2.11.

Let us again consider the regular involution semigroup $\mathcal{M} = \mathcal{M}(P(G, \mathcal{X}, \mathcal{Y}); I; P; \alpha_1, \alpha_2)$ of Example 1.7. Here we can always suppose that the matrix P is normalized [38]: for some fixed $i_0 \in I$, and all $i \in I$, $p_{ii_0} = p_{i_0 i} = 1$ is the identity of G. For $i, j \in I$, put

$$M_{ij} = \{(A, p_{ji}^{-1})_{ij} \mid A \in \mathscr{Y}\},$$

$$M_{i_0 i_0} = L,$$

$$\theta_{ij}: L \to L, \qquad (A, 1)_{i_0 i_0} \mapsto (p_{ij}^{-1} A, 1)_{i_0 i_0},$$

$$\phi_{ij}: M_{ij} \to L, \qquad (A, p_{ji}^{-1})_{ij} \mapsto (p_{ji} A, 1)_{i_0 i_0},$$

$$\psi_{ij}: M_{ij} \to L, \qquad (A, p_{ji}^{-1})_{ij} \mapsto (A, 1)_{i_0 i_0}.$$

Then $E = (L; M_{ij}; \phi_{ij}, \psi_{ij}; I, I)$ is the pseudo-semilattice which is determined by $\mathscr{M}$ and (2.10) is satisfied. On E we may consider the involution $\star$ which is induced by the involution $*$ on $\mathscr{M}$ (compare with (1.11)):

$$(A, p_{ji}^{-1})_{ij}^{\star} = (p_{ji}^{\alpha_2} A^{\alpha_1}, p_{ji}^{\alpha_2})_{ji} = (p_{ij}^{-1} A^{\alpha_1}, p_{ij}^{-1})_{ji}.$$

Consider the automorphism $^\circ$ of order 2 on L, which is given by

$$(A, 1)_{i_0 i_0}^{\circ} = (A^{\alpha_1}, 1)_{i_0 i_0}.$$

It is routine to check that (2.11) and (2.12) are satisfied. This exemplifies the fact that the pseudo-semilattice E with involution $\star$ which is determined by the regular involution semigroup $\mathscr{M}$, is constructed in the way described above.

If $(A, g) \in P(G, \mathscr{X}, \mathscr{Y})$, then $\alpha_{(A,g)}$ which is defined by

$$\operatorname{dom} \alpha_{(A,g)} = \{(B, 1)_{i_0 i_0} \mid B \leqslant A\},$$

and

$$(B, 1)_{i_0 i_0} \alpha_{(A,g)} = (g^{-1} B, 1)_{i_0 i_0} \quad \text{for all } B \leqslant A,$$

belongs to $T(L)$. The mapping

$$\text{(2.15)} \qquad \mathscr{M} \to \mathfrak{M}, \quad (A, g)_{ij} \to (\alpha_{(A,g)})_{ij}$$

is an idempotent separating homomorphism of $\mathscr{M}$ onto a full regular subsemigroup of $\mathfrak{M}$. One may verify that this homomorphism is in fact a $\star$-homomorphism. Since $\mathfrak{M}$ is, up to isomorphism, the fundamental involution semigroup $T(E)$, the above homomorphism may considered to be the fundamental representation of $\mathscr{M}$. This illustrates Theorem 2.2.

Let E be a special $\star$-biordered set. Then the special $\star$-semigroup $\bar{T}(E)$ of Theorem 2.9 may be constructed in an easier way. The construction in the following theorem immediately entails Munn's construction of $T(E)$ in case E is a semilattice. The ideas of the following theorem resemble Yamada's procedure in [45]. However, Yamada's starting point is a warp (in the sense of [6]) and not a biordered set.

Theorem 2.13. *Let E be a special $\star$-biordered set, and let P be the set of projections of E. Let $T(P)$ be the set of ω-isomorphisms of $T^\star(E)$, α: $\omega(e_\alpha) \to \omega(f_\alpha)$, with $e_\alpha, f_\alpha \in P$, which are also $\star$-isomorphisms. For all $e, f \in P$, let*

$$(2.16) \qquad \gamma_{e,f} = \tau^l(e(ef)^\dagger, (ef)^\dagger)\tau^r((ef)^\dagger, (ef)^\dagger f)$$

where $(ef)^\dagger$ is the unique element in $S(e,f)$ for which $e(ef)^\dagger$ and $(ef)^\dagger f$ are projections. On $T(P)$ we introduce a multiplication and an involution $\star$ by the following: for all $\alpha, \beta \in T(P)$,

$$(2.17) \qquad \begin{aligned} \alpha\beta &= \alpha \circ \gamma_{f_\alpha, e_\beta} \circ \beta, \\ \alpha^\star &= \alpha^{-1}, \end{aligned}$$

where the product on the right is a composition of one-to-one partial transformations. Then $T(P)$ is the greatest fundamental special $\star$-semigroup with biordered set E.

Proof. Let us again consider the semigroup $\bar{T}(E)$ of Theorem 2.9. An element of $\bar{T}(E)$ is of the form $\bar{\alpha}$, where $\alpha \in T^\star(E)$, $e_\alpha, f_\alpha \in P$, such that α is also a $\star$-isomorphism. The mapping ϕ: $\bar{\alpha} \to \alpha$ is then well defined, and is a bijection of $\bar{T}(E)$ onto $T(P)$.

For all $e, f \in P$, $(ef)^\dagger$ is well defined by Proposition 2.5. Since $\bar{T}(E)$ is a full regular subsemigroup of $T(E)$, $\bar{T}(E)$ contains the idempotents

$$\overline{\tau^l(e(ef)^\dagger, (ef)^\dagger)} \quad \text{and} \quad \overline{\tau^r((ef)^\dagger, (ef)^\dagger f)}$$

and consequently also $\overline{\gamma_{e,f}}$. Since $\gamma_{e,f}$: $\omega(e(ef)^\dagger) \to \omega((ef)^\dagger f)$, where $e(ef)^\dagger, (ef)^\dagger f \in P$, we have $\overline{\gamma_{e,f}}\phi = \gamma_{e,f} \in T(P)$. From this we have

$\alpha \circ \gamma_{f_\alpha, e_\beta} \circ \beta \in T(P)$ for all $\alpha, \beta \in T(P)$. One readily verifies that

$$\overline{\alpha\beta} = \overline{\alpha \circ \gamma_{f_\alpha, e_\beta} \circ \beta} = \bar{\alpha}\bar{\beta}$$

in $T(E)$, since $(f_\alpha e_\beta)^\dagger \in S(f_\alpha, e_\beta)$. Therefore the above considered mapping ϕ is a multiplicative morphism.

If $\bar{\alpha} \in \bar{T}(E)$, where $e_\alpha, f_\alpha \in P$, then $\bar{\alpha}^\star = \overline{\alpha^{-1}}$, from which $\bar{\alpha}^\star \phi = \alpha^{-1}$. Thus ϕ is a $\star$-isomorphism of $\bar{T}(E)$ onto $T(P)$.

Example 2.14 (see also Example 1.9). Let $(L, \wedge, \vee, 0, 1, {}^\perp)$ be an orthocomplemented modular lattice. If n and ν are any pair of complementary elements in L, then

$$(n; \nu)\colon\ L \to L,\ x \to \nu \wedge (n \vee x) = x(n; \nu)$$

is an idempotent order preserving mapping of L onto the principal ideal of L which is generated by ν. We denote by $P(L)$ the subsemigroup of the full transformation semigroup on the set L which is generated by the above considered mappings $(n; \nu)$. From [37] we have that

$$E(L) = \{(n; \nu) \mid n, \nu \in L,\ n \text{ and } \nu \text{ complementary in } L\}$$

is the set of idempotents of $P(L)$. In $(E(L), \omega^l, \omega^r, \tau^l, \tau^r)$ we have [37] $(n_1; \nu_1)\, \omega^l\, (n_2; \nu_2)$ if and only if $\nu_1 \leqslant \nu_2$ in L and then

$$(n_1; \nu_1)\, \tau^l\, (n_2; \nu_2) = (n_2 \vee (\nu_2 \wedge n_1); \nu_1),$$

$(n_1; \nu_1)\, \omega^r\, (n_2; \nu_2)$ if and only if $n_1 \geqslant n_2$ in L and then

$$(n_1; \nu_1)\, \tau^r\, (n_2; \nu_2) = (n_1; \nu_2 \wedge (n_2 \vee \nu_1)).$$

On $E(L)$ we can define an involution * by

$$(n; \nu)^\star = (\nu^\perp; n^\perp),\quad (n; \nu) \in E(L).$$

This involution * on the biordered set $E(L)$ extends in a unique way to an involution * on $P(L)$. The set of projections consists of the idempotents which are of the form $(n^\perp; n)$, $n \in P$. Obviously this set of so-called "Sasaki projections" forms for the natural partial order on the idempotents a lattice which is isomorphic to the given lattice L. With this involution *, the semigroup $P(L)$ becomes a Baer $\star$-semigroup in the sense of [9]. Since $P(L)$ is a fundamental regular semigroup [37], $P(L)$ may be identified with the idempotent generated part of $T(E(L))$.

Remark that $P(L)$ (and thus also $T(E(L))$) is then $\star$-regular, since for all $(n; \nu) \in E(L)$,

$$(2.18) \qquad \begin{pmatrix} (n;\nu) & (n;n^{\perp}) \\ (\nu^{\perp};\nu) & (\nu^{\perp};n^{\perp}) \end{pmatrix}$$

is an E-square. Given two projections $(n_1; n_1^{\perp})$ and $(n_2; n_2^{\perp})$, we see that $(n_2 \vee (n_2^{\perp} \wedge n_1); n_1^{\perp} \wedge (n_1 \vee n_2^{\perp}))$ is the unique element in the sandwich set of $(n_1; n_1^{\perp})$ and $(n_2; n_2^{\perp})$ which satisfies the condition expressed in the statement of Proposition 2.5:

$$(n_2 \vee (n_2^{\perp} \wedge n_1); n_1^{\perp} \wedge (n_1 \vee n_2^{\perp}))\, \tau^{l}\, (n_1; n_1^{\perp}) =$$
$$= (n_1 \vee (n_1^{\perp} \wedge n_2); n_1^{\perp} \wedge (n_1 \vee n_2^{\perp})),$$
$$(n_2 \vee (n_2^{\perp} \wedge n_1); n_1^{\perp} \wedge (n_1 \vee n_2^{\perp}))\, \tau^{r}\, (n_2; n_2^{\perp}) =$$
$$= (n_2 \vee (n_2^{\perp} \wedge n_1); n_2^{\perp} \wedge (n_2 \vee n_1^{\perp}))$$

are projections, and in $P(L)$ we have

$$(n_2 \vee (n_2^{\perp} \wedge n_1); n_1^{\perp} \wedge (n_1 \vee n_2^{\perp})) = ((n_1; n_1^{\perp})(n_2; n_2^{\perp}))^{\dagger}.$$

In order to see that the $\star$-regular biordered set $E(L)$ is not always the biordered set of some special $\star$-semigroup, we take L to be the orthocomplemented modular lattice of subspaces of an Euclidean vector space. Let ν be any plane and n any line such that $n \neq \nu$ and $n \not\subseteq \nu$. Then $(n; \nu) \in E(L)$ and the E-square (2.18) consists of four different elements. If x is any line such that $x \subseteq \nu$, $x \neq \nu \wedge n^{\perp}$, $x \not\subseteq (\nu \wedge n^{\perp})^{\perp}$, then $x(n; \nu)(\nu^{\perp}; n^{\perp}) \neq x(n; n^{\perp})$. Thus the E-square (2.18) cannot be τ-commutative, and $E(L)$ does not satisfy the condition (B′) of Corollary 2.7.

