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# Special elements of the lattice of epigroup varieties

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Abstract. We study special elements of three types (namely, neutral, modular and upper-modular elements) in the lattice of all epigroup varieties. Neutral elements are completely determined (it turns out that only four varieties have this property). We find a strong necessary condition for modular elements that completely reduces the problem of description of corresponding varieties to nilvarieties satisfying identities of some special type. Modular elements are completely classified within the class of commutative varieties, while upper-modular elements are completely determined within the wider class of strongly permutative varieties.

## 1. Introduction

**1.1. Semigroup pre-history.** In this article, we study the lattice of all epigroup varieties. Our considerations are motivated by some earlier investigations of the lattice of semigroup varieties and closely related to these investigations. We start with a brief explanation of the 'semigroup pre-history' of the present work.

One of the main branches of the theory of semigroup varieties is an examination of lattices of semigroup varieties (see the survey [16]). If $\mathcal{V}$ is a variety, then $L(\mathcal{V})$ stands for the subvariety lattice of $\mathcal{V}$ under the natural order (the class-theoretical inclusion). The lattice operations in $L(\mathcal{V})$ are the (class-theoretical) intersection denoted by $\mathcal{X} \wedge \mathcal{Y}$ and the join $\mathcal{X} \vee \mathcal{Y}$, i.e., the least subvariety of $\mathcal{V}$ containing both $\mathcal{X}$ and $\mathcal{Y}$.

There are a number of articles devoted to examination of identities (first of all, the distributive, modular or Arguesian laws) and some related restrictions (such as semimodularity or semidistributivity) in lattices of semigroup varieties, and many important results are obtained there. In particular, the semigroup varieties with modular, Arguesian or semimodular subvariety lattices were completely classified, and deep results concerning the semigroup varieties with distributive subvariety lattices were obtained. An overview of all these results may be found in [16, Section 11].

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The results mentioned above specify, so to say, 'globally' modular or distributive parts of the lattice of semigroup varieties. The following natural step is to examine varieties that guarantee modularity or distributivity, so to say, in their 'neighborhood'. Saying this, we have in mind special elements in the lattice of semigroup varieties. There are many types of special elements that are considered in lattice theory. Recall some of them. An element $x$ of a lattice $\langle L; \vee, \wedge\rangle$ is called

- *neutral* if, for all $y, z \in L$, the sublattice of $L$ generated by $x$, $y$ and $z$ is distributive;
- *modular* if $(x \vee y) \wedge z = (x \wedge z) \vee y$ for all $y, z \in L$ with $y \leq z$;
- *upper-modular* if $(z \vee y) \wedge x = (z \wedge x) \vee y$ for all $y, z \in L$ with $y \leq x$.

*Lower-modular* elements are defined dually to upper-modular ones. Note that special elements play an essential role in abstract lattice theory. For instance, if an element $x$ of a lattice $L$ is neutral, then $L$ can be decomposed into subdirect product of its intervals $(x] = \{y \in L \mid y \leq x\}$ and $[x) = \{y \in L \mid x \leq y\}$ (see [2, Theorem 254 on p. 226]). Thus, the knowledge of which elements of a lattice are neutral gives important information on the structure of the lattice as a whole.

All these types of elements as well as some other types of elements of the lattice **SEM** of all semigroup varieties have been intensively and successfully studied. Briefly, a semigroup variety that is a neutral element of the lattice **SEM** is called a *neutral in* **SEM** variety. An analogous convention is applied to all other types of special elements. Results about special elements in **SEM** are overviewed in the recent survey [24]. In particular,

- neutral in **SEM** varieties were completely determined in [31];
- strong necessary conditions for modular in **SEM** varieties were discovered in [6] and [19] (and reproved in a simpler way in [11]);
- commutative modular in **SEM** varieties are completely determined in [19];
- commutative upper-modular in **SEM** varieties were completely classified in [22]; it is noted in [21] that this result may be extended to strongly permutative varieties without any change (for a definition of strongly permutative varieties, see in Subsection 1.2).

**1.2. Epigroups.** Considerable attention in semigroup theory is devoted to semigroups equipped with an additional unary operation. Such algebras are said to be *unary semigroups*. As concrete types of unary semigroups, we mention completely regular semigroups (see [8]), inverse semigroups (see [7]), semigroups with involution, etc.

One more natural type of unary semigroups is epigroups. A semigroup $S$ is called an *epigroup* if for any element $x$ of $S$, there is a natural $n$ such that $x^n$ is a *group element* (this means that $x^n$ lies in some subgroup of $S$). Extensive information about epigroups may be found in the fundamental work [14] by L. N. Shevrin and the survey [15] by the same author. The class of epigroups is very wide. In particular, it includes all periodic semigroups (because some power of each element in such a semigroup lies in some finite cyclic subgroup) and all completely regular semigroups (in which all elements are group elements). The unary operation on an epigroup is defined in the following way. If $S$ is an epigroup and $x \in S$, then some power of $x$ lies in a maximal subgroup of $S$. We denote this subgroup by $G_x$. The unit element of $G_x$ is denoted by $x^\omega$. It is well known (see [14], for instance) that the element $x^\omega$ is well defined and $xx^\omega = x^\omega x \in G_x$. We denote the element inverse to $xx^\omega$ in $G_x$ by $\overline{x}$. The map $x \longmapsto \overline{x}$ is the just mentioned unary operation on an epigroup $S$. The element $\overline{x}$ is called the *pseudoinverse* of $x$. Throughout this paper, we consider epigroups as algebras with the operations of multiplication and pseudoinversion. This naturally leads to the concept of varieties of epigroups as algebras with these two operations. The idea to examine epigroups in the framework of the theory of varieties was promoted by L. N. Shevrin in [14] (see also [15]). An overview of the first results obtained here may be found in [16, Section 2]. The class of all epigroup varieties forms the lattice under the class-theoretical inclusion. As usual, operations in this lattice are defined as follows: the join of two varieties of epigroups $\mathcal{X}$ and $\mathcal{Y}$ is the least variety that contains both $\mathcal{X}$ and $\mathcal{Y}$, and the meet of two varieties of epigroups is merely their class-theoretical intersection.

If $S$ is a *completely regular* semigroup (i.e., a union of groups) and $x \in S$, then $\overline{x}$ is the element inverse to $x$ in the maximal subgroup containing $x$. Thus, the operation of pseudoinversion on a completely regular semigroup coincides with the unary operation traditionally considered on completely regular semigroups. We see that varieties of completely regular semigroups (considered as unary semigroups) are varieties of epigroups in the sense defined above. Further, it is well known and may be easily checked that in every periodic epigroup, the operation of pseudoinversion may be expressed in terms of multiplication (see [14], for instance). This means that periodic varieties of epigroups may be identified with periodic varieties of semigroups.

It seems to be very natural to examine all restrictions on semigroup varieties mentioned in Subsection 1.1 for epigroup varieties. This is realized in [26, 28] for identities and related restrictions to the subvariety lattice. In particular, epigroup varieties with modular, Arguesian or semimodular subvariety lattices are completely classified and epigroup analogs of results concerning semigroup varieties with distributive subvariety lattices are obtained there. In the present article, we start with an examination of special elements in the lattice **Epi** of all epigroup varieties. We consider here neutral, modular and upper-modular elements in **Epi**. For brevity, we call an epigroup variety *neutral* if it is a neutral element of the lattice **Epi**. An analogous convention will be applied for all other types of special elements. Our main results give:

- a complete description of neutral varieties;
- a strong necessary condition for modular varieties;

- a description of commutative modular varieties;
- a classification of strongly permutative upper-modular varieties.

We denote by $\mathcal{T}$, $\mathcal{SL}$ and $\mathcal{ZM}$ the trivial variety, the variety of all semilattices and the variety of all semigroups with zero multiplication, respectively. The first main result of the article is the following theorem.

**Theorem 1.1.** *For an epigroup variety $\mathcal{V}$, the following are equivalent:*

(a) *$\mathcal{V}$ is a neutral element of the lattice* **Epi***;*

(b) *$\mathcal{V}$ is simultaneously a modular, lower-modular and upper-modular element of the lattice* **Epi***;*

(c) *$\mathcal{V}$ coincides with one of the varieties $\mathcal{T}$, $\mathcal{SL}$, $\mathcal{ZM}$, or $\mathcal{SL} \vee \mathcal{ZM}$.*

In contrast, we note that the lattice of completely regular semigroup varieties contains infinitely many neutral elements including all band varieties, the varieties of all groups, all completely simple semigroups, all orthodox semigroups and some other (this readily follows from [17, Corollary 2.9]).

Theorem 1.1 is analogous to the description of neutral elements of the lattice **SEM** obtained by Volkov [31, Proposition 4.1].

To formulate the second result, we need some definitions. We follow the agreement that an adjective indicating a property shared by all semigroups of a given variety is applied to the variety itself; the expressions like 'completely regular variety', 'periodic variety', 'nilvariety', etc. are understood in this sense. A pair of identities $wx = xw = w$ where the letter $x$ does not occur in the word $w$ is usually written as the symbolic identity $w = 0$. This notation is justified, because a semigroup with such identities has a zero element and all values of the word $w$ in this semigroup are equal to zero. We will refer to the expression $w = 0$ as a single identity and call such identities 0-*reduced*. We call an identity $u = v$ *substitutive* if $u$ and $v$ are plain semigroup words (i.e., they do not contain the operation of pseudoinversion), these words depend on the same letters, and $v$ may be obtained from $u$ by renaming of letters. The second main result of the article is the following theorem.

**Theorem 1.2.** *If an epigroup variety $\mathcal{V}$ is a modular element of the lattice* **Epi***, then $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and $\mathcal{N}$ is a nilvariety given by* 0*-reduced and substitutive identities only.*

It is easy to see that this theorem completely reduces the problem of the description of modular varieties to nilvarieties defined by substitutive and 0-reduced identities only (see Corollary 3.2 below).

Theorem 1.2 is analogous to [19, Theorem 2.5] (that is a slightly stronger version of [6, Proposition 1.6]). Note that Theorem 2.5 of [19] was proved directly and in a simple way in [11].

The third result of the article is the following theorem.

**Theorem 1.3.** *A commutative epigroup variety $\mathcal{V}$ is a modular element of the lattice* **Epi** *if and only if $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or*

*$\mathcal{SL}$ and $\mathcal{N}$ is a nilvariety that satisfies the commutative law and the identity*

$$x^2y = 0. \tag{1.1}$$

Theorem 1.3 is analogous to the description of commutative semigroup varieties that are modular elements of the lattice **SEM** obtained in [19, Theorem 3.1].

Let $\Sigma$ be an identity system written in the language of one associative binary operation and one unary operation. The class of all epigroups that satisfy $\Sigma$ (where the unary operation is treated as pseudoinversion) is denoted by $K_\Sigma$. The class $K_\Sigma$ is not obliged to be a variety, because it may not be closed under taking of (infinite) direct products (see [15, Subsection 2.3], for instance). Note that the identity systems $\Sigma$ with the property that $K_\Sigma$ is a variety are completely determined in [4, Proposition 2.15]. If $K_\Sigma$ is a variety, then we use for this variety the standard notation $\operatorname{var}\Sigma$. It is evident that if the class $K_\Sigma$ consists of periodic epigroups, then it is a periodic semigroup variety and therefore, is an epigroup variety. Therefore, the notation $\operatorname{var}\Sigma$ is correct in this case. Put $\mathcal{C}_m = \operatorname{var}\{x^m = x^{m+1},\, xy = yx\}$ for an arbitrary natural number $m$. In particular, $\mathcal{C}_1 = \mathcal{SL}$. It will be convenient for us also to assume that $\mathcal{C}_0 = \mathcal{T}$. Recall that an identity of the form

$$x_1x_2\cdots x_n = x_{1\pi}x_{2\pi}\cdots x_{n\pi}$$

where $\pi$ is a non-trivial permutation on the set $\{1, 2, \dots, n\}$ is called *permutative*; if $1\pi \ne 1$ and $n\pi \ne n$, then this identity is said to be *strongly permutative*. A variety that satisfies a [strongly] permutative identity is also called [*strongly*] *permutative*. The fourth main result of the article is the following theorem.

**Theorem 1.4.** *A strongly permutative epigroup variety $\mathcal{V}$ is an upper-modular element of the lattice* **Epi** *if and only if one of the following holds:*

(i) *$\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$, and $\mathcal{N}$ is a nilvariety satisfying the commutative law and the identity*

$$x^2y = xy^2; \tag{1.2}$$

(ii) *$\mathcal{V} = \mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}$ where $\mathcal{G}$ is an abelian group variety, $0 \le m \le 2$ and $\mathcal{N}$ satisfies the commutative law and the identity* (1.1).

Note that, according to Theorem 1.2, all modular (in particular, all neutral) varieties are periodic. In contrast with this fact, Theorem 1.4 shows that there exist non-periodic upper-modular varieties. In particular, the variety of all abelian groups has this property. However, Theorem 1.4 can be considered as an analogue of the description of commutative semigroup varieties that are upper-modular elements of the lattice **SEM** given by [22, Theorem 1.2].

This article consists of five sections. In Section 2, we collect definitions, notation and auxiliary results used in what follows. Section 3 is devoted to the proof of Theorems 1.1–1.4. In Section 4, we collect several corollaries from our main results. Finally, in Section 5, we formulate several open questions.

## 2. Preliminaries

**2.1. Some properties of the operation of pseudoinversion.** It is well known (see [14, 15], for instance) that if $S$ is an epigroup and $x \in S$, then $x\,\overline{x} = x^\omega$. This allows us to replace expressions of the form $u\,\overline{u}$ in epigroup identities by $u^\omega$, for brevity. The following three lemmas are well known and may be easily checked.

