INDUCTIVE GROUPOIDS

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CONTENTS

INTRODUCTION

By an *invariant* of a semigroup S, we mean an object $A(S)$ (such as a group, a partially ordered set or some other mathematical structure) associated with S in some natural way such that the assignment $S \mapsto A(S)$ is functorial. For example, the partially ordered set Λ_S of principal left ideals, the partially ordered set I_S of principal right ideals and the set $E(S)$ of idempotents are invariants of the semigroup S. Many existing structure theorems for semigroups give an explicit procedure for contructing the required semigroup from a given set of its invariants. The semigroup so constructed depends not only on the invariants themselves, but also on certain relations between them. The construction of the maximum fundamental regular semigroup $U = U(I, \Lambda; \Gamma, \Delta)$ from the invariants $I = I_U$ and $\Lambda = \Lambda_U$ is an example of such a result (see [6, 7, 8]). Here I and Λ are regular partially ordered sets and the relation between them is specified by the *cross-connection* Γ, Λ.

In this article, we wish to present another analysis of the structure of regular semigroups using *inductive groupoids* which can be described as a combination of a groupoid representing the local structure of the semigroup and a *biordered set* that determine the global structure. In this paper by a *groupoid* we mean a small category in which every morphism is an isomorphism. More information about groupoids and some of the natural examples of groupoids can be found in [9, 12, 14].A brief description of biordered sets is given in the second section of this paper. More details can be found in [14].

It was Schein [21] who introduced inductive groupoids to determine the structure of inverse semigroups. The first section discusses the work of Schein on inductive groupoids of inverse semigroups. The second section discuss the generalization of Schein's definition of inductive groupoids of inverse semigroups to inductive groupoids of regular semigroups [14, 15]. The last section discuss some applications of inductive groupoids to congruences on regular semigroups, extensions, etc. Also some concrete examples of inductive groupoids are given.

The notation and terminology used here regarding semigroup theory follow [2]. detailed descriptions of the concepts mentioned can be found in [2] or [10]. Notation and terminology regarding category theory are as in [14]. Details can be had from [9] and [15] also.

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1. INVERSE SEMIGROUPS

A semigroup is defined as a set with an associative binary operation and as such is a generalization of a group. It is difficult, if not impossible, to have a reasonable theory of so general a structure and hence to develop a satisfactory structure theory, we must restrict ourselves to semigroups satisfying some additional conditions. With this in mind, we first consider the class of inverse semigroups, which is closest to the class of groups. For notation and terminology we follow [2] and for groupoids, we follow [9, 14].

By definition, a semigroup S is said to be an inverse semigroup, if for each x in S, there exists a unique element x^{-1} in S such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$ (see [2]). We first note that just as the set of all bijections of a set onto itself is a "universal" model of a group, the set of all 'partial injections' of a set is such a model of an inverse semigroup. Let us briefly consider the details of this assertion.

By a partial injection (or a partial symmetry) of a set X, we mean an injection from a subset of X to X. Thus a partial injection α of X is a bijection $\alpha : Y \to Y'$ of a subset Y of X onto a subset Y' of X. In the following, we denote the domain of α by $\operatorname{dom}(\alpha)$ and the range of α by $\operatorname{ran}(\alpha)$. Also we denote the set of all partial injections of X by $\mathscr{I}(X)$. Note in particular that $\emptyset$ is a subset of X and consequently, the empty relation $\emptyset$ is a member of $\mathscr{I}(X)$.

Now for α and β in $\mathscr{I}(X)$, we define $\alpha \circ \beta$ by $x(\alpha \circ \beta) = y$ iff there exists $z \in X$ such that $x\alpha = z$ and $z\beta = y$. It is not difficult to see that this is equivalent to the condition $y = (x\alpha)\beta$ where $x \in (\operatorname{ran}(\alpha) \bigcap \operatorname{dom}(\beta))\alpha^{-1}$. Thus

$$\operatorname{dom}(\alpha \circ \beta) = \big(D_{\alpha\beta}\big)\alpha^{-1} \quad \operatorname{ran}(\alpha \circ \beta) = \big(D_{\alpha\beta}\big)\beta \tag{1.1}$$

where

$$D_{\alpha\beta} = \operatorname{ran}(\alpha) \bigcap \operatorname{dom}(\beta). \tag{1.2}$$

It is not difficult to show that $\mathscr{I}(X)$ with this binary operation is an inverse semigroup.

Note that the binary operation $\circ$ defined on $\mathscr{I}(X)$ is just the usual composition of relations, if partial injections are regarded as binary relations on the set X. Note also that the usual composition of α and β as functions (defined as $\big(\alpha \circ \beta\big)(x) = \alpha\big(\beta(x)\big)$ is the left-right dual of the relational composition $\alpha \circ \beta$ defined above. We use the relational composition As in [14], unless stated otherwise.

Conversely, if S is an inverse semigroup, then for each $a \in S$, the map $\alpha_a \colon Saa^{-1} \to Sa^{-1}a$ defined by $x\alpha_a = xa$ is a partial injection of S and it can be shown that the map $a \mapsto \alpha_a$ is a monomorphism of S to $\mathscr{I}(S)$. Thus we have the following analogue of Cayley's theorem [2] for groups.

THEOREM 1.1 (Vagner–Preston Representation Theorem). *Any inverse semigroup S is isomorphic to a subsemigroup of $\mathscr{I}(X)$ for a suitable set X.*

Recall that a *groupoid* is a small category in which every morphism is an isomorphism (see [14] or [9] for details). As in [14], when we say that G is a groupoid, we mean that G is the morphism set of a groupoid and that the set vG of vertices (objects) of the groupoid is identified with the set of identities in G. Thus each x in G has a unique inverse x^{-1} in G such that $e_x = xx^{-1}$ and $f_x = x^{-1}x$ are identities in G corresponding to the domain and codomoain of x.

Now since the elements of $\mathscr{I}(X)$ are partial maps of subsets of X, it naturally has the structure of (the morphism set of) a category with (partial) composition defined by

$$\alpha * \beta = \begin{cases} \alpha \circ \beta & \text{if } \operatorname{ran}(\alpha) = \operatorname{dom}(\beta); \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{1.3}$$

Moreover, since each element of $\mathscr{I}(X)$ is a bijection onto its range, this category is in fact a groupoid. We shall denote this groupoid by $\mathscr{I}_*(X)$.

We next see how this groupoid structure of $\mathscr{I}_*(X)$ can be described in semigroup theoretic terms. For this, first note that the product $\alpha * \beta$ is defined if and only if $\alpha^{-1}\alpha = \beta\beta^{-1}$. Since $\alpha^{-1}\alpha$ is the unique idempotent in the $\mathscr{L}$ class of α and $\beta\beta^{-1}$ is the unique idempotent in the $\mathscr{R}$ class of β(see [2]), the above condition holds if and only if the $\mathscr{H}$ class $L_\alpha \bigcap R_\beta$ contains an idempotent and by Clifford–Miller theorem (see [2], page 59), this is equivalent to the condition that the product $\alpha \circ \beta$ in the semigroup $\mathscr{I}(X)$ is in the $\mathscr{H}$ class $R_\alpha \bigcap L_\beta$. Now in any semigroup, and in any inverse semigroup in particular, we can define the trace product as the partial product

$$x * y = \begin{cases} xy & \text{if } xy \in R_x \bigcap L_y \\ \text{undefined} & \text{otherwise} \end{cases} \tag{1.4}$$

If S is a semigroup, then the partial algebra on S with the trace product is called the trace of S and is denoted by S_*. It follows that the product $\alpha * \beta$ in the groupoid $\mathscr{I}_*(X)$ is just the trace product of α and β in the semigroup $\mathscr{I}(X)$. More generally, it is not difficult to show that the trace any inverse semigroup is a groupoid. In fact, in view of the discussions above it is hardly surprising that we have the following

PROPOSITION 1.2. *Let S be an inverse semigroup. Then the trace S_* of S is a groupoid. Moreover, if $\theta\colon S \to \mathscr{I}(X)$ is the Vagner–Preston representation of S, then θ is also an embedding of the groupoid S_* in the groupoid $\mathscr{I}_*(X)$.*

We next see how the product $\circ$ in the semigroup $\mathscr{I}(X)$ can be recovered from the partial product $*$ in the category $\mathscr{I}_*(X)$. Since we are using the relational composition, it is natural to write the restriction of α to a subset D of $\operatorname{dom}(\alpha)$ by $D|\alpha$; thus $D|\alpha$ is the element of $\mathscr{I}_*(X)$ with

$$\operatorname{dom}(D|\alpha) = D \quad \text{and} \quad y(D|\alpha) = y\alpha \quad \text{for all } y \in D. \tag{1.5a}$$

For $E \subseteq \operatorname{ran}(\alpha)$ we define the range-restriction of α to E, denoted $\alpha|E$, by

$$\alpha|E = \left(E|\alpha^{-1}\right)^{-1} \tag{1.5b}$$

Note that for any α and β in $\mathscr{I}(X)$, we can write

$$\alpha \circ \beta = \alpha_1 * \beta_1 \quad \text{where} \quad \alpha_1 = \alpha|D_{\alpha\beta} \quad \text{and} \quad \beta_1 = D_{\alpha\beta}|\beta \tag{1.6}$$

using the notations introduced in Equations (1.1), (1.5a) and (1.5b). The equation above shows that the the product of any α and β in the semigroup $\mathscr{I}(X)$ is equal to the composite of α_1 and β_1 in the groupoid $\mathscr{I}_*(X)$ where α_1 is the range-restriction of α to the subset $D_{\alpha\beta} \subseteq \operatorname{ran}(\alpha)$ and β_1 is the (domain) restriction of β to $D_{\alpha\beta} \subseteq \operatorname{dom}(\beta)$.