The orthocomplemented modular lattice L of length 2 which is depicted in Example 1.9 yields a 14-element biordered set $E(L)$. The special $\star$-semigroup $\bar{T}(E(L))$ is properly contained in $T(E(L))$, and $\bar{T}(E(L))$ is a combinatorial completely 0-simple semigroup with a unit group adjoined.

Let L be any orthocomplemented lattice. Then L can be coordinatized by some strongly regular Baer $\star$-semigroup which is also a special $\star$-semigroup if and only if $P(L)$ is a special $\star$-semigroup. In view of Proposition 2.5 and Theorem 2.6, this is the case if and only if the product of any two projections in $P(L)$ yields an idempotent. Thus, L can be coordinatized by a strongly regular Baer $\star$-semigroup which is a special $\star$-semigroup if and only if the composition of any two Sasaki projections on L is an idempotent transformation on L. This latter condition can be expressed in the form of an identity in three variables which must be satisfied in L:

$$z \wedge (z^\perp \vee (y \wedge (y^\perp \vee (z \wedge (z^\perp \vee (y \wedge (y^\perp \vee x)))))) =$$

$$= z \wedge (z^\perp \vee (y \wedge (y^\perp \vee x))).$$

The lattices under consideration thus form a variety of orthocomplemented modular lattices (which contains the variety of Boolean algebras properly).

3. CROSS-CONNECTIONS OF REGULAR INVOLUTION SEMIGROUPS

In this section we assume that the reader is familiar with the concepts of regular partially ordered sets, cross-connections and related notions. For definitions of these concepts and their basic properties, we refer the reader to [13], [14], [15], [31].

Recall from [13] or from [31] that if I is a regular partially ordered set, then the set of all normal equivalences on I is again a regular partially ordered set under the reverse of inclusion. We denote this partially ordered set by I°.

Lemma 3.1. *Let* $\alpha: I \to I'$ *be an isomorphism of regular partially ordered sets. For each* $\nu \in I^\circ$, *define*

$$\nu\alpha^\circ = \{(x, y) \in I' \times I' \mid (x\alpha^{-1}, y\alpha^{-1}) \in \nu\}.$$

Then $\alpha^\circ: I^\circ \to I'^\circ$ *is an isomorphism such that for all* $\nu \in I^\circ$,

$$A(\nu)\alpha = A(\nu\alpha^\circ) \quad \textit{and} \quad M(\nu)\alpha = M(\nu\alpha^\circ).$$

Furthermore, the mapping $\bar{\alpha}$: $S(I) \to S(I')$ *defined by*

$$f\bar{\alpha} = \alpha^{-1} f \alpha$$

for all $f \in S(I)$ *is an isomorphism of the semigroup* $S(I)$ *onto* $S(I')$, *such that for all* $f \in S(I)$,

$$\operatorname{im} f\bar{\alpha} = (\operatorname{im} f)\alpha \quad \textit{and} \quad \ker f\bar{\alpha} = (\ker f)\alpha^{\circ}.$$

Proof. Recall that $S(I)$ denotes the semigroup of all normal mappings of I written as right operators. Since α is an isomorphism, it is easy to see that $f: I \to I$ is a normal mapping if and only if $f\bar{\alpha} = \alpha^{-1} f \alpha$ is a normal mapping of I'. $\bar{\alpha}$: $S(I) \to S(I')$ is clearly an isomorphism. Since $I(x)\alpha = I'(x\alpha)$ for all $x \in I$, it follows that $\operatorname{im} f\bar{\alpha} = I(x)\alpha = (\operatorname{im} f)\alpha$ if $\operatorname{im} f = I(x)$. Also

$$(u, v) \in \ker f\bar{\alpha} \Leftrightarrow u\alpha^{-1} f\alpha = v\alpha^{-1} f\alpha \Leftrightarrow$$
$$\Leftrightarrow (u\alpha^{-1})f = (v\alpha^{-1})f \Leftrightarrow$$
$$\Leftrightarrow (u\alpha^{-1}, v\alpha^{-1}) \in \ker f \Leftrightarrow$$
$$\Leftrightarrow (u, v) \in (\ker f)\alpha^{\circ}.$$

Since I is a regular partially ordered set, for all $v \in I^{\circ}$, there exists at least one $f \in S(I)$ such that $\ker f = v$; the result proved above then shows that $v\alpha^{\circ} \in I'^{\circ}$. Since the partial order on I° is the reverse of inclusion, α° is an order isomorphism. Now, if $v \in I^{\circ}$, $u \in A(v)$ if and only if for some $f \in S(I)$ with $\ker f = v$, $f \mid I(u)$ is an isomorphism onto $I(uf)$. Since α is an isomorphism, this is true if and only if $f\bar{\alpha} \mid I'(u\alpha) = \alpha^{-1} f\alpha \mid I'(u\alpha)$ is an isomorphism onto $I'((u\alpha)f\bar{\alpha}) = I'((uf)\alpha)$, that is if and only if $u\alpha \in A(f\bar{\alpha})$. The equality $M(v)\alpha = M(v\alpha^{\circ})$ is proved similarly.

Recall from [14] (see also [31]) that a cross-connection $[I, \Lambda; \Gamma, \Delta]$ consists of two regular partially ordered sets I and Λ and two mappings Γ: $\Lambda \to I^{\circ}$ and Δ: $I \to \Lambda^{\circ}$ satisfying the conditions (C1) and (C2) of [31]. If S is a regular semigroup then $I_S = S/\mathscr{R}$ and $\Lambda_S = S/\mathscr{L}$ are regular partially ordered sets and there exist mappings $\Gamma_S: \Lambda_S \to I_S^{\circ}$ and $\Delta_S: I_S \to \Lambda_S^{\circ}$ such that $[I_S, \Lambda_S; \Gamma_S, \Delta_S]$ is a cross-connection. We refer to this cross-connection as the cross-connection which is induced by S.

It follows from [31] that the mappings Γ_S and Δ_S are defined as follows. For each $e \in E(S)$, (that is, for each $L_e \in \Lambda_S$), we have

(3.1) $\qquad \Gamma_S(L_e) = \ker \lambda_e$

where λ_e is the normal retraction of I_S which sends $R_g \in I_S$ to R_{ek} with $k \in S(e, g)$ (see equations (2.2) and (2.3) of [31]). Now in S, $eg \mathscr{R} ek$ and so λ_e is the mapping which sends R_g to R_{eg} so that λ_e is the image of e under the representation λ defined by Hall [17]). Similarly for each $R_f \in I_S$ (with $f \in E(S)$),

(3.2) $\qquad \Delta_S(R_f) = \ker \rho_f$

where ρ_f is the image of f under the representation ρ of [17], that is, the mapping sending $L_h \in \Lambda_S$ to L_{hf}.

Conversely, let $[I, \Lambda; \Gamma, \Delta]$ be a cross-connection. Then the subset $U = U(I, \Lambda; \Gamma, \Delta)$ of $S^{op}(I) \times S(\Lambda)$ (where $S^{op}(I)$ denotes the left-right dual of $S(I)$ in which mappings are written as left operators) consisting of all pairs (f, g) satisfying the conditions (C3) and (C4) of [31], is a fundamental regular subsemigroup of $S^{op}(I) \times S(\Lambda)$ such that U is isomorphic to $T(E(U))$. Also

$$E(U) = \{(e^1_{x,y}, e^2_{x,y}) \mid x \in M(\Gamma(y))\}$$

where $e^1_{x,y}$ $[e^2_{x,y}]$ is the canonical projection along $\Gamma(y)$ $[\Delta(x)]$ upon $I(x)$ $[\Lambda(y)]$. Further, there exist natural isomorphisms $\eta_R : I \to I_U$ defined by $x\eta_R = R_{(e^1_{x,y}, e^2_{x,y})}$ and $\eta_L : \Lambda \to \Lambda_U$ defined by $y\eta_L = L_{(e^1_{x,y}, e^2_{x,y})}$ (where $x \in M(\Gamma(y))$) such that up to these isomorphisms the cross-connections $[I, \Lambda; \Gamma, \Delta]$ and $[I_U, \Lambda_U; \Gamma_U, \Delta_U]$ are the same. This means that if we identify I with I_U and Λ with Λ_U by the isomorphisms η_R and η_L respectively, then Γ becomes equal to Γ_U and Δ to Δ_U. Equivalently, the following diagrams commute:

$$\begin{array}{ccc} I & \xrightarrow{\eta_R} & I_U \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta_U} \\ \Lambda^\circ & \xrightarrow[\eta_L^\circ]{} & \Lambda_U^\circ \end{array} \qquad\qquad \begin{array}{ccc} \Lambda & \xrightarrow{\eta_L} & \Lambda_U \\ \downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \Gamma_U} \\ I^\circ & \xrightarrow[\eta_R^\circ]{} & I_U^\circ \end{array}$$

By an isomorphism from a cross-connection $[I, \Lambda; \Gamma, \Delta]$ to a cross-connection $[I', \Lambda'; \Gamma', \Delta']$ we mean a pair of order isomorphisms $\phi_1: I \to I'$ and $\phi_2: \Lambda \to \Lambda'$ making the diagrams (D1) and (D2) commute. In this case we write (ϕ_1, ϕ_2): $[I, \Lambda; \Gamma, \Delta] \approx [I', \Lambda'; \Gamma', \Delta']$.

$$\begin{array}{ccc} I & \xrightarrow{\phi_1} & I' \\ \downarrow{\Delta} & & \downarrow{\Delta'} \\ \Lambda^\circ & \xrightarrow{\phi_2^\circ} & \Lambda'^\circ \end{array} \qquad\qquad \begin{array}{ccc} \Lambda & \xrightarrow{\phi_2} & \Lambda' \\ \downarrow{\Gamma} & & \downarrow{\Gamma'} \\ I^\circ & \xrightarrow{\phi_1^\circ} & I'^\circ \end{array}$$

(D1) (D2)

Thus (η_R, η_L): $[I, \Lambda; \Gamma, \Delta] \approx [I_U, \Lambda_U; \Gamma_U, \Delta_U]$.

