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RESEARCH ARTICLE

DUALITIES IN LATTICES OF SEMIGROUP VARIETIES

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INTRODUCTION

The subvariety lattice $L(\mathcal{V})$ of a variety of universal algebras $\mathcal{V}$ is one of the main objects investigated in the theory of varieties. The first results about lattices of semigroup varieties see in the survey article [4].

Recently a lot of papers have appeared in which semigroup varieties with restrictions on subvariety lattice were investigated (see, e.g., [1,9,15,16,19-22]). Many various lattice conditions are considered in this context. Conditions connecting with lattice duality however are not among them.

We say that varieties of universal algebras $\mathcal{V}$ and $\mathcal{V}'$ are <u>dual one to another</u> if $L(\mathcal{V})$ and $L(\mathcal{V}')$ are dual. A variety $\mathcal{V}$ is said to be <u>selfdual</u> if it is dual to itself, i.e. $L(\mathcal{V})$ is selfdual. Further let X be an arbitrary class of semigroup varieties. A semigroup variety $\mathcal{V}$ is called <u>admitting duality in class</u> X if a semigroup variety $\mathcal{V}'$ exists such that $\mathcal{V}' \in$ X and $\mathcal{V}'$ is dual to $\mathcal{V}$. We shall omit the words "in class X" and say that a variety $\mathcal{V}$ <u>admits duality</u> if X is the class of all semigroup varieties.

The investigation of dualities is a traditional aspect of the consideration of derivative lattices of algebraic objects (see, e.g., [14] or [17]). Moreover an additional stimul exists to investigate dualities in lattices of semigroup varieties. Namely, semigroup varieties of two types examined earlier admit duality: varieties which subvariety lattices are finite chains [15] or Boolean algebras [1,20]. If a semigroup variety $\mathcal{V}$ belongs to one of these classes then $\mathcal{V}$ is selfdual; moreover it is clear that each subvariety of $\mathcal{V}$ is selfdual (in particular admits duality) too. If $\mathcal{V}$ is an arbitrary semigroup variety such that each subvariety of $\mathcal{V}$ admits duality in class X then we say that $\mathcal{V}$ <u>hereditarily admits duality in class</u> X (<u>h.a.d. in</u> X for short). As above we omit the words "in class X" and say that $\mathcal{V}$ <u>hereditarily admits duality</u> (<u>h.a.d.</u>) if X is the class of all semigroup varieties. Finally, a variety of universal algebras $\mathcal{V}$ is said to be <u>hereditarily selfdual</u> if each subvariety of $\mathcal{V}$ is selfdual.

H.a.d. and hereditarily selfdual semigroup varieties are investigated just in the present paper. It consists of two sections.

Section 1 is devoted to hereditarily selfdual varieties. A necessary condition for hereditary selfduality of arbitrary varieties of universal algebras is found and a characterisation of hereditarily selfdual varieties of two large classes of universal algebras is obtained in the paper [18]; these two classes include varieties of all "classical" algebras except semigroups. These universal algebraic results are completed and made more precise in Section 1. Proposition 1 gives a necessary condition for hereditary selfduality which is stronger that the corresponding result of [18]. Proposition 2 yields a characterisation of hereditarily selfdual varieties of very large class of universal algebras embracing as varieties considered in [18] as semigroup varieties. Finally, the main result of Section 1 (Theorem 1) gives a description of hereditarily selfdual semigroup varieties "modulo groups". Moreover, combining Theorem 1 with the results of [2] we obtain a complete description of hereditarily selfdual semigroup varieties in which every periodic group is locally finite (in particular, a complete description of locally finite hereditarily selfdual semigroup varieties).

Section 2 is devoted to h.a.d. semigroup varieties. We obtain a necessary condition on arbitrary semigroup variety being h.a.d. (Theorem 2). This result reduces in some sence the question of complete description of h.a.d. semigroup varieties to examination of two cases: varieties in which every nilsemigroup is a zero semigroup and nilpotent varieties. Unfortunately, all our attempts to obtain further information about arbitrary h.a.d. varieties are not successful for the present. The main cause of this situation is the following circumstance. If $\mathcal{V}$ is an arbitrary h.a.d. semigroup variety then it is possible that an arbitrary semigroup variety dual to $\mathcal{V}$ is a variety of periodic but not locally finite groups. However, practically there is no any positive information about subvariety lattices of periodic not locally finite group varieties at this time. That's why after proving of Theorem 2 we investigate semigroup varieties h.a.d. in class K only where K is the class of all semigroup varieties in which every periodic group is locally finite. Theorem 3 gives a complete description of semigroup varieties h.a.d. in K with the assumption that one of the following statement holds:

(i) $\mathcal{V}$ contains a nilsemigroup which is not a zero semigroup;

(ii) $\mathcal{V}$ satisfies an identity of the kind $x_1 \ldots x_n = x_{1\tau} \ldots x_{n\tau}$ where τ is a permutation, $1\tau \neq 1$ and $n\tau \neq n$.