To describe this equation in terms of category theory, we first note that the operation of restricting a map to a subset of its domain induces a partial order on the groupoid $\mathscr{I}_*(X)$ defined by

$$\alpha \leq \beta \iff \operatorname{dom}(\alpha) \subseteq \operatorname{dom}(\beta) \text{ and } \alpha = \operatorname{dom}(\alpha)|\beta. \tag{1.7}$$

In view of Equation (1.5b), the range-restriction also defines the same relation on $\mathscr{I}_*(X)$. Note that if subsets of X are identified with the corresponding identity maps in $\mathscr{I}_*(X)$, then the inclusion of subsets of X is just the restriction of this partial order to the collection of identities in the groupoid $\mathscr{I}_*(X)$. Moreover, this partial order satisfies the following conditions:

(1) If $\alpha \leq \beta$ and $\lambda \leq \mu$ and if $\alpha * \lambda$ and $\beta * \mu$ are defined, then $\alpha * \lambda \leq \beta * \mu$
(2) If $\alpha \leq \beta$, then $\alpha^{-1} \leq \beta^{-1}$

Thus Equation (1.6) shows that product in the semigroup $\mathscr{I}(X)$ can be recovered from the composition of the groupoid $\mathscr{I}_*(X)$ and the restriction order on $\mathscr{I}_*(X)$. This motivates the following definition.

DEFINITION 1.1. Let G be a groupoid and $\leq$ be a partial order on the set G. Then G is said to be an ordered groupoid with respect to $\leq$ if the following conditions are satisfied.

(a) If $x \leq u$ and $y \leq v$ and if xy and uv are defined in G, then $xy \leq uv$

(b) If $x \leq y$, then $x^{-1} \leq y^{-1}$

(c) For each $x \in G$ and for each identity e in G with $e \leq e_x$, there exists a unique element $e|x$ such that $e|x \leq x$ and $(e|x)(e|x)^{-1} = e$.

For $e \leq e_x$, the unique element $e|x$ is called *restriction* of x to e. For $f \leq f_x$, the *corestriction* of x to f is defined by:

$$x|f = \left(f|x^{-1}\right)^{-1} \tag{1.8}$$

It is clear that axioms (a) and (b) are (left-right) self-dual. The dual of (c) can be stated as follows:

(c)* For each $x \in G$ and for each identity f in G with $f \leq f_x$, there exists a unique element $f|x$ such that $f|x \leq x$ and $(f|x)^{-1}(f|x) = f$.

It can be shown that for any partial order on a groupoid satisfying (a) and (b), statements (c) and (c)* are equivalent.

A morphism $\phi : G \to H$ of an ordered groupoids is a functor which is also order preserving.Such a morphism preserves restriction and corestriction; that is, for each $x \in G$, $e \leq e_x$ and $f \leq f_x$, we have

$$\phi(e|x) = \phi(e)|\phi(x), \qquad \phi(x|f) = \phi(x)|\phi(f).$$

In particular, the restriction $\mathsf{v}\phi$ of ϕ to $\mathsf{v}G$ is an order preserving map to $\mathsf{v}H$. The morphism $\phi : G \to H$ is an embedding of ordered groupoids if ϕ is a faithful functor which is an order isomorphism onto its range and ϕ is an isomorphism if ϕ is an embedding whose morphism map is surjective (see [14]). We thus have a category $\mathfrak{OG}$ with objects as ordered groupoids and morphisms defined above.

Ordered groupoids are interesting objects on their own right. Any groupoid G is an ordered groupoid with respect to the trivial partial order (identity relation on G). More generally, ordered groupoids arise as the set of all partial symmetries of structures having appropriate concept of subobjects and isomorphisms. For example, isomorphisms of subgroups of a group and analytic isomorphisms of regions of the complex plane are examples of ordered groupoids.

It follows from our discussion above that $\mathscr{I}_*(X)$ is an ordered groupoid with respect to restriction. The ordered groupoid $\mathscr{I}_*(X)$ has a "universal" property similar to the property of the inverse semigroup $\mathscr{I}(X)$ given by Theorem 1.1. It can be shown that given any ordered groupoid G, there is an embedding $\theta : G \to \mathscr{I}_*(G)$ of ordered groupoids.

Remark* 1.1*: The definition of an ordered groupoid G can also be formulated in terms of the primitive operation *restriction* instead of the partial order on G. In this approach, the definition is based on a groupoid G, a partially ordered set E and an identification of E as the set $\mathsf{v}G$ (a bijection $\phi : E \to \mathsf{v}G$). The relation between E and G can be stated in terms of restrictions. Also, the partial order on G can be defined in terms of the restriction by

$$x \leq y \iff e_x \leq e_y, \quad \text{and} \quad x = e_x|y.$$

Morphisms between ordered groupoid can then be defined as functors that preserve restrictions.

We have noted that the ordered groupoid $\mathscr{I}_*(X)$ is the trace of the inverse semigroup $\mathscr{I}(X)$. However, not all ordered groupoids arise as trace of inverse semigroups; for example, the ordered groupoid of analytic isomorphisms of regions of the complex plane is not the trace of any inverse semigroup. To characterize those ordered groupoids that arises this way, we again look at the structure of the universal ordered groupoid $\mathscr{I}_*(X)$. Note that the set of identity maps in $\mathscr{I}_*(X)$ with this partial order is isomorphic to the poset of subsets of X under set inclusion and so is a semilattice. Now Schein defines an *inductive groupoid* as an ordered groupoid in which the identities form a semilattice [21]. Since the definition of an inductive groupoid as given by Nambooripad in his study on regular semigroups [14] is more general, we will refer to an inductive groupoid as defined by Schein as a *Schein groupoid*.

DEFINITION 1.2. An ordered groupoid G in which the set of identities is a semilatiice with respect to the order induced by G is called a Schein groupoid. A morphism $\phi : G \to H$ of Schein groupoids is a morphism of ordered groupoids such that $\mathsf{v}\phi$: $\mathsf{v}G \to \mathsf{v}H$ is a homomorphism of semilattices.

Thus the category $\mathscr{I}_*(X)$ is a Schein groupoid. Now for each α in $\mathscr{I}_*(X)$, let us denote the identity maps on the domain and range of α by ϵ_α and ϕ_α respectively. Also, for identities ϵ_1 and ϵ_2 in G, we denote the identity map on $\operatorname{dom}(\epsilon_1) \bigcap \operatorname{dom}(\epsilon_2)$, which is in fact the meet of ϵ_1 and ϵ_2, by $\epsilon_1 \wedge \epsilon_2$. By the definition of relational composition, we have

$$\epsilon_1 \wedge \epsilon_2 = 1_{\operatorname{dom}(\epsilon_1) \bigcap \operatorname{dom}(\epsilon_2)} = \epsilon_1 \circ \epsilon_2 = \epsilon_2 \circ \epsilon_1. \tag{1.9}$$

Again for each α in $\mathscr{I}_*(X)$ and an identity ϵ with $\epsilon \leq \epsilon_\alpha$, we denote the restriction of α to $\operatorname{dom}(\epsilon)$ by $\epsilon|\alpha$. Further, for an identity ϕ in $\mathscr{I}_*(X)$ with $\phi \leq \phi_\alpha$, the corestriction of α to ϕ is defined by

$$\alpha|\phi = \big(\phi|\alpha^{-1}\big)^{-1}.$$

Hence $\alpha|\phi = \alpha|\operatorname{dom}(\phi)$ by Equation (1.5b). Thus corestriction is the same as range-restriction. Using these notations and Equation (1.9), we can rewrite Equation (1.6) as

$$\alpha \circ \beta = \big(\alpha|(\phi_\alpha \wedge \epsilon_\beta)\big) * \big((\phi_\alpha \wedge \epsilon_\beta)|\beta\big) = \big(\alpha \circ (\phi_\alpha \circ \epsilon_\beta)\big) * \big((\phi_\alpha \circ \epsilon_\beta) \circ \beta\big) \tag{1.10}$$

In other words, for α and β in $\mathscr{I}(X)$, if we define

$$\psi = \phi_\alpha \circ \epsilon_\beta = \phi_\alpha \wedge \epsilon_\beta$$

then we have

$$\alpha \circ \beta = (\alpha \circ \psi) * (\psi \circ \beta) = \big(\alpha|\psi\big) * \big(\psi|\beta\big) \tag{1.11}$$

Note also that $\phi_a l = \alpha^{-1}\alpha$ and $\epsilon_\beta = \beta\beta^{-1}$.

Next we see how these ideas can be generalized to an arbitrary inverse semigroup. First note that for α and β in $\mathscr{I}_*(X)$, we have $\alpha \leq \beta$ if and only if $\alpha = \alpha\alpha^{-1}\beta$ in $\mathscr{I}(X)$. Now in any inverse semigroup S, the relation $\leq$ defined by $x \leq y$ if and only if $x = xx^{-1}y$, is a partial order with the property that the set $\mathsf{E}(S)$ of idempotents of S is a semilattice in which the meet $e \wedge f$ of two idempotents is the product $ef = fe$. It is called the natural partial order on S. We have also observed that the trace S_* of S is a groupoid. The following result describes the relation between inverse semigroups and inductive groupoids.