If

$$(\phi_1, \phi_2): [I, \Lambda; \Gamma, \Delta] \approx [I', \Lambda'; \Gamma', \Delta']$$

and

$$(\phi_1', \phi_2'): [I', \Lambda'; \Gamma', \Delta'] \approx [I'', \Lambda''; \Gamma'', \Delta'']$$

are isomorphisms of cross-connections, then it is clear that

$$(\phi_1\phi_1', \phi_2\phi_2'): [I, \Lambda; \Gamma, \Delta] \approx [I'', \Lambda''; \Gamma'', \Delta'']$$

is also an isomorphism.

Theorem 3.2. *Let* (ϕ_1, ϕ_2): $[I, \Lambda; \Gamma, \Delta] \approx [I', \Lambda'; \Gamma', \Delta']$ *be an isomorphism of cross-connections. Then the mapping*

$$(\bar{\phi}_1, \bar{\phi}_2): (f, g) \mapsto (f\bar{\phi}_1, g\bar{\phi}_2)$$

is an isomorphism of $U = U(I, \Lambda; \Gamma, \Delta)$ *onto* $U' = U(I', \Lambda'; \Gamma', \Delta')$. *Conversely, if* $\psi: U \to U'$ *is an isomorphism, then there exists an isomorphism of cross-connections* (ψ_R, ψ_L): $[I, \Lambda; \Gamma, \Delta] \approx [I', \Lambda'; \Gamma', \Delta']$ *such that* $\psi = (\bar{\psi}_R, \bar{\psi}_L)$.

Proof. By Lemma 3.1 it is clear that $(\bar{\phi}_1, \bar{\phi}_2)$ is an isomorphism of U onto a subsemigroup of $S^{op}(I') \times S(\Lambda')$. It is therefore sufficient to show that $\operatorname{im}(\bar{\phi}_1, \bar{\phi}_2) = U'$, that is, $(f, g) \in U$ if and only if $(f\bar{\phi}_1, g\bar{\phi}_2) \in$ $\in U'$. So, assume $(f, g) \in U$, $\operatorname{im} f\bar{\phi}_1 = I'(x')$ and $\operatorname{im} g\bar{\phi}_2 = \Lambda'(y')$. Then there exist $x \in I$ and $y \in \Lambda$ such that $x\phi_1 = x'$, $y\phi_2 = y'$, $\operatorname{im} f = I(x)$ and $\operatorname{im} g = \Lambda(y)$. By condition (C3) of [31] and Lemma 3.1, $\ker f\bar{\phi}_1 = (\ker f)\phi_1^\circ = \Gamma(y)\phi_1^\circ = \Gamma'(y\phi_2) = \Gamma'(y')$, since diagram (D2) commutes. Similarly $\ker g\bar{\phi}_2 = \Delta'(x')$, and so the pair $(f\bar{\phi}_1, g\bar{\phi}_2)$ satisfies the condition (C3). To prove (C4), consider $u' \in I'$. Since (f, g) satisfies (C4), by the commutativity of the diagram (D1) we have

$$\Delta'((f\bar{\phi}_1)(u')) = \Delta'((f(u'\phi_1^{-1}))\phi_1) = (\Delta((f(u'\phi_1^{-1}))))\phi_2^\circ =$$
$$= ((\Delta(u'\phi_1^{-1}))g^{-1})\phi_2^\circ = (((\Delta'(u'))(\phi_2^\circ)^{-1})g^{-1})\phi_2^\circ .$$

Now,

$$(s', t') \in (((\Delta'(u'))(\phi_2^\circ)^{-1})g^{-1})\phi_2^\circ \Leftrightarrow$$
$$\Leftrightarrow (s'\phi_2^{-1}g\phi_2, t'\phi_2^{-1}g\phi_2) = (s'g\bar{\phi}_2, t'g\bar{\phi}_2) \in \Delta'(u').$$

Thus for all $u' \in I'$, $\Delta'((f\bar{\phi}_1)(u')) = (\Delta'(u'))(g\bar{\phi}_2)^{-1}$. Similarly for all $v' \in \Lambda'$, $\Gamma'((v')(g\bar{\phi}_2)) = (f\bar{\phi}_1)^{-1}(\Gamma'(v'))$. Hence $(f\bar{\phi}_1, g\bar{\phi}_2) \in U'$. It can be shown similarly that for all $(f', g') \in U'$, $(f'\bar{\phi}_1^{-1}, g'\bar{\phi}_2^{-1}) = (f'\overline{\phi_1^{-1}}, g'\overline{\phi_2^{-1}}) \in U$ and so $(\bar{\phi}_1, \bar{\phi}_2)$ is an isomorphism of U onto U'.

Suppose now that $\psi\colon U \to U'$ is an isomorphism. Define $\phi_1'\colon I_U \to I_{U'}$ and $\phi_2'\colon \Lambda_U \to \Lambda_{U'}$ by

$$R_e\phi_1' = R_{e\psi}, \quad L_f\phi_2' = L_{f\psi}$$

for all $R_e \in I_U$ and $L_f \in \Lambda_U$. Clearly ϕ_1' and ϕ_2' are order isomorphisms. Let $e \in E = E(U)$ and $g' \in E' = E(U')$. Then

$$\lambda_{e\psi} R_{g'} = R_{(e(g'\psi^{-1}))\psi} = R_{e(g'\psi^{-1})}\phi_1' =$$
$$= (\lambda_e(R_{g'}\phi_1'^{-1}))\phi_1' = (\lambda_e\overline{\phi_1'})(R_{g'})$$

and hence $\lambda_{e\psi} = \lambda_e\overline{\phi_1'}$. Similarly $\rho_{e\psi} = \rho_e\overline{\phi_2'}$. Therefore

$$\Gamma_{U'}(L_e\phi_2') = \Gamma_{U'}(L_{e\psi}) = \ker \lambda_{e\psi} = (\ker \lambda_e)\phi_1'^\circ = \Gamma_U(L_e)\phi_1'^\circ$$

by Lemma 3.1. This proves that diagram (D2) commutes. Similarly diagram (D1) also commutes and so

$$(\phi_1', \phi_2')\colon [I_U, \Lambda_U; \Gamma_U, \Delta_U] \approx [I_{U'}, \Lambda_{U'}; \Gamma_{U'}, \Delta_{U'}]$$

is an isomorphism. If

$$(\eta_R, \eta_L)\colon [I, \Lambda; \Gamma, \Delta] \approx [I_U, \Lambda_U; \Gamma_U, \Delta_U]$$

and

$$(\eta'_R, \eta'_L)\colon [I', \Lambda'; \Gamma', \Delta'] \approx [I_{U'}, \Lambda_{U'}; \Gamma_{U'}, \Delta_{U'}]$$

are natural isomorphisms, it follows that

$$(\psi_R, \psi_L)\colon (\eta_R \phi'_1 \eta_R'^{-1}, \eta_L \phi'_2 \eta_L'^{-1})$$

is an isomorphism of $[I, \Lambda; \Gamma, \Delta]$ to $[I', \Lambda'; \Gamma', \Delta']$. From the direct part it follows that $(\bar{\psi}_R, \bar{\psi}_L)$ is an isomorphism of U onto U'. Now an automorphism of a fundamental regular semigroup coincides with the identity transformation if and only if it fixes the idempotents. Therefore, to show that $\psi = (\bar{\psi}_R, \bar{\psi}_L)$, it is sufficient to show that the two isomorphisms coincide on E. Accordingly, choose $e \in E$. Then for some $x \in I$, $y \in \Lambda$ with $x \in M(\Gamma(y))$, $e = (e^1_{x,y}, e^2_{x,y})$. By Lemma 3.1, $e^1_{x,y}\bar{\eta}_R$ is the projection along $\Gamma(y)\eta^\circ_R = \Gamma_U(y\eta_L) = \Gamma_U(L_{(e^1_{x,y}, e^2_{x,y})}) = \Gamma_U(L_e)$ upon $I_U(x\eta_R) = I_U(R_e)$ and so $e^1_{x,y}\bar{\eta}_R = \lambda_e$. Similarly $e^2_{x,y}\bar{\eta}_L = \rho_e$. Hence

$$e(\bar{\psi}_R, \bar{\psi}_L) = e(\bar{\eta}_R, \bar{\eta}_L)(\bar{\phi}'_1, \bar{\phi}_2)(\bar{\eta}_R'^{-1}, \bar{\eta}_L'^{-1}) =$$
$$= (\lambda_e \bar{\phi}'_1, \rho_e \bar{\phi}'_1)(\bar{\eta}_R'^{-1}, \bar{\eta}_L'^{-1}) = (\lambda_{e\psi}, \rho_{e\psi})(\bar{\eta}_R'^{-1}, \bar{\eta}_L'^{-1}) = e\psi.$$

Remark. It may be noted that the isomorphism $(\bar{\eta}_R, \bar{\eta}_L)\colon U = U(I, \Lambda; \Gamma, \Delta) \to U(I_U, \Lambda_U; \Gamma_U, \Delta_U)$ corresponding to the natural isomorphism $(\eta_R, \eta_L)\colon [I, \Lambda; \Gamma, \Delta] \approx [I_U, \Lambda_U; \Gamma_U, \Delta_U]$ is the same as the representation (λ, ρ) of Hall [17].

If S^{op} denotes the left-right dual of the regular semigroup S, then it is clear that the cross-connection $[I_{S^{op}}, \Lambda_{S^{op}}; \Gamma_{S^{op}}, \Delta_{S^{op}}]$ is isomorphic to $[\Lambda_S, I_S; \Delta_S, \Gamma_S]$. In particular if $U = U(I, \Lambda; \Gamma, \Delta)$, then the mapping

(3.3) $\quad \zeta\colon (f, g) \to (g, f)$

is an isomorphism of $U(I, \Lambda; \Gamma, \Delta)$ onto $U(\Lambda, I; \Delta, \Gamma)$. We use these observations in the following characterization of cross-connections of a regular involution semigroup.

Theorem 3.3. *Let* $U = U(I, \Lambda; \Gamma, \Delta)$ *where* $[I, \Lambda; \Gamma, \Delta]$ *is a cross-connection. Suppose that* $\alpha: I \to \Lambda$ *is an isomorphism making the following diagram commute.*

$$\begin{array}{ccc} I & \xrightarrow{\ \alpha\ } & \Lambda \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Gamma} \\ \Lambda^{\circ} & \xrightarrow{\ \alpha^{\circ}\ } & I^{\circ} \end{array}$$

(D3)

Then α *induces an involution* $^{\star}: U \to U$ *defined by:*

(3.4) $$(f, g)^{\star} = (g\bar{\alpha}^{-1}, f\bar{\alpha})$$

for all $(f, g) \in U$.

