Chains and anti-chains in the lattice of epigroup varieties*

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Abstract

Let $\mathcal{E}_n$ be the variety of all epigroups of index $\le n$. We prove that, for an arbitrary natural number n, the interval $[\mathcal{E}_n, \mathcal{E}_{n+1}]$ of the lattice of epigroup varieties contains a chain isomorphic to the chain of real numbers with the usual order and an anti-chain of the cardinality continuum.

Key words: epigroup, variety, lattice of subvarieties.

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A semigroup S is called an *epigroup* if for any element x of S some power of x lies in some subgroup of S. For an element a of a given epigroup, let e_a be the unit element of the maximal subgroup G that contains some power of a. It is known that $ae_a = e_a a$ and this element lies in G. We denote by $\overline{a}$ the element inverse to ae_a in G. This element is called the *pseudo-inverse* of a. The mapping $a \longmapsto \overline{a}$ defines a unary operation on an epigroup. The idea to treat epigroups as unary semigroups (that is semigroups with an additional unary operation of pseudo-inversion) was promoted by Shevrin in [2]. A systematic overview of the material accumulated in the theory of epigroups by the beginning of the 2000s was given in the survey [3].

By epigroup variety we mean a variety of epigroups treated just as unary semigroups. Results about epigroup varieties that are known so far mainly concern with equational and structural aspects (see corresponding results in [2, 3]). As to considerations of the varietal lattices, there are only a few results about such a type (see Sections 2 and 3 in the recent survey [4]). In [2] several open questions about lattices of epigroup varieties were formulated; some of them are reproduced in [3] and [4]. The aim of this note is to answer one of these questions and obtain an information closely related with one more of them.

An epigroup S has *index* n if the nth power of every element of S lies in some of its subgroups and n is the least number with this property. The class of all epigroups of index $\le n$ is denoted by $\mathcal{E}_n$. For each n, the class $\mathcal{E}_n$ is known to be a variety of epigroups; it is given by the identities

$$(xy)z = x(yz),\ x\overline{x} = \overline{x}x,\ x\overline{x}^{\,2} = \overline{x},\ x^{n+1}\overline{x} = x^n$$

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(see [2]). The chain $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \mathcal{E}_n \subset \cdots$ can be regarded as the "spine" of the lattice of all epigroup varieties, since for any epigroup variety $\mathcal{V}$ there exists n such that $\mathcal{V} \subseteq \mathcal{E}_n$.

The following questions have been formulated in [2] and repeated in [3, 4]:

1) *What are the order types of maximal chains in the intervals* $[\mathcal{E}_n, \mathcal{E}_{n+1}]$ *of the lattice of epigroup varieties*?

2) *What are the cardinalities of maximal anti-chains in these intervals*?

The first question is still open. But the following theorem shows that the intervals $[\mathcal{E}_n, \mathcal{E}_{n+1}]$ contain rather complicated chains.

Theorem 1. *For an arbitrary natural number n, the interval $[\mathcal{E}_n, \mathcal{E}_{n+1}]$ contains a chain isomorphic to the chain of real numbers with the usual order.*

Note that chains we construct in the proof of Theorem 1 are not maximal in the intervals of the kind $[\mathcal{E}_n, \mathcal{E}_{n+1}]$ (see Remark 4 below).

The complete answer on the second question is given by the following

Theorem 2. *For an arbitrary natural number n, the interval $[\mathcal{E}_n, \mathcal{E}_{n+1}]$ contains an anti-chain of cardinality continuum.*

There are two results that play the key role in the proof of both theorems. The first of them was proved by Ježek in [1]. To formulate this result, we recall that a word u is said to be *applicable to a word* v if v may be presented in the form $a\xi(u)b$ where a and b are (maybe empty) words, while ξ is an endomorphism on the free semigroup under a countably infinite alphabet. The mentioned result by Ježek is that there are a countably infinite set of semigroup words $\{w_i \mid i \in I\}$ such that w_i is not applicable to w_j for any $i, j \in I$, $i \neq j$, and x^2 is not applicable to w_i for any $i \in I$. For our aim, it is convenient to enumerate these words by rational numbers. In what follows we will refer to these words as to the words Z_α where α runs over the set of all rational numbers. For each rational α, the first letter of Z_α will be denoted by x_α.

To formulate the second result, we need some definitions and notation. A pair of identities $wx = xw = w$ where the letter x does not occur in the word w is usually written as the symbolic identity $w = 0$. (This notation is justified because a semigroup with the identities $wx = xw = w$ has a zero element and all values of the word w in this semigroup are equal to zero.) An identity of the form $w = 0$ as well as a variety given by identities of such a form are called 0-*reduced*. A semigroup variety is called a *nil-variety* if it consists of nil-semigroups; this takes place if and only if it satisfies the identity $x^n = 0$ for some n. It is evident that every 0-reduced variety is a nil-variety. It is clear that every nil-semigroup is an epigroup and every nil-variety of semigroups may be considered as a variety of epigroups.