**Lemma 2.1.** *The following identity holds in an epigroup $S$ if and only if $S$ is completely regular:*

$$x = \overline{\overline{x}} \tag{2.1}$$

**Lemma 2.2.** *If an epigroup variety $\mathcal{V}$ satisfies the identity $x^m = x^{m+1}$ for some natural $m$ then the identities $x^\omega = \overline{x} = \overline{\overline{x}} = x^m$ hold in $\mathcal{V}$.*

**Lemma 2.3.** *The following identity holds in an epigroup $S$ if and only if $S$ is a nil-semigroup:*

$$\overline{x} = 0 \tag{2.2}$$

**2.2. Identities of certain varieties.** We denote by $F$ the free unary semigroup over a countably infinite alphabet (with the operations $\cdot$ and $^{-}$). Elements of $F$ are called *words*. The symbol $\equiv$ stands for the equality relation on the unary semigroup $F$. If $w \in F$, then we denote by $c(w)$ the set of all letters occurring in $w$ and by $t(w)$ the last letter of $w$. A letter is called *simple* [*multiple*] *in a word* $w$ if it occurs in $w$ ones [at least twice]. We call a word a *semigroup word* if it does not include the operation of pseudoinversion. An identity is called a *semigroup identity* if both its parts are semigroup words. Put

$$\mathcal{P} = \mathrm{var}\{xy = x^2y,\, x^2y^2 = y^2x^2\} \quad \text{and} \quad \overleftarrow{\mathcal{P}} = \mathrm{var}\{xy = xy^2,\, x^2y^2 = y^2x^2\}.$$

The first two claims of the following lemma are well known and may be easily verified; the third one was proved in [1, Lemma 7].

**Lemma 2.4.** *A non-trivial semigroup identity $v = w$ holds:*

(i) *in the variety $\mathcal{SL}$ if and only if $c(v) = c(w)$;*

(ii) *in the variety $\mathcal{C}_2$ if and only if $c(v) = c(w)$ and every letter from $c(v)$ is either simple both in $v$ and $w$ or multiple both in $v$ and $w$;*

(iii) *in the variety $\mathcal{P}$ if and only if $c(v) = c(w)$ and either the letters $t(v)$ and $t(w)$ are multiple in $v$ and $w$, respectively, or $t(v) \equiv t(w)$ and the letter $t(v)$ is simple both in $v$ and $w$.*

If $w$ is a semigroup word, then $\ell(w)$ stands for the length of $w$; otherwise, we put $\ell(w) = \infty$. We also need the following three remarks about the identities of nil-semigroups.

**Lemma 2.5.** *Let $\mathcal{V}$ be a nilvariety.*

(i) *If the variety $\mathcal{V}$ satisfies an identity $u = v$ with $c(u) \neq c(v)$, then $\mathcal{V}$ also satisfies the identity $u = 0$.*
(ii) *If the variety $\mathcal{V}$ satisfies an identity of the form $u = vuw$ where the word $vw$ is non-empty, then $\mathcal{V}$ also satisfies the identity $u = 0$.*
(iii) *If the variety $\mathcal{V}$ satisfies an identity of the form $x_1x_2\cdots x_n = v$ and $\ell(v) \neq n$, then $\mathcal{V}$ also satisfies the identity $x_1x_2\cdots x_n = 0$.*

*Proof.* (i) and (ii): These are well known and can be easily verified.

(iii): If $v$ is a non-semigroup word, it suffices to refer to Lemma 2.3. Now let $v$ be a semigroup word. If $\ell(v) < n$, then $c(v) \neq \{x_1, x_2, \dots, x_n\}$, and the desired conclusion follows from the claim (i). Finally, if $\ell(v) > n$, the claim we want to prove readily follows from [9, Lemma 1]. □

**2.3. Decomposition of commutative varieties into the join of subvarieties.** As usual, we denote by $\mathrm{Gr}\,S$ the set of all group elements of an epigroup $S$. For an arbitrary epigroup variety $\mathcal{X}$, we put $\mathrm{Gr}(\mathcal{X}) = \mathcal{X} \wedge \mathcal{GR}$ where $\mathcal{GR}$ is the variety of all groups. Put

$$\mathcal{LZ} = \mathrm{var}\{xy = x\} \quad \text{and} \quad \mathcal{RZ} = \mathrm{var}\{xy = y\}.$$

By $\mathrm{var}\,S$, we denote the variety of epigroups generated by an epigroup $S$. The following two facts play an important role in the proof of Theorem 1.4. 'Semigroup prototypes' of Proposition 2.6 and Lemma 2.7 were given in [29, Proposition 1] and [5], respectively.

**Proposition 2.6.** *If $\mathcal{V}$ is an epigroup variety and $\mathcal{V}$ does not contain the varieties $\mathcal{LZ}$, $\mathcal{RZ}$, $\mathcal{P}$, and $\overleftarrow{\mathcal{P}}$, then $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is a variety generated by a monoid, and $\mathcal{N}$ is a nilvariety.*

*Proof.* It is verified in [29, Lemma 2] that if a semigroup variety does not contain the varieties $\mathcal{LZ}$, $\mathcal{RZ}$, $\mathcal{P}$, and $\overleftarrow{\mathcal{P}}$, then $\mathcal{V}$ satisfies the quasiidentity

$$e^2 = e \to ex = xe. \tag{2.3}$$

Repeating literally the proof of this claim (while using a term 'subepigroup' rather than 'subsemigroup'), one can establish that the similar claim is true for epigroup varieties. Thus, $\mathcal{V}$ satisfies the quasiidentity (2.3). The rest of the proof is quite similar to the proof of Proposition 1 in [29].

Let $S$ be an epigroup that generates the variety $\mathcal{V}$, $x \in S$, and $E$ the set of all idempotents from $S$. In view of (2.3), $ES$ is an ideal in $S$. By the definition of an epigroup, there is a natural $n$ such that $x^n \in \mathrm{Gr}\,S$. Then $x^n = x^\omega x^n$ and $x^\omega \in E$. We see that $x^n \in ES$. Therefore, the Rees quotient semigroup $S/ES$ is a nil-semigroup and therefore, is an epigroup. The natural homomorphism $\rho$ from $S$ onto $S/ES$ separates elements from $S \setminus ES$.

Now let $e \in E$. In view of (2.3), we have that $eS$ is a subsemigroup in $S$. It is well known and easy to see that every epigroup satisfies the identity $\overline{x} = x(\overline{x})^2$ (see [14, 15], for instance). Hence, the equality $\overline{ex} = ex(\overline{ex})^2$ holds. We have verified that for any $e \in E$, the set $eS$ is a subepigroup in

$S$. Put $S^* = \prod_{e\in E} eS$. Then $S^*$ is an epigroup with unit $(\dots, e, \dots)_{e\in E}$. It follows from (2.3) that the map $\varepsilon$ from $S$ into $S^*$ given by the rule $\varepsilon(x) = (\dots, ex, \dots)_{e\in E}$ is a semigroup homomorphism. It is well known (see [14, 15], for instance) that an arbitrary semigroup homomorphism $\xi$ from an epigroup $S_1$ into an epigroup $S_2$ is also an epigroup homomorphism (i.e., $\xi(\overline{a}) = \overline{\xi(a)}$ for any $a \in S_1$). Therefore, $\varepsilon$ is an epigroup homomorphism from $S$ into $S^*$. One can verify that $\varepsilon$ separates elements of $ES$. Let $e, f \in E$, $x, y \in S$, and $\varepsilon(ex) = \varepsilon(fy)$. Then $e \cdot ex = e \cdot fy$ and $f \cdot ex = f \cdot fy$. Since $e, f \in E$, we have

$$ex = efy \quad \text{and} \quad fex = fy. \tag{2.4}$$

Therefore,

$$ex \stackrel{(2.4)}{=} efy \stackrel{(2.4)}{=} efex \stackrel{(2.3)}{=} feex \stackrel{e\in E}{=} fex \stackrel{(2.4)}{=} fy.$$

We see that $ex = fy$ whenever $\varepsilon(ex) = \varepsilon(fy)$. This means that $\varepsilon$ separates elements of $ES$.

Thus, $\varepsilon$ and $\rho$ are homomorphisms from $S$ into $S^*$ and $S/ES$, respectively, and the intersection of kernels of these homomorphisms is trivial. Therefore, the epigroup $S$ is decomposable into a subdirect product of the epigroups $S^*$ and $S/ES$, whence $\mathcal{V} \subseteq \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M} = \operatorname{var} S^*$ is a variety generated by a monoid and $\mathcal{N} = \operatorname{var}(S/ES)$ is a nilvariety. On the other hand, $S^*, S/ES \in \mathcal{V}$, whence $\mathcal{M} \vee \mathcal{N} \subseteq \mathcal{V}$. We have proved that $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$. □

Recall that an epigroup is called *combinatorial* if all its subgroups are trivial. For any natural $n$, we denote by $C_n$ the $n$-element combinatorial cyclic monoid. Since $C_n$ is finite, it may be considered as an epigroup. It is well known and may be easily verified that the variety $\mathcal{C}_m$ is generated by the epigroup $C_{m+1}$.

**Lemma 2.7.** *If an epigroup variety $\mathcal{M}$ is generated by a commutative epigroup with unit, then $\mathcal{M} = \mathcal{G} \vee \mathcal{C}_m$ for some abelian group variety $\mathcal{G}$ and some $m \geq 0$.*

*Proof.* It is well known that the variety of all abelian groups is the least non-periodic epigroup variety. This variety evidently contains the infinite cyclic group. Further, for each natural $m$, let $G_m$ denote the cyclic group of order $m$. It is evident that if $\mathcal{M}$ is periodic, then the set $\{m \in \mathbb{N} \mid G_m \in \mathcal{M}\}$ has the greatest element. We denote by $G$ the infinite cyclic group whenever the variety $\mathcal{M}$ is non-periodic, and the finite cyclic group of the greatest order among all cyclic groups in $\mathcal{M}$ otherwise. In both the cases, $G \in \mathcal{M}$. Further, let $c_m$ be a generator of the monoid $C_m$. Put $X = \{m \in \mathbb{N} \mid C_m \in \mathcal{M}\}$. If the set $X$ does not have the greatest element, then the semigroup $\prod_{m\in X} C_m$ is not an epigroup since, for example, no power of the element $(\dots, c_m, \dots)_{m\in X}$ belongs to a subgroup. Therefore, the set of numbers $X$ contains the greatest number. We denote this number by $n$. Repeating literally arguments from the proof of Theorem 1 in [5], we have that every epigroup from $\mathcal{M}$ is a homomorphic image of some subepigroup of the epigroup $G \times C_n$. Therefore, $\mathcal{M} = \mathcal{G} \vee \mathcal{C}_{n-1}$ where $\mathcal{G} = \operatorname{var} G$. □

It is evident that a strongly permutative variety does not contain the varieties $\mathcal{LZ}$, $\mathcal{RZ}$, $\mathcal{P}$, and $\overleftarrow{\mathcal{P}}$. Besides that, every monoid satisfying a permutative identity is commutative. Thus, we have the following corollary of Proposition 2.6 and Lemma 2.7.

**Corollary 2.8.** *If $\mathcal{V}$ is a strongly permutative epigroup variety, then we have $\mathcal{V} = \mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}$ where $\mathcal{G}$ is an abelian group variety, $m \geq 0$, and $\mathcal{N}$ is a nilvariety.*

**2.4. A direct decomposition of one varietal lattice.** We denote by $\mathcal{AG}$ the variety of all abelian groups. Put

$$\mathcal{H} = \operatorname{var}\{x^2y = xyx = yx^2 = 0\}.$$

The aim of this subsection is to prove the following fact.

**Proposition 2.9.** *The lattice $L(\mathcal{AG} \vee \mathcal{C}_2 \vee \mathcal{H})$ is isomorphic to the direct product of the lattices $L(\mathcal{AG})$ and $L(\mathcal{C}_2 \vee \mathcal{H})$.*

*Proof.* We need two auxiliary statements.

**Lemma 2.10.** *If $\mathcal{X} \subseteq \mathcal{AG} \vee \mathcal{C}_2 \vee \mathcal{H}$, then $\mathcal{X} = \mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}$ where $\mathcal{G}$ is some abelian group variety, $0 \leq m \leq 2$, and $\mathcal{N} \subseteq \mathcal{H}$.*

*Proof.* Being a subvariety of the variety $\mathcal{AG} \vee \mathcal{C}_2 \vee \mathcal{H}$, the variety $\mathcal{X}$ satisfies the identity $x^2y = yx^2$. It is evident that this identity fails in the varieties $\mathcal{LZ}$ and $\mathcal{RZ}$. Further, Lemma 2.4(iii) and the dual statement imply that this identity is false in the varieties $\mathcal{P}$ and $\overleftarrow{\mathcal{P}}$ as well. Therefore, none of the four mentioned varieties is contained in $\mathcal{X}$. Besides that, the variety $\mathcal{AG} \vee \mathcal{C}_2 \vee \mathcal{H}$ (and therefore, $\mathcal{X}$) satisfies the identity $x^2yz = x^2zy$. Substituting 1 for $x$, we have that all monoids in $\mathcal{X}$ are commutative. Proposition 2.6 and Lemma 2.7 imply now that $\mathcal{X} = \mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}$ for some abelian group variety $\mathcal{G}$, some $m \geq 0$, and some nilvariety $\mathcal{N}$. It is evident that $\mathcal{G} \subseteq \mathcal{AG}$. Lemmas 2.1, 2.2, and 2.3 imply that $\mathcal{AG}$, $\mathcal{C}_m$, and $\mathcal{H}$ satisfy the identities (2.1), $\overline{\overline{x}} = x^m$, and (2.2), respectively. Therefore, the identity $x^2y = \overline{\overline{x}}^{\,2}y$ holds in the variety $\mathcal{AG} \vee \mathcal{C}_2 \vee \mathcal{H}$. But this identity is false in the variety $\mathcal{C}_m$ with $m > 2$. Hence, $m \leq 2$. This implies that the variety $\mathcal{AG} \vee \mathcal{C}_m \vee \mathcal{H}$ satisfies the identities $x^2y = yx^2 = \overline{\overline{x}}^{\,2}y$ and $xyx = xy\overline{\overline{x}}$. Since $\mathcal{N} \subseteq \mathcal{X} \subseteq \mathcal{AG} \vee \mathcal{C}_2 \vee \mathcal{H}$, Lemma 2.3 implies that the variety $\mathcal{N}$ satisfies the identities $x^2y = xyx = yx^2 = 0$, whence $\mathcal{N} \subseteq \mathcal{H}$. □

Note that if $\mathcal{X}$ is an arbitrary epigroup variety, then the class of all epigroups in $\mathcal{X}$ satisfying the identity (2.2) is the greatest nilsubvariety of $\mathcal{X}$. We denote this subvariety by $\operatorname{Nil}(\mathcal{X})$.