Conversely if * *is an involution of* U, *then there exists an isomorphism* $\alpha: I \to \Lambda$ *making the diagram* (D3) *commute and such that the involution on* U *induced by* α *coincides with* *.

Proof. Let $\alpha: I \to \Lambda$ be an isomorphism making the diagram (D3) commute. Then $(\alpha, \alpha^{-1}): [I, \Lambda; \Gamma, \Delta] \approx [\Lambda, I; \Delta, \Gamma]$ is an isomorphism and so, if ζ is the anti-isomorphism defined by (3.3), by Theorem 3.2 $(\alpha, \alpha^{-1})\zeta^{-1}$ is an anti-isomorphism of U onto itself. Since the mapping defined by (3.4) is the same as $(\bar{\alpha}, \bar{\alpha}^{-1})\zeta^{-1}$, it follows that (3.4) defines an anti-isomorphism of U onto U. Since

$$(f, g)^{\star\star} = (f\bar{\alpha}\bar{\alpha}^{-1}, g\bar{\alpha}^{-1}\bar{\alpha}) = (f, g),$$

we conclude that (3.4) defines an involution on U.

Conversely, let * be an involution on U. Then $\psi = {}^{\star}\zeta$ is an isomorphism of U onto $U(\Lambda, I; \Delta, \Gamma)$. By Theorem 3.2 there exists an isomorphism (ψ_R, ψ_L) of $[I, \Lambda; \Gamma, \Delta]$ to $[\Lambda, I; \Delta, \Gamma]$ such that $\psi = (\bar{\psi}_R, \bar{\psi}_L)$. Now for all $(f, g) \in U$, $(f, g)^{\star\star} = (f, g)$ and since $^{\star} = \psi\zeta^{-1}$, we get

$$(f, g) = (f, g)^{\star\star} = (f\bar{\psi}_R\bar{\psi}_L, g\bar{\psi}_L\bar{\psi}_R).$$

This implies that $\psi_L = \psi_R^{-1}$ and so the diagram (D3) commutes (with $\alpha = \psi_R$). Since $^* = \psi\zeta^{-1}$, the involution induced by ψ_R coincides with the given involution.

The existence of the isomorphism $\alpha\colon I \to \Lambda$ suggests that the structure of a fundamental regular involution semigroup may be expressed in terms of a regular partially ordered set I and a mapping $\eta\colon I \to I^\circ$. We have the following.

Theorem 3.4. *Let I be a regular partially ordered set and $\eta\colon I \to I^\circ$ a mapping satisfying the following conditions.*

(I1) *For all $x, y \in I$, $x \in M(\eta(y))$ implies $y \in M(\eta(x))$.*

(I2) *Let $x \in M(\eta(y))$ and let $p(x, y)$ denote the projection along $\eta(x)$ upon $I(y)$. Then for all $z \in I$,*

$$\eta(zp(x, y)) = (\eta(z))(p(y, x))^{-1}.$$

Let $U = U(I, \eta)$ denote the subset of $S^{op}(I) \times S(I)$ consisting of all pairs (f, g) satisfying the following conditions.

(I3) $\operatorname{im} f = I(x)$, $\operatorname{im} g = I(y)$ *implies* $\ker f = \eta(y)$, $\ker g = \eta(x)$.

(I4) *For all $z \in I$,*

$$\eta(fz) = (\eta(z))g^{-1}, \quad \eta(zg) = f^{-1}(\eta(z)).$$

Then U is a fundamental regular subsemigroup of $S^{op}(I) \times S(I)$ and the mapping defined by

$$(3.5) \qquad (f, g)^* = (g, f)$$

for all $(f, g) \in U$ is an involution in U.

Conversely every fundamental regular involution semigroup U determines a unique mapping $\eta_U\colon I_U \to I_U^\circ$ satisfying (I1) *and* (I2), *and U is $\star$-isomorphic to a full subsemigroup of $U(I_U, \eta_U)$.*

The fundamental regular involution semigroups $U(I, \eta)$ and $U(I', \eta')$ are $\star$-isomorphic if and only if there exists an isomorphism $\phi\colon I \to I'$ such that the following diagram commutes.

$$\begin{array}{ccc} I & \xrightarrow{\ \phi\ } & I' \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta'} \\ I^{\circ} & \xrightarrow{\ \phi^{\circ}\ } & I'^{\circ} \end{array}$$

(D4)

The unique $\star$*-isomorphism determined by* ϕ *is* $(\bar{\phi}, \bar{\phi})$.

Proof. If the mapping $\eta\colon I \to I^{\circ}$ satisfies (I1) and (I2), it is clear that $[I, I; \eta, \eta]$ is a cross-connection. Further, $(f, g) \in S^{op}(I) \times S(I)$ satisfies conditions (C3) and (C4) of [31] if and only if (f, g) satisfies (I3) and (I4). Hence $U(I, \eta) = U(I, I; \eta, \eta)$ and so $U(I, \eta)$ is a fundamental regular subsemigroup of $S^{op}(I) \times S(I)$. Further the isomorphism $\alpha = \iota_I\colon I \to I$ satisfies the condition of Theorem 3.3 and so $(f, g) \to (g\bar{\alpha}^{-1}, f\bar{\alpha}) = (g, f)$ is an involution of $U(I, \eta)$.

Conversely let U be a fundamental regular involution semigroup and let $[I, \Lambda; \Gamma, \Delta]$ be the cross-connection induced by U. It follows from Theorems 2.1 and 2.2 that $U(I, \Lambda; \Gamma, \Delta)$ is an involution semigroup and U is $\star$-isomorphic to a full subsemigroup of $U(I, \Lambda; \Gamma, \Delta)$. By Theorem 3.3, there exists an isomorphism $\alpha\colon I \to \Lambda$ making diagram (D3) commute. We shall show that the mapping

(3.6) $$\eta = \Delta(\alpha^{\circ})^{-1} = \alpha\Gamma$$

satisfies conditions (I1) and (I2). To prove (I1) suppose that $x \in M(\eta(y))$. Then by Lemma 3.1, $x\alpha \in M(\eta(y))\alpha = M((\eta(y))\alpha^{\circ}) = M(\Delta(y))$ and so by condition (C1) of [31], $y \in M(\Gamma(x\alpha))$. Since diagram (D3) commutes, we have $M(\Gamma(x\alpha)) = M(\Delta(x)\alpha^{\circ -1}) = M(\eta(x))$ and so the condition (I1) holds. Now by Lemma 3.1, $p(x, y)\bar{\alpha}$ is the projection along $(\eta(x))\alpha^{\circ} = \Delta(x)$ upon $I(y)\alpha = I(y\alpha)$ and so $p(x, y)\bar{\alpha} = e^2_{x, y\alpha}$. Similarly $e^1_{x, y\alpha}$ is the projection along $\Gamma(y\alpha) = \Delta(y)\alpha^{\circ -1} = \eta(y)$ upon $I(x)$, and so $e^1_{x, y\alpha} = p(y, x)$. Hence for all $z \in I$, since (D3) commutes, we have

$$\eta(zp(x, y)) = \eta(((z\alpha)e^2_{x, y\alpha})\alpha^{-1}) = \Gamma((z\alpha)e^2_{x, y\alpha}) =$$

$$= (e^1_{x, y\alpha})^{-1}(\Gamma(z\alpha)) = \qquad \text{(by (C2) of [31])}$$

$$= (\eta(z))p(y, x)^{-1}.$$

Hence (I2) also holds and so $[I, I; \eta, \eta]$ is a cross-connection and $(1, \alpha)$: $[I, I; \eta, \eta] \approx [I, \Lambda; \Gamma, \Delta]$ is an isomorphism of cross-connections. Therefore, by Theorem 3.2, $(1, \bar{\alpha})$ is an isomorphism of $U(I, \eta)$ onto $U(I, \Lambda; \Gamma, \Delta)$. If $(f, g) \in U(I, \eta)$, then $((f, g)^{\star})(1, \bar{\alpha}) = (g, f)(1, \bar{\alpha}) =$ $= (g, f\bar{\alpha})$ and $((f, g)(1, \bar{\alpha}))^{\star} = (f, g\bar{\alpha})^{\star} = (g\bar{\alpha}\bar{\alpha}^{-1}, f\bar{\alpha}) = (g, f\bar{\alpha})$ by (3.4) and (3.5). This proves that $(1, \bar{\alpha})$ is a $\star$-isomorphism and we conclude that U is $\star$-isomorphic to a full subsemigroup of $U(I, \eta)$.

Let ψ: $U(I, \eta) \to U(I', \eta')$ be a $\star$-isomorphism. By Theorem 3.2, there exists a unique isomorphism (ψ_R, ψ_L): $[I, I; \eta, \eta] \approx [I', I'; \eta', \eta']$ such that $\psi = (\bar{\psi}_R, \bar{\psi}_L)$. We shall show that ψ_R makes diagram (D4) commute. Since (D1) and (D2) commute, it is sufficient to show that $\psi_R = \psi_L$. Since ψ is a $\star$-isomorphism, for all $(f, g) \in U(I, \eta)$, we have

$$(g\bar{\psi}_R, f\bar{\psi}_L) = ((f, g)^{\star})\psi = ((f, g)\psi)^{\star} = (g\bar{\psi}_L, f\bar{\psi}_R)$$

and so $g\bar{\psi}_R = g\bar{\psi}_L$, $f\bar{\psi}_R = f\bar{\psi}_L$. Now for all $x \in I$, we can find $(f, g) \in$ $\in U(I, \eta)$ such that $\operatorname{im} g = I(x)$, and from Lemma 3.1 and the result proved above we obtain $I(x\psi_R) = \operatorname{im} g\bar{\psi}_R = \operatorname{im} g\bar{\psi}_L = I(x\psi_L)$ which implies $x\psi_R = x\psi_L$. Hence we conclude that $\psi_R = \psi_L$. Conversely, if ϕ: $I \to I'$ is an isomorphism making (D4) commute, then (ϕ, ϕ): $[I, I; \eta, \eta] \approx [I', I'; \eta', \eta']$ is an isomorphism. It is easy to see that $(\bar{\phi}, \bar{\phi})$ is a $\star$-isomorphism.

Corollary 3.5. *A regular partially ordered set I is the poset of principal right ideals of some regular involution semigroup if and only if there exists a mapping η: $I \to I^{\circ}$ such that* (I1) *and* (I2) *are satisfied.*

Let P be a regular partially ordered set and σ: $P \to P^{\circ}$ be a mapping. We say that the pair (P, σ) is a *P-set* if it satisfies axioms (I1), (I2) and the following.

(I0) For all $e \in P$, $e \in M(\sigma(e))$.