THEOREM 1.3 (Schein [21]). *Let S be an inverse semigroup. Then the trace S_* of S is a Schein groupoid with respect to the trace product $*$ and the natural partial order $\leq$. Also, for x and y in S,*

$$xy = (xh) * (hy)$$

where

$$h = (x^{-1}x)(yy^{-1})$$

Conversely, if $(G, *, \leq)$ is a Schein groupoid, then G is an inverse semigroup with respect to the product defined by

$$xy = (x|h) * (h|y)$$

where

$$h = x^{-1}x \wedge yy^{-1}$$

Now for an inverse semigroup S, we denote the Schein groupoid arising from the trace product and the natural partial order by $\boldsymbol{G}(S)$. If $h : S \to S'$ is a homomorphism of inverse semigroups, then h preserves trace products, natural partial order and the semilattice product on its set of idempotents. Hence h is also a morphism of Schein groupoids; we denote this morphism by $\boldsymbol{G}(h)$. Again, we denote by $\boldsymbol{S}(G)$, the inverse semigroup determined by the Schein groupoid G. Definition 1.2 and the definition of products in $\boldsymbol{S}(G)$, it follows that any morphism $\phi : G \to H$ of Schein groupoids induces a unique homomorphism $\boldsymbol{S}(\phi)$ of $\boldsymbol{S}(G)$ to $\boldsymbol{S}(H)$. Thus we have

THEOREM 1.4. *For each inverse semigrouo S let $\boldsymbol{G}(S)$ denote the Schein groupoid defined in Theorem 1.3 and for each homomorphism $h\colon S \to S'$ of inverse semigroups, let $\boldsymbol{G}(h)\colon \boldsymbol{G}(S) \to \boldsymbol{G}(S')$ be the morphism of Schein groupoids as above. Then the assignments*

$$\boldsymbol{G} : S \mapsto \boldsymbol{G}(S), \qquad h \mapsto \boldsymbol{G}(h) \tag{1.12a}$$

is a functor $\boldsymbol{G}\colon \mathfrak{IS} \to \mathfrak{SG}$ of the category $\mathfrak{IS}$ of inverse semigroups to the category $\mathfrak{SG}$ of Schein groupoids. On the other hand, for a Schein groupoid G, let $\boldsymbol{S}(G)$ denote the inverse semigroup defined in Theorem 1.3 above and for each morphism $\phi : G \to G'$ of Schein groupoid let $\boldsymbol{S}(\phi)\colon \boldsymbol{S}(G) \to \boldsymbol{S}(G')$ denote the honomorphism of the inverse semigroups determined by ϕ. Then the assignments

$$\boldsymbol{S} : G \mapsto \boldsymbol{S}(G), \qquad \phi \mapsto \boldsymbol{S}(\phi) \tag{1.12b}$$

is a functor $\boldsymbol{S} : \mathfrak{SG} \to \mathfrak{IS}$.

It can be shown that for any inverse semigroup S and any homomorphism $h : S \to S'$ of inverse semigroups, we have

$$\boldsymbol{S}\big(\boldsymbol{G}(S)\big) = S, \qquad \boldsymbol{S}\big(\boldsymbol{G}(h)\big) = h. \tag{1.12c}$$

Similarly, for any Schein groupoid G and any morphism $\phi : G \to H$ of Schein groupoids, we have

$$\boldsymbol{G}\big(\boldsymbol{S}(G)\big) = G, \qquad \boldsymbol{G}\big(\boldsymbol{S}(\phi)\big) = \phi. \tag{1.12d}$$

In fact we have following relation between the category $\mathfrak{IS}$ of inverse semigroups and the category $\mathfrak{SG}$ of Schein groupoids.

THEOREM 1.5. *The functor $\boldsymbol{G} : \mathfrak{IS} \to \mathfrak{SG}$ defined by assignments of Equation* (1.12a) *is an isomorphism of categories such that*

$$\boldsymbol{G}^{-1} = \boldsymbol{S} : \mathfrak{SG} \to \mathfrak{IS}$$

is the functor defined by the assignments in Equation (1.12b)*.*

The discussion above, in particular, Theorem 1.3 shows that, as mathematical structures, inverse semigroups and Schein groupoids are equivalent. The representation of an inverse semigroup as its Schein groupoid is quite useful in analyzing its structure. We mention one example to illustrate the fact.

Let E be a semilattice and let $T^*(E)$ denote the set of all isomorphisms of principal ideals of E. Then $T^*(E))$ is easily seen to be an ordered subgroupoid of $\mathscr{I}_*(E)$ and by definition, the identitities of $T^*(E)$ are the identity maps on principal ideals of E. For $e \in E$, if we denote by $\omega(e)$, the principal ideal generated by e, then $e \mapsto \omega(e)$ is an order isomorphism of E onto the set of identities $T^*(E)$. It follows that $T^*(E)$ is a Schein groupoid. We denote the inverse semigroup $\boldsymbol{S}(T^*(E))$ by $T(E)$.

Now an inverse semigroup S is said to be fundamental, if every non trivial congruence on S identifies at least one pair of distinct idempotents. Since a congruence on S is idempotent separating iff it is contained in the Green's relation $\mathscr{H}$, it follows that S is fundamental iff the relation $\mathscr{H}$ on S contains no non trivial congruences. It is not difficult to show that the semigroup $T(E)$ constructed above is fundamental. To see this, let ρ be a congruence in $T(E)$ with $\rho \subseteq \mathscr{H}$ and let $\alpha \;\rho\; \beta$ in $T(E)$. Then $\alpha \;\mathscr{H}\; \beta$ from which it follows that $\mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$. If we write $\mathrm{dom}(\alpha) = \mathrm{dom}(\beta) = \omega(e)$ and $\epsilon = 1_{\omega(e)}$, then we have $\epsilon = \alpha\alpha^{-1} \;\rho\; \beta\alpha^{-1}$. Let $\psi = \beta\alpha^{-1}$. Then $\psi \;\mathscr{H}\; \epsilon$, since $\rho \subseteq \mathscr{H}$, and so $\mathrm{dom}(\psi) = \mathrm{dom}(\epsilon) = \omega(e)$. Also, for $f \in \omega(e)$, if we write $\phi = 1_{\omega(f)}$, then $\phi\epsilon = \phi$, since $f \leq e$. Hence $\phi\psi \;\rho\; \phi\epsilon = \phi$, from which we have $\omega(f\psi) = \mathrm{ran}(\phi\psi) = \mathrm{ran}(\phi) = \omega(f)$ and so $f\psi = f$. It now follows that $\beta\alpha^{-1} = \psi = \epsilon$ so that $\alpha = \beta$.

Now in any inverse semigroup S and for each $x \in S$, the map $e \mapsto x^{-1}ex$ is an isomorphism of $\omega(e_x)$ onto $\omega(f_x)$. If we denote this map by $\mathfrak{a}_S(x)$, then $\mathfrak{a}_S(x) \in T(E)$ for each $x \in S$ and the map $\mathfrak{a}_S \colon x \mapsto \mathfrak{a}_S(x)$ is in fact a homomorphism of S into $T(E)$. Now idempotents in $T(E)$ (identites in $T^*(E)$) are of the form $1_{\omega(e)} = \mathfrak{a}_S(e)$ for some $e \in \mathsf{E}(S)$. Hence $\mathsf{E}(T(E)) \subseteq \mathrm{im}\,\mathfrak{a}_S = \mathfrak{a}_S(S)$ and so $\mathrm{im}\,\mathfrak{a}_S$ is a full subsemigroup of $T(E)$. (We say that S is a *full subsemigroup* of T if S is a subsemigroup of T with $\mathsf{E}(T) = \mathsf{E}(S)$). It follows from the definition of $\mathfrak{a}_S$ that if $\mathfrak{a}_S(x) = \mathfrak{a}_S(y)$ for $x, y \in S$, then $e_x = e_y$ and $f_x = f_y$ and so, $x \;\mathscr{H}\; y$. Thus the kernel of the homomorphism $\mathfrak{a}_S$ is contained in $\mathscr{H}$. In other words, if we denote the kernel of $\mathfrak{a}_S$ by μ_S, then we have

$$\mu_S = \{(x, y) \in S \times S : \mathfrak{a}_S(x) = \mathfrak{a}_S(y)\} \subseteq \mathscr{H}.$$

In particular, if S is fundamental, then $\mu_S = 1_S$ and so $\mathfrak{a}_S$ is injective. Thus, as a consequence of Theorem 1.3, we have the following result.

THEOREM 1.6 (Munn [13]). *Let $T(E)$ be the set of all isomorphisms of principal order ideals of a semilattice E. Then $T(E)$ is a fundamental inverse subsemigroup of $\mathscr{I}(E)$ whose semilattice of idempotents is isomorphic to E. Moreover, if S is any fundamental inverse semigroup whose semilattice is isomorphic to E, then S is isomorphic to a full subsemigroup of $T(E)$.*

2. INDUCTIVE GROUPOIDS OF REGULAR SEMIGROUPS

We thus see that the structure theory of inverse semigroups can be reduced to the theory of Schein groupoids. We have also seen that the Schein groupoid of an inverse semigroup is determined by the groupoid S_* (the trace of S) representing the local structure of S and the semilattice $\mathsf{E}(S)$ of idempotents of S with the relation between these structures specified by axioms in Definition 1.1

There are several difficulties in trying to extend the notion of Schein groupoids to regular semigroups. The first thing that we note is that a regular semigroup with the trace product is not a category, since left and right identities of an element cannot be uniquely defined. This is due to the fact that in a regular semigroup, the $\mathscr{L}$-class and the $\mathscr{R}$-class of an element may contain more than one idempotent. One way out is to blow up the semigroup S by considering for each element x of S, all pairs (x, x') where x' is an inverse of x in S. Note that every idempotent of S is uniquely determined by one such pair. Also, the trace product of x and y in S is defined if and only if there exist inverses x' and y' of x and y such that $x'x = yy'$. We thus have the following result [14].

PROPOSITION 2.1. *Let S be a regular semigroup and define*

$$\boldsymbol{G}(S) = \{(x, x') : x \in S, \quad x' \in \mathscr{V}()(x)\} \tag{2.1}$$

where $\mathscr{V}()(x)$ denote the set of all inverses of x in S. Define a partial binary operation on $\boldsymbol{G}(S)$ by

$$(x, x')(y, y') = \begin{cases} (xy, y'x') & \text{if } x'x = yy'; \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{2.2}$$

Then $\boldsymbol{G}(S)$ is a groupoid whose identities are pairs (e, e) where e is an idempotent of S.