**Lemma 2.11.** *Let $\mathcal{G}$ be an abelian group variety, $0 \leq m \leq 2$, and $\mathcal{N}_1, \mathcal{N}_2$ nilvarieties with $\operatorname{Nil}(\mathcal{C}_m) \subseteq \mathcal{N}_i \subseteq \mathcal{H}$ for $i = 1, 2$. If $\mathcal{N}_1 \neq \mathcal{N}_2$, then*

$$\mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}_1 \neq \mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}_2. \tag{2.5}$$

*Proof.* We say that varieties $\mathcal{X}_1$ and $\mathcal{X}_2$ *differ by an identity* $u = v$ if this identity holds in one of the varieties $\mathcal{X}_1$ or $\mathcal{X}_2$ but fails in another one. Since $\mathcal{N}_1 \neq \mathcal{N}_2$, the varieties $\mathcal{N}_1$ and $\mathcal{N}_2$ differ by some identity. We may assume without loss of generality that this identity holds in $\mathcal{N}_1$ but fails in $\mathcal{N}_2$. First, suppose that the identity we mention is a 0-reduced identity $u = 0$. Since $\mathcal{N}_2 \subseteq \mathcal{H}$, this identity fails in $\mathcal{H}$. In view of Lemma 2.3 and the definition of the variety $\mathcal{H}$, there are only two possibilities for the word $u$: either $u \equiv x_1x_2\cdots x_n$ for some natural $n$ or $u \equiv x^2$. Suppose that $u \equiv x_1x_2\cdots x_n$ for some $n$. Then the identity $x_1x_2\cdots x_n = 0$ holds in $\mathcal{N}_1$ but fails in $\mathcal{N}_2$. If $m = 2$, then the variety $\mathcal{N}_1$ contains the variety $\mathrm{Nil}(\mathcal{C}_2) = \mathrm{var}\{x^2 = 0,\, xy = yx\}$. The latter variety does not satisfy the identity $x_1x_2\cdots x_n = 0$. Therefore, $m \leq 1$. Then the variety $\mathcal{G} \vee \mathcal{C}_m$ is completely regular. Lemmas 2.1 and 2.3 imply now that the variety $\mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}_1$ satisfies the identity $x_1x_2\cdots x_n = \overline{\overline{x_1}}x_2\cdots x_n$. But Lemma 2.3 implies that this identity is false in $\mathcal{N}_2$ and therefore, in $\mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}_2$. Thus, (2.5) holds. Now let $u \equiv x^2$. Then Lemmas 2.1, 2.2, and 2.3 imply that the identity $x^2 = \overline{\overline{x}}^2$ holds in the variety $\mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}_1$ but is false in the variety $\mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}_2$. We see again that the inequality (2.5) holds.

It remains to consider the case when $\mathcal{N}_1$ and $\mathcal{N}_2$ differ by some non-0-reduced identity $u = v$. Suppose that $c(u) \neq c(v)$. Lemma 2.5(i) then implies that the variety $\mathcal{N}_1$ satisfies both the identities $u = 0$ and $v = 0$. Then the variety $\mathcal{N}_2$ does not satisfy at least one of them, because $\mathcal{N}_1$ and $\mathcal{N}_2$ do not differ with $u = v$ otherwise. We see that $\mathcal{N}_1$ and $\mathcal{N}_2$ differ by some 0-reduced identity, and we go to the situation considered in the previous paragraph. Now let $c(u) = c(v)$. Suppose that the identity $u = 0$ holds in $\mathcal{H}$. Since $\mathcal{N}_1, \mathcal{N}_2 \subseteq \mathcal{H}$, we have that the identity $u = 0$ holds in both the varieties $\mathcal{N}_1$ and $\mathcal{N}_2$. Then $\mathcal{N}_1$ satisfies also the identity $v = 0$. But $v = 0$ fails in $\mathcal{N}_2$, because $u = v$ holds in $\mathcal{N}_2$ otherwise. We see that the varieties $\mathcal{N}_1$ and $\mathcal{N}_2$ differ by some 0-reduced identity. This case has already been considered in the previous paragraph. Thus, the identity $u = 0$ fails in $\mathcal{H}$. Analogously, $v = 0$ fails in $\mathcal{H}$. A semigroup word $w$ is called *linear* if each letter from $c(w)$ is simple in $w$. All non-semigroup words as well as all non-linear semigroup words except $x^2$ equal to 0 in $\mathcal{H}$. Since the identity $u = v$ is non-trivial and $c(u) = c(v)$, the words $u$ and $v$ are linear. Using the fact that $c(u) = c(v)$ again, we have that the identity $u = v$ is permutative. Then it is evident that this identity holds in $\mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}_1$ but fails in $\mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}_2$. We have proved that the inequality (2.5) holds. □

Now we start the direct proof of Proposition 2.9. Let $\mathcal{V} \subseteq \mathcal{AG} \vee \mathcal{C}_2 \vee \mathcal{H}$. In view of Lemma 2.10, $\mathcal{V} = \mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}$ for some abelian group variety $\mathcal{G}$, some $0 \leq m \leq 2$, and some variety $\mathcal{N}$ with $\mathcal{N} \subseteq \mathcal{H}$. Put $\mathcal{U} = \mathcal{C}_m \vee \mathcal{N}$. We have that $\mathcal{V} = \mathcal{G} \vee \mathcal{U}$ where $\mathcal{G} \subseteq \mathcal{AG}$ and $\mathcal{U} \subseteq \mathcal{C}_2 \vee \mathcal{H}$. It remains to establish that this decomposition of the variety $\mathcal{V}$ into the join of some subvariety of the variety $\mathcal{AG}$ and some subvariety of the variety $\mathcal{C}_2 \vee \mathcal{H}$ is unique.

Let $\mathcal{V} = \mathcal{G}' \vee \mathcal{U}'$ where $\mathcal{G}' \subseteq \mathcal{AG}$ and $\mathcal{U}' \subseteq \mathcal{C}_2 \vee \mathcal{H}$. We need to verify that $\mathcal{G} = \mathcal{G}'$ and $\mathcal{U} = \mathcal{U}'$. Let $u = v$ be an arbitrary identity satisfied by $\mathcal{G}$. The variety $\mathcal{U}$ satisfies the identity $x^3 = x^4$. Then the identity $u^4v^3 = u^3v^4$ holds in the variety $\mathcal{G} \vee \mathcal{U}$. Let us cancel this identity on $u^3$ from the left and on $v^3$ from the right, thus concluding that $u = v$ holds in $\mathrm{Gr}(\mathcal{G} \vee \mathcal{U})$. Therefore, $\mathrm{Gr}(\mathcal{G} \vee \mathcal{U}) \subseteq \mathcal{G}$. The opposite inclusion is evident. Thus, $\mathrm{Gr}(\mathcal{G} \vee \mathcal{U}) = \mathcal{G}$. Analogously, $\mathrm{Gr}(\mathcal{G}' \vee \mathcal{U}') = \mathcal{G}'$. We see that

$$\mathcal{G} = \mathrm{Gr}(\mathcal{G} \vee \mathcal{U}) = \mathrm{Gr}(\mathcal{V}) = \mathrm{Gr}(\mathcal{G}' \vee \mathcal{U}') = \mathcal{G}'.$$

It remains to check that $\mathcal{U} = \mathcal{U}'$. Recall that $\mathcal{U} = \mathcal{C}_m \vee \mathcal{N}$ where $0 \leq m \leq 2$ and $\mathcal{N} \subseteq \mathcal{H}$, while $\mathcal{U}' \subseteq \mathcal{C}_2 \vee \mathcal{H}$. It is evident that the variety $\mathcal{C}_2 \vee \mathcal{H}$ (and therefore, $\mathcal{U}'$) is combinatorial. Thus, Lemma 2.10 implies that $\mathcal{U}' = \mathcal{C}_k \vee \mathcal{N}'$ for some $0 \leq k \leq 2$ and some variety $\mathcal{N}'$ with $\mathcal{N}' \subseteq \mathcal{H}$.

Suppose that $m \neq k$. Without loss of generality, we may assume that $m < k$, i.e., either $m = 0$ and $1 \leq k \leq 2$, or $m = 1$ and $k = 2$. First, suppose that $m = 0$ and $1 \leq k \leq 2$. It is evident that any group satisfies the identity $x^\omega = y^\omega$. Lemma 2.3 implies that this identity holds in $\mathcal{N}$ and therefore, in $\mathcal{V} = \mathcal{G} \vee \mathcal{T} \vee \mathcal{N} = \mathcal{G} \vee \mathcal{N}$. But Lemma 2.4(ii) implies that this identity fails in the variety $\mathcal{SL}$. However, this is impossible, because $\mathcal{SL} \subseteq \mathcal{G}' \vee \mathcal{C}_k \vee \mathcal{N}' = \mathcal{V}$. Suppose now that $m = 1$ and $k = 2$. Then Lemmas 2.1, 2.2, and 2.3 imply that the identity $x^2y = x^2\,\overline{\overline{y}}$ holds in the variety $\mathcal{V} = \mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N} = \mathcal{G} \vee \mathcal{SL} \vee \mathcal{N}$. But this identity is false in the variety $\mathcal{C}_2$ (and therefore, in $\mathcal{G}' \vee \mathcal{C}_k \vee \mathcal{N}' = \mathcal{V}$) by Lemmas 2.2 and 2.4(ii). To avoid a contradiction, we conclude that $m = k$. Thus $\mathcal{U} = \mathcal{C}_m \vee \mathcal{N}$ and $\mathcal{U}' = \mathcal{C}_m \vee \mathcal{N}'$.

Put $\overline{\mathcal{N}} = \mathrm{Nil}(\mathcal{U})$ and $\overline{\mathcal{N}'} = \mathrm{Nil}(\mathcal{U}')$. Then $\mathcal{U} = \mathcal{C}_m \vee \overline{\mathcal{N}}$ and $\mathcal{U}' = \mathcal{C}_m \vee \overline{\mathcal{N}'}$, whence $\mathcal{G} \vee \mathcal{C}_m \vee \overline{\mathcal{N}} = \mathcal{G} \vee \mathcal{U} = \mathcal{V} = \mathcal{G} \vee \mathcal{U}' = \mathcal{G} \vee \mathcal{C}_m \vee \overline{\mathcal{N}'}$. Then Lemma 2.11 applies and we conclude that $\overline{\mathcal{N}} = \overline{\mathcal{N}'}$, whence $\mathcal{U} = \mathcal{U}'$. □

Analog of Proposition 2.9 for semigroup varieties was proved in [18, Proposition 2a] (namely, it was checked there that $L(\mathcal{G} \vee \mathcal{C}_2 \vee \mathcal{H}) \cong L(\mathcal{G}) \times L(\mathcal{C}_2 \vee \mathcal{H})$ where $\mathcal{G}$ is an abelian periodic group variety). The proof of Proposition 2.9 given above is quite similar to the proof of the mentioned result from [18]. But the results of [18] were not used directly above. Therefore, the mentioned result from [18] may be considered now to be a consequence of Proposition 2.9.

**2.5. Varieties of finite degree.** If $n$ is a natural number, then a variety $\mathcal{X}$ is called a *variety of degree* $n$ if all nil-semigroups in $\mathcal{X}$ are nilpotent of degree $\leq n$ and $n$ is the least number with such a property. If $\mathcal{X}$ is not a variety of degree $\leq n$, we will say that $\mathcal{X}$ is a variety of *degree* $> n$. A variety is said to be a *variety of finite degree* if it is a variety of degree $n$ for some $n$. If $\mathcal{V}$ is a variety of finite degree, we denote the degree of $\mathcal{V}$ by $\deg(\mathcal{V})$; otherwise, we put $\deg(\mathcal{V}) = \infty$. We need the following result.

**Proposition 2.12** ([4, Corollary 1.3])**.** *Let $n$ be an arbitrary natural number. For an epigroup variety $\mathcal{V}$, the following are equivalent:*

1) $\deg(\mathcal{V}) \le n$;
2) $\mathcal{V} \not\supseteq \operatorname{var}\{x^2 = x_1x_2\cdots x_{n+1} = 0,\ xy = yx\}$;
3) $\mathcal{V}$ *satisfies an identity of the form*

$$x_1\cdots x_n = x_1\cdots x_{i-1}\cdot\overline{\overline{x_i\cdots x_j}}\cdot x_{j+1}\cdots x_n \tag{2.6}$$

*for some* $i$ *and* $j$ *with* $1 \le i \le j \le n$.

Proposition 2.12 readily implies the following assertion.