By an isomorphism of a P-set (P, σ) to a P-set (P', σ') we mean an order isomorphism α: $P \to P'$ making the following diagram commute:

$$\begin{array}{ccc} P & \xrightarrow{\ \alpha\ } & P' \\ \big\downarrow \sigma & & \big\downarrow \sigma' \\ P^{\circ} & \xrightarrow{\ \alpha^{\circ}\ } & P'^{\circ} \end{array}$$

(D5)

It may be noted that our definition of a P-set given above is more general than the concept of a P-set introduced by Yamada in [45] in order to characterize the set of projections of a special $\star$-semigroup. Indeed, it follows from Yamada's result and Theorem 3.6 below that a P-set in the sense of Yamada is also a P-set in the sense above, but not conversely. Imaoka [23] gave an alternate characterization of the set of projections of a special $\star$-semigroup using the concept of a P-groupoid. A P-groupoid is a pair (P, θ) consisting of a set P and a mapping $\theta \colon P \to \mathscr{I}_P$ satisfying some axioms [23]. The following proposition, whose proof is straightforward, shows that P-sets may also be defined in a similar fashion.

Proposition 3.6. *Let* (P, σ) *be a* P*-set. Define* $\theta \colon P \to S(P)$ *as follows:*

(3.7) $\qquad \theta(e) = p(e, e), \qquad e \in P$

where, for $x, y \in P$, $p(x, y)$ *denotes the projection along* $\sigma(x)$ *upon* $P(y)$. *Then* θ *satisfies the following conditions.*

(P0) *For all* $e \in P$, $\theta(e)$ *is a normal retraction upon* $P(e)$.

(P1) $e \in M(\theta(f))$ *implies* $f \in M(\theta(e))$.

(P2) *Let* $e \in M(\theta(f))$ *and let* $p(e, f)$ *denote the projection along* $\ker \theta(e)$ *upon* $P(f)$. *Then for all* $g \in P$,

$$\ker(\theta(gp(e, f))) = \ker(p(f, e)\theta(g)).$$

Conversely, if P *is a regular partially ordered set, and* $\theta \colon P \to S(P)$ *is a mapping satisfying* (P0), (P1) *and* (P2), *then the mapping* σ *defined by*

(3.8) $\qquad \sigma(e) = \ker \theta(e), \qquad e \in P$

satisfies axioms (I0), (I1) *and* (I2) *so that* (P, σ) *is a* P*-set.*

If (P, σ) *and* (P', σ') *are P-sets, an order isomorphism* $\phi\colon P \to P'$ *is an isomorphism of P-sets if and only if the following diagram commutes.*

$$\begin{array}{ccc} P & \xrightarrow{\ \phi\ } & P' \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta'} \\ S(P) & \xrightarrow{\ \bar{\phi}\ } & S(P') \end{array}$$

(D6)

Here θ *and* θ' *denote mappings defined by* (3.7).

In the following it will be convenient to denote P-sets by P, P' etc. If P is a P-set, the underlying regular partially ordered set will also be denoted by P and the associated mappings of P into P° and into $S(P)$ will be denoted by σ and θ respectively (or by σ_P and θ_P respectively, if it is necessary to specify P).

Theorem 3.7. *Let* P *be the partially ordered set of projections of a* $\star$*-regular semigroup* S. *Then there exists a mapping* $\sigma\colon P \to P^\circ$ *such that* $P(S) = (P, \sigma)$ *is a P-set.*

Conversely if P *is a P-set then there exists a fundamental* $\star$*-regular semigroup* $T = T(P)$ *such that* $P(T)$ *is isomorphic to* P. *Furthermore a fundamental* $\star$*-regular semigroup* S *is* $\star$*-isomorphic to a full subsemigroup of* T *if and only if* $P(S)$ *is isomorphic to* P *as a P-set.*

Proof. Let P be the partially ordered set of projections and $[I, \Lambda; \Gamma, \Delta]$ be the cross-connection of a $\star$-regular semigroup S. By the remarks made at the end of Section 1 and by Theorems 2.1 and 2.2, $U = U(I, \Lambda; \Gamma, \Delta)$ is $\star$-regular and the representation $(\lambda, \rho)\colon x \mapsto (\lambda_x, \rho_x)$ is a $\star$-homomorphism of S into U. Hence for all $e \in E(S)$, $(\lambda_e, \rho_e)^\star = (\lambda_{e^\star}, \rho_{e^\star})$ and by (3.3), $(\lambda_e, \rho_e)^\star = (\rho_e \bar{\alpha}^{-1}, \lambda_e \bar{\alpha})$, where $\alpha\colon I \to \Lambda$ is the isomorphism making (D3) commute. Hence by Lemma 3.1, $\Lambda(R_e\alpha) = \operatorname{im} \lambda_e \bar{\alpha} = \operatorname{im} \rho_{e^\star} = \Lambda(L_{e^\star})$ and so $R_e \alpha = L_{e^\star}$ for all $e \in E(S)$. Now if $\eta = \Delta(\alpha^\circ)^{-1} = \alpha\Gamma$ is the map defined by (3.6), then for all $g \in P$, λ_g is the projection along $\Gamma(L_g) = \Gamma(R_g\alpha) = \eta(R_g)$ upon $I(R_g)$ and so $R_g \in M(\eta(R_g))$. Thus η satisfies axiom (I0). Axioms (I1) and (I2) hold by Theorem 3.4. Therefore (I, η) is a P-set. Now the mapping

$$\nu\colon R_g \to g, \quad g \in P$$

is clearly an order isomorphism and so, $P(S) = (P, \sigma)$ is a P-set isomorphic to (I, η) if we define σ as $\sigma = \nu^{-1}\eta\nu^{\circ}$.

Conversely let $P = (P, \sigma)$ be a P-set. Then by Theorem 3.4, $T = T(P) = U(P, \sigma)$ is a fundamental regular involution semigroup. Now, idempotents of T are of the form $(p(x, y), p(y, x))$ with $x \in M(\sigma(y))$. Since axiom (I0) holds, $(p(x, x), p(x, x))$ is an idempotent in T which, by (3.5), is a projection. Also if $x \in M(\sigma(y))$, $(p(x, x), p(x, x))$ is clearly $\mathscr{L}$-equivalent to $(p(x, y), p(y, x))$. Hence T is $\star$-regular.

Now let $\eta_T = \Delta_T\alpha^{\circ -1}$ where $\alpha\colon I_T \to \Lambda_T$ is the isomorphism induced by the involution in T. Then by the first part of the proof, (I_T, η_T) is a P-set. Also by the fundamental result on cross-connections [14] we have isomorphisms $T \cong U(I_T, \Lambda_T; \Gamma_T, \Delta_T) \cong U(I_T, \eta_T)$ and so there exists an isomorphism $\beta\colon P \to I_T$ making (D4) commute (by Theorem 3.4). Then β is an isomorphism of the P-set P to (I_T, η_T). Since $\nu\colon R_g \to g$ (g is a projection in T) is an isomorphism of the P-set (I_T, η_T) to $P(T)$, it follows that $\beta\nu$ is an isomorphism of P to $P(T)$.

Finally let S be a fundamental $\star$-regular semigroup. Since S is isomorphic to a full subsemigroup of $T(P(S))$, S is isomorphic to a full subsemigroup of T if and only if $T(P(S))$ is isomorphic to T. By Theorem 3.4, this is true if and only if $P(S)$ is isomorphic to P.

Theorem 3.8. *Let P be a P-set and $\theta = \theta_P$. Then P is the P-set of a special $\star$-semigroup if and only if the following condition holds.*

(P3) *For all $e, f \in P$ with $e \in M(\theta(f))$, $\theta(e)\theta(f)\theta(e) = \theta(e)$.*

Proof. We note that the condition (P3) is equivalent to the statement that for $e, f \in P$ with $e \in M(\theta(f))$, the projections $(p(e, e), p(e, e)) = (\theta(e), \theta(e))$ and $(\theta(f), \theta(f))$ are mutually inverse, that is, the E-square

$$\begin{pmatrix} (p(e,f), p(f,e)) & (p(f,f), p(f,f)) \\ (p(e,e), p(e,e)) & (p(f,e), p(e,f)) \end{pmatrix}$$

is τ-commutative. Now P is the P-set of a special $\star$-semigroup if and only if $T(P)$ contains a full special $\star$-subsemigroup. By Corollary 2.7 and the remark above, this is true if and only if P satisfies (P3).

Remark. The theorem above shows that P-sets satisfying axiom (P3) characterize the sets of projections of special $\star$-semigroups. Since the concept of P-groupoids [23] also does the same, it is of interest to find the explicit relation between the two. Indeed, let S be a special $\star$-semigroup, $\sigma_S = \sigma_{P(S)}$ and $\theta_S = \theta_{P(S)}$. We have seen that the map ν: $R_g \to g$ is an isomorphism of the P-set $I_S = (I_S, \eta_S)$ to $P(S)$. Hence if $\theta' = \theta_{I_S}$, we have $\nu\theta_S = \theta'\bar{\nu}$. Now for each $R_g \in I_S$ with $g \in P(S)$, it is easy to see that $\theta'(g) = \lambda_g$ and hence for all $h \in P(S)$, $\nu(\lambda_g(R_h)) = \nu(R_{gh})) = h\theta_S(g)$. Since S is a special $\star$-semigroup, the projection in R_{gh} is ghg and so for all $h \in P(S)$, $h\theta_S(g) = ghg$. Hence it follows from the construction of the P-groupoid of S in the proof of Theorem 3.2 of [23] that $\theta_S(g)$ is the same as the map θ_g of Imaoka. Thus the map θ_S is the same as the map θ constructed in the proof of Theorem 3.2 of [23]. Thus $(P(S), \theta_S)$ is a P-groupoid. It is easy to see that if two P-sets P and P' are isomorphic (as P-sets) and if (P, θ_P) is a P-groupoid, so is $(P', \theta_{P'})$. Similarly if the P-groupoids (P, θ) and (P', θ') are isomorphic (as P-groupoids) and if θ satisfies axioms (P0), (P1), (P2) and (P3) then so does θ'. Hence it follows from the above and Theorem 3.2 of [23] that every P-set satisfying (P3) is a P-groupoid and conversely.

We have seen that fundamental regular involution semigroups may be constructed from a regular partially ordered set I and a mapping η: $I \to I^\circ$ satisfying axioms (I1) and (I2). In general, the semigroup $U(I, \eta)$ constructed from the pair (I, η) does not admit a faithful representation into $S(I)$. However in several interesting cases, such a representation is possible. It is therefore of interest to characterize those regular involution semigroups that can be so represented.

Lemma 3.9. *Every congruence on a regular semigroup* S *contained in* $\mathscr{L}$ *is idempotent separating if and only if* Δ_S *is injective. When* S *satisfies this condition,* S *is left reductive.*

Proof. Recall from [17] that the kernel congruence of the representation ρ is the maximum congruence $\mathscr{L}_c$ contained in $\mathscr{L}$. Therefore it is sufficient to prove that $\mathscr{L}_c \subseteq \mathscr{H}$ if and only if Δ_S is one-to-one.