***Remark* 2.1:** The idea of considering the set $\boldsymbol{G}(S)$ of pairs of the form (x, x') with $x' \in \mathscr{V}()(x)$ occures both in [21] and [**?**]. It was noted by both Schein and Rielly-Scheliblish that the relation $\boldsymbol{G}(S)$ is a subsemigroup of $S \times S$ if and only if S is orthodox (that is, if and only if $\boldsymbol{E}(S)$ is a band). However, as far as we are aware, the first systematic study of the structure of $\boldsymbol{G}(S)$ for arbitrary regular semigroups and its use in structure theory appeared in [14].

We call $\boldsymbol{G}(S)$ the groupoid of S. Because of the bijection $e \mapsto (e, e)$, the set of identities of $\boldsymbol{G}(S)$ can be identified with the set $\boldsymbol{E}(S)$ of idempotents of S. Thus $\boldsymbol{G}(S)$ is a groupoid whose set of identities is equal to $\boldsymbol{E}(S)$. In fact, $\boldsymbol{G}(S)$ is an ordered groupoid. To describe the order in $\boldsymbol{G}(S)$, first note that as in the case of an inverse semigroup, the set $\boldsymbol{E}(S)$ is a poset with order defined by $e \le f$ if and only if $ef = fe = e$. In the case of regular semigroups, we denote this order by ω, following the notation in [14]. Also, in the case of an inverse semigroup S, for $x \in S$ and $e \in \boldsymbol{E}(S)$ with $e \le xx^{-1}$, the restriction of x to e in the Schein groupoid of S is defined by $e|x = ex$. Analogously, we can define for (x, x') in the groupoid $\boldsymbol{G}(S)$ of a regular semigroup S and $e \; \omega \; xx'$, the restriction of (x, x') to e by

$$e|(x, x') = (ex, ex') \tag{2.3}$$

The partial order on $\boldsymbol{G}(S)$ induced by this operation is given by

$$(x, x') \le (y, y') \iff x = (xx')y, \; x' = y'(xx') \text{ and } xx' \; \omega \; yy'. \tag{2.4}$$

We then have the following

PROPOSITION 2.2. *Let S be a regular semigroup and let $\boldsymbol{G}(S)$ be the groupoid of S. Then $\boldsymbol{G}(S)$ is an ordered groupoid with composition defined by Equation* (2.1) *and order defined by Equation* (2.4)*.*

When we try to recover the semigroup products from the groupoid products as in the case of inverse semigroups, we encounter several difficulties. First note that in the case of a regular semigroup S, the underlying set of the groupoid $G = \boldsymbol{G}(S)$ is not S itself, as in the case of an inverse semigroup. However, this is easily remedied by taking the quotient of G by the equivalence relation which identifies all elements of G with the same first coordinate. Next recall that in the case of an inverse semigroup, the semigroup product xy is equal to the groupoid product $(xh) * (hy)$, where $h = (x^{-1}x)(yy^{-1}) = x^{-1}x \wedge yy^{-1}$; but in the case of a regular semigroup, the set of idempotents is not a lattice. This difficulty is overcome in [14], by considering for two idempotents e and f in a regular semigroup S, their *sandwich set* defined by

$$\mathsf{S}(e, f) = \{\, fxe : x \text{ is an inverse of } ef \,\}. \tag{2.5}$$

Note that if S is an inverse semigroup, then $\mathsf{S}(e, f) = \{ef\}$ for any pair of idempotents in S. In the case of a regular semigroup S, it can be easily seen that $\mathsf{S}(e, f) \subseteq \boldsymbol{E}(S)$. Moreover, for x and y in S and any of their inverses x' and y', it can be shown that if $h \in \mathsf{S}(x'x, yy')$, then

$$xy = (xh) * (hy)$$

and also that $y'hx'$ is an inverse of xy with

$$y'hx' = (y'h) * (hx')$$

We thus have the following result.

THEOREM 2.3. *Let S be a regular semigroup and $G = \boldsymbol{G}(S)$. Define the relation p on G by*

$$(x, x') \, p \, (y, y') \iff x = y. \tag{2.6}$$

Then p is an equivalence relation on G. and if the equivalence class containing (x, x') is denoted by $\overline{(x, x')}$, then the quotient set G/p with product defined by

$$\overline{(x, x')}\,\overline{(y, y')} = \overline{(xh, hx')(hy, hy')} \tag{2.7}$$

where $h \in \mathsf{S}(x'x, yy')$, is a semigroup isomorphic with S.

We next see how the groupoid $\boldsymbol{G}(S)$ can be abstractly characterized and the above construction of a regular semigroup can be effected in such a general setting. For this, we will have to first give an abstract characterization of the algebraic structure of the set of idempotents of a regular semigroup. (Note that in the case of an inverse semigroup S, the algebraic structure of $E = \mathsf{E}(S)$ is completely determined by its order structure, since $ef = e \wedge f$ for any e and f in E). Also, the relation between this structure of $E = \mathsf{E}(S)$ and the structure of the groupoid $\boldsymbol{G}(S)$ must be made explicit. For example, the product of two idempotents in a regular semigroup may not be an idempotent (which is not the case in an inverse semigroup) and such products cannot be described in terms of the products in the groupoid, since no two identities in a category are composable.

An abstract characterization of the set of idempotents of a regular semigroup, as a partial algebra satisfying certain axioms, is given by one of the authors [14], where such a structure is called a *regular biordered set*. More generally, Easdown [5] has shown that a partial algebra E satisfying axioms of Definition 1.1 of [14] (that is, a biordered set) if and only if E can be embedded as the set $\mathsf{E}(S)$ of all idempotents of a semigroup S and it is shown in [14] that $\mathsf{E}(S)$ is a regular biordered set if and only if S is a regular semigroup. We may therefore assume with out loss of generality that every [regular] biordered set E is $\mathsf{E}(S)$ for a suitable [regulsr] semigroup S. Note that in the set $\mathsf{E}(S)$ of idempotents of a regular semigroup S, the partial order ω is equal to

$$\omega = \omega^l \bigcap \omega^r \tag{2.8}$$

where ω^l and ω^r are defined by

$$\omega^l = \{(e, f) \in E \times E \colon ef = e\} \quad \text{and} \quad \omega^r = \{(e, f) \in E \times E \colon fe = e\} \tag{2.9}$$

These relations can easily seen to be quasiorders on $\mathsf{E}(S)$. Also, the Green's relations $\mathscr{L}$ and $\mathscr{R}$ in $\mathsf{E}(S)$ are related to these orders by

$$\mathscr{L} = \omega^l \bigcap (\omega^l)^{-1} \quad \text{and} \quad \mathscr{R} = \omega^r \bigcap (\omega^r)^{-1} \tag{2.10}$$

Further, for any relation R on E and $e \in E$, we write $R(e)$ for $\{f \in E : (f, e) \in R\} \subseteq E$. In particular, we write

$$\omega^r(e) = \{f : f \ \omega^r \ e\}, \quad \omega^l(e) = \{f : f \ \omega^l \ e\}, \quad \text{and} \quad \omega(e) = \{f : f \ \omega \ e\}. \tag{2.11}$$

These subsets are called ω^r, ω^l and ω ideals respectively. Note that these are regular biordered subsets of E (cf. [14]).

One of the axioms for a biordered set E is that the relations ωl and ωr in E defined by Equation (2.9) are quasiorders and that the product of two elements of E is defined in the biordered set E if and only if they are comparable under one of these relations. Moreover, in a biordered set we can define the partial order $\omega$by Equation (2.8) and the equivalence relations $\mathscr{L}$and $\mathscr{R}$by Equation (2.10). Moreover the sandwich set of two idempotents can be described in terms of the partial product or equivalently, in terms of the quasiorders ω^l and ω^r in the biordered set [14].

Next we see how we can describe in the language of groupoids, the fact that certain elements of S are products of idempotents. For this, we make use of the fact that any product of idempotents in S is also equal to the product of a finite sequence of idempotents in which successive terms are $\mathscr{L}$ or $\mathscr{R}$ related. Also, the length of any such sequence can be "minimized" without affecting the product. These ideas can be precisely formulated as follows.

Let S be a regular semigroup and let $E = \mathsf{E}(S)$ be the set of idempotents of S. We define an *E-sequence* as a finite sequence $\sigma = (e_1, e_2, \ldots, e_n)$ of elements of E such that $e_i(\mathscr{L} \bigcup \mathscr{R})e_{i+1}$ for $i = 1, 2, \ldots, n-1$. Two E-sequences of the form (e, f) and (g, h) are similar if these pairs are related by the same relation ($\mathscr{L}$ or $\mathscr{R}$). In the E-sequence $s = (e_1, e_2, \ldots, e_n)$, elements e_i are called vertices and subsequences (e_{i-1}, e_i), $i = 2, 3, \ldots, n$ are called edges of

s. In an E-sequence $(e_1, e_2, \ldots, e_n)$, a vertex e_i is said to be inessential if edges (e_{i-1}, e_i) and (e_i, e_{i+1}) are similar, (that is, $e_{i-1} \mathscr{L} e_i \mathscr{L} e_{i+1}$ or $e_{i-1} \mathscr{R} e_i \mathscr{R} e_{i+1}$). Given an E-sequence s, the product $e_1 e_2 \ldots e_n$ of vertices of s in S is called the *the product* of s. Notice that we can introduce or remove an inessential vertex without affecting the product of an E-sequence. An introduction [or removal] of an inessential vertex into [or from] an E-sequence σ is called an elementary reduction of σ. Define the relation $\sim$ on the set of all finite E-sequences in E by $\sigma \sim \sigma'$ if σ' is obtained from σ by a finite sequence of elementary reductions. Then it is clear that $\sim$ is an equivalence relation on the set of all E-sequences of E. The equivalence classes of E-sequences are called *E-chains*. The E-chain determined by the E-sequence $\sigma = (e_1, e_2, \ldots, e_n)$ is denoted by $c(e_1, e_2, \ldots, e_n)$. Also, every E-sequence is $\sim$-equivalent to a unique *reduced* E-sequence having no inessential vertex and we may assume that $c(e_1, \ldots, e_n)$ represents the reduced E-sequence determined by the E-sequence $\sigma = (e_1, \ldots, e_n)$. Clearly edges (e_{i-1}, e_i), $i = 2, \ldots, n$ of σ are reduced. Hence the edges are E-chains. An E-cycle in E is an E-chain c with $e_c = f_c$. An E-cycle with four edges (or four distinct vertices) is called an E-square. If $e, f \in \omega^r(g)$ and $e \mathscr{L} f$, then it follows from biorder axioms that

$$e \mathscr{L} f \mathscr{R} fg \mathscr{L} eg \mathscr{R} e \quad \text{and so,} \quad c(e, f, fg, eg, e)$$

is an E-square. Similarly if $e, f \in \omega^l(g)$ and $e \mathscr{R} f$, then $c(e, f, gf, ge, e)$ is an E-square. E-squares formed in this way are called *singular* E-squares.