**Corollary 2.13.** *If* $\mathcal{X}$ *and* $\mathcal{Y}$ *are arbitrary epigroup varieties, then the following equality holds:* $\deg(\mathcal{X}\wedge\mathcal{Y}) = \min\{\deg(\mathcal{X}),\deg(\mathcal{Y})\}$.

The following corollary may be proved quite analogously to Corollary 2.13 of [22] by referring to Proposition 2.12 rather than Proposition 2.11 of [22].

**Corollary 2.14.** *If* $\mathcal{V}$ *is an arbitrary epigroup variety and* $\mathcal{N}$ *is a nilvariety, then the following equality holds:* $\deg(\mathcal{V}\vee\mathcal{N}) = \max\{\deg(\mathcal{V}),\deg(\mathcal{N})\}$.

Note that the analog of Corollary 2.14 for arbitrary epigroup varieties is wrong even in the periodic case. For instance, it is easy to deduce from Lemma 2.4(iii), the dual fact, and Proposition 2.12 that $\deg(\mathcal{P}) = \deg\big(\overleftarrow{\mathcal{P}}\big) = 2$ but $\deg\big(\mathcal{P}\vee\overleftarrow{\mathcal{P}}\big) = 3$.

Proposition 2.12 and Lemma 2.1 easily imply the following fact.

**Corollary 2.15.** *If* $\mathcal{V}$ *is an arbitrary epigroup variety and* $\mathcal{X}$ *is a completely regular variety, then* $\deg(\mathcal{V}\vee\mathcal{X}) = \deg(\mathcal{V})$.

**2.6. Some properties of special elements of lattices.** An element of a lattice $L$ is called *distributive* if $x\vee(y\wedge z) = (x\vee y)\wedge(x\vee z)$ for all $y,z\in L$. *Codistributive* elements are defined dually. The following claim is well known. It may be obtained by a combination of [3, Lemma 1 in Section 1 of Chapter II], the dual statement and [2, Theorem 255 on p. 228].

**Lemma 2.16.** *An element of a lattice* $L$ *is neutral in* $L$ *if and only if it is distributive, codistributive, and modular in* $L$.

The following assertion is a very special case of [10, Corollary 2.1].

**Lemma 2.17.** *Let* $L$ *be a lattice with* $0$ *and* $a$ *an atom which is a neutral element of the lattice* $L$. *Then* $x\in L$ *is an* [*upper-, lower-*]*modular element in* $L$ *if and only if* $x\vee a$ *has the same property.*

**2.7. $\mathcal{SL}$ and $\mathcal{ZM}$ are atoms.** It is evident that the atoms of the lattice **Epi** coincide with the atoms of the lattice **SEM**. The list of the atoms of the latter lattice is generally known (see [16], for instance). In particular, the following is valid.

**Lemma 2.18.** *The varieties* $\mathcal{SL}$ *and* $\mathcal{ZM}$ *are the atoms of the lattice* **Epi**.

## 3. Main results

This section contains the proofs of all four theorems. It is divided into five subsections. We start with a verification of a special case of Theorem 1.1, namely the claim that the variety $\mathcal{SL}$ is neutral (Subsection 3.1). This fact is used in the proofs of Theorems 1.1, 1.3, and 1.4. After that, we prove Theorem 1.2 (Subsection 3.2). We start with this theorem, because we need it to verify Theorem 1.1. Since we have started to examine modular varieties, we consider them also in the following subsection. Namely, in Subsection 3.3, we verify Theorem 1.3. After that, we prove Theorems 1.1 and 1.4 in Subsections 3.4 and 3.5, respectively.

**3.1. The variety $\mathcal{SL}$ is neutral.** Here we prove the following fact.

**Proposition 3.1.** *The variety $\mathcal{SL}$ is a neutral element of the lattice* **Epi**.

*Proof.* In view of Lemma 2.16, it suffices to verify that the variety $\mathcal{SL}$ is distributive, codistributive, and modular. Let $\mathcal{X}$ and $\mathcal{Y}$ be arbitrary epigroup varieties.

*Distributivity.* We need to verify the inclusion

$$(\mathcal{SL} \vee \mathcal{X}) \wedge (\mathcal{SL} \vee \mathcal{Y}) \subseteq \mathcal{SL} \vee (\mathcal{X} \wedge \mathcal{Y}),$$

because the opposite inclusion is evident. Suppose that an identity $u = v$ holds in $\mathcal{SL} \vee (\mathcal{X} \wedge \mathcal{Y})$. In particular, it holds in $\mathcal{SL}$, whence $c(u) = c(v)$ by Lemma 2.4(i). Let $u \equiv w_0, w_1, \ldots, w_n \equiv v$ be a deduction of this identity from the identities of the varieties $\mathcal{X}$ and $\mathcal{Y}$. Further considerations are given by induction on $n$.

*Induction base.* If $n = 1$, then the identity $u = v$ holds in one of the varieties $\mathcal{X}$ or $\mathcal{Y}$. Whence, it holds in one of the varieties $\mathcal{SL} \vee \mathcal{X}$ or $\mathcal{SL} \vee \mathcal{Y}$, and therefore $(\mathcal{SL} \vee \mathcal{X}) \wedge (\mathcal{SL} \vee \mathcal{Y})$ satisfies $u = v$.

*Induction step.* Now let $n > 1$. Consider the words $w'_1, \ldots, w'_{n-1}$ obtained from the words $w_1, \ldots, w_{n-1}$, respectively, by equating all the letters that do not not occur in $u$ to some letter in $c(u)$. Clearly, the sequence of words $u, w'_1, \ldots, w'_{n-1}, v$ is also a deduction of the identity $u = v$ from the identities of the varieties $\mathcal{X}$ and $\mathcal{Y}$. Thus, we may assume that $c(w_1), \ldots, c(w_{n-1}) \subseteq c(u) = c(v)$. If $c(w_0) = c(w_1) = \cdots = c(w_n)$, then the sequence $w_0, w_1, \ldots, w_n$ is a deduction of the identity $u = v$ from the identities of the varieties $\mathcal{SL} \vee \mathcal{X}$ and $\mathcal{SL} \vee \mathcal{Y}$, and we are done. Suppose now that $c(w_k) \neq c(w_{k+1})$ for some $0 \leq k \leq n-1$. Let $i$ be the least index with $c(w_i) \neq c(w_{i+1})$ and $j$ be the greatest index with $c(w_j) \neq c(w_{j-1})$. Suppose that $i > 0$. Then $c(w_i) = c(u) = c(v)$ and consequences of words $w_0, w_1, \ldots, w_i$ and $w_i, w_{i+1}, \ldots, w_n$ are deductions of the identities $u = w_i$ and $w_i = v$, respectively, from the identities of the varieties $\mathcal{X}$ and $\mathcal{Y}$. Lemma 2.4(i) implies now that the identities $u = w_i$ and $w_i = v$ hold in the variety $\mathcal{SL} \vee (\mathcal{X} \wedge \mathcal{Y})$. By induction assumption, these identities hold also in the variety $(\mathcal{SL} \vee \mathcal{X}) \wedge (\mathcal{SL} \vee \mathcal{Y})$. Whence, the last variety satisfies the identity $u = v$ too. The case when $j < n$ may be considered quite analogously. Thus, we may suppose that $i = 0$ and $j = n$. In other words, $c(u) \ne c(w_1)$ and $c(v) \ne c(w_{n-1})$.

The identity $u = w_1$ holds in one of the varieties $\mathcal{X}$ and $\mathcal{Y}$. Suppose that it holds in $\mathcal{X}$. Since $c(u) \ne c(w_1)$, Lemma 2.4(i) implies that $\mathcal{SL} \nsubseteq \mathcal{X}$. Let $S$ be an epigroup in $\mathcal{X}$ and $\zeta$ a homomorphism from $F$ to $S$. For a word $w$, we denote by $w^\zeta$ the image of $w$ under $\zeta$. It is well known (see [14, 15], for instance) that a variety that does not contain $\mathcal{SL}$ consists of archimedean epigroups. Further, the set of group elements in an archimedean epigroup is an ideal of this epigroup. In particular, this is the case for the epigroup $S$. Now we are going to check that $u^\zeta \in \mathrm{Gr}\, S$. Since $c(w_1) \subset c(u)$, there is a letter $x \in c(u) \setminus c(w_1)$. Substituting $x^\omega$ for $x$ in the identity $u = w_1$, we obtain the identity $u' = w_1$ that holds in $\mathcal{X}$. Therefore, $\mathcal{X}$ satisfies the identity $u = u'$. The word $u'$ contains a subword $x^\omega$. Since $(x^\omega)^\zeta \in \mathrm{Gr}\, S$ and $\mathrm{Gr}\, S$ is an ideal in $S$, we have that $u^\zeta \in \mathrm{Gr}\, S$. Therefore, $\mathcal{X}$ satisfies the identity $u = uu^\omega$. Similar arguments show that the identity $u = uu^\omega$ holds in $\mathcal{Y}$ whenever $\mathcal{Y}$ satisfies $u = w_1$, and that one of the varieties $\mathcal{X}$ and $\mathcal{Y}$ satisfies the identity $v = vv^\omega$. Therefore, the sequence of words

$$u, uu^\omega, w_1u^\omega, \ldots, w_{n-1}u^\omega, vu^\omega, vw_1^\omega, \ldots, vw_{n-1}^\omega, vv^\omega, v$$

is a deduction of the identity $u = v$ from the identities of the varieties $\mathcal{SL} \vee \mathcal{X}$ and $\mathcal{SL} \vee \mathcal{Y}$. Hence, this identity holds in $(\mathcal{SL} \vee \mathcal{X}) \wedge (\mathcal{SL} \vee \mathcal{Y})$.

*Codistributivity.* In view of Lemma 2.18, if $\mathcal{W}$ is an arbitrary epigroup variety, then either $\mathcal{W} \supseteq \mathcal{SL}$ or $\mathcal{W} \wedge \mathcal{SL} = \mathcal{T}$. We need to verify that

$$\mathcal{SL} \wedge (\mathcal{X} \vee \mathcal{Y}) = (\mathcal{SL} \wedge \mathcal{X}) \vee (\mathcal{SL} \wedge \mathcal{Y}).$$

Clearly, each part of this equality coincides with $\mathcal{SL}$ whenever at least one of the varieties $\mathcal{X}$ or $\mathcal{Y}$ contains $\mathcal{SL}$. It remains to verify that if $\mathcal{X} \nsupseteq \mathcal{SL}$ and $\mathcal{Y} \nsupseteq \mathcal{SL}$, then $\mathcal{X} \vee \mathcal{Y} \nsupseteq \mathcal{SL}$. This claim immediately follows from the fact that there is a non-trivial identity $u = v$ such that an epigroup variety $\mathcal{W}$ does not contain the variety $\mathcal{SL}$ if and only if $\mathcal{W}$ satisfies the identity $u = v$ (in particular, the identity $(x^\omega y^\omega x^\omega)^\omega = x^\omega$ has such property, see [15, Corollary 3.2], for instance).

*Modularity.* Let $\mathcal{X} \subseteq \mathcal{Y}$. We need to verify that

$$(\mathcal{SL} \vee \mathcal{X}) \wedge \mathcal{Y} \subseteq (\mathcal{SL} \wedge \mathcal{Y}) \vee \mathcal{X},$$

because the opposite inclusion is evident. If $\mathcal{SL} \subseteq \mathcal{Y}$, then each part of the inclusion evidently coincides with $\mathcal{SL} \vee \mathcal{X}$. Now let $\mathcal{SL} \nsubseteq \mathcal{Y}$. Then Lemma 2.18 implies that $\mathcal{SL} \wedge \mathcal{Y} = \mathcal{T}$, whence $(\mathcal{SL} \wedge \mathcal{Y}) \vee \mathcal{X} = \mathcal{X}$. Suppose that an identity $u = v$ holds in the variety $\mathcal{X}$. It suffices to verify that this identity holds in $(\mathcal{SL} \vee \mathcal{X}) \wedge \mathcal{Y}$, too. If $c(u) = c(v)$, then $u = v$ holds in $\mathcal{SL}$ by Lemma 2.4(i). Whence it is satisfied in $\mathcal{SL} \vee \mathcal{X}$, and we are done. Now let $c(u) \ne c(v)$. Since $\mathcal{SL} \nsubseteq \mathcal{Y}$, Lemma 2.4(i) implies that $\mathcal{Y}$ satisfies an identity $s = t$ with $c(s) \ne c(t)$. Without loss of generality, we may assume that there is a letter

$y \in c(t) \setminus c(s)$. Moreover, we may assume that $c(s) = \{x\}$ and $c(t) = \{x, y\}$ (if this is not the case, then we equate all letters but $y$ to $x$ in the identity $s = t$ and multiply the resulting identity by $x$ on the right). Let $s_1$ (respectively, $t_1$) be the word obtained from the word $s$ (respectively, $t$) by swapping $x$ and $y$. Clearly, the identity $s_1 = t_1$ follows from the identity $s = t$.

Consider the case when the word $v$ can be obtained from $u$ by replacing all occurrences of some letter $x \in c(u)$ by a different letter $y$. Let $u_1$, $u_2$, $u_3$, and $u_4$ be the words obtained from $u$ by substituting the words $s$, $s_1$, $t$, and $t_1$, respectively, for the letter $x$. Since $x \notin c(v)$, the identities $u_1 = v$, $u_2 = v$, $u_3 = v$, and $u_4 = v$ are obtained from $u = v$ by the same substitutions. Therefore, the words $u$, $v$, $u_1$, $u_2$, $u_3$, and $u_4$ are equal each to other in the variety $\mathcal{X}$. Further, $c(u) = c(u_1)$, $c(v) = c(u_2)$, and $c(u_3) = c(u_4)$, whence the identities $u = u_1$, $v = u_2$, and $u_3 = u_4$ hold in $\mathcal{SL} \vee \mathcal{X}$ by Lemma 2.4(i). The identities $u_1 = u_3$ and $u_2 = u_4$ follow from $s = t$ and $s_1 = t_1$, respectively. Hence these identities hold in $\mathcal{Y}$. Thus, the sequence of words $u, u_1, u_3, u_4, u_2, v$ is a deduction of the identity $u = v$ from the identities of the varieties $\mathcal{SL} \vee \mathcal{X}$ and $\mathcal{Y}$.