So first assume that $\mathscr{L}_c \subseteq \mathscr{H}$ and $\Delta_S(R_e) = \Delta_S(R_f)$ for $e, f \in E(S)$. Then $M(\Delta_S(R_e)) = M(\Delta_S(R_f))$ and it follows from equations (2.2), (2.3) and Theorem C of [31] that for every $\mathscr{L}$-class L of S, $L \cap R_e$ is a subgroup of S if and only if $L \cap R_f$ is also a subgroup. In particular there exists an idempotent f' such that $e \mathscr{L} f' \mathscr{R} f$. Then $\ker \rho_e = \Delta_S(R_e) = \Delta_S(R_{f'}) = \ker \rho_{f'}$ and $\operatorname{im} \rho_e = \Lambda_S(L_e) = \Lambda_S(L_{f'}) = \operatorname{im} \rho_{f'}$. Since ρ_e and $\rho_{f'}$ are normal retractions of Λ_S, we conclude that $\rho_e = \rho_{f'}$ and so $(e, f') \in \mathscr{L}_c$. Since $\mathscr{L}_c \subseteq \mathscr{H}$, we have $e = f'$ and hence $R_e = R_{f'} = R_f$. Thus Δ_S is one-to-one. Conversely, if Δ_S is one-to-one and if $(e, f) \in \mathscr{L}_c \cap (E(S) \times E(S))$, then $\rho_e = \rho_f$ and so $\Delta_S(R_e) = \ker \rho_e = \ker \rho_f = \Delta_S(R_f)$. Hence $R_e = R_f$ and so $e \mathscr{R} f$. Since $e \mathscr{L} f$, we conclude that $e = f$. Hence $\mathscr{L}_c \subseteq \mathscr{H}$. The last statement of the lemma follows from the fact that the kernel of the right regular representation of S is contained in $\mathscr{L}$ and it intersects every $\mathscr{H}$-class in a single element.

Theorem 3.10. *For a regular semigroup S, the following statements are equivalent.*

(i) *The representation ρ is faithful.*

(ii) *S is fundamental and Δ_S is injective.*

(iii) *S can be isomorphically embedded in $S(\Lambda_S)$ in such a way that S intersects every $\mathscr{L}$-class of $S(\Lambda_S)$.*

Proof. Obviously (i) $\Rightarrow$ (iii), and by Lemma 3.9 (i) $\Leftrightarrow$ (ii). To prove (iii) $\Rightarrow$ (i), we may assume that $S \subseteq S(\Lambda_S)$. Consider $f, g \in S$ such that $\rho_f = \rho_g$. Then for all $h \in S$, $hf \mathscr{L} hg$ and so (since $S \subseteq \mathscr{I}_{\Lambda_S}$), $\Lambda_S(xf) = \operatorname{im} hf = \operatorname{im} hg = \Lambda_S(xg)$, where $\operatorname{im} h = \Lambda_S(x)$, that is, $xf = xg$. Now for every $x \in \Lambda_S$, there exists $h \in S$ such that $\operatorname{im} h = \Lambda_S(x)$ and so $xf = xg$ for all $x \in \Lambda_S$. Thus $f = g$.

The theorem above implies that every subsemigroup of $S(\Lambda)$ (for some regular partially ordered set Λ) that intersects every $\mathscr{L}$-class of $S(\Lambda)$ is fundamental. In particular $S(\Lambda)$ is fundamental.

Theorems 3.4 and 3.10 yield the following.

Theorem 3.11. *Let S be a fundamental regular involution semi-*

group. Then ρ *is a faithful representation of* S *as a subsemigroup of* $S(I_S)$ *if and only if* η_S *(the map defined by* (3.6)*) is injective.*

Remark that, if the fundamental regular involution semigroup S satisfies the condition of Theorem 3.11, then η_S must be an order embedding of I_S into the poset (under the reverse of inclusion) of the normal equivalences on I_S.

Before ending this section we shall briefly consider fundamental $\star$-regular semigroups satisfying $\star$-cancellation (that is, condition (1.9)). We first show that condition (1.9) is a regular partial band condition.

Lemma 3.12. *Let* S *be a* $\star$*-regular semigroup. Then* S *satisfies condition* (1.9) *if and only if it satisfies the following.*

(C) *If an* E*-square*

$$\begin{pmatrix} e & g \\ h & e^\star \end{pmatrix}$$

is a rectangular band, then $g = h$.

Proof. First, let S satisfy (1.9). If

$$\begin{pmatrix} e & g \\ h & e^\star \end{pmatrix}$$

is a rectangular band, then g and h are projections and $e^\star h = e^\star e = h^\star h = h^\star e = h$ and so $e = h$ by (1.9). Hence $g \mathscr{R} h$ and since g and h are projections, we have $g = h$. Conversely let S satisfy (C). Consider $a, b \in S$ such that $a^\star a = a^\star b = b^\star a = b^\star b$. Then $R_a \cap L_{b^\star}$ contains an idempotent, say e. Since $R_a \cap L_{a^\star}$ and $R_b \cap L_{b^\star}$ contain projections, say g and h, the idempotent in $R_b \cap L_{a^\star}$ is $e^\star$. Let a' and b' be the inverses of a and b in the $\mathscr{H}$-classes $H_{b^\star}$ and $H_{a^\star}$ respectively. From $a^\star a = a^\star b$, we obtain $e = ge = (a^\star)^\dagger a^\star aa' = (a^\star)^\dagger a^\star ba' = gba'$ and similarly $e^\star = hab'$. Since $a'a = k$ is the projection in $H_{a^\star a}$, we have

$$ee^\star = gba'hab' = gba'ab' = gbb' = ge^\star = g.$$

This proves that

$$\begin{pmatrix} e & g \\ h & e^\star \end{pmatrix}$$

is a rectangular band and so $g = h$ by (3.9). Therefore $a \mathscr{H} b$. From $a^\star a = a^\star b$ we conclude that $a = b$. Thus S satisfies (1.9).

The result proved above shows that $\star$-cancellation is a regular partial band condition. Hence, if a $\star$-regular semigroup S contains a full subsemigroup satisfying (1.9), then every $\mathscr{H}$-coextension of S also satisfies (1.9) (if the coextension is $\star$-regular). But an idempotent separating homomorphic image of S need not satisfy (1.9).

Since (1.9) is not a biordered set condition, it cannot be characterized in terms of the P-set of projections. In particular, there is no relation between injectivity of σ_S and condition (1.9).

Indeed, let $Z_2 = \{0, 1\}$ be a group of order 2, $I = \Lambda = \{1, 2\}$, $P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. On $S = \mathscr{M}(Z_2; P; I, \Lambda)$ we define $\star$ by $g^\star_{ij} = (-g)_{ji}$. Then S becomes a completely simple $\star$-regular semigroup which satisfies (1.9). Yet, σ_S is a constant map.

However, for fundamental $\star$-regular semigroups satisfying (1.9), we have the following result.

Theorem 3.13. *Every fundamental $\star$-regular semigroup S satisfying* (1.9) *admits a faithful representation into* $S(P(S))$.

Proof. In view of Theorem 3.9 it is sufficient to show that the congruence $\mathscr{L}_c$ is idempotent separating. So consider $(e,f) \in \mathscr{L}_c \cap$ $\cap\, (E(S) \times E(S))$. Since the congruence classes of $\mathscr{L}_c$ containing idempotents are completely simple subsemigroups of S, it follows from [28] that e and f have inverses in the same $\mathscr{H}$-classes. Let g be the projection in R_e. Since $g \in V(e)$, H_g contains an inverse of f, and so $R_f \cap L_g$ contains an idempotent, say f_1. It is easy to see that $(g, f_1) \in \mathscr{L}_c$ and so $\rho_g = \rho_{f_1}$. Hence $\Delta(R_{f_1^\star}) = \Delta(R_g) = \Delta(R_{f_1})$, so that $\rho_{f_1} \mathscr{R}\, \rho_{f_1^\star}$ in $S(\Lambda_S)$. Also, since $(g, f_1) \in \mathscr{L}_c$, $(g, f_1^\star) \in \mathscr{R}_c$ and so $\Gamma(L_{f_1}) = \Gamma(L_{f_1^\star})$, that is

$\lambda_{f_1} \mathscr{L} \lambda_{f_1^*}$ in $S^{op}(I_S)$. Hence

$$(\lambda_{f_1^*}, \rho_{f_1^*})(\lambda_{f_1}, \rho_{f_1}) = (\lambda_{f_1^*}\lambda_{f_1}, \rho_{f_1^*}\rho_{f_1}) = (\lambda_{f_1^*}, \rho_{f_1}) = (\lambda_g, \rho_g).$$

Since the representation (λ, ρ) is faithful, we have $f_1^* f_1 = g$. Hence the E-square

$$\begin{pmatrix} f_1 & h \\ g & f_1^* \end{pmatrix}$$

is a rectangular band and so by Lemma 2.12, $g = h$. It follows that $e \mathscr{R} f$ and so we conclude that $e = f$. Thus $\mathscr{L}_c$ is idempotent separating.

Example 3.14 (see also Examples 1.9 and 2.14). Let $(L, \wedge, \vee; 0, 1, ^{\perp})$ be an orthocomplemented modular lattice. Obviously L is a regular partially ordered set. Let $\eta\colon L \to L^{\circ}$ be defined by the following. For $\nu \in L$, $\eta(\nu)$ is the normal equivalence on L which is given by $x\, \eta(\nu)\, y$ if and only if $\nu^{\perp} \vee x = \nu^{\perp} \vee y$. Obviously η is injective, and η is an order embedding of L into the poset of normal equivalences on L (under the reverse of inclusion).

For all $\nu \in L$, we have that $M(\eta(\nu))$ consists of the complements of $\nu^{\perp}$ in L. Therefore (I1) is obviously satisfied for the pair (L, η). Remark that for $n, \nu \in L$, we have that n and ν are complementary in L if and only if $n^{\perp} \in M(\eta(\nu))$, and if this is the case, then

$$p(n^{\perp}, \nu) = (n; \nu)\colon L \to L, \quad x \to \nu \wedge (n \vee x)$$

is the projection along $\eta(n^{\perp})$ upon $L(\nu)$. But then, for $z \in L$,

$$(x, y) \in \eta(zp(n^{\perp}, \nu)) \Leftrightarrow x \vee (\nu \wedge (n \vee z))^{\perp} = y \vee (\nu \wedge (n \vee z))^{\perp} \Leftrightarrow$$

$$\Leftrightarrow x \vee (\nu^{\perp} \vee (n^{\perp} \wedge z^{\perp})) = y \vee (\nu^{\perp} \vee (n^{\perp} \wedge z^{\perp})) \Leftrightarrow$$

$$\Leftrightarrow (n^{\perp} \wedge z^{\perp}) \vee (n^{\perp} \wedge (\nu^{\perp} \vee x)) = (n^{\perp} \wedge z^{\perp}) \vee (n^{\perp} \wedge (\nu^{\perp} \vee y)) \Leftrightarrow$$

$$\Leftrightarrow z^{\perp} \vee (n^{\perp} \wedge (\nu^{\perp} \vee x)) = z^{\perp} \vee (n^{\perp} \wedge (\nu^{\perp} \vee y)) \Leftrightarrow$$

$$\Leftrightarrow (n^{\perp} \wedge (\nu^{\perp} \vee x), n^{\perp} \wedge (\nu^{\perp} \vee y)) \in \eta(z) \Leftrightarrow$$

$$\Leftrightarrow (x, y) \in \eta(z)p(\nu, n^{\perp})^{-1}.$$

Thus (L, η) satisfies (I2). Further, for all $\nu \in L$, $\nu \in M(\eta(\nu))$, and so (L, η) satisfies (I0). Thus (L, η) is a P-set. The mapping

$$\theta\colon L \to S(L), \quad \nu \to p(\nu, \nu) = (\nu^{\perp}; \nu)$$

satisfies (P0), (P1), (P2) by Proposition 3.6, and by Theorem 3.7, $U(L, \eta)$ is a fundamental $\star$-regular semigroup.