(Note that the case (ii) embraces all commutative varieties). In particular we state the fairly surprising fact that every semigroup variety h.a.d. in K and satisfying (i) or (ii) is hereditarily selfdual. At the conclusion of Section 2 we give examples of h.a.d. in K (in particular, h.a.d.) but not hereditarily selfdual varieties as well as an example of admitting duality but not h.a.d. in K variety. Unfortunately, we have no examples of varieties which admit duality but not h.a.d. It seems that there exists such a variety but it is very difficult to find it in view of above mentioned causes.

We say that a variety $\mathcal{V}$ is <u>small</u> if $L(\mathcal{V})$ is finite and <u>chain</u> if this lattice is a chain. Varieties $\mathcal{V}_1$ and $\mathcal{V}_2$ of the same similarity type are called <u>disjoint</u> if $\mathcal{V}_1 \wedge \mathcal{V}_2 = \mathcal{O}$ where $\mathcal{O}$ is the trivial variety. Finally, a <u>semichain</u> variety is a join of a finite number of pairwise disjoint small chain varieties. The semigroup variety defining by a system of identities Σ (generating by a semigroup S) is written as var$\{\Sigma\}$ (var S respectively). Recall that a lattice L is called <u>0-distributive</u> if it satisfies the implication $x \wedge z = 0$ & $y \wedge z = 0 \rightarrow (x \vee y) \wedge z = 0$. We say that a lattice is <u>weakly 0-semimodular</u> if the join of two its arbitrary different atoms covers both of this atoms. Finally, a lattice L is said to be <u>lower weakly semimodular</u> if for any $x, y \in L$ the following holds: $x \vee y$ covers both x and y implies x and y cover $x \wedge y$. All other lattice notions used below may be found in the monographs [3] or [6].

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be varieties of periodic groups. Then $\mathcal{G}_1\mathcal{G}_2$ denotes the H.Neumann's product of $\mathcal{G}_1$ and $\mathcal{G}_2$ [11]. In other words, $\mathcal{G}_1\mathcal{G}_2$ consists of groups and a group G belongs to $\mathcal{G}_1\mathcal{G}_2$ if and only if there exists a normal subgroup G_1 of G such that $G_1 \in \mathcal{G}_1$ and $G/G_1 \in \mathcal{G}_2$. As it is shown in [11] $\mathcal{G}_1\mathcal{G}_2$ is a group variety.

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§1. HEREDITARILY SELFDUAL VARIETIES

PROPOSITION 1. *Let* $\mathcal{V}$ *be a hereditarily selfdual variety of universal algebras. Then* $\mathcal{V}$ *is small and semichain and the lattice* $L(\mathcal{V})$ *is* 0*-distributive.*

Proof. As it is shown in [18], every hereditarily selfdual varieties of universal algebras with 0-distributive subvariety lattice is small and semichain. Hence we need only to prove that $L(\mathcal{V})$ is 0-distributive. Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \leq \mathcal{V}$ and $\mathfrak{A} \wedge \mathfrak{C} = \mathfrak{B} \wedge \mathfrak{C} = \mathcal{O}$. Put $\mathcal{D} = (\mathfrak{A} \vee \mathfrak{B}) \wedge \mathfrak{C}$ and $\mathcal{E} = \mathfrak{A} \vee \mathfrak{B}$. It is clear that $\mathfrak{A} \wedge \mathcal{D} = \mathfrak{B} \wedge \mathcal{D} = \mathcal{O}$ and $\mathfrak{A}, \mathfrak{B}, \mathcal{D} \subseteq \mathcal{E}$. As it is prove in [8] the lattice $L(\mathfrak{M})$ satisfies the implication $(\&_{i \in I} x_i \vee y = z) \,\&\, \bigwedge_{i \in I} x_i = 0 \rightarrow y = z$ for any variety of universal algebras $\mathfrak{M}$. The lattice $L(\mathcal{E})$ is selfdual and hence it satisfies the implication $x_1 \wedge y = x_2 \wedge y = 0 \,\&\, x_1 \vee x_2 = 1 \rightarrow y = 0$. But $\mathfrak{A} \wedge \mathcal{D} = \mathfrak{B} \wedge \mathcal{D} = \mathcal{O}$ and $\mathfrak{A} \vee \mathfrak{B} = \mathcal{E}$. We see that $\mathcal{D} = \mathcal{O}$. Thus $\mathfrak{A} \wedge \mathfrak{C} = \mathfrak{B} \wedge \mathfrak{C} = \mathcal{O}$ imply $(\mathfrak{A} \vee \mathfrak{B}) \wedge \mathfrak{C} = \mathcal{O}$ for any $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \subseteq \mathcal{V}$, i.e. $L(\mathcal{V})$ is 0-distributive. The proposition is proved.

