MODULAR AND CANCELLABLE ELEMENTS OF THE LATTICE OF SEMIGROUP VARIETIES

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The collection of all semigroup varieties forms a lattice under class-theoretical inclusion. We denote this lattice by $\mathbb{SEM}$.

The structure of $\mathbb{SEM}$ is extremely complicated. In particular, it contains an anti-isomorphic copy of the partition lattice over a countably infinite set (Burris and Nelson, 1971). Whence $\mathbb{SEM}$ does not satisfy any non-trivial lattice (quasi)identity.

Since the lattice $\mathbb{SEM}$ does not satisfy lattice identities, it is natural to examine varieties with distributive (modular etc.) subvariety lattice.

In 1989–1992 Volkov completely classify semigroup varieties with modular subvariety lattices and describe varieties with distribuitive subvariety lattices modulo groups and without some very special class of varieties.

In 2002–2004 Volkov and V. describe varieies with Arguesian, upper semimodular or lower semimodular subvariety lattice.

The next natural step is to consider varieties that guarantee, so to speak, "nice lattice behaviour" in their neighborhood.

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We say about special elements of diferent types in $\mathbb{SEM}$.

Let ε be a lattice identity with variables $x_0, x_1, \ldots, x_n$:

$$p(x_0, x_1, \ldots, x_n) = q(x_0, x_1, \ldots, x_n).$$

This is the first-order formula without free variables, and all variables are subject to universal quantification. The identity $p = q$ holds in a lattice L if

$$(\forall x_0, x_1, \ldots, x_n \in L) \quad (p(x_0, x_1, \ldots, x_n) = q(x_0, x_1, \ldots, x_n)).$$

Now we consider the situation when all variables but one are subject to universal quantification, while one of them, say, x_0, is left free. Then we have the following first order formula:

$$(\forall x_1, \ldots, x_n \in L) \quad (p(x_0, x_1, \ldots, x_n) = q(x_0, x_1, \ldots, x_n)).$$

We say that an element x of a lattice L is an *(ε, x_0)-element* of L if the latest formula holds true whenever $x_1, \ldots, x_n$ are evaluated by arbitrary elements of L.

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There are three identities that define distributive lattices:

1) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$,
2) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,
3) $(x \vee y) \wedge (y \vee z) \wedge (z \vee x) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$.

These three identities permit to define five types of special elements.

(i) An element $x \in L$ is called *distributive* in L if

$$(\forall y, z) \quad (x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)).$$

(ii) *Codistributive* elements are defined dually.

(iii) An element $x \in L$ is called *standard* in L if

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(iv) *Costandard* elements are defined dually.

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An element $x \in L$ is neutral in L if and only if, for all $y, z \in L$, elements x, y and z generate a distributive sublattice of L.

Special elements of distributive types

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As usual, we write the modular law in the form of quasi-identity:

$$y \leq z \longrightarrow (x \vee y) \wedge z = (x \wedge z) \vee y.$$

This permits to define three types of special elements:

(i) An element $x \in L$ is called *modular* in L if

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One more type of special elements is defined on the base of the following quasi-identity:

$$x \vee y = x \vee z \ \& \ x \wedge y = x \wedge z \rightarrow y = z.$$

An element $x \in L$ is called *cancellable* in L if

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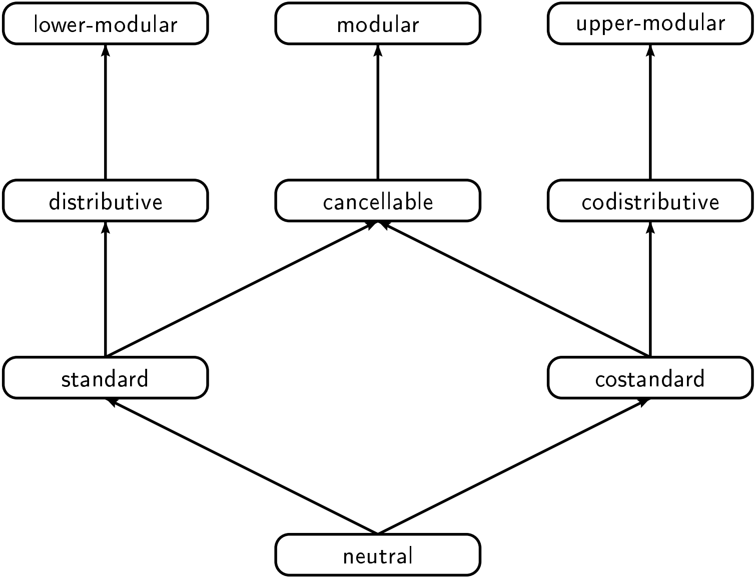
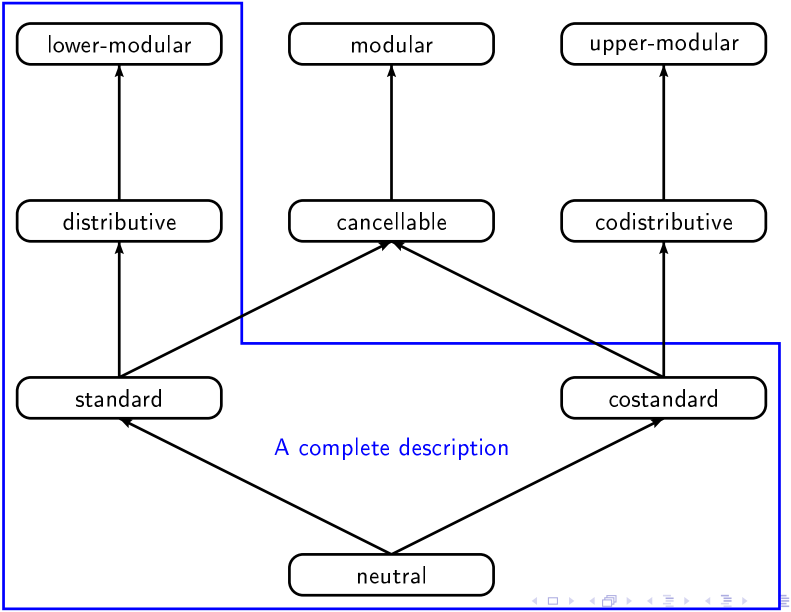
Relationships between special element in abstract lattices



What we know about special elements in the lattice $\mathbb{SEM}$ so far



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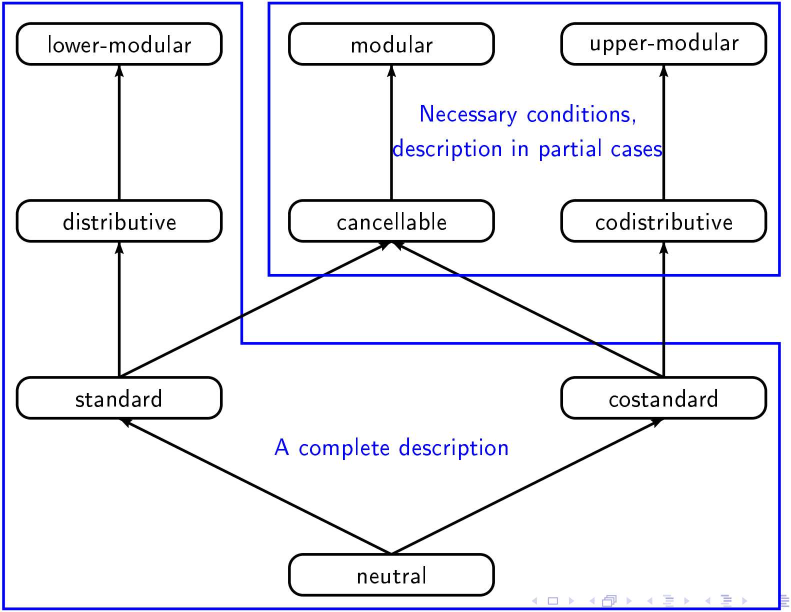

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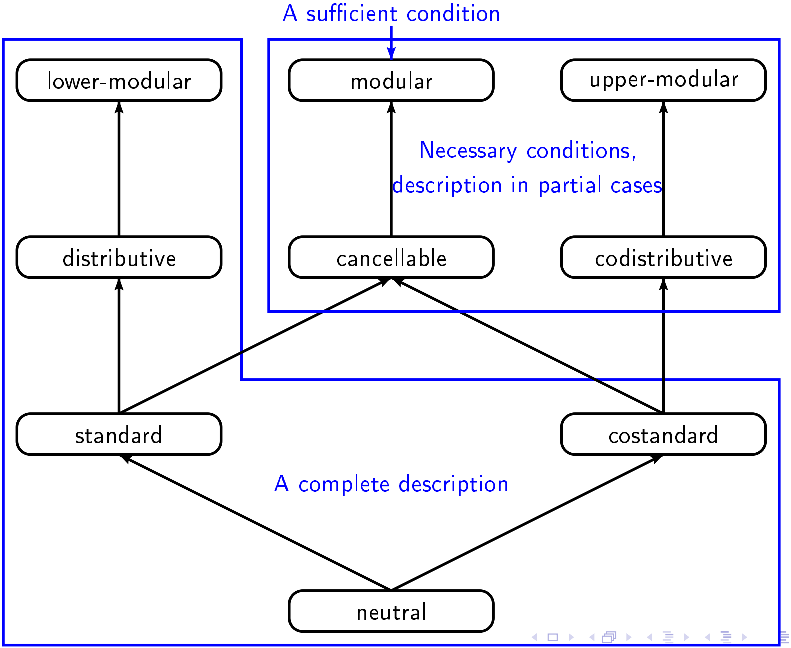