It can be shown that any product of idempotents in S is the product of elements in an E-chain (see [14], Theorem 1.2). Note also that if $x = e_1 e_2 \cdots e_m$ and $y = f_1 f_2 \cdots f_n$ in S, where $c_1 = c(e_1, e_2, \ldots, e_m)$ and $c_2 = c(f_1, f_2, \ldots, f_n)$ are E-chains with $e_m = f_1$, then $xy = e_1 g_2 \cdots f_n$, where $(e_1, g_2, \ldots, f_n)$ is the reduced E-chain corresponding to the juxtaposed E-chain $(e_1, e_2, \ldots, e_m, f_1, f_2, \ldots, f_n)$. In view of this, it is natural to define a product in the set $\mathfrak{C}(E)$ of all E-chains on an arbitrary biordered set E as follows: for $c_1 = c(e_1, \ldots, e_m), c_2 = c(f_1, \ldots, f_n) \in \mathfrak{C}(E)$ let

$$c_1 c_2 = \begin{cases} c(e_1, \ldots, e_m, f_1, \ldots, f_n) & \text{if } e_m = f_1; \\ \text{undefined} & \text{otherwise} \end{cases} \tag{2.12}$$

where the right-hand side represents the chain determined by the juxtaposition of sequences $(e_1, \ldots, e_m)$ and $(f_1, \ldots, f_n)$. It is easy to see that $\mathfrak{C}(E)$ with this product is a groupoid.

Also, the identities in $\mathfrak{C}(E)$ are E-chains of the form $c(e, e)$ with $e \in E$ and these can be identified with the elements of E itself.

Again, if $x = e_0 e_1 \cdots e_n$ in S, where $c(e_0, e_1, \ldots, e_n)$ is an E-chain and $e \ \omega \ e_0$, then $ex = ee_1 \cdots e_n$, but $(e, e_1, \ldots, e_n)$ is not an E-chain. However, it is not difficult to see that if we define

$$h_0 = e \ and \ h_i = e_i h_{i-1} e_i \quad \text{for} \quad i - 1, 2, \ldots, n,$$

then $c(h_0, h_1, \ldots, h_n)$ is an E-chain with $ex = h_0 h_1 \cdots h_n$. Moreover, it can be shown that if we define

$$e | c(e_1, \ldots, e_n) = c(h_0, h_1, \ldots, h_n) \tag{2.13}$$

then this defines a restriction operation in $\mathfrak{C}(E)$ under which it is an ordered groupoid.

The *ordered groupoid of E-chains* $\mathfrak{C}(E)$ can also be characterized in the following way. Since $\mathscr{L}$ and $\mathscr{R}$ are equivalence relations on E, we may regard these as groupoids with vertex set E. Morphisms of $\mathscr{L}$ are edges in $\mathfrak{C}(E)$ of the form $c(e, f)$ with $e \mathscr{L} f$ (see [9]). Similarly, $\mathscr{R}$ is a groupoid with $v\mathscr{R} = E$ and morphisms as edges $c(e, f)$ with $e \mathscr{R} f$. These becomes ordered subgroupoids of $\mathfrak{C}(E)$ if we define restrictions in $\mathscr{R}$ by

$$g | (e, f) = (g, gf) \quad \text{for} \quad g \ \omega \ e \mathscr{R} f. \tag{2.14a}$$

and the restriction in $\mathscr{L}$ by

$$g | (e, f) = (g, fg) \quad \text{for} \quad g \ \omega \ e \mathscr{L} f. \tag{2.14b}$$

The above construction of $\mathfrak{C}(E)$ can be given a 'categorical' description. Not that the definition of products of E-chains above shows that

$$c(e_0, e_1, \ldots, e_n) = c(e_0, e_1)c(e_1, e_2)\ldots c(e_{n-1}, e_n).$$

This fact that every E-chain is a finite product of edges implies that $\mathfrak{C}(E)$ is "generated" by $\mathscr{L}$ and $\mathscr{R}$ in the (rather strong) sense that the diagram (D1) below is a push-out in the category $\mathfrak{O}\mathfrak{G}$.

(D1)
$$\begin{tikzcd} 1_E \arrow[r, "j_r"] \arrow[d, "j_l"'] & \mathscr{R} \arrow[d, "\eta_r"] \\ \mathscr{L} \arrow[r, "\eta_l"'] & \mathfrak{C}(E) \end{tikzcd}$$

Here, the various arrows indicate the corresponding embeddings. Thus $\mathfrak{C}(E)$ is the *amalgamated product* of $\mathscr{L}$ and $\mathscr{R}$, amalgamating 1_E.

Since the construction of $\mathfrak{C}(E)$ involves only the vocabulary and syntaxof a biordered set it can be done on any abstract regular biordered set. (The precise details of such construction within category theory can be found in [14]). Recall that we have a category $\mathfrak{R}\mathfrak{B}$ of regular biordered sets [14] with morphisms as regular bimorphisms. The construction of $\mathfrak{C}(E)$ described above constructs the object function of a functor $\mathfrak{C} : \mathfrak{R}\mathfrak{B} \to \mathfrak{O}\mathfrak{G}$. For each regular bimorphism $\theta : E \to E'$ and E-chain $c = c(e_0, \ldots, e_n)$, let

$$c\theta = c(e_0\theta, e_1\theta, \ldots, e_n\theta).$$

It is clear that $c\theta \in \mathfrak{C}(E')$ and so

(2.15) $$\mathfrak{C}(\theta)(c) = c\theta \quad \text{for all} \quad c \in \mathfrak{C}(E).$$

is a well defined map of the morphism set of $\mathfrak{C}(E)$ to the morphism set of $\mathfrak{C}(E')$. From Equation (2.12) and the definition of $c\theta$ above, it is clear that

$$\mathfrak{C}(\theta)(c_1c_2) = \mathfrak{C}(\theta)(c_1)\mathfrak{C}(\theta)(c_2) \quad \text{and} \quad \mathfrak{C}(\theta)\big(c(e,e)\big) = c(e\theta, e\theta).$$

Setting $\boldsymbol{v}\mathfrak{C}(\theta) = \theta$, we have a functor $\mathfrak{C}(\theta)\colon \mathfrak{C}(E) \to \mathfrak{C}(E')$. Since $\theta\colon E \to E'$ preserves biorder products, it follows from Equation (2.13) that the functor $\mathfrak{C}(\theta)$ preserves restrictions and hence $\mathfrak{C}(\theta)$ is a morphism of $\mathfrak{O}\mathfrak{G}$. It is easy to verify that the assignments

(2.16) $$\mathfrak{C}\colon E \mapsto \mathfrak{C}(E), \quad \theta \mapsto \mathfrak{C}(\theta)$$

is a functor $\mathfrak{C}\colon \mathfrak{R}\mathfrak{B} \to \mathfrak{O}\mathfrak{G}$.

Next note that if in a regular semigroup S, $x = e_1 \cdots e_n$, where $c(e_1, e_2, \ldots, e_n)$ is an E-chain in $E = \boldsymbol{E}(S)$, then $x' = e_ne_{n-1}\cdots e_1$ is an inverse of x in S. Hence we have a map $\varepsilon\colon \mathfrak{C}(E) \to \boldsymbol{G}(S)$ defined by

(2.17) $$\varepsilon\big(c(e_1, e_2, \ldots, e_n)\big) = (e_1e_2\cdots e_n, e_ne_{n-1}\cdots e_1)$$

and it is easy to see that ε induces an order isomorphism of the identities of $\mathfrak{C}(E)$ onto the set of identities of $\boldsymbol{G}(S)$. Moreover, it can be shown that $\varepsilon : \mathfrak{C}(E) \to \boldsymbol{G}(S)$ preserves composition and partial order so that it is an order-preserving function which is a $\boldsymbol{v}$–isomorphism.

Thus for any regular semigroup S, the set of idempotents $E = \boldsymbol{E}(S)$ is a regular biordered set and there are two ordered groupoids $\boldsymbol{G}(S)$ and $\mathfrak{C}(E)$ with a morphism $\varepsilon\colon \mathfrak{C}(E) \to \boldsymbol{G}(S)$ of ordered groupoids. Also, the relations between the algebraic structure of E and the groupoid structure of $\boldsymbol{G}(S)$ can be described using the map ε.