PROPOSITION 2. *Let* $\mathcal{V}$ *be a variety of universal algebras and the lattice* $L(\mathcal{V})$ *is weakly* 0*-semimodular The following are equivalent:*

a) $\mathcal{V}$ *is hereditarily selfdual;*

b) $L(\mathcal{V})$ *is a direct product of a finite number of finite chains.*

Proof. The implication b)$\rightarrow$a) is evident.

a)$\rightarrow$b). Let $\mathfrak{A} \subseteq \mathcal{V}$ and φ is an arbitrary dualism of $L(\mathfrak{A})$ onto itself, and let $\mathfrak{B} \subset \mathfrak{A}$. The lattice $L(\mathfrak{A})$ is atomic as well as an arbitrary subvariety lattice. Hence $\varphi(\mathfrak{B}) \supseteq \mathfrak{C}$ where $\mathfrak{C}$ is an atom of $L(\mathfrak{A})$ and $\mathfrak{B} \subseteq \mathcal{D}$ where $\mathcal{D} = \varphi^{-1}(\mathfrak{C})$ is a coatom of $L(\mathfrak{A})$. We see that the lattice $L(\mathcal{V})$ is strongly coatomic.

Further let $\mathcal{E}, \mathcal{F} \subseteq \mathcal{V}$ and $\mathcal{G} = \mathcal{E} \vee \mathcal{F}$ covers both $\mathcal{E}$ and $\mathcal{F}$, and ψ is an arbitrary dualism of $L(\mathcal{G})$ onto itself. Then $\mathcal{O} = \psi(\mathcal{G})$ is covered by $\psi(\mathcal{E})$ and $\psi(\mathcal{F})$, i.e. $\psi(\mathcal{E})$ and $\psi(\mathcal{F})$ are atoms of $L(\mathcal{V})$. It is clear that $\psi(\mathcal{E}) \neq \neq \psi(\mathcal{F})$. The $\mathcal{O}$-semimodularity of $L(\mathcal{V})$ implies that $\psi(\mathcal{E}) \vee \psi(\mathcal{F})$ covers both $\psi(\mathcal{E})$ and $\psi(\mathcal{F})$. Hence $\mathcal{E}$ and $\mathcal{F}$ cover $\mathcal{E} \wedge \mathcal{F}$. Thus we prove that the lattice $L(\mathcal{V})$ is lower weakly semimodular.

Besides that $L(\mathcal{V})$ is coalgebraic as well as any subvariety lattice. By the statement dual to the Theorem 3.7 [3] a coalgebraic, strongly coatomic and lower weakly semimodular lattice is lower semimodular. Hence $L(\mathcal{V})$ is lower semimodular. The selfduality of $L(\mathcal{V})$ implies that it is semimodular. But a semimodular, lower semimodular and coalgebraic lattice is modular by the statement dual to the Theorem 3.6 [3]. We see that $L(\mathcal{V})$ is modular.

Suppose that $L(\mathcal{V})$ is not distributive. Then it contains the 5-element modular non-distributive sublattice. Let $\mathcal{Y}$ be the gratest element of this sublattice. The selfduality of $L(\mathcal{Y})$ implies that it is not $\mathcal{O}$-distributive. But it is impossible by Proposition 1.