Finally, consider an arbitrary identity $u = v$ that holds in $\mathcal{X}$. Replacing one by one all the letters from $c(u) \setminus c(v)$ by some letters of $c(v)$, we obtain the sequence $u, w_1, \dots, w_m$, in which any adjacent words differ by replacing one letter. In the sequence of identities $u = v, w_1 = v, \dots, w_m = v$, every identity (except the first one) is obtained from the previous one by replacing one letter. Therefore, the words $u, w_1, \dots, w_m$ are equal each to other in $\mathcal{X}$. As we have proved above, this implies that the identities $u = w_1 = \cdots = w_m$ hold in $(\mathcal{SL} \vee \mathcal{X}) \wedge \mathcal{Y}$. Analogously, if we replace in the identity $v = w_m$ all letters from $c(v) \setminus c(w_m)$ by an arbitrary letter from $c(w_m)$, then we obtain a sequence of identities $v = w'_1 = \cdots = w'_n$ that hold in $(\mathcal{SL} \vee \mathcal{X}) \wedge \mathcal{Y}$ as well. In particular, these identities hold in $\mathcal{X}$. Moreover, since $c(w_m) = c(w'_n)$, Lemma 2.4(i) implies that the identity $w_m = w'_n$ holds in $\mathcal{SL} \vee \mathcal{X}$. Therefore, the identity $u = v$ holds in $(\mathcal{SL} \vee \mathcal{X}) \wedge \mathcal{Y}$. □

For convenience of references, we formulate the following fact that immediately follows from Lemmas 2.17, 2.18, and Proposition 3.1.

**Corollary 3.2.** *An epigroup variety $\mathcal{X}$ is an* [*upper-, lower-*]*modular element of the lattice* **Epi** *if and only if the variety $\mathcal{X} \vee \mathcal{SL}$ has the same property.*

**3.2. Modular varieties: the necessary condition.** Here we are going to prove Theorem 1.2. We need several auxiliary facts.

**Proposition 3.3.** *If an epigroup variety $\mathcal{V}$ is a modular element of the lattice* **Epi**, *then $\mathcal{V}$ is periodic.*

*Proof.* Let $\mathcal{V}$ be a modular epigroup variety. Suppose that $\mathcal{V}$ is non-periodic. Being an epigroup variety, $\mathcal{V}$ satisfies the identity $x^n = x^n x^\omega$ for some natural

$n$. Consider varieties

$$\mathcal{N}_1 = \mathrm{var}\{x_1x_2\cdots x_{n+3} = 0\},$$
$$\mathcal{N}_2 = \mathrm{var}\{x_1x_2\cdots x_{n+3} = 0,\ x^{n+1}y = x^ny^2\}.$$

To prove that $\mathcal{V}$ is non-modular, we are going to check that the varieties $\mathcal{V}$, $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{V} \vee \mathcal{N}_1$, and $\mathcal{V} \wedge \mathcal{N}_1$ form the 5-element non-modular sublattice $N_5$. Note that $\mathcal{N}_2 \subseteq \mathcal{N}_1$. Whence, to achieve our aim, it suffices to verify the equalities $\mathcal{V} \vee \mathcal{N}_1 = \mathcal{V} \vee \mathcal{N}_2$ and $\mathcal{V} \wedge \mathcal{N}_1 = \mathcal{V} \wedge \mathcal{N}_2$.

The inclusion $\mathcal{V} \vee \mathcal{N}_2 \subseteq \mathcal{V} \vee \mathcal{N}_1$ is evident. It is also evident that a non-trivial identity $u = v$ holds in $\mathcal{N}_2$ if and only if either $\ell(u), \ell(v) \geq n+3$ or $u = v$ coincides with the identity $x^{n+1}y = x^ny^2$. Let $u = v$ be a non-trivial identity that is satisfied by the variety $\mathcal{V} \vee \mathcal{N}_2$. Substituting $y^2$ for $y$ in the identity $x^{n+1}y = x^ny^2$, we have $x^{n+1}y^2 = x^ny^4$, which implies $x^{n+3} = x^{n+4}$. Therefore, a variety satisfying $x^{n+1}y = x^ny^2$ is periodic. Since the identity $u = v$ holds in a non-periodic variety $\mathcal{V}$, it differs from the identity $x^{n+1}y = x^ny^2$. Therefore, $\ell(u), \ell(v) \geq n+3$. This implies that $u = v$ holds in $\mathcal{N}_1$, and therefore in $\mathcal{V} \vee \mathcal{N}_1$. Thus, $\mathcal{V} \vee \mathcal{N}_1 \subseteq \mathcal{V} \vee \mathcal{N}_2$. The equality $\mathcal{V} \vee \mathcal{N}_1 = \mathcal{V} \vee \mathcal{N}_2$ is proved.

The inclusion $\mathcal{V} \wedge \mathcal{N}_2 \subseteq \mathcal{V} \wedge \mathcal{N}_1$ is evident. The variety $\mathcal{V} \wedge \mathcal{N}_1$ is a nilvariety and is contained in $\mathcal{V}$. Since $\mathcal{V}$ satisfies $x^n = x^nx^\omega$, Lemma 2.5(ii) implies that $x^n = 0$ holds in $\mathcal{V} \wedge \mathcal{N}_1$. Therefore, $x^{n+1}y = 0 = x^ny^2$ in $\mathcal{V} \wedge \mathcal{N}_1$. We see that $\mathcal{V} \wedge \mathcal{N}_1 \subseteq \mathcal{N}_2$, whence $\mathcal{V} \wedge \mathcal{N}_1 \subseteq \mathcal{V} \wedge \mathcal{N}_2$. The equality $\mathcal{V} \wedge \mathcal{N}_1 = \mathcal{V} \wedge \mathcal{N}_2$ is proved as well. □

We denote by **Per** the lattice of all periodic semigroup varieties. It is evident that **Per** is a sublattice of **Epi**.

**Lemma 3.4.** *Let $\mathcal{V}$ be a nilvariety that is a modular element of the lattice* **Epi**. *If $\mathcal{V}$ satisfies a non-substitutive identity $u = v$, then it also satisfies the identity $u = 0$.*

*Proof.* If the identity $u = v$ is not a semigroup one, then Lemma 2.3 is applied with the desirable conclusion. So, we may assume that $u = v$ is a semigroup identity. Note that the variety $\mathcal{V}$ is periodic, whence it may be considered as a semigroup variety. Clearly, $\mathcal{V}$ is a modular element of the lattice **Per**. It is verified in [19, Proposition 2.2] that if a semigroup variety is modular in the lattice **SEM**, then it has the property we are verifying. All varieties that appear in the proof of this claim are periodic. Therefore, the desirable conclusion is true for modular elements of the lattice **Per**, and we are done. □

The formulation of the following statement and its proof are closely related to the formulation and proof of Lemma 3.1 of the article [11]. But we need to modify slightly some terminology from this article. Lemma 3.1 of [11] deals with the notions of equivalent and non-stable pairs of (semigroup) words defined in [11]. Here we need some modification of the first notion and do not require the second one at all. So, we call semigroup words $u$ and $v$ *equivalent* if $u \equiv \xi(v)$ for some automorphism $\xi$ of $F$. Clearly, if words $u$ and $v$ are equivalent semigroup words and $c(u) = c(v)$, then $u = v$ is a substitutive identity.

**Lemma 3.5.** *Let $\mathcal{V}$ be an epigroup variety that is a modular element of the lattice* **Epi** *and let $u$, $v$, $s$, and $t$ be pairwise non-equivalent words of the same length depending on the same letters. If the variety $\mathcal{V}$ satisfies the identities $u = v$ and $s = t$, then it also satisfies the identity $u = s$.*

*Proof.* In view of Proposition 3.3, the variety $\mathcal{V}$ is periodic. Whence, it may be considered as a semigroup variety. Clearly, $\mathcal{V}$ is a modular element of the lattice **Per**. The proof of [11, Lemma 3.1] readily implies that if $u$, $v$, $s$, and $t$ are pairwise non-equivalent words of the same length depending on the same letters, $\mathcal{V}$ satisfies the identities $u = v$ and $s = t$, and $\mathcal{V}$ does not satisfy the identity $u = s$, then there are periodic varieties (in actual fact, even nilvarieties) $\mathcal{U}$ and $\mathcal{W}$ such that $\mathcal{U} \subseteq \mathcal{W}$ but $(\mathcal{V} \vee \mathcal{U}) \wedge \mathcal{W} \ne (\mathcal{V} \wedge \mathcal{W}) \vee \mathcal{U}$. This contradicts the claim that $\mathcal{V}$ is a modular element of the lattice **Per**. □

Now we are ready to complete the proof of Theorem 1.2.

*Proof of Theorem 1.2.* Let $\mathcal{V}$ be a modular epigroup variety. According to Proposition 3.3, the variety $\mathcal{V}$ is periodic. It follows immediately from [11, Proposition 3.3] that if a periodic semigroup variety is a modular element of the lattice **SEM**, then it is the join of one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and a nilvariety. Repeating literally arguments from the proof of this statement with references to Proposition 2.6 and Lemma 2.7 of the present work rather than Lemma 2.6 of the article [11] and to Lemma 3.5 of the present work rather than Lemma 3.1 of the article [11], we obtain that the variety $\mathcal{V}$ has the same property. Thus, $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and $\mathcal{N}$ is a nilvariety. It remains to verify that if $\mathcal{N}$ satisfies a non-substitutive identity $u = v$, then $\mathcal{N}$ also satisfies the identity $u = 0$. If the identity $u = v$ is not a semigroup one, then Lemma 2.3 is applied with the conclusion that $\mathcal{N}$ satisfies the identity $u = 0$. So, we may assume that $u = v$ is a semigroup identity. Note that the variety $\mathcal{N}$ is periodic, whence it may be considered as a semigroup variety. In this situation, the desired conclusion directly follows from [19, Proposition 2.2]. Theorem 1.2 is proved. □

### 3.3. Modular commutative varieties.

*Proof of Theorem 1.3. Necessity.* Let $\mathcal{V}$ be a commutative modular epigroup variety. By Theorem 1.2, $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and $\mathcal{N}$ is a nilvariety. Corollary 3.2 implies that the variety $\mathcal{N}$ is modular. Since every commutative variety satisfies the identity $x^2y = yx^2$, Lemma 3.4 implies that the identity (1.1) holds in $\mathcal{N}$.

*Sufficiency.* In view of Corollary 3.2, it suffices to verify that a commutative epigroup variety satisfying the identity (1.1) is modular. This fact may be verified by the same arguments as in the proof of the 'if' part of Theorem 1 in [27]. Theorem 1.3 is proved. □

**3.4. Neutral varieties.** This subsection is devoted to the proof of Theorem 1.1.

*Proof of Theorem 1.1.* The implication (a) ⇒ (b) is evident.

(b) ⇒ (c): Since the variety $\mathcal{V}$ is modular, we may apply Theorem 1.2 and conclude that $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ and $\mathcal{N}$ is a nilvariety. It remains to verify that $\mathcal{N}$ is one of the varieties $\mathcal{T}$ or $\mathcal{ZM}$. By Lemma 2.18, we have to check that $\mathcal{N} \subseteq \mathcal{ZM}$. In other words, we need to verify that $\mathcal{N}$ is a variety of degree $\leq 2$. Arguing by contradiction, we suppose that $\mathcal{N}$ is a variety of degree $> 2$. The variety $\mathcal{V}$ is lower-modular and upper-modular. Corollary 3.2 implies that the variety $\mathcal{N}$ is lower-modular and upper-modular, too.

A variety is called 0-*reduced* if it may be given by 0-reduced identities only. One can verify that the variety $\mathcal{N}$ is 0-reduced. Arguing by contradiction, we suppose that this is not the case. The 'only if' part of the proof of Theorem 3.1 in [31] implies that there is a periodic group variety $\mathcal{G}$ such that $\mathrm{Nil}(\mathcal{G} \vee \mathcal{N}) \supset \mathcal{N}$. Put $\mathcal{N}' = \mathrm{Nil}(\mathcal{G} \vee \mathcal{N})$. Since $\mathcal{N} \subseteq \mathcal{N}'$ and the variety $\mathcal{N}$ is lower-modular, we have

$$\mathcal{N}' = (\mathcal{G} \vee \mathcal{N}) \wedge \mathcal{N}' = (\mathcal{G} \wedge \mathcal{N}') \vee \mathcal{N} = \mathcal{T} \vee \mathcal{N} = \mathcal{N},$$

contradicting the claim that $\mathcal{N} \subset \mathcal{N}'$.

Further, $\mathcal{N}$ is periodic, whence it may be considered as a semigroup variety. Therefore, $\mathcal{N}$ is an upper-modular element of the lattice **Per**. A semigroup variety is called *proper* if it differs from the variety of all semigroups. It follows from [23, Theorem 1] that if a proper semigroup variety of degree $> 2$ is upper-modular in **SEM**, then it is commutative. All varieties that appear in the proof of this fact are periodic. So, we have that a variety of degree $> 2$ is commutative whenever it is an upper-modular element in **Per**. In particular, the variety $\mathcal{N}$ is commutative.

Thus, $\mathcal{N}$ is a 0-reduced and commutative nilvariety. Therefore, $\mathcal{N} \subseteq \mathcal{ZM}$.