For this, we first observe that we are justified in using the category terminology in groupoids. Thus if have morphisms in a groupoid G forming the following diagram,

$$\begin{tikzcd} e \arrow[r, "\alpha"] \arrow[d, "\beta"'] & f \arrow[d, "\gamma"] \\ g \arrow[r, "\delta"'] & h \end{tikzcd}$$

then the diagram is commutative if morphisms $\alpha \circ \gamma$ and $\beta \circ \delta$ are equal. Given a regular semigroup S we may thus consider diagrams in $\mathfrak{C}(E)$ where $E = \boldsymbol{\mathsf{E}}(S)$ as well as in $G = \boldsymbol{\mathsf{G}}(S)$. In the following, we adopt the convention of using vertical [horizontal] arrows to denote ε-images $\varepsilon(c(e,f)) = \varepsilon(e,f)$ of edges $c(e,f)$ with $e \mathscr{L} f$ [$e \mathscr{R} f$]. Other morphisms will be denoted by dotted arrows. Note that for $\alpha = (x, x') \in G$ and $e \in E$ with $e\ \omega\ xx' = e_\alpha$, we have

$$f_{e|\alpha} = x'ex \in E \text{ with } x'ex\ \omega\ x'x = f_\alpha$$

so that $f_{e|\alpha} \in \omega(f_\alpha)$. Also, the map

$$\mathfrak{a}_S(\alpha)\colon \omega(e_\alpha) \to \omega(f_\alpha), \quad e \mapsto f_{e|\alpha} \tag{2.18}$$

is easily shown to be a biorder isomorphism of $\omega(e_\alpha)$ onto $\omega(f_\alpha)$ (see [14]). Moreover, for $e_1,\ e_2 \in E$ and $f_1 = e_1\alpha,\ f_2 = e_2\alpha$, if $e_1\ \omega^r\ e_2$, then $f_1\ \omega^r\ f_2$ with $(e_1e_2)\alpha = f_1f_2$, since α is a biorder isomorphism. Also in this case, if we write $g = e_1e_2$ and $h = f_1f_2$, then for any $k\ \omega\ e_1$, using the definition of restriction Equation (2.3), we have

$$\begin{aligned}(k)\varepsilon(e_1, g)(g|\alpha) &= x'kgx = (x'kx)(x'gx)\\ &= (k)(e_1|\alpha)\varepsilon(f_1, h)\\ &= (k)(\alpha|f_1)\varepsilon(f_1, h).\end{aligned}$$

Dually, if $e_1\omega^l e_2$, and if $g = e_2e_1,\ h = f_2f_1$, then we can similarly show that

$$\varepsilon(e_1, g)(g|\alpha) = (\alpha|f_1)\varepsilon(f_1, h).$$

Thus we have:

(IG1) For $\alpha \in G$ and $e_1, e_2 \in \omega(e_\alpha)$, let $f_i = f_{e_i|\alpha}$, $i = 1, 2$. Suppose that either $e_1\omega^r e_2$ or $e_1\omega^l e_2$ and let $g = e_2e_1e_2,\ h = f_2f_1f_2$. Then the following equality

$$\varepsilon(e_1, g)(g|\alpha) = (\alpha|f_1)\varepsilon(f_1, h)$$

holds in G. That is, if $e_1\omega^r e_2$, then the first diagram below commutes in G and if $e_1\ \omega\ [l]e_2$, then the second diagram commutes:

$$\begin{tikzcd} e_1 \arrow[rr, "\varepsilon(e_1,g)"] \arrow[dr, dotted, "\alpha|f_1"'] & & e_2 \arrow[dr, dotted, "g|\alpha"] & \\ & f_1 \arrow[rr, "\varepsilon(f_1,h)"'] & & h \end{tikzcd}
\qquad
\begin{tikzcd} e_1 \arrow[dd, "\varepsilon(e_1,g)"'] \arrow[dr, dotted, "\alpha|f_1"] & \\ & f_1 \arrow[dd, "\varepsilon(f_1,h)"] \\ g \arrow[dr, dotted, "g|\alpha"'] & \\ & h \end{tikzcd}$$

A diagram in $\mathfrak{C}(E)$ is said to be ε-commutative if its image under ε is commutative in G. It can be seen that an E-square ε-commutative in G if and only if it is a rectangular band in S. In particular it can be easily seen that:

(IG2) For all $f, g \in \omega^r(e)$ with $f \mathscr{L} g$ [or $f, g \in \omega^l(e)$ with $f \mathscr{R} g$] the singular E-square $c(f, g, ge, fe, f)$ [$c(f, ef, eg, g, f)$], represented as the first [second] diagram below, is commutative:

$$\begin{tikzcd} f \arrow[r, leftrightarrow, "\varepsilon(f,fe)"] \arrow[d, leftrightarrow, "\varepsilon(f,g)"'] & f \arrow[d, leftrightarrow, "\varepsilon(fe,ge)"] \\ g \arrow[r, leftrightarrow, "\varepsilon(g,ge)"'] & ge \end{tikzcd} \qquad \begin{tikzcd} f \arrow[r, leftrightarrow, "\varepsilon(f,g)"] \arrow[d, leftrightarrow, "\varepsilon(f,ef)"'] & g \arrow[d, leftrightarrow, "\varepsilon(g,eg)"] \\ ef \arrow[r, leftrightarrow, "\varepsilon(ef,eg)"'] & ge \end{tikzcd}$$

Thus every singular E-square in $\mathfrak{C}(E)$ is ε-commutative.

The foregoing discussion shows that given an ordered groupoid G and a biordered set E, in order that there exists a regular semigroup S such that $\boldsymbol{G}(S)$ is isomorphic to G and $\boldsymbol{E}$ is isomorphic to E, it is necessary that there is a v–isomorphism$\varepsilon : \mathfrak{C}(E) \to G$ satisfying (IG1) and (IG2). These conditions can also be shown to be sufficient. To describe this, we first make the following

DEFINITION 2.1. Suppose that G is an arbitrary ordered groupoid and that $\varepsilon\colon \mathfrak{C}(E) \to G$ is a v–isomorphism where E is a biordered set. Then the pair (G, ε) is called an *inductive groupoid* or that G is an inductive groupoid with respect to the *evaluation* ε if the conditions (IG1) and (IG2) hold.

The discussion preceding the definition above shows that the ordered groupoid $\boldsymbol{G}(S)$ is an inductive groupoid with respect to the evaluation $\varepsilon = \varepsilon_S$ defined by Equation (2.17). Conversely given any abstract inductive groupoid G with evaluation ε, we can construct a regular semigroup S by a method suggested by Theorem 2.3. First define the relation p on G as follows:

(2.19a) $$\alpha \, p \, \beta \iff e_\alpha \mathscr{R} e_\beta, \quad f_\alpha \mathscr{L} f_\beta \quad \text{and} \quad \varepsilon(e_\alpha, e_\beta)\beta = \alpha\varepsilon(f_\alpha, f_\beta).$$

Then p is an equivalence relation on G. Notice that in the case of the inductive groupoid $\boldsymbol{G}(S)$, $(x, x') \; p \; (y, y')$ if and only if $x = y$ so that there is a natural bijection of S with the quotient set $\boldsymbol{G}(S)/p$. In the case of arbitrary inductive groupoid G, we see that the desired semigroup is the set G/p with suitable definition of binary operation [14]: For $\bar{\alpha}, \bar{\beta} \in G/p$, define

(2.19b) $$\bar{\alpha} \cdot \bar{\beta} = \overline{(\alpha * h)(h * \beta)}$$

where $h \in \mathsf{S}(f_\alpha, e_\beta)$ and

(2.19c) $$\alpha * h = (\alpha | f_\alpha h)\varepsilon(f_\alpha h, h), \quad \text{and} \quad h * \beta = \varepsilon(h, he_\beta)(he_\beta | \beta)$$

(see [14], § 4 for a detailed discussion of these including the proof of the following).

THEOREM 2.4. *Let S be a regular semigroup. Then the set $\boldsymbol{G}(S)$ defined by Equation* (2.1) *is an inductive groupoid in which composition, partial order and evaluation are defined by Equations* (2.1), (2.4) *and* (2.17) *respectively.*

Conversely, let G be an inductive groupoid with $\mathsf{v}G = E$ and evaluation $\varepsilon\colon \mathfrak{C}(E) \to G$. Define the relation p on G by Equation (2.19a)*. Then Equation* (2.19b) *defines a single valued binary operation $\cdot$ on G/p such that $\boldsymbol{S}(G) = (G/p, \cdot)$ is a regular semigroup and $\boldsymbol{E}(\boldsymbol{S}(G))$ is isomorphic to E.*

The theorem above associates with each regular semigroup S an inductive groupoid $\boldsymbol{G}(S)$ and a regular semigroup $\boldsymbol{S}(G)$ with every inductive groupoid G. By Theorem 2.3, the regular semigroup $\boldsymbol{S}\big(\boldsymbol{G}(S)\big)$ constructed from the inductive groupoid $\boldsymbol{G}(S)$ is isomorphic to S.

Let $h : S \to S'$ be a homomorphism of regular semigroups. By Theorem 1.1 of [14], $\theta = \boldsymbol{E}(h) : E \to E'$ is a regular bimorphism where $E = \boldsymbol{E}(S)$ and $E' = \boldsymbol{E}(S')$. For any $(x, x') \in \boldsymbol{G}(S)$, it is clear that $((x)h, (x')h) \in \boldsymbol{G}(S')$ and it is easy to see from Equation (2.1) that the map

$$\boldsymbol{G}(h) : (x, x') \mapsto (xh, x'h) \tag{2.20}$$

is a functor $\boldsymbol{G}(h) : \boldsymbol{G}(S) \to \boldsymbol{G}(S')$ such that the vertex map of $\boldsymbol{G}(h)$ is $v\boldsymbol{G}(h) = \theta$. By Equation (2.3) (or Equation (2.4)), $\boldsymbol{G}(h)$ is an order preserving functor. Let ε and ε' denote evaluations of $\boldsymbol{G}(S)$ and $\boldsymbol{G}(S')$ respectively as defined by Equation (2.17). For any $c = c(e_0, \dots, e_n) \in \mathfrak{C}(E)$, by Equation (2.15) and (2.17), we have

$$\begin{aligned}
\varepsilon'\big(\mathfrak{C}(\theta)(c)\big) &= \varepsilon'\big(c(e_0\theta, \dots, e_n\theta)\big) \\
&= \big((e_0)\theta \dots (e_n)\theta, (e_n)\theta \dots (e_0)\theta\big) \\
&= \big((e_0 \dots e_n)h, (e_n \dots e_0)h\big) \\
&= \boldsymbol{G}(h)\big(e_0 \dots e_n, e_n \dots e_0\big) \\
&= \boldsymbol{G}(h)\big(\varepsilon\big(c(e_0 \dots e_n)\big)\big).
\end{aligned}$$

Hence the diagram (D3) commutes (with $\phi = \boldsymbol{G}(h)$. We are thus led to the following definition of morphisms of inductive groupoids.