Thus $L(\mathcal{V})$ is distributive. Finally by Proposition 1 $\mathcal{V}$ is semichain, i.e. $\mathcal{V} = \bigvee_{i=1}^{n} \mathcal{C}_i$ where $\mathcal{C}_1, \ldots, \mathcal{C}_n$ are pairwise disjoint varieties and $L(\mathcal{C}_1), \ldots, L(\mathcal{C}_n)$ are finite chains. It is clear that $L(\mathcal{V}) \cong L(\mathcal{C}_1) \times \ldots \times L(\mathcal{C}_n)$. The proposition is proved.

As it is noted in [8], there exists a variety of universal algebras $\mathcal{V}$ such that $L(\mathcal{V})$ is the 5-element non-modular lattice. It means, in particular, that an analogue of Proposition 2 does not valid for arbitrary varieties of universal algebras.

We finish a universal algebraic part of our work. Further we deal with semigroup varieties only.

Let us introduce the notations for some concrete semigroup varieties:

$$\mathcal{A}_r = \mathrm{var}\,\{x^r y = y,\ xy = yx\},\quad r \geq 2;$$
$$\mathcal{L} = \mathrm{var}\,\{xy = x\};$$
$$\mathcal{R} = \mathrm{var}\,\{xy = y\};$$
$$\mathcal{Y} = \mathrm{var}\,\{x^2 = x,\ xy = yx\};$$
$$\mathcal{N}_k = \mathrm{var}\,\{x_1 \ldots x_k = t^2,\ xy = yx\},\quad k \geq 2;$$
$$\mathcal{N}_3^2 = \mathrm{var}\,\{xyz = t^2\};$$
$$\mathcal{N}_3^c = \mathrm{var}\,\{xyz = t^3,\ xy = yx\}.$$

Recall that a semigroup variety is called Cliffordian (or completely regular) if it consists of Cliffordian semigroups (unions of groups) only.

THEOREM 1. For an arbitrary semigroup variety $\mathcal{V}$ the following are equivalent:

a) $\mathcal{V}$ is hereditarily selfdual;

b) $L(\mathcal{V})$ is a direct product of a finite number of finite chains;

c) $\mathcal{V}$ is semichain and $L(\mathcal{V})$ is modular;

d) $\mathcal{V}$ is contained either in a variety of the kind $\mathcal{G} \vee \mathcal{L} \vee \mathcal{R} \vee \mathcal{Y} \vee \mathcal{N}_2$ where $\mathcal{G}$ is a semichain periodic group variety or in a variety of the kind $\mathcal{A}_r \vee \mathcal{Y} \vee \mathcal{N}$ where $\mathcal{N} \in \{\mathcal{N}_k, \mathcal{N}_3^2, \mathcal{N}_3^c\}$.

Proof. It is well known that the lattice of all semigroup varieties is weakly $\mathcal{O}$-semimodular (see, e.g., [4]). Hence the equivalence of a) and b) follows immediately from Proposition 2. The implication b)$\to$c) is evident.

c)$\to$d). It is easy to verify that the lattice $L(\mathcal{C} \vee \mathcal{N})$ is non-modular if $\mathcal{C}$ is a non-commutative Cliffordian variety and $\mathcal{N}$ is a nilvariety, $\mathcal{N} \not\subseteq \mathcal{N}_2$. (This is an easy consequence of Lemma 1 [22], e.g.). Hence either each Cliffordian subvariety of $\mathcal{V}$ is commutative or each nil-subvariety of $\mathcal{V}$ is contained in $\mathcal{N}_2$. Further $\mathcal{V}$ is a join of a finite number of pairwise disjoint small chain varieties. By Corollary 1 of [15] varieties $\mathcal{L}, \mathcal{R}, \mathcal{Y}, \mathcal{N}_k, \mathcal{N}_3^2, \mathcal{N}_3^c$ and only they are nongroup small chain varieties. We see that $\mathcal{V} \subseteq \mathcal{G} \vee \mathcal{L} \vee \mathcal{R} \vee \mathcal{Y} \vee \mathcal{N}_2$ where $\mathcal{G}$ is a semichain group variety if each nil-subvariety of $\mathcal{V}$ is contained in $\mathcal{N}_2$ and $\mathcal{V} \subseteq \mathcal{A}_r \vee \mathcal{Y} \vee \mathcal{N}$ where $\mathcal{N} \in \{\mathcal{N}_k, \mathcal{N}_3^2, \mathcal{N}_3^c\}$ if each Cliffordian subvariety of $\mathcal{V}$ is commutative.

d)$\longrightarrow$b). The proof of this implication is naturally divided into two cases.