(c) ⇒ (a): It is well known that the set of all neutral elements of a lattice $L$ forms a sublattice in $L$ (see [2, Theorem 259 on p. 230]). In view of Proposition 3.1 and the equality $\mathcal{T} = \mathcal{SL} \wedge \mathcal{ZM}$, it remains to verify that the variety $\mathcal{ZM}$ is neutral. By Lemma 2.16, it suffices to check that $\mathcal{ZM}$ is distributive, codistributive, and modular. Let $\mathcal{X}$ and $\mathcal{Y}$ be arbitrary epigroup varieties.

*Distributivity.* We need to verify that

$$(\mathcal{ZM} \vee \mathcal{X}) \wedge (\mathcal{ZM} \vee \mathcal{Y}) \subseteq \mathcal{ZM} \vee (\mathcal{X} \wedge \mathcal{Y}),$$

because the opposite inclusion is evident. Suppose that an identity $u = v$ holds in $\mathcal{ZM} \vee (\mathcal{X} \wedge \mathcal{Y})$. We aim to check that this identity is satisfied by the variety $(\mathcal{ZM} \vee \mathcal{X}) \wedge (\mathcal{ZM} \vee \mathcal{Y})$. The identity $u = v$ holds in $\mathcal{ZM}$ and there is a deduction of this identity from the identities of the varieties $\mathcal{X}$ and $\mathcal{Y}$.

In other words, there are words $u_0, u_1, \dots, u_n$ such that $u_0 \equiv u$, $u_n \equiv v$, and for each $i = 0, 1, \dots, n-1$, the identity $u_i = u_{i+1}$ holds either in $\mathcal{X}$ or in $\mathcal{Y}$. Let $u_0, u_1, \dots, u_n$ be the shortest sequence of words with these properties. If all the words $u_0, u_1, \dots, u_n$ are not letters, then $u_0 = u_1 = \cdots = u_n$ holds in $\mathcal{ZM}$. This means that the sequence of words $u_0, u_1, \dots, u_n$ is a deduction of the identity $u = v$ from the identities of the varieties $\mathcal{ZM} \vee \mathcal{X}$ and $\mathcal{ZM} \vee \mathcal{Y}$, whence $u = v$ holds in $(\mathcal{ZM} \vee \mathcal{X}) \wedge (\mathcal{ZM} \vee \mathcal{Y})$. Now let $i$ be an index such that $u_i \equiv x$ for some letter $x$. Clearly, $0 < i < n$, because the variety $\mathcal{ZM}$ satisfies the identity $u_0 = u_n$ but does not satisfy any non-trivial identity of the kind $x = w$. The identity $u_{i-1} = x$ holds in one of the varieties $\mathcal{X}$ or $\mathcal{Y}$, say in $\mathcal{X}$. Then $\mathcal{Y}$ satisfies the identity $x = u_{i+1}$. Since both identities $u_{i-1} = x$ and $x = u_{i+1}$ fail in $\mathcal{ZM}$, we have that $\mathcal{ZM}$ is contained neither in $\mathcal{X}$ nor in $\mathcal{Y}$. Therefore, the varieties $\mathcal{X}$ and $\mathcal{Y}$ are completely regular. By Lemma 2.1, each of the identities $u_0 = \overline{\overline{u_0}}$ and $\overline{\overline{u_n}} = u_n$ holds in one of the varieties $\mathcal{X}$ or $\mathcal{Y}$. Further, for each $i = 0, 1, \dots, n-1$, one of the varieties $\mathcal{X}$ and $\mathcal{Y}$ satisfies the identity $\overline{\overline{u_i}} = \overline{\overline{u_{i+1}}}$. The words $u_0$ and $u_n$ are not letters, whence the variety $\mathcal{ZM}$ satisfies the identities $u_0 = \overline{\overline{u_0}} = \overline{\overline{u_1}} = \cdots = \overline{\overline{u_n}} = u_n$. We summarize that the sequence of words $u_0, \overline{\overline{u_0}}, \overline{\overline{u_1}}, \dots, \overline{\overline{u_n}}, u_n$ is a deduction of the identity $u = v$ from the identities of the varieties $\mathcal{ZM} \vee \mathcal{X}$ and $\mathcal{ZM} \vee \mathcal{Y}$. Therefore, this identity holds in $(\mathcal{ZM} \vee \mathcal{X}) \wedge (\mathcal{ZM} \vee \mathcal{Y})$.

*Codistributivity.* In view of Lemma 2.18, if $\mathcal{W}$ is an arbitrary epigroup variety, then either $\mathcal{W} \supseteq \mathcal{ZM}$ or $\mathcal{W} \wedge \mathcal{ZM} = \mathcal{T}$. We need to verify that

$$\mathcal{ZM} \wedge (\mathcal{X} \vee \mathcal{Y}) = (\mathcal{ZM} \wedge \mathcal{X}) \vee (\mathcal{ZM} \wedge \mathcal{Y}).$$

Clearly, both parts of this equality equal to $\mathcal{ZM}$ whenever at least one of the varieties $\mathcal{X}$ or $\mathcal{Y}$ contains $\mathcal{ZM}$. Otherwise, $\mathcal{X} \vee \mathcal{Y} \not\supseteq \mathcal{ZM}$, whence each part of the equality coincides with $\mathcal{T}$.

*Modularity.* For any epigroup variety $\mathcal{X}$, we put $\mathrm{CR}(\mathcal{X}) = \mathcal{CR} \wedge \mathcal{X}$ where $\mathcal{CR}$ is the variety of all completely regular epigroups. Suppose that $\mathcal{X} \subseteq \mathcal{Y}$. We need to prove that

$$(\mathcal{ZM} \vee \mathcal{X}) \wedge \mathcal{Y} \subseteq (\mathcal{ZM} \wedge \mathcal{Y}) \vee \mathcal{X}$$

because the opposite inclusion is evident. If $\mathcal{ZM} \subseteq \mathcal{Y}$, then each part of the inclusion coincides with $\mathcal{ZM} \vee \mathcal{X}$. Now let $\mathcal{ZM} \nsubseteq \mathcal{Y}$. Then $\mathcal{ZM} \wedge \mathcal{Y} = \mathcal{T}$, whence the right part of the inclusion coincides with $\mathcal{X}$. Clearly, $\mathcal{ZM} \nsubseteq \mathcal{X}$. Then the varieties $\mathcal{X}$ and $\mathcal{Y}$ are completely regular. Therefore,

$$(\mathcal{ZM} \vee \mathcal{X}) \wedge \mathcal{Y} \subseteq \mathrm{CR}(\mathcal{ZM} \vee \mathcal{X}).$$

Let $u = v$ be an identity that holds in $\mathcal{X}$. Lemmas 2.1 and 2.3 imply that $\mathcal{ZM} \vee \mathcal{X}$ satisfies the identity $\overline{\overline{u}} = \overline{\overline{v}}$. Therefore, $u = v$ in $\mathrm{CR}(\mathcal{ZM} \vee \mathcal{X})$. We have proved that $\mathrm{CR}(\mathcal{ZM} \vee \mathcal{X}) \subseteq \mathcal{X}$, whence

$$(\mathcal{ZM} \vee \mathcal{X}) \wedge \mathcal{Y} \subseteq \mathrm{CR}(\mathcal{ZM} \vee \mathcal{X}) \subseteq \mathcal{X} = (\mathcal{ZM} \wedge \mathcal{Y}) \vee \mathcal{X}.$$

We have completed the proof of Theorem 1.1. $\square$

**3.5. Upper-modular varieties.** Here we verify Theorem 1.4. To do this, we need several auxiliary statements.

**Lemma 3.6.** *If a strongly permutative epigroup variety $\mathcal{V}$ is an upper-modular element of the lattice* **Epi***, then $\mathcal{V}$ is commutative.*

*Proof.* In view of Corollary 2.8, $\mathcal{V} = \mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}$ where $\mathcal{G}$ is an abelian group variety, $m \geq 0$, and $\mathcal{N}$ is a nilvariety. If $\deg(\mathcal{V}) \leq 2$, then $\mathcal{N} \subseteq \mathcal{ZM}$, and we are done. Now let $\deg(\mathcal{V}) > 2$. By Proposition 2.12, $\mathcal{V}$ contains the variety $\mathcal{X} = \mathrm{var}\{x^2 = xyz = 0,\, xy = yx\}$. Suppose that $\mathcal{V}$ is not commutative. Let $\mathcal{G}'$ be a non-abelian group variety. Since $\mathcal{V}$ is strongly permutative, every group in $\mathcal{V}$ is abelian. Therefore, the variety $(\mathcal{G}' \wedge \mathcal{V}) \vee \mathcal{X}$ is commutative. Since $\mathcal{X} \subseteq \mathcal{V}$ and the variety $\mathcal{V}$ is upper-modular, we have that $(\mathcal{G}' \wedge \mathcal{V}) \vee \mathcal{X} = (\mathcal{G}' \vee \mathcal{X}) \wedge \mathcal{V}$. We see that the variety $(\mathcal{G}' \vee \mathcal{X}) \wedge \mathcal{V}$ is commutative. Hence, there is a deduction of the identity $xy = yx$ from the identities of the varieties $\mathcal{G}' \vee \mathcal{X}$ and $\mathcal{V}$. In particular, there is a word $v$ such that $v \not\equiv xy$ and the identity $xy = v$ holds either in $\mathcal{G}' \vee \mathcal{X}$ or in $\mathcal{V}$. The claims (i) and (iii) of Lemma 2.5 imply that a variety with the identity $xy = v$ is either commutative or a variety of degree $\leq 2$. The variety $\mathcal{G}' \vee \mathcal{X}$ is neither commutative (because $\mathcal{G}'$ is non-abelian) nor a variety of degree $\leq 2$ (because $\deg(\mathcal{X}) > 2$). Since $\deg(\mathcal{V}) > 2$, we have that $\mathcal{V}$ is commutative. $\square$

It is proved in [22, Theorem 1.1] that if $\mathcal{V}$ is a proper upper-modular in **SEM** variety, then first, $\mathcal{V}$ is periodic (note that this is a very special case of [12, Theorem 1]), and second, every nilsubvariety of $\mathcal{V}$ is commutative and satisfies the identity (1.2). As we have already mentioned in Subsection 1.2, the epigroup analog of the first claim is not true. Our next step is the following partial epigroup analog of the second claim.

**Proposition 3.7.** *If a strongly permutative epigroup variety $\mathcal{V}$ is an upper-modular element of the lattice* **Epi***, then every nil-semigroup in $\mathcal{V}$ satisfies the identity* (1.2)*.*

*Proof.* According to Lemma 3.6, the variety $\mathcal{V}$ is commutative. If all nil-semigroups in $\mathcal{V}$ are singletons, then the desirable conclusion is evident. Suppose now that $\mathcal{V}$ contains a non-singleton nil-semigroup $N$. Let $\mathcal{N}$ be the variety generated by $N$. Clearly, this variety is commutative. It is evident that $\mathcal{ZM} \subseteq \mathcal{N}$. We need to verify that $\mathcal{N}$ satisfies the identity (1.2). Put

$$\mathcal{I} = \mathrm{var}\{x^2y = xy^2,\, xy = yx,\, x^2yz = 0\}$$

and $\mathcal{N}' = \mathcal{N} \wedge \mathcal{I}$. It is clear that $\mathcal{ZM} \subseteq \mathcal{I}$, whence $\mathcal{ZM} \subseteq \mathcal{N}'$.

A semigroup analog of the proposition that we are verifying here is proved in [22] (see the last paragraph of Section 3 in that article). The arguments used there are based on the fact that there is a variety $\mathcal{X}$ such that the following two claims are valid:

(i) $(\mathcal{X} \vee \mathcal{N}') \wedge \mathcal{V} \subseteq \mathcal{I}$;

(ii) if $v \in \{x^2y, xyx, yx^2\}$ and $w \in \{xy^2, yxy, y^2x\}$, then the identity $v = w$ fails in $\mathcal{X}$.

In [22], a certain periodic group variety plays the role of $\mathcal{X}$. Here we should take another $\mathcal{X}$. Namely, put $\mathcal{X} = \mathcal{LZM} \vee \mathcal{RZM}$ where

$$\mathcal{LZM} = \mathrm{var}\{xyz = xy\} \quad \text{and} \quad \mathcal{RZM} = \mathrm{var}\{xyz = yz\}.$$

The variety $\mathcal{X}$ satisfies the identity $xyzxy = xy$. Therefore, Lemma 2.4(i) implies that $\mathcal{SL} \nsubseteq \mathcal{X}$. Further, substituting 1 for $x$ and $y$ in the identity $xyzxy = xy$, we obtain that all groups in $\mathcal{X}$ are singletons. Hence every commutative semigroup in $\mathcal{X}$ is a nil-semigroup. Further, $\mathcal{X}$ satisfies the identity $xy = (xy)^2$, whence all nil-semigroups in $\mathcal{X}$ lie in $\mathcal{ZM}$ by Lemma 2.5(ii). Since the variety $\mathcal{X} \wedge \mathcal{V}$ is commutative, $\mathcal{X} \wedge \mathcal{V} \subseteq \mathcal{ZM}$. The variety $\mathcal{V}$ is upper-modular and $\mathcal{N}' \subseteq \mathcal{V}$. Therefore,

$$(\mathcal{X} \vee \mathcal{N}') \wedge \mathcal{V} = (\mathcal{X} \wedge \mathcal{V}) \vee \mathcal{N}' \subseteq \mathcal{ZM} \vee \mathcal{N}' = \mathcal{N}' \subseteq \mathcal{I}.$$

We have proved claim (i). To verify claim (ii), we note that if $\mathcal{X}$ satisfies a semigroup identity $v = w$, then the words $v$ and $w$ have the same prefix of length 2 and the same suffix of length 2. Clearly, this is not the case whenever $v \in \{x^2y, xyx, yx^2\}$ and $w \in \{xy^2, yxy, y^2x\}$. Now we can complete the proof by using the same arguments as in the last paragraph of [22, Section 3]. □

The proof of the following statement repeats almost literally the 'only if' part of the proof of Theorem 2 in [27].