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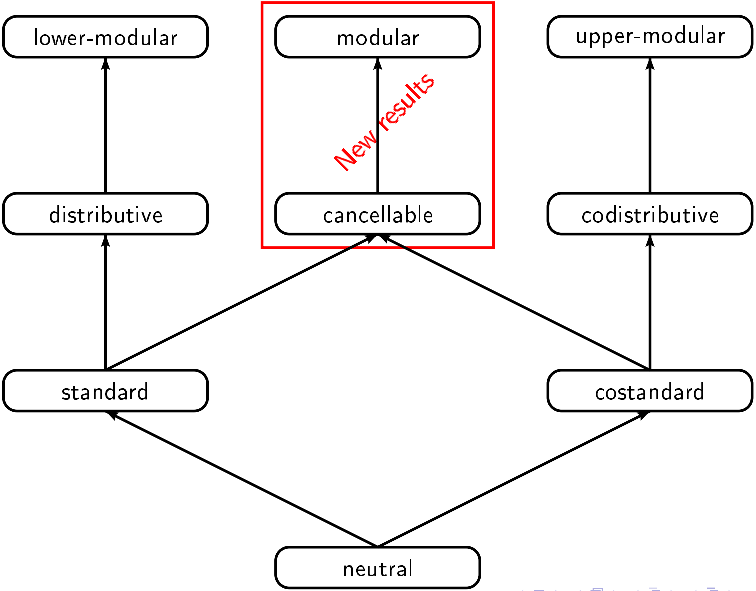
What we will discuss below



Let $\mathbf{T}$ be the trivial variety, $\mathbf{SL} = \mathrm{var}\{x \approx x^2, xy \approx yx\}$ (the variety of semilattices), $\mathbf{SEM}$ be the variety of all semigroups.

$\mathbf{w} \approx 0$ is a short form for the identities $\mathbf{w}x \approx x\mathbf{w} \approx \mathbf{w}$ where x is a letter that does not occur in the word $\mathbf{w}$. (A semigroup S satisfies these identities if and only if S contains 0 and all values of $\mathbf{w}$ in S equals 0.)

A *nilvariety* is a variety with an identity of the form $x^n \approx 0$ for some n.

Proposition (Ježek and McKenzie, 1993; reproved in a simpler way by Shaprynskii, 2012)

If a semigroup variety $\mathbf{V}$ *is a modular element of the lattice* $\mathbb{SEM}$ *then either* $\mathbf{V} = \mathbf{SEM}$ *or* $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$ *and* $\mathbf{SL}$, *while* $\mathbf{N}$ *is a nilvariety.*

Since a cancellable element of a lattice is modular, the conclusion of this Proposition is true for cancellable elements of the lattice $\mathbb{SEM}$ too.

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Necessary condition for modular and cancellable elements

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Reduction to the nil-case

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Lemma

1) (Shaprynskii, 2011) *Let* ε *be a non-trivial lattice identity and* x_0 *be a letter that occurs in* ε. *A semigroup variety* $\mathbf{V}$ *is an* (ε, x_0)*-element of the lattice* $\mathbb{SEM}$ *if and only if the variety* $\mathbf{V} \vee \mathbf{SL}$ *has this property.*

2) (Gusev, Skokov and V.) *A semigroup variety* $\mathbf{V}$ *is a cancellable element of the lattice* $\mathbb{SEM}$ *if and only if the variety* $\mathbf{V} \vee \mathbf{SL}$ *has this property.*

Thus, the problem of classification of modular or cancellable elements of $\mathbb{SEM}$ is completely reduced to nilvarieties.

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If a proper semigroup variety $\mathbf{V}$ *is a modular (a cancellable) element of the lattice* $\mathbb{SEM}$ *then either* $\mathbf{V} = \mathbf{SEM}$ *or* $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$ *and* $\mathbf{SL}$, *while* $\mathbf{N}$ *is a nilvariety.*

Lemma

1) (Shaprynskii, 2011) *Let* ε *be a non-trivial lattice identity and* x_0 *be a letter that occurs in* ε*. A semigroup variety* $\mathbf{V}$ *is an* (ε, x_0)*-element of the lattice* $\mathbb{SEM}$ *if and only if the variety* $\mathbf{V} \vee \mathbf{SL}$ *has this property.*

2) (Gusev, Skokov and V.) *A semigroup variety* $\mathbf{V}$ *is a cancellable element of the lattice* $\mathbb{SEM}$ *if and only if the variety* $\mathbf{V} \vee \mathbf{SL}$ *has this property.*

Thus, the problem of classification of modular or cancellable elements of $\mathbb{SEM}$ is completely reduced to nilvarieties.