(i) $\mathcal{V} \subseteq \mathcal{G} \vee \mathcal{L} \vee \mathcal{R} \vee \mathcal{Y} \vee \mathcal{N}_2$ where $\mathcal{G}$ is a semichain group variety, i.e. $\mathcal{G} = \bigvee_{i=1}^{n} \mathcal{G}_i$ where $\mathcal{G}_1, \dots, \mathcal{G}_n$ are pairwise disjoint small chain group varieties. It is easy to verify that $L(\mathcal{G}) \cong L(\mathcal{G}_1) \times \dots \times L(\mathcal{G}_n)$ (stronger results see [18], Lemma 3, or [19], Lemma 1). By results of [12] $L(\mathcal{G} \vee \mathcal{L} \vee \mathcal{R} \vee \mathcal{Y} \vee \mathcal{N}_2) \cong L(\mathcal{G}) \times L(\mathcal{L}) \times L(\mathcal{R}) \times L(\mathcal{Y}) \times L(\mathcal{N}_2)$. It remains to account that $\mathcal{L}, \mathcal{R}, \mathcal{Y}$ and $\mathcal{N}_2$ are small chain varieties.

(ii) $\mathcal{V} \subseteq \mathcal{A}_r \vee \mathcal{Y} \vee \mathcal{N}$ where $\mathcal{N} \in \{\mathcal{N}_k, \mathcal{N}_3^2, \mathcal{N}_3^c\}$. By [10] $L(\mathcal{K} \vee \mathcal{Y}) \cong L(\mathcal{K}) \times L(\mathcal{Y})$ for any semigroup variety $\mathcal{K} \not\supseteq \mathcal{Y}$. Further by Proposition 2 [19] $L(\mathcal{A}_r \vee \mathcal{N}^*) \cong L(\mathcal{A}_r) \times L(\mathcal{N}^*)$ where $\mathcal{N}^* = \mathrm{var}\{x^2y = ztz = yx^2\}$. But $\mathcal{N}_k, \mathcal{N}_3^2, \mathcal{N}_3^c \subseteq \mathcal{N}^*$. Hence $L(\mathcal{A}_r \vee \mathcal{Y} \vee \mathcal{N}) \cong L(\mathcal{A}_r \vee \mathcal{N}) \times L(\mathcal{Y}) \cong L(\mathcal{A}_r) \times L(\mathcal{N}) \times L(\mathcal{Y})$ for any $\mathcal{N} \in \{\mathcal{N}_k, \mathcal{N}_3^2, \mathcal{N}_3^c\}$. Finally $L(\mathcal{A}_r)$ is a direct product of a finite number of finite chains and $\mathcal{N}_k, \mathcal{N}_3^2, \mathcal{N}_3^c$ and $\mathcal{Y}$ are small chain varieties. The theorem is proved.

Locally finite chain varieties of groups were described in [2]. Combining this result with Theorem 1 we obtain a complete description of hereditarily selfdual locally finite semigroup varieties.

It is interesting to compare Theorem 1 with the description of hereditarily selfdual congruence-permutable varieties obtained in [18].

PROPOSITION 3 ([18], Theorem 2). *For a congruence-permutable variety of universal algebras* $\mathcal{V}$ *the following are equivalent*:

a) $\mathcal{V}$ *is hereditarily selfdual*;

b) $\mathcal{V}$ *is semichain*;

c) $L(\mathcal{V})$ *is a direct product of a finite number of finite chains*.

We see that the semigroup case differs essentially from the congruence-permutable one: a semichain variety of semigroups often is not hereditarily selfdual. However Theorem 1 and Proposition 3 have an essential "common part" too: equivalence of hereditary selfduality of $\mathcal{V}$ and decomposability of $L(\mathcal{V})$ into direct product of a finite number of finite chains. Of course, this coincidence is not accidentally. This equivalence is guaranteed by Proposition 2 in both cases.

§2. HEREDITARILY ADMITTING DUALITY VARIETIES

We start this section from the statement which gives some information about arbitrary admitting duality semigroup varieties. We do not use it below. It seems however that it is of some independent interest.

It is well known that a semigroup variety $\mathcal{V}$ is *periodic* (i.e. consists of periodic semigroups) if and only if $\mathcal{V} \not\supseteq \mathcal{A}$ where $\mathcal{A} = \mathrm{var}\{xy = yx\}$.