**Proposition 3.8.** *If a nilvariety of epigroups $\mathcal{X}$ satisfies the identities* (1.2) *and $xy = yx$, then $\mathcal{X}$ is an upper-modular element of the lattice* **Epi**.

*Proof.* It is easy to prove (see [27, Lemma 2.7], for instance) that $\mathcal{X}$ satisfies the identity $x^2yz = 0$. Thus, $\mathcal{X} \subseteq \mathcal{I}$. Put

$$U = \{x^2, x^3, x^2y, x_1x_2\cdots x_n \mid n \in \mathbb{N}\}.$$

It is evident that any subvariety of $\mathcal{I}$ may be given in $\mathcal{I}$ only by identities of the type $u = v$ or $u = 0$ where $u, v \in U$. Lemma 2.5 implies that if $u, v \in U$ and $u \not\equiv v$, then $u = v$ implies in $\mathcal{I}$ the identity $u = 0$. Now it is very easy to check that the lattice $L(\mathcal{I})$ has the form shown on Fig. 1 where

$$\begin{aligned}
\mathcal{I}_n &= \mathrm{var}\{x^2yz = x_1x_2\cdots x_n = 0,\ x^2y = xy^2,\ xy = yx\} \ \text{ where } n \ge 4,\\
\mathcal{J} &= \mathrm{var}\{x^2yz = x^3 = 0,\ x^2y = xy^2,\ xy = yx\},\\
\mathcal{J}_n &= \mathrm{var}\{x^2yz = x^3 = x_1x_2\cdots x_n = 0,\ x^2y = xy^2,\ xy = yx\} \ \text{ where } n \ge 4,\\
\mathcal{K} &= \mathrm{var}\{x^2y = 0,\ xy = yx\},\\
\mathcal{K}_n &= \mathrm{var}\{x^2y = x_1x_2\cdots x_n = 0,\ xy = yx\} \ \text{ where } n \ge 3,\\
\mathcal{L} &= \mathrm{var}\{x^2 = 0,\ xy = yx\},\\
\mathcal{L}_n &= \mathrm{var}\{x^2 = x_1x_2\cdots x_n = 0,\ xy = yx\} \ \text{ where } n \in \mathbb{N}.
\end{aligned}$$

Note that $\mathcal{L}_1 = \mathcal{T}$ and $\mathcal{L}_2 = \mathcal{ZM}$.

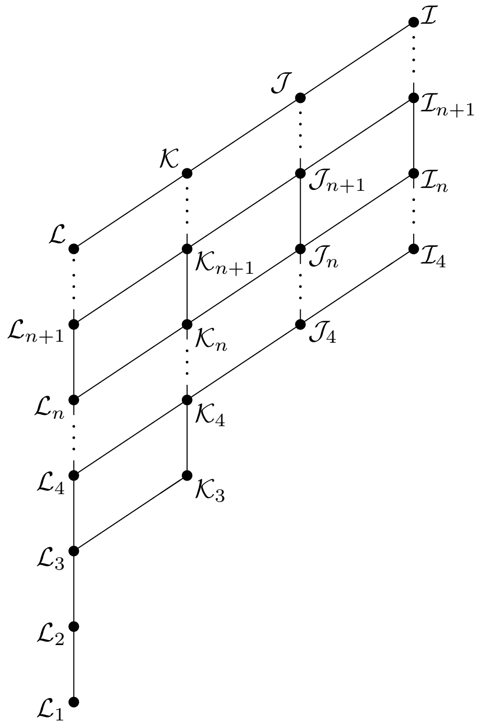

Figure 1. The lattice $L(\mathcal{I})$

Let $\mathcal{X} \subseteq \mathcal{I}$. We have to check that if $\mathcal{Y} \subseteq \mathcal{X}$ and $\mathcal{Z}$ is an arbitrary epigroup variety, then $(\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{X} = (\mathcal{Z} \wedge \mathcal{X}) \vee \mathcal{Y}$. For a variety $\mathcal{M}$ with $\mathcal{M} \subseteq \mathcal{I}$, we denote by $\mathcal{M}^*$ the least of the varieties $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$, and $\mathcal{L}$ that contains $\mathcal{M}$. Fig. 1 shows that if $\mathcal{M}_1, \mathcal{M}_2 \subseteq \mathcal{I}$, then $\mathcal{M}_1 = \mathcal{M}_2$ if and only if $\deg(\mathcal{M}_1) = \deg(\mathcal{M}_2)$ and $\mathcal{M}_1^* = \mathcal{M}_2^*$. Therefore, we have to verify the following two equalities:

$$\deg\bigl((\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{X}\bigr) = \deg\bigl((\mathcal{Z} \wedge \mathcal{X}) \vee \mathcal{Y}\bigr), \tag{3.1}$$

$$\bigl((\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{X}\bigr)^* = \bigl((\mathcal{Z} \wedge \mathcal{X}) \vee \mathcal{Y}\bigr)^*. \tag{3.2}$$

*The equality* (3.1). Put $\deg(\mathcal{X}) = k$, $\deg(\mathcal{Y}) = \ell$, and $\deg(\mathcal{Z}) = m$. According to Corollaries 2.13 and 2.14, we have

$$\deg\bigl((\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{X}\bigr) = \min\bigl\{\max\{m, \ell\}, k\bigr\},$$
$$\deg\bigl((\mathcal{Z} \wedge \mathcal{X}) \vee \mathcal{Y}\bigr) = \max\bigl\{\min\{m, k\}, \ell\bigr\}.$$

Clearly, $\ell \le k$, because $\mathcal{Y} \subseteq \mathcal{X}$. It is then evident that

$$\min\bigl\{\max\{m, \ell\}, k\bigr\} = \max\bigl\{\min\{m, k\}, \ell\bigr\} = \begin{cases} \ell & \text{if } m \le \ell \le k, \\ m & \text{if } \ell \le m \le k, \\ k & \text{if } \ell \le k \le m. \end{cases}$$

The equality (3.1) is proved.

*The equality* (3.2). Clearly, this equality is equivalent to the following claim: if $u$ is one of the words $x^3$, $x^2y$ and $x^2$, then the variety $(\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{X}$ satisfies the identity $u = 0$ if and only if the variety $(\mathcal{Z} \wedge \mathcal{X}) \vee \mathcal{Y}$ does so. It suffices to verify that $u = 0$ holds in $(\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{X}$ whenever it is so in $(\mathcal{Z} \wedge \mathcal{X}) \vee \mathcal{Y}$, because the opposite claim immediately follows from the evident inclusion $(\mathcal{Z} \wedge \mathcal{X}) \vee \mathcal{Y} \subseteq (\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{X}$. Further considerations are divided into two cases.

*Case* 1: $u \equiv x^n$ where $n \in \{2, 3\}$. Then $x^n = 0$ in $(\mathcal{Z} \wedge \mathcal{X}) \vee \mathcal{Y}$. This means that $x^n = 0$ in $\mathcal{Y}$ and there is a deduction of the identity $x^n = 0$ from the identities of the varieties $\mathcal{Z}$ and $\mathcal{X}$. In particular, there is a word $v$ such that $v \not\equiv x^n$ and $x^n = v$ holds in either $\mathcal{Z}$ or $\mathcal{X}$. If $x^n = v$ in $\mathcal{X}$, then the fact that $\mathcal{X}$ is a nilvariety together with the claims (i) and (ii) of Lemma 2.5 imply that $x^n = 0$ in $\mathcal{X}$, and moreover in $(\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{X}$. Now let $x^n = v$ in $\mathcal{Z}$. The case when $v$ is a semigroup word may be considered by the same arguments as in the Case 1 in the 'only if' part of the proof of Theorem 2 in [27]. Now let $v$ be a non-semigroup word. Then, in view of Lemma 2.3, the identity $v = 0$ holds in the variety $\mathcal{X}$, and therefore in $\mathcal{Y}$. Recall that $x^n = 0$ in $\mathcal{Y}$. Therefore, the identity $x^n = v$ holds in $\mathcal{Y}$. Since this identity holds in $\mathcal{Z}$ as well, we obtain that it holds in $\mathcal{Y} \vee \mathcal{Z}$. We see that the sequence $x^n$, $v$, $0$ is a deduction of the identity $x^n = 0$ from the identities of the varieties $\mathcal{Y} \vee \mathcal{Z}$ and $\mathcal{X}$, whence $x^n = 0$ holds in $(\mathcal{Y} \vee \mathcal{Z}) \wedge \mathcal{X}$.

*Case* 2: $u \equiv x^2y$. We have to check that if the variety $(\mathcal{Z} \wedge \mathcal{X}) \vee \mathcal{Y}$ satisfies the identity (1.1), then the variety $(\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{X}$ also satisfies this identity. Put $W = \{x^2y, xyx, yx^2, y^2x, yxy, xy^2\}$. The variety $(\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{X}$ is commutative. Therefore, it suffices to verify that this variety satisfies an identity $w = 0$ for some word $w \in W$. By the hypothesis, the variety $(\mathcal{Z} \wedge \mathcal{X}) \vee \mathcal{Y}$ satisfies the identity (1.1). This means that this identity holds in $\mathcal{Y}$ and there is a deduction of the identity from the identities of the varieties $\mathcal{X}$ and $\mathcal{Z}$. Let $x^2y \equiv u_0, u_1, \ldots, u_n, 0$ be such a deduction. The case when $u_n \in W$ may be considered by the same way as in the 'only if' part of the proof of Theorem 2 in [27].

Now let $u_n \notin W$. Since $u_0 \in W$, there is an index $i > 0$ such that $u_i \notin W$ while $u_{i-1} \in W$. The identity $u_{i-1} = u_i$ holds in one of the varieties $\mathcal{Z}$ or $\mathcal{X}$. If $u_{i-1} = u_i$ holds in $\mathcal{X}$, then $\mathcal{X}$ satisfies the identity $u_{i-1} = 0$ (this follows from [27, Lemma 2.5] whenever $u_i$ is a semigroup word and from Lemma 2.3, otherwise). Therefore, $u_{i-1} = 0$ holds in $(\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{X}$. Since $u_{i-1} \in W$, we are done.

Finally, suppose that $u_{i-1} = u_i$ holds in $\mathcal{Z}$. If $u_i$ is a semigroup word, then we may complete the proof by the same arguments as in the 'only if' part of the proof of Theorem 2 in [27]. Now suppose that $u_i$ is not a semigroup word. Lemma 2.3 implies then that the variety $\mathcal{Y}$ satisfies the identity $u_i = 0$. Hence, the identity $u_{i-1} = u_i$ holds in $\mathcal{Y}$, and therefore the variety $\mathcal{Y} \vee \mathcal{Z}$ satisfies this identity. Applying Lemma 2.3 again, we conclude that $u_i = 0$

holds in $\mathcal{X}$. Whence, the sequence $u_{i-1}$, $u_i$, $0$ is a deduction of the identity $u_{i-1} = 0$ from the identities of the varieties $\mathcal{Y} \vee \mathcal{Z}$ and $\mathcal{X}$. Thus, $u_{i-1} = 0$ holds in $(\mathcal{Y} \vee \mathcal{Z}) \wedge \mathcal{X}$, and we are done.

The equality (3.2) is proved. Thus, we have proved Proposition 3.8. □

*Proof of Theorem 1.4. Necessity.* Let $\mathcal{V}$ be a strongly permutative upper-modular epigroup variety. In view of Corollary 2.8, $\mathcal{V} = \mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}$ where $\mathcal{G}$ is an abelian group variety, $m \geq 0$, and $\mathcal{N}$ is a nilvariety. Lemma 3.6 and Proposition 3.7 imply, respectively, that $\mathcal{N}$ is commutative and satisfies the identity (1.2). The variety $\mathcal{C}_m$ contains a nilsubvariety $\mathrm{var}\{x^m = 0,\, xy = yx\}$. Clearly, this variety does not satisfy the identity (1.2) whenever $m \geq 3$. Now Proposition 3.7 applies again and we conclude that $m \leq 2$. If the variety $\mathcal{N}$ satisfies the identity (1.1), then the claim (ii) of Theorem 1.4 holds. Suppose now that the identity (1.1) fails in $\mathcal{N}$. By [29, Lemma 7], this implies that $\mathcal{N}$ contains the variety $\mathcal{J}$. We need to verify that $\mathcal{G} = \mathcal{T}$ and $m \leq 1$. Arguing by contradiction, suppose that either $\mathcal{G} \neq \mathcal{T}$ or $m \geq 2$. Then $\mathcal{V}$ contains a variety of the form $\mathcal{X} \vee \mathcal{J}$ where $\mathcal{X}$ is either a non-trivial abelian group variety or the variety $\mathcal{C}_2$. The variety $\mathcal{C}_2$ is generated by a 3-element monoid (see a remark before Lemma 2.7). Thus, $\mathcal{M}$ is generated by an epigroup with unit, in any case. Suppose that $\mathcal{X}$ satisfies the identity (1.2). Substituting 1 for $y$ in this identity, we have that $x^2 = x$ holds in $\mathcal{X}$. But this identity is false both in a non-trivial group variety and in the variety $\mathcal{C}_2$. As is verified in the proof of [29, Lemma 8], this implies that (1.2) is false in any nil-semigroup in $\mathcal{X} \vee \mathcal{J}$. But this contradicts Proposition 3.7.