The set $E(U(L, \eta))$ of idempotents of $U(L, \eta)$ consists of the elements $(p(\nu, n^{\perp}), p(n^{\perp}, \nu)) = ((\nu^{\perp}; n^{\perp}), (n; \nu)) \in S^{op}(L) \times S(L)$. It follows that the biordered set $E(L)$ which was considered in Example 2.14 is biorder isomorphic to the biordered set of idempotents of $U(L, \eta)$, and so the fundamental $\star$-regular semigroup $U(L, \eta)$ is $\star$-isomorphic to the fundamental $\star$-regular semigroup $T(E(L))$ [31].

Using Theorem 3.10 (and Theorem 3.4) we can say that the mapping $\pi\colon U(L, \eta) \to S(L)$, $(f, g) \mapsto g$ is a monomorphism. Obviously π maps $E(U(L, \eta))$ onto the set $E(L)$ which was considered in 2.14, and π maps $\langle E(U(L, \eta))\rangle$ isomorphically onto $P(L)$, i.e. the subsemigroup of $S(L)$ which is generated by the mappings $(n; \nu)$, n, ν complementary in L.

Let us consider $(g, f) \in U(L, \eta)$. We shall put $g1 = a$ and $1f = b$. Then in $U(L, \eta)$ we must have

$$(g, f) \ \mathscr{L} \ ((b^{\perp}; b), (b^{\perp}; b)) \ \mathscr{R} \ (f, g) =$$

$$= (g, f)^{\star} \ \mathscr{L} \ ((a^{\perp}; a), (a^{\perp}; a)) \ \mathscr{R} \ (g, f),$$

and so in $\pi(U(L))$ we must have

$$f \ \mathscr{L} \ (b^{\perp}, b) \ \mathscr{R} \ g \ \mathscr{L} \ (a^{\perp}; a) \ \mathscr{R} \ f.$$

If $y \in L$, then $b \wedge y \in \operatorname{im} f$ since f is a normal mapping. Thus there exists an element $x \in L$ such that $xf = b \wedge y$. From $(a^{\perp}; a) \ \mathscr{R} \ f$ follows $\ker (a^{\perp}; a) = \ker f$. Hence $x \vee a^{\perp}$ is the greatest element in the ff^{-1}-class of x. Thus the mapping

$$\overset{+}{f}\colon L \to L, \quad y \mapsto y\overset{+}{f} = \bigvee \{x \mid xf = y \wedge b\}$$

is well defined, and for all $x, y \in L$ we then have

$$xf\overset{+}{f} = x \vee a^{\perp} \geqslant x, \quad y\overset{+}{f}f = y \wedge b \leqslant y.$$

It is easy to see that $\overset{+}{f}$ is order preserving. This means that f is a residuated mapping, and that $\overset{+}{f}$ is its residual [4]. It follows immediately from the definition of a normal mapping that f maps principal ideals onto principal ideals. Hence f is a totally range-closed residuated mapping [4]. Let y be any element of L. Then, since $\overset{+}{f}$ is order preserving, we see that $\overset{+}{f}$ maps the principal filter of L which is generated by y into the principal filter which is generated by $y\overset{+}{f}$. Let z be any element of the latter principal filter: $z \geqslant y\overset{+}{f}$. Therefore $y \wedge b \leqslant zf \leqslant b$, from which $(y \vee zf) \wedge b = zf = zf \wedge b$, by the modularity of L. Hence $(y \vee zf)\overset{+}{f} = z$. We conclude that $\overset{+}{f}$ maps principal filters onto principal filters. Thus the residuated mapping f is strongly range-closed [4].

Let us now suppose that f is a strongly range-closed residuated mapping of the lattice L, and let $\overset{+}{f}$ be the residual of f. From Theorem 13.5 of [4] we know that for all $x, y \in L$

(3.9) $\quad (x\overset{+}{f} \wedge y)f = x \wedge yf,$

(3.10) $\quad (xf \vee y)\overset{+}{f} = x \vee y\overset{+}{f}.$

Let us put $a = 0\overset{+}{f}$ and $b = 1f$. Then it follows from (3.10) that $xf\overset{+}{f} =$ $= x \vee a^{\perp}$ for all $x \in L$. Hence $xf \doteq xf\overset{+}{f}f = (x \vee a^{\perp})f$ for all $x \in L$, and so

$$(x, y) \in \ker f \Leftrightarrow x \vee a^{\perp} = y \vee a^{\perp} \Leftrightarrow (x, y) \in \ker (a^{\perp}; a).$$

Thus $\ker f$ is a normal equivalence on L. If $x \in L$, then $(x \wedge (x^{\perp} \vee a, x) \in$ $\in \ker f$, and the principal ideal of L which is generated by $x \wedge (x^{\perp} \vee a)$ intersects the ff^{-1}-classes in at most one element. Since f maps principal ideals onto principal ideals, we see that f maps the principal ideal $L(x \wedge (x^{\perp} \vee a))$ isomorphically onto the principal ideal $L(xf)$. Hence, f is a normal mapping. In an analogous way one shows that $^{\perp}\overset{+}{f}^{\perp}$ is a normal mapping.

Again, let f be the strongly range-closed residuated mapping of the lattice L which was considered above, and take $(^{\perp}\overset{+}{f}^{\perp}, f) \in S^{op}(L) \times S(L)$.

Then $\operatorname{im} f = L(1f) = L(b)$, $\operatorname{im}({}^{\perp}\overset{+}{f}{}^{\perp}) = L((\overset{+}{f}0)^{\perp}) = L(a)$. Then

$$\ker f = \{(x,y) \in L \times L \mid x \vee a^{\perp} = y \vee a^{\perp}\} = \eta(a),$$

and

$$\begin{aligned} \ker {}^{\perp}\overset{+}{f}{}^{\perp} &= \{(x,y) \in L \times L \mid (x^{\perp}, y^{\perp}) \in \ker \overset{+}{f}\} = \\ &= \{(x,y) \in L \times L \mid x^{\perp} \wedge b = y^{\perp} \wedge b\} = \qquad \text{(by (3.9))} \\ &= \eta(b). \end{aligned}$$

Thus the pair $({}^{\perp}\overset{+}{f}{}^{\perp}, f)$ satisfies (I3). Further, if $z \in L$,

$$\begin{aligned} &(x,y) \in \eta(z)f^{-1} \Leftrightarrow xf \vee z^{\perp} = yf \vee z^{\perp} \Rightarrow \\ &\Rightarrow x \vee \overset{+}{f}(z^{\perp}) = \overset{+}{f}(xf \vee z^{\perp}) = \overset{+}{f}(yf \vee z^{\perp}) = y \vee \overset{+}{f}(z^{\perp}) \Leftrightarrow \quad \text{(by (3.10)} \\ &\Leftrightarrow (x,y) \in \eta({}^{\perp}\overset{+}{f}{}^{\perp}(z)). \end{aligned}$$

Conversely, if $(x,y) \in \eta({}^{\perp}\overset{+}{f}{}^{\perp}(z))$, then $(xf \vee z^{\perp}, yf \vee z^{\perp}) \in \ker \overset{+}{f}$. Consequently,

$$xf \vee (z^{\perp} \wedge b) = (xf \vee z^{\perp}) \wedge b = (yf \vee z^{\perp}) \wedge b = yf \vee (z^{\perp} \wedge b),$$

from which it follows that $xf \vee z^{\perp} = yf \vee z^{\perp}$, and thus $(x,y) \in \eta(z)f^{-1}$. We conclude that $\eta(z)f^{-1} = \eta({}^{\perp}\overset{+}{f}{}^{\perp}(z))$ for all $z \in L$. Analogously, $\eta(zf) = ({}^{\perp}\overset{+}{f}{}^{\perp})^{-1}\eta(z)$ for all $z \in L$. Thus also (I4) is satisfied, and we may conclude that $({}^{\perp}\overset{+}{f}{}^{\perp}, f) \in U(L, \eta)$.

From the foregoing it follows that the map $\pi\colon U(L,\eta) \to B(L)$, $(g,f) \longmapsto f$ is an isomorphism of $U(L,\eta)$ onto the semigroup of strongly range-closed residuated mappings on the lattice L. Further, if $(g,f) \in$ $\in U(L,\eta)$, then g must be well defined by f, and is in fact $g = {}^{\perp}\overset{+}{f}{}^{\perp}$. Since ${}^{*}\colon U(L,\eta) \to U(L,\eta)$, $({}^{\perp}\overset{+}{f}{}^{\perp}, f) \longmapsto ({}^{\perp}\overset{+}{f}{}^{\perp}, f)^{*} = (f, {}^{\perp}\overset{+}{f}{}^{\perp})$ is the appropriate involution on $U(L,\eta)$, the mapping ${}^{*}\colon B(L) \to B(L)$, $f \longmapsto {}^{\perp}\overset{+}{f}{}^{\perp}$ is an involution on $B(L)$. The latter involution on $B(L)$ is exactly the one which was considered by Foulis [9], [10], [11] (see Example 1.9), and π is a $\star$-isomorphism. Therefore also the fundamental $\star$-regular semigroup $T(E(L))$ will be $\star$-isomorphic to $B(L)$ [31]. Further, $E(L)$ is the set of idempotents of $B(L)$, and $P(L)$ is the idempotent generated part of $B(L)$.

Let S be any strongly regular Baer $\star$-semigroup which coordinatizes the orthocomplemented modular lattice L. Then the fundamental representation of S which was considered in [30] yields a representation of S by a full $\star$-subsemigroup of $T(E(S))$ (see Theorem 2.2). Also, following Grillet's approach [13], [14], [15], S may be represented by a full $\star$-subsemigroup of $U(L, \eta)$. Finally, S may be represented by a semigroup of strongly range-closed residuated mappings of L [9], [10], [11]. By the above, and also by [31], one can show that the three representations which are considered here are equivalent.