LEMMA 1. *An arbitrary admitting duality semigroup variety* $\mathcal{V}$ *is periodic*.

Proof. Let $\mathcal{V}'$ be an arbitrary semigroup variety dual to $\mathcal{V}$. The lattice $L(\mathcal{V}')$ is coalgebraic and hence the lattice $L(\mathcal{V})$ is algebraic. But the lattice $L(\mathcal{A})$ is not algebraic. Indeed $\mathcal{A}_3 = \mathcal{A}_3 \wedge \mathcal{A} = \mathcal{A}_3 \wedge (\bigvee_{n=1}^{\infty} \mathcal{A}_{2^n}) \neq \bigvee_{n=1}^{\infty} (\mathcal{A}_3 \wedge \mathcal{A}_{2^n}) = \mathcal{O}$, i.e. $L(\mathcal{A})$ is not continious, but every algebraic lattice is continious.

Now we start to investigate h.a.d. semigroup varieties. As we mentioned above, the lattice of all semigroup varieties is weakly $\emptyset$-semimodular. A simple modification of first, second and third paragraphs of proving of the implication a)$\rightarrow$b) of Proposition 2 permits to obtain the following two lemmas.

LEMMA 2. *If $\mathcal{V}$ is a h.a.d. semigroup variety then $L(\mathcal{V})$ is strongly coatomic.*

LEMMA 3. *If $\mathcal{V}$ is a h.a.d. semigroup variety then $L(\mathcal{V})$ is lower semimodular.*

Following [13] we shall say that a semigroup variety $\mathcal{V}$ has a *finite index* if there exists a natural number n such that an arbitrary nilsemigroup of $\mathcal{V}$ is nilpotent of step $\leq n$; the least n with this property is called the *index* of $\mathcal{V}$. It is easy to verify that a variety $\mathcal{V}$ has an index ≤ 2 if and only if $\mathcal{V} \not\supseteq \mathcal{N}_3$ (see [5], Lemma 3).

LEMMA 4. *Let $\mathcal{C}$ be a non-commutative semigroup variety of an index ≤ 2. Then $\mathcal{C} \vee \mathcal{N}_3$ is not a h.a.d. variety.*

Proof. The non-commutativity of $\mathcal{C}$ implies easily that $\mathcal{C} \vee \mathcal{N}_3 \supseteq \mathcal{N}_3^2$ (see [22], Lemma 1). Further $\mathcal{C} \not\supseteq \mathcal{N}_3$ implies that $\mathcal{C} \vee \mathcal{N}_2 \neq \mathcal{C} \vee \mathcal{N}_3$. Suppose that $\mathcal{C} \vee \mathcal{N}_3$ is h.a.d. By Lemma 2 there exists a coatom $\mathcal{M}$ of the interval $[\mathcal{C} \vee \mathcal{N}_2, \mathcal{C} \vee \mathcal{N}_3]$. (May be that $\mathcal{M} = \mathcal{C} \vee \mathcal{N}_2$ but it is not essential for the proof). It is easy to verify that $\mathcal{M} \vee \mathcal{N}_3^2 = \mathcal{C} \vee \mathcal{N}_3$ and $\mathcal{M} \wedge \mathcal{N}_3^2 = \mathcal{N}_2$. We see that $\mathcal{M} \vee \mathcal{N}_3^2$ covers $\mathcal{M}$ but $\mathcal{N}_3^2$ does not cover $\mathcal{M} \wedge \mathcal{N}_3^2$. Thus the lattice $L(\mathcal{C} \vee \mathcal{N}_3)$ is not lower semimodular that is impossible by Lemma 3. The lemma is proved.

Let P be the semigroup $\{e, a, \emptyset\}$ where $e^2 = e$, $ea = a$ and all other products are equal $\emptyset$ and $\overleftarrow{P}$ be the semigroup dual to P. Recall that a semigroup variety is said to be a *variety with central idempotents* if it satisfies the implication $e^2 = e \to ex = xe$.

LEMMA 5 ([22], Lemma 2). *If $\mathcal{V}$ is a semigroup variety containing no varieties $\mathcal{L}, \mathcal{R}$, var P and var $\overleftarrow{P}$ then $\mathcal{V}$ is a variety with central idempotents.*

LEMMA 6. *An arbitrary h.a.d. semigroup variety $\mathcal{V}$ has a finite index.*

Proof. By Theorem 2 of [13] it is sufficient to verify that $\mathcal{V} \not\supseteq \mathcal{N}_\omega$ where $\mathcal{N}_\omega = \mathrm{var}\{x^2y = y,\ xy = yx\}$. It is well known that $L(\mathcal{N}_\omega)$ is a chain of the type $\omega + 1$ (see, e.g., [4]). It remains to refer to Lemma 2.