An element x of a lattice $\langle L; \vee, \wedge \rangle$ is called *lower-modular* if

$$\forall\, y, z \in L\colon \quad x \leq y \longrightarrow (z \vee x) \wedge y = (z \wedge y) \vee x.$$

Upper-modular elements are defined dually. It was verified in [5, Corollary 3] that a 0-reduced semigroup variety is a lower-modular element of the lattice of all semigroup varieties. The proof of this fact given in [5] is based on the following two statements: 1) the fully invariant congruence on the free semigroup corresponding to a 0-reduced variety has exactly one non-singleton class; 2) an equivalence relation π on a set S has at most one non-singleton class if and only if π is an upper-modular element of the equivalence lattice of S (this observation was checked in [5, Proposition 3]). It is evident that these arguments are applicable for epigroup varieties as well. Thus we have

Lemma 3. *A* 0*-reduced epigroup variety is a lower-modular element of the lattice of all epigroup varieties.* □

A semigroup variety given by an identity system Σ is denoted by $\mathrm{var}\,\Sigma$.

Now we are ready to prove both theorems.

Proof of Theorem 1. Let n be a natural number and ξ a real number. Put

$$\mathcal{C}_\xi^n = \mathrm{var}\,\{x^{n+1} = x_\alpha^{n-1} Z_\alpha = 0 \mid \alpha \ge \xi\}$$

(if $n = 1$ then x_α^0 is the empty word) and $\mathcal{D}_\xi^n = \mathcal{E}_n \vee \mathcal{C}_\xi^n$. It is clear that $\mathcal{C}_\xi^n \subseteq \mathcal{E}_{n+1}$, whence $\mathcal{D}_\xi^n \in [\mathcal{E}_n, \mathcal{E}_{n+1}]$. Let now ξ_1 and ξ_2 be real numbers with $\xi_1 \le \xi_2$. Then $\mathcal{C}_{\xi_1}^n \subseteq \mathcal{C}_{\xi_2}^n$ and therefore $\mathcal{D}_{\xi_1}^n \subseteq \mathcal{D}_{\xi_2}^n$. To prove Theorem 1, it suffices to verify that $\mathcal{D}_{\xi_1}^n \ne \mathcal{D}_{\xi_2}^n$ whenever $\xi_1 \ne \xi_2$. Arguing by contradiction, suppose that $\xi_1 < \xi_2$ (and therefore $\mathcal{C}_{\xi_1}^n \subset \mathcal{C}_{\xi_2}^n$) but $\mathcal{D}_{\xi_1}^n = \mathcal{D}_{\xi_2}^n$ (see Fig. 1).

Note that all varieties of the kind $\mathcal{C}_\xi^n$ are 0-reduced. Further, for any ξ, the variety $\mathcal{E}_n \wedge \mathcal{C}_\xi^n$ is a nil-variety of index $\le n$, whence it satisfies the identity $x^n = 0$. Therefore

$$\mathcal{E}_n \wedge \mathcal{C}_{\xi_2}^n \subseteq \mathcal{C}_{\xi_1}^n. \tag{1}$$

We have

$$\begin{aligned}
\mathcal{C}_{\xi_1}^n &= (\mathcal{E}_n \wedge \mathcal{C}_{\xi_2}^n) \vee \mathcal{C}_{\xi_1}^n && \text{by (1)}\\
&= (\mathcal{E}_n \vee \mathcal{C}_{\xi_1}^n) \wedge \mathcal{C}_{\xi_2}^n && \text{by Lemma 3}\\
&= \mathcal{D}_{\xi_1}^n \wedge \mathcal{C}_{\xi_2}^n && \text{by the definition of } \mathcal{D}_{\xi_1}^n\\
&= \mathcal{D}_{\xi_2}^n \wedge \mathcal{C}_{\xi_2}^n && \text{because } \mathcal{D}_{\xi_1}^n = \mathcal{D}_{\xi_2}^n\\
&= \mathcal{C}_{\xi_2}^n && \text{by the definition of } \mathcal{D}_{\xi_2}^n.
\end{aligned}$$

Thus $\mathcal{C}_{\xi_1}^n = \mathcal{C}_{\xi_2}^n$. A contradiction. □

Let $C = \{\mathcal{D}_\xi^n \mid \xi \in \mathbb{R}\}$. If $\xi \in \mathbb{R}$ then $\mathcal{E}_n \ne \mathcal{D}_\xi^n$ because $\mathcal{C}_\xi^n \nsubseteq \mathcal{E}_n$, and $\mathcal{E}_{n+1} \ne \mathcal{D}_\xi$ because $\mathcal{D}_\xi \subset \mathcal{D}_\lambda \subseteq \mathcal{E}_{n+1}$ for any $\lambda \in \mathbb{R}$ with $\xi < \lambda$. Thus we may ajoin $\mathcal{E}_n$ [respectively $\mathcal{E}_{n+1}$] as the least [the greatest] element to the chain C and obtain a chain C^* in $[\mathcal{E}_n, \mathcal{E}_{n+1}]$ with $C \subset C^*$. We have the following