DEFINITION 2.2. Let G and G' be inductive groupoids with evaluations $\varepsilon \colon \mathfrak{C}(E) \to G$ and $\varepsilon' \colon \mathfrak{C}(E') \to G'$ respectively. Suppose that $\theta \colon E \to E'$ is a regular bimorphism. An an order preserving functor $\phi : G \to G'$ is called an *inductive functor* with respect to θ (or that the pair (ϕ, θ) is inductive) if the diagram

$$\begin{tikzcd}
\mathfrak{C}(E) \arrow[r, "\varepsilon"] \arrow[d, "\mathfrak{C}(\theta)"'] & G \arrow[d, "\phi"] \\
\mathfrak{C}(E') \arrow[r, "\varepsilon'"'] & G'
\end{tikzcd} \tag{D3}$$

commutes; that is, $\varepsilon\phi = \mathfrak{C}(\theta)\varepsilon'$. The map $\phi : G \to G'$ is an *inductive isomorphism* if ϕ is an inductive functor (with respect to some $\theta \colon E \to E'$) which is an isomorphism of ordered groupoids (which implies that θ is a biorder isomorphism as well).

Since $v\varepsilon = 1_E$ and $v\varepsilon' = 1_{E'}$ it follows from the diagram (D3) that $\theta = v\phi$. Thus for any inductive functor ϕ, the map $v\phi$ is always a regular bimorphism. If $\phi \colon G \to H$ and $\psi \colon H \to K$ are inductive functors with respect to $\theta \colon vG \to vH$ and $\eta \colon vH \to vK$ respectively, then $\phi \circ \psi \colon vG \to vK$ is easily seen to be inductive with respect to $\theta \circ \eta$. Also, 1_G is inductive with respect to 1_{vG}. Hence there is a category $\mathfrak{IG}$ with objects as inductive groupoids and morphisms as inductive functors. The definition of $\mathfrak{IG}$ implies that there are two *forgetful* functors

$$\mathscr{G} \colon \mathfrak{IG} \to \mathfrak{OG} \quad \text{and} \quad v \colon \mathfrak{IG} \to \mathfrak{RB}$$

The functor $\mathscr{G}$ sends each inductive groupoid to the underlying ordered groupoid and each inductive functor to the corresponding order preserving functor. Similarly $vG \colon \mathfrak{IG} \to \mathfrak{RB}$ is a functor which sends inductive groupoid G to the biordered set vG and inductive functor ϕ to the bimorphism $v\phi$ so that ϕ is inductive with respect to $v\phi$.

If $\phi \colon G \to H$ any inductive functor then

$$\alpha \; p \; \beta \quad \text{in } G \text{ implies} \quad \phi(\alpha) \; p \; \phi(\beta)$$

in H. Hence ϕ induces a map

$$\boldsymbol{S}(\phi)\colon \bar{\alpha} \mapsto \overline{\phi(\alpha)} \tag{2.21}$$

of $\boldsymbol{S}(G)$ to $\boldsymbol{S}(H)$ which is a homomorphism. We can now extend the construction of Theorem 2.4 to morphisms of inductive groupoids and regular semigroups (see [14], § 3,4 for details of the construction and proofs).

THEOREM 2.5. *Let $h\colon S \to S'$ be a homomorphism of regular semigroups. Then Equation* (2.20) *defines an inductive functor $\boldsymbol{G}(h)\colon \boldsymbol{G}(S) \to \boldsymbol{G}(S')$. Also, the assignments*

$$\boldsymbol{G}\colon S \mapsto \boldsymbol{G}(S), \quad h \mapsto \boldsymbol{G}(h)$$

is a functor $\boldsymbol{G}\colon \mathfrak{IG} \to \mathfrak{RS}$.

Similarly, if $\phi\colon G \to H$ is an inductive functor, then the map defined by Equation (2.21)*is a homomorphism $\boldsymbol{S}(\phi)\colon \boldsymbol{S}(G) \to \boldsymbol{S}(H)$ of regular semigroups. Moreover, the assignments*

$$\boldsymbol{S}\colon G \mapsto \boldsymbol{S}(G), \quad \phi \mapsto \boldsymbol{S}(\phi)$$

is a functor $\boldsymbol{S}\colon \mathfrak{IG} \to \mathfrak{RS}$.

The functors $\boldsymbol{G}\colon \mathfrak{RS} \to \mathfrak{IG}$ and $\boldsymbol{S}\colon \mathfrak{IG} \to \mathfrak{RS}$ are mutually inverse up to equivalence. We have (cf. [14], § 4)

THEOREM 2.6. *For any regular semigroup S, define*

$$x\Phi_S = \overline{(x, x')} \quad \textit{for all} \quad x \in S,\ x' \in \mathscr{V}()(x). \tag{2.22a}$$

Then $\Phi_S\colon S \to \boldsymbol{S}(\boldsymbol{G}(S))$ is an isomorphism and the map $S \mapsto \Phi_S$ is a natural isomorphism

$$\Phi\colon 1_{\mathfrak{RS}} \approxeq \boldsymbol{G} \circ \boldsymbol{S}.$$

Similarly, for any inductive groupoid G, let

$$\nu_G(\alpha) = (\bar{\alpha}, \overline{\alpha^{-1}}) \quad \textit{for all} \quad \alpha \in G. \tag{2.22b}$$

Then $\nu_G : G \to \boldsymbol{G}(\boldsymbol{S}(G))$ is an inductive isomorphism and the map $G \to \nu_G$ is a natural isomorphism

$$\nu\colon 1_{\mathfrak{IG}} \approxeq \boldsymbol{S} \circ \boldsymbol{G}.$$

In particular categories $\mathfrak{IG}$ and $\mathfrak{RS}$ are naturally equivalent.

Recall that every semilattice E is a regular biordered in which the relations ω^r and ω^l defined by Equation (2.9) coincide so that we have

$$\omega r = \omega l = \omega. \tag{2.23}$$

Conversely every regular biordered set satisfying this condition is a semilattice. In this case the groupoid $\mathfrak{C}(E)$ coincides with the trivial groupoid and an evaluation of a semilattice E in an ordered groupoid G is simply a an order isomorphism of E onto vG. Hence an inductive groupoid G (in the sense of Definition 2.1) is a Schein groupoid (cf. Definition 1.2) if and only if vG is a semilattice. It follows that Theorems 1.3, 1.4 and 1.5 are particular cases of Theorems 2.4, 2.5 and 2.6 respectively.

3. Some applications of Inductive groupoids

We now discuss the significance and the use of the representation of regular semigroups as inductive groupoids and the consequences of the category equivalence of the category $\mathfrak{RS}$ of regular semigroups and the category $\mathfrak{IG}$ of inductive groupoids. The category equivalence implies that any statement about regular semigroups can be suitably translated as a statement regarding inductive groupoids and vice-versa. Consequently, in any discussion, it is possible to replace one by the other according to convenience.

Recall that an element in a semigroup is regular if and only if it has at least one (semigroup) inverses. Regularity is a significant concept both in the theory of semigroups and the applications of this theory. An inverse semigroup has the addditional property that every element has a *unique* inverse. This fact has strong influence on the structure of inverse semigroups. Some of the consequences of the uniqueness are that the trace S_* of an inverse semigroup is a groupoid and that the map $x \mapsto x^{-1}$ is an involution on S One may say that the uniqueness of inverses gives the structure of inverse semigroup an intrinsic symmetry. Many of the existing results about inverse semigroups exploit this symmetry significantly.

Though every element in a regular semigroup has an inverse, the inverse may not be unique. Consequently, the symmetry that exists in the case of inverse semigroup, does not exist in the case of arbitrary regular semigroups. This presents substantial problems in formulating and proving results on regular semigroups. On the other hand, inductive groupoid does possess some of the symmetry that is lacking in semigroups. Therefore many results can be formulated and proved much more simply and elegantly in terms of inductive groupoids rather than semigroups. We discuss some examples below to illustrate this.

Let E be a regular biordered set and let $T^*(E)$ denote the set of all ω-isomorphisms (biorder isomorphisms of ω-ideals § 2). As in § 1, it is clear that $T^*(E)$ is an ordered subgroupoid of the symmetric ordered groupoid $\mathscr{I}_*(E)$ in which identities are identity maps on ω-ideals. For all $e, f \in E$ with $e \mathscr{R} f$ or $e \mathscr{L} f$, let

$$g\tau(e,f) = \begin{cases} gf & \text{if } e \mathscr{R} f; \\ fg & \text{if } e \mathscr{L} f. \end{cases} \tag{3.1}$$

Then it follows directly from biorder axioms that $\tau(e, f)$ is an ω-isomorphism of $\omega(e)$ to $\omega(f)$. Also we have v–isomorphisms of ordered groupoids $\tau_R : \mathscr{R} \to T^*(E)$ and $\tau_L : \mathscr{L} \to T^*(E)$ defined by

$$\tau_R : (e,f) \in \mathscr{R} \mapsto \tau(e,f); \quad \mathsf{v}\tau_R = 1_E$$
$$\tau_L : (e,f) \in \mathscr{L} \mapsto \tau(e,f); \quad \mathsf{v}\tau_L = 1_E$$

and so, since the diagram (D1) is a push out, there is a unique v–isomorphism

$$\tau : \mathfrak{C}(E) \to T^*(E) \quad \text{such that} \quad \tau_R = \eta_r \circ \tau, \quad \tau_L = \eta_l \circ \tau.$$

For $c = c(e_0, \dots, e_n) \in \mathfrak{C}(E)$, we have

$$\tau(c) = \tau(e_0, e_1) \circ \tau(e_1, e_2) \dots \tau(e_{n-1}, e_n)$$

Hence from Equation (3.1) and the definition of restriction in $\mathfrak{C}(E)$, for all $c \in \mathfrak{C}(E)$ and $g \in \omega(e_c)$ we have

$$g\tau(c) = f_{g|c} \tag{3.2}$$

It is also easy to verify that the v–isomorphism $\tau : \mathfrak{C}(E) \to T^*(E)$ satisfies axioms (IG1) and (IG2). Hence the ordered groupoid $T^*(E)$ is inductive with respect to τ. By arguments similar to those in § 1, we can prove (see § 5, [14]):

Theorem 3.1. *Let E be a (regular) biordered set and let $T^*(E)$ denote the groupoid of all ω-isomorphisms of E. Then Equation* (3.2) *defines a v–isomorphism $\tau : \mathfrak{C}(E) \to T^*(E)$ and $T^*(E)$ is inductive with respect to τ. The semigroup $T(E) = \mathbf{S}\big(T^*(E)\big)$ is fundamental and* *regular. Moreover, if S is any fundamental regular semigroup with $\mathsf{E}(S)$ isomorphic to E, then S is isomorphic to a full subsemigroup of $T(E)$.*

We observe that the theorem above has close similarity with Munn's theorem (Theorem 1.6); in fact the above statement is formally obtained from Munn's theorem by replacing semilattice by biordered sets.