It is clear that an arbitrary semigroup variety of a finite index is periodic. Recall that a periodic variety $\mathcal{V}$ contains the gratest group subvariety and the gratest nil-subvariety. We shall denote these varieties as $G(\mathcal{V})$ and $N(\mathcal{V})$ respectively.

LEMMA 7. *Let $\mathcal{V}$ be a semigroup variety of a finite index with central idempotents. Then $\mathcal{V} = \mathcal{G} \vee \mathcal{K} \vee \mathcal{N}$ where $\mathcal{G}$ is a variety of periodic groups, $\mathcal{K} \subseteq \mathcal{Y}$ and $\mathcal{N}$ is a nilpotent variety. Moreover, if $G(\mathcal{V})$ is a small variety then $\mathcal{V}$ is a small one too.*

Proof. The first statement of the lemma immediately follows from following three facts:

1) as it is follows from the proof of Proposition 1 of [22] an arbitrary periodic variety with central idempotents is the join of a variety generated by a semigroup with the unit and a nilvariety;

2) Lemma 9 of [22] easily implies that an arbitrary non-Cliffordian semigroup with the unit generates a variety which has no a finite index;

3) it is easy to verify that an arbitrary Cliffordian variety with central idempotents is the join of a periodic group variety and a variety $\mathcal{K} \subseteq \mathcal{Y}$.

Thus $\mathcal{V} = \mathcal{G} \vee \mathcal{K} \vee \mathcal{N}$ where $\mathcal{G}$ is a periodic group variety, $\mathcal{K} \subseteq \mathcal{Y}$ and $\mathcal{N}$ is a nilpotent variety. We may assume without loss of generality that $\mathcal{G} = G(\mathcal{V})$ and $\mathcal{N} = N(\mathcal{V})$. Let now $\mathcal{M} \subseteq \mathcal{V}$. Then $\mathcal{M}$

is a variety of a finite index with central idempotents. Hence $\mathcal{M} = \mathcal{G}' \vee \tilde{\mathcal{K}}' \vee \mathcal{N}'$, where $\mathcal{G}'$ is a periodic group variety, $\tilde{\mathcal{K}}' \subseteq \mathcal{T}$ and $\mathcal{N}'$ is nilpotent. It is clear that $\mathcal{G}' \subseteq \mathcal{G}$ and $\mathcal{N}' \subseteq \mathcal{N}$. Hence $|L(\mathcal{V})| \leq$ $\leq |L(\mathcal{G})| \times |L(\mathcal{T})| \times |L(\mathcal{N})|$. The lattices $L(\mathcal{T})$ and $L(\mathcal{N})$ are finite. Hence if $L(\mathcal{G})$ is finite then $L(\mathcal{V})$ is finite too. The lemma is proved.

Now we ready to establish

THEOREM 2. *Let $\mathcal{V}$ be a h.a.d. semigroup variety and one of the following conditions holds:*

(i) *$\mathcal{V}$ contains a nilsemigroup which is not a zero semigroup;*

(ii) *$\mathcal{V}$ satisfies an identity of the kind $x_1 \ldots x_n = x_{1\tau} \ldots x_{n\tau}$ where τ is a permutation, $1\tau \neq 1$ and $n\tau \neq n$.*

Then $\mathcal{V} \subseteq \mathcal{A}_r \vee \mathcal{T} \vee \mathcal{N}$ where $\mathcal{N}$ is a nilpotent h.a.d. variety.

Proof. Lemmas 6 and 7 show that it is suffucient to verify that $\mathcal{V}$ is a variety with central idempotents and each group of $\mathcal{V}$ is abelian. If $\mathcal{V}$ satisfies condition (i) then it follows from Lemmas 4 and 5. Finally, in the case (ii) it is easy to see that $\mathcal{V}$ has the mentioned properties.