A sufficient condition and a necessary condition for modular elements in the nil-case

Identities of the form $\mathbf{w} \approx 0$ and varieties given by such identities are called *0-reduced*. It is evident that any 0-reduced variety is a nilvariety.

Proposition

Every 0-reduced semigroup variety is a modular element of the lattice $\mathbb{SEM}$.

This claim follows from a result by Ježek, 1981. It is mentioned explicitly first by V. and Volkov, 1988, and independently by Ježek and McKenzie, 1993.

Substitutive identity is an identity of the form $\mathbf{u} \approx \mathbf{v}$ where the words $\mathbf{u}$ and $\mathbf{v}$ depend on the same letters and $\mathbf{v}$ is obtained from $\mathbf{u}$ by renaming of letters.

Examples: $xy \approx yx$, $xyzt \approx zxyt$, $x^2y \approx y^2x$, $xyxz \approx yzyx$.

Proposition (V., 2007)

If a nilvariety $\mathbf{V}$ *is a modular element of the lattice* $\mathbb{SEM}$ *then* $\mathbf{V}$ *can be given by 0-reduced and substitutive identities only.*

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If a nilvariety $\mathbf{V}$ *is a modular element of the lattice* $\mathbb{SEM}$ *then* $\mathbf{V}$ *can be given by 0-reduced and substitutive identities only.*

The simplest partial case of substitutive identities are *permutative* identities, that is identities of the form

$$x_1x_2\cdots x_n \approx x_{1\pi}x_{2\pi}\cdots x_{n\pi}$$

where π is a non-trivial permutation on the set $\{1, 2, \ldots, n\}$. The number n is called a *length* of this identity.

In turn, the simplest partial case of permutative identities is the commutative law.

Theorem (V., 2007)

A commutative semigroup variety $\mathbf{V}$ *is a modular element of the lattice* $\mathbb{SEM}$ *if and only if* $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$ *or* $\mathbf{SL}$*, while* $\mathbf{N}$ *satisfies the identities* $x^2y \approx 0$*,* $xy \approx yx$*.*

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Theorem (Gusev, Skokov and V.)

For a commutative semigroup variety $\mathbf{V}$*, the following are equivalent:*

a) $\mathbf{V}$ *is a modular element of the lattice* $\mathbb{SEM}$*;*

b) $\mathbf{V}$ *is a cancellable element of the lattice* $\mathbb{SEM}$*;*

c) $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$ *or* $\mathbf{SL}$*, while* $\mathbf{N}$ *satisfies the identities* $x^2y \approx 0$*,* $xy \approx yx$*.*

Are the properties to be modular and cancellable element of $\mathbb{SEM}$ equivalent for arbitrary semigroup varieties?

To answer this question, we consider slightly wider class of semigroup varieties than the class of commutative varieties. The commutative law is a permutative identity of length 2. We consider varieties that satisfy a permutative identity of length 3.

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Modular and cancellable elements in the case of a permutative identity of length 3

Theorem (Skokov and V.)

A semigroup variety $\mathbf{V}$ *satisfying a permutative identity of length* 3 *is a modular element of the lattice* $\mathbb{SEM}$ *if and only if* $\mathbf{V} = \mathbf{M} \vee \mathbf{N}$ *where* $\mathbf{M}$ *is one of the varieties* $\mathbf{T}$ *or* $\mathbf{SL}$, *while the variety* $\mathbf{N}$ *satisfies one of the following identity systems:*

$$xyz \approx zyx,\ x^2y \approx 0;$$
$$xyz \approx yzx,\ x^2y \approx 0;$$
$$xyz \approx yxz,\ xyzt \approx xzty,\ xy^2 \approx 0;$$
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Theorem (V., May 27, 2018)

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The variety $\mathrm{var}\{xyz \approx yxz \approx xzy,\ x^2y \approx xyx \approx yx^2 \approx 0\}$ satisfies all permutative identities of length 3.

Proposition (V.)

Let $\mathbf{V}$ *be a semigroup variety that is a cancellable element of the lattice* $\mathbb{SEM}$ *and* n *be a natural number. If* $\mathbf{V}$ *satisfies some permutative identity of length* n *then it satisfies all permutative identities of length* n*.*

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Thank you very much!