Another class of regular semigroups whose inductive groupoids can be characterized naturally in terms of their biordered sets are idempotent generated regular semigroups. In this case inductive groupoids of such semigroups can be obtained as quotients of the ordered groupoid $\mathfrak{C}(E)$ (see [14], § 6 for details). In particular cases, these inductive groupoids have interesting properties. Thus if E is the biordered set of all idempotent $n \times n$-matrices then the semigroup S generated by E (under matrix multiplication) is the set of all singular $n \times n$-matrices [4, 11]. In this case E is a finite dimensional manifold and the inductive groupoid of S is the groupoid of all polygonal paths in E.

Often results about inverse semigroups can be extended naturally to regular semigroups using inductive groupoids. For example, consider the construction of essential and normal extension of inverse semigroups. Recall that an essential extension of an inverse [regular] semigroup S is an inverse [regular] semigroup $T \supseteq S$ such that any homomorphism $\phi : T \to U$ which is injective on S is injective on the whole of T;again, T is a conjugate extension if $t^{-1}St \subseteq S$ for all $t \in T$ and it is a normal extension of S if T is a full (that is, $\mathsf{E}(S) = \mathsf{E}(T)$) and conjugate extension of S [20]. T is also maximal if it is not properly contained in any extension T' of the same type as T. Petrich [19] has given a construction of the maximal, essential-normal extension of inverse semigroups. In [16] the inductive groupoid of the maximal, essential-normal extension of regular semigroup was constructed by a method quite analogous to Petrich construction for inverse semigroups. Petrich and Pastijn later generalized the construction to obtain a class of extensions. They did not use inductive groupoids explicitly and their methods are quite involved. Radhkrishnan Chettiyar [1] later obtained a much more intuitive and elegant construction of inductive groupoids of extensions. Notice that once we have the inductive groupoids, we can always obtain the corresponding semigroups using Theorem 2.4 (see [14], § 4 for details).

We give a brief description of the inductive groupoids of extensions as follows (see [1], Ch. 4). Let S be a regular semigroup and let $\mathcal{F} = \{S_e : e \in \mathsf{E}(S)\}$ be a family of regular subsemigroups of S such that

- **(EX1)** $\mathsf{E}(S_e) = \omega(e)$ for all $e \in \mathsf{E}(S)$;
- **(EX2)** $S_e \subseteq S_f$ if $e \; \omega \; f$; and
- **(EX3)** for each $x \in S$ and $x' \in \mathscr{V}()(x)$, the map $\theta(x, x') : s \mapsto x'sx$ is an isomorphism of $S_{xx'}$ onto $S_{x'x}$.

Then $\mathcal{F}$ is called an extensive family [18] of S. An isomorphism $\sigma : S_e \to S_f$ is called an $\mathcal{F}$-isomorphism if for all $g \; \omega \; e$, the map $\sigma|S_g$ is an isomorphism of S_g onto $S_{g\sigma}$. It is easy to see that the set $\mathsf{A}_{\mathcal{F}}(S)$ of all $\mathcal{F}$-isomorphisms is an ordered groupoid under the obvious composition and partial order. As shown in [1], there is a natural evaluation of $\mathsf{E}(S)$ in $\mathsf{A}_{\mathcal{F}}(S)$ making it an inductive groupoid. Furthermore, there exists an identity separating inductive functor $q : \boldsymbol{G}(S) \to \mathsf{A}_{\mathcal{F}}(S)$. The semigroup $\hat{\mathsf{A}}_{\mathcal{F}}(S) = \boldsymbol{S}\big(\mathsf{A}_{\mathcal{F}}(S)\big)$ is an essential extension of S if and only if q is injective or equivalently, the homomorphism $\theta_{\mathcal{F}} = \boldsymbol{S}(q) : S \to \hat{\mathsf{A}}_{\mathcal{F}}(S)$ is injective.

It is well-known that a congruence ρ on a regular semigroup S is determined uniquely by the set of congruence classes $K_\rho = \{\rho_\lambda : \lambda \in \Lambda\}$ that contain idempotents or that K_ρ is the kernel-system of ρ [3]. As in [3], we shall refer to K_ρ as a kernel-normal system in S. If S is inverse, each ρ-class $\rho_\lambda \in K_\rho$ that contain idempotents is an inverse subsemigroup of S and so, the kernel-normal system K_ρ is a collection of inverse subsemigroups of S. Kernel-normal systems for inverse semigroups can be characterized axiomatically (see [3], Chapter 7, page 60). For regular semigroups it is known that subsemigroups belonging to K_ρ may not be regular and this makes a direct characterization of kernel-normal systems for regular semigroups difficult. However, if we consider congruences on inductive groupoids, then congruence classes containing identities are inductive subgroupoids and the collection of all such inductive subgroupoids form a *normal* system. In this case a characterization kernel-normal system for inductive groupoids, similar to those for inverse semigroups, is possible.

Other applications of the method of inductive groupoids include the construction of an important class of subdirect products, called S^*-direct products; see [17] for definitions and various applications of the construction. Inductive groupoids are also useful in studying congruences, homomorphism, extensions and co-extensions of regular semigroups (see [15] for more details).

REFERENCES

[1] S.Radhakrishnan Chettiyar. *A Study on Extensions of Regular Semigroups*. PhD thesis, Department of Mathematics, University of Kerala, Department of Mathematics, Univewrsity of Kerala, Kariavattom, October 1996. The thesis was submitted in October 1996 and was awarded in March 1997.

[2] A. H. Clifford and G. B. Preston. *Algebraic Theory of Semigroups*. Math. Surveys No.7. American Mathmatical Society, Providence, Rhode Island, 1961.

[3] A. H. Clifford and G. B. Preston. *Algebraic Theory of Semigroups*. Math. Surveys No.7. American Mathmatical Society, Providence, Rhode Island, 1967.

[4] R. J. H. Dawlings. The semigroup of singular endomorphisms of a finite dimensional vector space. In T. E. Hall, P. R. Jones and G. B. Preston, editor, *Semigroups*, pages 121–132. Academic Press, 1980.

[5] D. Easdown. Biordered sets come from semigroups. *J. Algebra*, 96:581–591, 1985.

[6] P. A. Grillet. Structure of regular semigroups–I: a representation. *Semigroup Forum*, 8:177–183, 1974.

[7] P. A. Grillet. Structure of regular semigroups–II: cross-connections. *Semigroup Forum*, 8:254–259, 1974.

[8] P. A. Grillet. Structure of regular semigroups–III: the reduced case. *Semigroup Forum*, 8:260–265, 1974.

[9] P. Higgins. *Categories and Groupoids*. Van Nostrand Reinhold, London, 1971.

[10] J. M. Howie. *An Introduction to Semigroup Theory*. Academic Press, London, 1976.

[11] V. N. Krishnachandran and K. S. S. Nambooripada. Topology od singular endomorphisms. To appear in Semigroup Forum where it was accepted for publication. To appear in 2000.

[12] S. MacLane. *Categories for the Working Mathematicians*. Springer-Verlag, New York, 1971.

[13] W. D. Munn. Fundamental inverse semigroups. *Quart. J. Math., Oxford series (2)*, 21:157–170, 1970.

[14] K. S. S. Nambooripad. Structure of regular semigroups–I. *Memoirs of the Amer. Math. Soc.*, 22, No. 224 (Second of 3 numbers), November 1979.

[15] K. S. S. Nambooripad. Topics in structure theory of regular semigroups. *Colloquia Mathematica Societatis János Bolyai*, 39, Semigroups, Szeged(Hungary):175–197, 1981.

[16] K. S. S. Nambooripad and Radhakrishnan Chettiyar. Essential and normal extensions. *Indian Journal of Pure and Applied Mathematics*, 29:283–302, 1985.

[17] K. S. S. Nambooripad and R. Veeramony. Subdirect products of regular semigroups. *Semigroup Forum*, 27:265–307, 1983.

[18] F. Pastijn and M. Petrich. *Extensions of Regular Semigroups*. John Wiley & Sons, New York, 1985.

[19] M. Petrich. Extensions normales de semigroup inverses. *Fund. Math.*, 112:187–203, 1981.

[20] M. Petrich. *Inverse Semigroups*. John Wiley & Sons, New York, 1984.

[21] B. M. Schein. On the theory of generalized groups and generalized heaps. In *The Theory Semigroups and its Applications, I*, Izv. dat. Saratov Univ., pages 286–324. Saratov University (In Russian), Sartov, 1966. English Translation:Amer. Math. Soc. Transl. (2), 89–122 (1979).