Remark 4. *The chain C is not the maximal chain in the interval $[\mathcal{E}_n, \mathcal{E}_{n+1}]$.* □

Proof of Theorem 2. As in the proof of Theorem 1, let n be a natural number and ξ a real number. Now we put

$$\mathcal{A}_\xi^n = \mathrm{var}\,\{x^{n+1} = x_\alpha^{n-1} Z_\alpha = 0 \mid \xi - 1 < \alpha < \xi + 1\}$$

and $\mathcal{B}_\xi^n = \mathcal{E}_n \vee \mathcal{A}_\xi^n$. It is clear that $\mathcal{A}_\xi^n \subseteq \mathcal{E}_{n+1}$ and $\mathcal{B}_\xi^n \in [\mathcal{E}_n, \mathcal{E}_{n+1}]$. Let ξ_1 and ξ_2 be different real numbers. Then the varieties $\mathcal{A}_{\xi_1}^n$ and $\mathcal{A}_{\xi_2}^n$ are non-comparable.

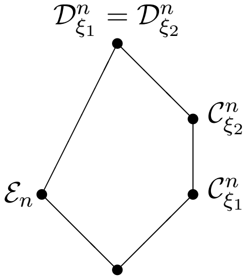

Figure 1

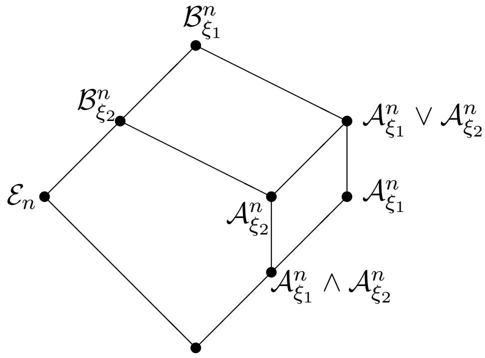

Figure 2

To prove Theorem 2, it suffices to verify that the varieties $\mathcal{B}^n_{\xi_1}$ and $\mathcal{B}^n_{\xi_2}$ are non-comparable too. Arguing by contradiction, suppose that $\mathcal{B}^n_{\xi_2} \subseteq \mathcal{B}^n_{\xi_1}$ (see Fig. 2).

Note that all varieties of the kind $\mathcal{A}^n_\xi$ are 0-reduced. Further, the variety $\mathcal{E}_n \wedge (\mathcal{A}^n_{\xi_1} \vee \mathcal{A}^n_{\xi_2})$ is a nil-variety of index $\le n$, whence it satisfies the identity $x^n = 0$. Therefore

$$\mathcal{E}_n \wedge (\mathcal{A}^n_{\xi_1} \vee \mathcal{A}^n_{\xi_2}) \subseteq \mathcal{A}^n_{\xi_1}. \tag{2}$$

Furthermore, $\mathcal{B}^n_{\xi_1} \supseteq \mathcal{A}^n_{\xi_1}$ and $\mathcal{B}^n_{\xi_1} \supseteq \mathcal{B}^n_{\xi_2} \supseteq \mathcal{A}^n_{\xi_2}$, whence

$$\mathcal{B}^n_{\xi_1} \supseteq \mathcal{A}^n_{\xi_1} \vee \mathcal{A}^n_{\xi_2}. \tag{3}$$

We have

$$\begin{aligned}
\mathcal{A}^n_{\xi_1} &= \big(\mathcal{E}_n \wedge (\mathcal{A}^n_{\xi_1} \vee \mathcal{A}^n_{\xi_2})\big) \vee \mathcal{A}^n_{\xi_1} && \text{by (2)}\\
&= (\mathcal{E}_n \vee \mathcal{A}^n_{\xi_1}) \wedge (\mathcal{A}^n_{\xi_1} \vee \mathcal{A}^n_{\xi_2}) && \text{by Lemma 3}\\
&= \mathcal{B}^n_{\xi_1} \wedge (\mathcal{A}^n_{\xi_1} \vee \mathcal{A}^n_{\xi_2}) && \text{by the definition of } \mathcal{B}^n_{\xi_1}\\
&= \mathcal{A}^n_{\xi_1} \vee \mathcal{A}^n_{\xi_2} && \text{by (3).}
\end{aligned}$$

Thus $\mathcal{A}^n_{\xi_1} = \mathcal{A}^n_{\xi_1} \vee \mathcal{A}^n_{\xi_2}$, whence $\mathcal{A}^n_{\xi_2} \subseteq \mathcal{A}^n_{\xi_1}$. A contradiction. □

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