*Sufficiency.* If $\mathcal{V}$ satisfies the claim (i) of Theorem 1.4, then $\mathcal{V}$ is upper-modular by Proposition 3.8 and Corollary 3.2. Suppose now that $\mathcal{V}$ satisfies the claim (ii). In other words, $\mathcal{V} = \mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}$ where $\mathcal{G}$ is an abelian group variety, $0 \leq m \leq 2$, and $\mathcal{N}$ satisfies the identities $xy = yx$ and (1.1). Let $\mathcal{Y} \subseteq \mathcal{V}$ and $\mathcal{Z}$ an arbitrary epigroup variety. We aim to verify that

$$(\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{V} = (\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y}.$$

As we have already mentioned in the proof of Lemma 2.7, the variety $\mathcal{C}_m$ is generated by the $(m+1)$-element combinatorial cyclic monoid $C_m$, and the set $X = \{m \in \mathbb{N} \mid C_m \in \mathcal{X}\}$ has the greatest element. Indeed, for any $m \geq 0$, let $c_m$ be a generator of $C_m$. Put $C = \prod_{m \in X} C_m$. Then the semigroup $C$ is not an epigroup, because no power of the element $(\dots, c_m, \dots)_{m \in X}$ belongs to a subgroup of $C$. Thus, the set $X$ has the greatest element. We denote this element by $m$ and put $C(\mathcal{X}) = \mathcal{C}_m$. It is clear that the varieties $(\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{V}$ and $(\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y}$ are commutative. In view of Proposition 2.6 and Lemma 2.7, it suffices to verify the following three equalities:

$$\mathrm{Gr}\big((\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{V}\big) = \mathrm{Gr}\big((\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y}\big), \tag{3.3}$$

$$C\big((\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{V}\big) = C\big((\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y}\big), \tag{3.4}$$

$$\mathrm{Nil}\big((\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{V}\big) = \mathrm{Nil}\big((\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y}\big). \tag{3.5}$$

*The equality* (3.3). If $\mathcal{G}$ is a periodic group variety, then we denote by $\exp(\mathcal{G})$ the *exponent* of $\mathcal{G}$, that is, the least number $n$ such that $\mathcal{G}$ satisfies the identity $x = x^{n+1}$. For a non-periodic group variety $\mathcal{G}$, we put $\exp(\mathcal{G}) = \infty$. As usual, we denote by $\mathrm{lcm}\{m,n\}$ (respectively, $\gcd\{m,n\}$) the least common multiple (the greatest common divisor) of positive integers $m$ and $n$. To simplify further considerations, we will assume that any natural number divides $\infty$; in particular, $\gcd\{n,\infty\} = n$ and $\mathrm{lcm}\{n,\infty\} = \infty$ for arbitrary natural $n$. We will assume also that $\gcd\{\infty,\infty\} = \mathrm{lcm}\{\infty,\infty\} = \infty$. Put $\mathcal{G}_1 = \mathrm{Gr}\big((\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{V}\big)$ and $\mathcal{G}_2 = \mathrm{Gr}\big((\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y}\big)$. Since $\mathcal{G}_1, \mathcal{G}_2 \subseteq \mathcal{V}$, we have that $\mathcal{G}_1$ and $\mathcal{G}_2$ are abelian group varieties. To prove the equality (3.3), it suffices to verify that $\exp(\mathcal{G}_1) = \exp(\mathcal{G}_2)$. This claim is verified by the same arguments as the analogous claim in the proof of Theorem 1.2 in [22].

*The equality* (3.4). Here and below we need the following easy remark. It is evident that if $m \geq 3$, then the identity (1.1) fails in the variety $\mathrm{Nil}(\mathcal{C}_m) = \mathrm{var}\{x^m = 0,\, xy = yx\}$. This means that each part of the equality (3.4) coincides with the variety $\mathcal{C}_m$ for some $0 \leq m \leq 2$. Then we may complete the proof of equality (3.4) by the same arguments as in the proof of the equality (4.2) in [22].

*The equality* (3.5). The varieties $\mathcal{G}$, $\mathcal{C}_2$, and $\mathcal{N}$ satisfy the identity $x^2y = \overline{\overline{x}}^{\,2}y$. By Lemma 2.3, the variety $\mathrm{Nil}(\mathcal{V})$ satisfies the identity (1.1). Fig. 1 shows that it suffices to check the following two claims: first, the varieties $(\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{V}$ and $(\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y}$ have the same degree, and second, the variety $\mathrm{Nil}\big((\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{V}\big)$ satisfies the identity

$$x^2 = 0 \tag{3.6}$$

if and only if the variety $\mathrm{Nil}\big((\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y})\big)$ satisfies this identity. The former claim may be verified by the same way as in the proof of the equality (4.3) in [22] with references to Proposition 2.12, Corollary 2.14, and Corollary 2.15 of the present article rather than Proposition 2.11, Lemma 2.13, and Lemma 2.12 of [22], respectively.

It remains to verify that the variety $\mathrm{Nil}\big((\mathcal{Z} \vee \mathcal{Y}) \wedge \mathcal{V}\big)$ satisfies the identity (3.6) whenever $\mathrm{Nil}\big((\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y}\big)$ does (because the opposite claim is evident). Suppose that the identity (3.6) holds in $\mathrm{Nil}\big((\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y}\big)$. Proposition 2.6 and Lemma 2.7 imply that the variety $(\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y}$ is the join of some group variety, the variety $\mathcal{C}_m$ for some $m \geq 0$, and the variety $\mathrm{Nil}\big((\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y}\big)$. Here $m \leq 2$, because $(\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y} \subseteq \mathcal{V}$. Hence, the variety $(\mathcal{Z} \wedge \mathcal{V}) \vee \mathcal{Y}$ satisfies the identity $x^2 = \overline{\overline{x^2}}$. In particular, this identity holds in the varieties $\mathcal{Y}$ and $\mathcal{Z} \wedge \mathcal{V}$. Therefore, there is a sequence of words $u_0, u_1, \dots, u_k$ such that $u_0 \equiv x^2$, $u_k \equiv \overline{\overline{x^2}}$, and for each $i = 0, 1, \dots, k-1$, the identity $u_i = u_{i+1}$ holds in one of the varieties $\mathcal{Z}$ or $\mathcal{V}$. We may assume that $u_i \not\equiv u_{i+1}$ for each $i = 0, 1, \dots, k-1$. Arguments from the proof of the equality (4.3) in [22] show that it suffices to check that the identity (3.6) holds in one of the varieties $\mathrm{Nil}(\mathcal{Z})$ or $\mathrm{Nil}(\mathcal{V})$. This fact follows from Lemma 2.5 whenever $u_1$ is a semigroup word and from Lemma 2.3 otherwise.

We have completed the proof of Theorem 1.4. □

## 4. Corollaries

Comparing Proposition 4.1 in [31] with Theorem 1.1 leads to the following fact.

**Corollary 4.1.** *Let $\mathcal{V}$ be a periodic semigroup variety. Then $\mathcal{V}$ is a neutral element of* **SEM** *if and only if $\mathcal{V}$ is a neutral element of* **Epi**.

Analogously, Theorem 3.1 in [19] and Theorem 1.3 show that the following is true.

**Corollary 4.2.** *Let $\mathcal{V}$ be a periodic commutative semigroup variety. Then $\mathcal{V}$ is a modular element of* **SEM** *if and only if $\mathcal{V}$ is a modular element of* **Epi**.

Our next corollary is not so immediate a consequence of earlier results as the previous two statements.

**Corollary 4.3.** *Let $\mathcal{V}$ be a periodic strongly permutative semigroup variety. Then $\mathcal{V}$ is an upper-modular element of* **SEM** *if and only if $\mathcal{V}$ is an upper-modular element of* **Epi**.

*Proof. Necessity.* Let $\mathcal{V}$ be a periodic strongly permutative semigroup variety and $\mathcal{V}$ is an upper-modular element of **SEM**. It follows from results of [5] and the proof of Proposition 1 in [29] that $\mathcal{V} = \mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}$ for some Abelian periodic group variety $\mathcal{V}$, some $m \geq 0$ and some nilvariety $\mathcal{N}$. By [22, Theorem 1.1], if $\mathcal{X}$ is a proper semigroup variety that is an upper-modular element of **SEM** and $\mathcal{Y}$ is a nilsubvariety of $\mathcal{X}$ then $\mathcal{Y}$ is commutative. Thus, $\mathcal{N}$ is commutative. This implies that the variety $\mathcal{V}$ is commutative too. Now we may apply [22, Theorem 1.2] and conclude that $\mathcal{V}$ satisfies one of the claims (i) or (ii) of Theorem 1.4. Therefore, $\mathcal{V}$ is upper-modular in **Epi**.

*Sufficiency.* Let $\mathcal{V}$ be a periodic strongly permutative semigroup variety and $\mathcal{V}$ an upper-modular element of **Epi**. By Theorem 1.4, either $\mathcal{V}$ satisfies the claim (i) of this theorem or $\mathcal{V}$ satisfies the claim (ii) of this theorem and the variety $\mathcal{G}$ from this claim is periodic. In both these cases, $\mathcal{V}$ is an upper-modular element of **SEM** by [22, Theorem 1.2]. □

Theorems 1.2 and 1.4 evidently imply the following assertion.

**Corollary 4.4.** *If a commutative epigroup variety is a modular element of the lattice* **Epi**, *then it is an upper-modular element of this lattice.*

Theorem 1.4 readily implies the following statement.

**Corollary 4.5.** *If a strongly permutative epigroup variety $\mathcal{V}$ is an upper-modular element of the lattice* **Epi**, *then the lattice $L(\mathcal{V})$ is distributive.*

*Proof.* If $\mathcal{V}$ satisfies the claim (i) of Theorem 1.4, then it is periodic, whence it may be considered as a semigroup variety. In this case, it suffices to take into account a description of commutative semigroup varieties with distributive subvariety lattice obtained in [30]. Suppose now that $\mathcal{V}$ satisfies the claim (ii) of Theorem 1.4. Then $\mathcal{V} \subseteq \mathcal{AG} \vee \mathcal{C}_2 \vee \mathcal{N}$ where $\mathcal{N}$ satisfies the commutative law and the identity (1.1). In view of Proposition 2.9, $L(\mathcal{V})$ is embeddable into the direct product of the lattices $L(\mathcal{AG})$ and $L(\mathcal{C}_2 \vee \mathcal{N})$. The former lattice is generally known to be distributive. Finally, the variety $\mathcal{C}_2 \vee \mathcal{N}$ is periodic, whence it may be considered as a semigroup variety. To complete the proof, it remains to note that the lattice $L(\mathcal{C}_2 \vee \mathcal{N})$ is distributive by the mentioned result of [30]. □

## 5. Open questions

We do not know whether it is possible to remove the word ‘commutative’ in Corollary 4.2 and the words ‘strongly permutative’ in Corollary 4.3. One can formulate the corresponding questions.

**Question 5.1.** Is it true that a periodic semigroup variety is a modular element of **SEM** if and only if it is a modular element of **Epi**?

**Question 5.2.** Is it true that a periodic semigroup variety is an upper-modular element of **SEM** if and only if it is an upper-modular element of **Epi**?

It was proved in [21, Corollary 3.5] that the following strengthened semigroup analog of Theorem 1.1 is true: a semigroup variety is neutral in **SEM** if and only if it is simultaneously lower-modular and upper-modular in **SEM**. We do not know, whether the epigroup analog of this claim is valid.

**Question 5.3.** Is it true that an epigroup variety is a neutral element of the lattice **Epi** if and only if it is simultaneously a lower-modular and upper-modular element of this lattice?

As we have already mentioned in Subsection 3.5, it is proved in [22, Theorem 1.1] that if $\mathcal{V}$ is a proper upper-modular in **SEM** semigroup variety, then every nilsubvariety of $\mathcal{V}$ is commutative and satisfies the identity (1.2). Proposition 3.7 gives a partial epigroup analog of this assertion. We do not know whether the full analog is true.

**Question 5.4.** Suppose that an epigroup variety $\mathcal{V}$ is an upper-modular element of the lattice **Epi** and let $\mathcal{N}$ be a nilsubvariety of $\mathcal{V}$.

(a) Is it true that $\mathcal{N}$ is commutative?
(b) Is it true that $\mathcal{N}$ satisfies the identity (1.2)?

Proposition 3.7 shows that an affirmative answer to Question 5.4(a) would immediately imply the same answer to Question 5.4(b).

Further, it is verified in [23, Theorem 1] that every proper upper-modular in **SEM** variety is either commutative or has a degree $\leq 2$. We do not know whether the epigroup analog of this alternative is valid. We formulate the corresponding question together with its weaker version.

**Question 5.5.** Suppose that an epigroup variety $\mathcal{V}$ is an upper-modular element of the lattice **Epi**.

(a) Is it true that $\mathcal{V}$ either is commutative or has a degree $\leq 2$?
(b) Is it true that $\mathcal{V}$ either is permutative or has a finite degree?

An affirmative answer to Question 5.5(a) together with Theorem 1.4 would immediately imply a complete description of upper-modular epigroup varieties of degree $> 2$.

At the conclusion of the article, we briefly mention lower-modular and distributive varieties. It is verified in [20] that every proper lower-modular in **SEM** variety is periodic (this fact is essentially generalized by [12, Theorem 1]). The question, whether the epigroup analog of this claim is the case, is open.

**Question 5.6.** Is it true that if an epigroup variety $\mathcal{V}$ is a lower-modular element of the lattice **Epi**, then the variety $\mathcal{V}$ is periodic?

Lower-modular in **SEM** varieties were completely determined in [13] (and this result is reproved in a simpler way in [11]). Further, it is evident that a distributive element of a lattice is lower-modular.Varieties distributive in **SEM** were completely classified in [25]. This allows us to hope that the affirmative answer to Question 5.6 would result in a complete description of the lower-modular and the distributive epigroup varieties.

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