The orthocomplemented modular lattice L of length 2 which is depicted in Example 1.9 yields a fundamental strongly regular Baer $\star$-semigroup $B(L)$ which does not satisfy the $\star$-cancellation (1.9). Therefore the converse of Theorem 3.13 does not hold. For a discussion of the $\star$-cancellation law in Baer $\star$-semigroups, we refer to Section 6 of [11].

REFERENCES

[1] C.L. Adair, Varieties of $\star$-Orthodox Semigroups, Ph. D. Thesis, University of South Carolina, 1979.

[2] C.L. Adair, Bands which admit an involution, *Symposium on Regular Semigroups,* Northern Illinois University, April 1979, 1–10.

[3] B. Banaschewski, Brandt groupoids and semigroups, *Semigroup Forum,* 18 (1979), 307–312.

[4] T.S. Blyth – M.F. Janowitz, *Residuation Theory,* Pergamon Press, London, 1972.

[5] A.H. Clifford, *The partial groupoid of idempotents of a regular semigroup,* Tulane University, 1974.

[6] A.H. Clifford, *The fundamental representation of a regular semigroup,* Tulane University, 1974.

[7] A.H. Clifford – G.B. Preston, *The Algebraic Theory of Semigroups,* Vol. I, Amer. Math. Soc., Providence, R. I., 1961.

[8] M.P. Drazin, Regular semigroups with involution, *Symposium on Regular Semigroups,* Northern Illinois University, April 1979, 29–46.

[9] D.J. Foulis, Baer ⋆-semigroups, *Proc. Amer. Math. Soc.*, 11 (1960), 648–654.

[10] D.J. Foulis, Conditions for the modularity of an orthomodular lattice, *Pacific J. Math.*, 11 (1961), 889–895.

[11] D.J. Foulis, Relative inverses in Baer ⋆-semigroups, *Michigan Math. J.*, 10 (1963), 65–84.

[12] K.R. Goodearl, *von Neumann Regular Rings,* Pitman, London, 1979.

[13] P.A. Grillet, The structure of regular semigroups, I: a representation, *Semigroup Forum,* 8 (1974), 177–183.

[14] P.A. Grillet, The structure of regular semigroups, II: cross-connections, *Semigroup Forum,* 8 (1974), 254–259.

[15] P.A. Grillet, The structure of regular semigroups, III: the reduced case, *Semigroup Forum,* 8 (1974), 260–265.

[16] T.E. Hall, On the lattice of congruences on a regular semigroup, *Bull. Austral. Math. Soc.*, 1 (1969), 231–235.

[17] T.E. Hall, On regular semigroups, *J. Algebra,* 24 (1973), 1–24.

[18] D. Hardy – F. Pastijn, The maximal regular ideal of the semigroup of binary relations, *Czechoslovak Math. J.*, 31 (1981), 194–197.

[19] R.E. Hartwig – I.J. Katz, Products of EP elements in reflexive semigroups, *Linear Algebra Appl.*, 14 (1976), 11–19.

[20] J.M. Howie, *An Introduction to Semigroup Theory*, Academic Press, London, 1976.

[21] J.M. Howie – G. Lallement, Certain fundamental congruences on a regular semigroup, *Proc. Glasgow Math. Assoc.*, 7 (1966), 145–159.

[22] T. Imaoka, On fundamental regular ⋆-semigroups, *Mem. Fac. Sci. Shimane Univ.*, 14 (1980), 19–23.

[23] T. Imaoka, On regular ⋆-semigroups, to appear.

[24] T. Imaoka, ⋆-Congruences on regular ⋆-semigroups, to appear.

[25] T. Imaoka, Some remarks on fundamental regular ⋆-semigroups, *Semigroups and related topics*, Proc. Symp. Kyoto, Japan, 1980, 82–84.

[26] G. Kalmbach, *Orthomodular lattices*, London Math. Soc., Monograph series 18, London, 1983.

[27] D.B. McAlister, Groups, semilattices and inverse semigroups, II, *Trans. Amer. Math. Soc.*, 196 (1974), 351–370.

[28] J. Meakin – K.S.S. Nambooripad, Coextensions of regular semigroups by rectangular bands, *Trans. Amer. Math. Soc.*, 269 (1982), 197–224.

[29] J. Meakin – F. Pastijn, The structure of pseudo-semilattices, *Algebra Universalis*, 13 (1981), 355–372.

[30] K.S.S. Nambooripad, Structure of regular semigroups, I, *Mem. Amer. Math. Soc.*, 224 (1979).

[31] K.S.S. Nambooripad, Relations between biordered sets and cross-connections, *Semigroup Forum*, 16 (1978), 67–81.

[32] K.S.S. Nambooripad, Pseudo-semilattices and biordered sets, I, *Simon Stevin*, 55 (1981), 103–110.

[6] A.H. Clifford, *The fundamental representation of a regular semigroup,* Tulane University, 1974.

[7] A.H. Clifford – G.B. Preston, *The Algebraic Theory of Semigroups,* Vol. I, Amer. Math. Soc., Providence, R. I., 1961.

[8] M.P. Drazin, Regular semigroups with involution, *Symposium on Regular Semigroups,* Northern Illinois University, April 1979, 29–46.

[9] D.J. Foulis, Baer ⋆-semigroups, *Proc. Amer. Math. Soc.,* 11 (1960), 648–654.

[10] D.J. Foulis, Conditions for the modularity of an orthomodular lattice, *Pacific J. Math.,* 11 (1961), 889–895.

[11] D.J. Foulis, Relative inverses in Baer ⋆-semigroups, *Michigan Math. J.,* 10 (1963), 65–84.

[12] K.R. Goodearl, *von Neumann Regular Rings,* Pitman, London, 1979.

[13] P.A. Grillet, The structure of regular semigroups, I: a representation, *Semigroup Forum,* 8 (1974), 177–183.

[14] P.A. Grillet, The structure of regular semigroups, II: cross-connections, *Semigroup Forum,* 8 (1974), 254–259.

[15] P.A. Grillet, The structure of regular semigroups, III: the reduced case, *Semigroup Forum,* 8 (1974), 260–265.

[16] T.E. Hall, On the lattice of congruences on a regular semigroup, *Bull. Austral. Math. Soc.,* 1 (1969), 231–235.

[17] T.E. Hall, On regular semigroups, *J. Algebra,* 24 (1973), 1–24.

[18] D. Hardy – F. Pastijn, The maximal regular ideal of the semigroup of binary relations, *Czechoslovak Math. J.,* 31 (1981), 194–197.

[19] R.E. Hartwig – I.J. Katz, Products of EP elements in reflexive semigroups, *Linear Algebra Appl.,* 14 (1976), 11–19.

[20] J.M. Howie, *An Introduction to Semigroup Theory,* Academic Press, London, 1976.

[21] J.M. Howie – G. Lallement, Certain fundamental congruences on a regular semigroup, *Proc. Glasgow Math. Assoc.,* 7 (1966), 145–159.

[22] T. Imaoka, On fundamental regular $\star$-semigroups, *Mem. Fac. Sci. Shimane Univ.,* 14 (1980), 19–23.

[23] T. Imaoka, On regular $\star$-semigroups, to appear.

[24] T. Imaoka, $\star$-Congruences on regular $\star$-semigroups, to appear.

[25] T. Imaoka, Some remarks on fundamental regular $\star$-semigroups, *Semigroups and related topics,* Proc. Symp. Kyoto, Japan, 1980, 82–84.

[26] G. Kalmbach, *Orthomodular lattices,* London Math. Soc., Monograph series 18, London, 1983.

[27] D.B. McAlister, Groups, semilattices and inverse semigroups, II, *Trans. Amer. Math. Soc.,* 196 (1974), 351–370.

[28] J. Meakin – K.S.S. Nambooripad, Coextensions of regular semigroups by rectangular bands, *Trans. Amer. Math. Soc.,* 269 (1982), 197–224.

[29] J. Meakin – F. Pastijn, The structure of pseudo-semilattices, *Algebra Universalis,* 13 (1981), 355–372.

[30] K.S.S. Nambooripad, Structure of regular semigroups, I, *Mem. Amer. Math. Soc.,* 224 (1979).

[31] K.S.S. Nambooripad, Relations between biordered sets and cross-connections, *Semigroup Forum,* 16 (1978), 67–81.

[32] K.S.S. Nambooripad, Pseudo-semilattices and biordered sets, I, *Simon Stevin,* 55 (1981), 103–110.

[33] K.S.S. Nambooripad, Pseudo-semilattices and biordered sets, II, Pseudo-inverse semigroups, *Simon Stevin,* 56 (1982) 143–159.

[34] K.S.S. Nambooripad, Pseudo-semilattices and biordered sets, III, Regular locally testable semigroups, *Simon Stevin,* 56 (1982), 239–256.

[35] T.E. Nordahl – H.E. Scheiblich, Regular ⋆-semigroups, *Semigroup Forum,* 16 (1978), 369–378.

[36] F. Pastijn, Structure theorems for pseudo-inverse semigroups, *Symposium on Regular Semigroups,* Northern Illinois University, April 1979, 128–138.

[37] F. Pastijn, Biordered sets and complemented modular lattices, *Semigroup Forum,* 21 (1980), 205–220.

[38] F. Pastijn, Rectangular bands of inverse semigroups, *Simon Stevin,* 56 (1982), 4–92.

[39] M. Petrich, *Lectures in Semigroups,* John Wiley and Sons, London, 1977.

[40] N.R. Reilly, A class of regular ⋆-semigroups, *Semigroup Forum,* 18 (1979), 385–386.

[41] N.R. Reilly – H.E. Scheiblich, Congruences on regular semigroups, *Pacific J. Math.,* 23 (1967), 349–360.

[42] B.M. Schein, On the theory of generalized groups and generalized heaps, *Theory of Semigroups and its Applications* I, Izv. dat. Saratov. Univ., Saratov, 1966, 286–324 (in Russian).

[43] B.M. Schein, Semigroups of rectangular binary relations, *Dokl. Akad. Nauk SSSR,* 165 (1965), 1563–1566.

[44] L.A. Skornjakov, *Complemented Modular Lattices and Regular Rings,* Oliver and Boyd Ltd, London, 1964.

[45] M. Yamada, On the structure of fundamental regular ⋆-semigroups, to appear.

[46] K.A. Zareckiĭ, Abstract characterization of the semigroup of all binary relations, *Proc. Leningrad Pedagogical Inst.*, 183 (1958), 251–263 (in Russian).

K.S.S. Nambooripad

Department of Mathematics, University of Kerala, Kariavattom 695581, India.

F.J.C.M. Pastijn

Dienst Hogere Meetkunde, Rijksuniversiteit te Gent, Krijgslaan 281, B-9000 Gent, Belgium.