Theorem 2 and the second statement of Lemma 7 imply

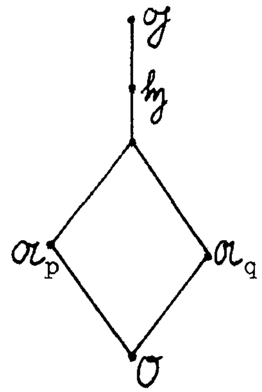
COROLLARY. *Let $\mathcal{V}$ be a h.a.d. semigroup variety and one of the conditions (i) and (ii) of Theorem 2 holds. Then $\mathcal{V}$ is a small variety.*

Recall that we denote as K the class of all semigroup varieties in which every periodic group is locally finite. Further we investigate semigroup varieties h.a.d. in K only. Our aim is to describe such varieties with the assumption that either (i) or (ii) holds.

Recall that a finite group is called *critical* if it does not belong to the variety generating by all its proper subgroups and all its proper factor-groups. It is well known that each critical group G contains the unique minimal non-trivial normal subgroup which is called the monolith of G (see, e.g., Theorem 51.32 of [11]). The following lemma communicated to the author by M.V.Sapir.

LEMMA 8. There exists no locally finite group variety which subvariety lattice is the lattice on the Figure 1.

FIGURE 1



Proof. Let $\mathfrak{X}$ be a counterexample to the lemma, $\mathfrak{Y}$ be the greatest proper subvariety of $\mathfrak{X}$, $\mathfrak{A}_p$ and $\mathfrak{A}_q$ be atoms of $L(\mathfrak{X})$. (Of course p and q are different primes). It is convinient to divide further considerations on two steps.

Step 1. Let us prove that $\mathfrak{Y} = \mathfrak{A}_p\mathfrak{A}_q$ (without loss of generality). The variety $\mathfrak{Y}$ is locally finite and hence it is generated by its finite groups. It means that there exist finite groups of $\mathfrak{Y} \setminus (\mathfrak{A}_p \vee \mathfrak{A}_q)$. Let H be a group of the least order among these finite groups. It is clear that H is a critical group. Let N be the monolith of H. Suppose that $N = H$. Then H is a simple group. Moreover, $|H| = p^{\alpha}q^{\beta}$ and hence H is solvable (see, e.g., [7], Theorem 5.3.2). Thus H is a simple solvable group, i.e. a cyclic group of a prime order. But it is impossible because $\mathfrak{Y} = \operatorname{var} H \supsetneq \mathfrak{A}_p \vee \mathfrak{A}_q$. Thus $N \neq H$ and hence $N \in \mathfrak{A}_p \vee \mathfrak{A}_q$. Put $N_p = \{x \in N \mid x^p = 1\}$ and $N_q = \{x \in N \mid x^q = 1\}$. Then N_p and N_q are characteristic subgroups of N. But a monolith does not contain proper characteristic subgroups. That's why either $N_p = \{1\}$ and $N = N_q \in \mathfrak{A}_q$ or $N_q = \{1\}$ and $N = N_p \in \mathfrak{A}_p$. Suppose that $N \in \mathfrak{A}_p$ without loss of generality. It is clear that $|H/N| < |H|$ and hence $H/N \in \mathfrak{A}_p \vee \mathfrak{A}_q$, in particular H/N is abelian. Thus the Sylow p-subgroup of H/N is a normal subgroup of H/N. Since $N \in \mathfrak{A}_p$ it implies that the

Sylow p-subgroup K of H is a normal subgroup of H. Hence $H/K \in \mathfrak{A}_q$. We see that $H \in \mathfrak{A}_p\mathfrak{A}_q$ and $\mathfrak{H} = \mathrm{var}\, H \subseteq \mathfrak{A}_p\mathfrak{A}_q$. Finally, by Theorem 54.41 of [11] $\mathfrak{A}_p \vee \mathfrak{A}_q$ is the gratest proper subvariety of $\mathfrak{A}_p\mathfrak{A}_q$. Hence $\mathfrak{H} = \mathfrak{A}_p\mathfrak{A}_q$.

Step 2. Consider now the variety $\mathfrak{G}$. As above $\mathfrak{G} = \mathrm{var}\, G$ where G is a finite group of the least order of $\mathfrak{G}\setminus\mathfrak{H}$ and G is a critical group; further $M \neq G$ and $M \in \mathfrak{H} = \mathfrak{A}_p\mathfrak{A}_q$ where M is the monolith of G. Repeating the considerations of the Step 1 we prove that either $M \in \mathfrak{A}_p$ and hence $\mathfrak{G} = \mathfrak{A}_p\mathfrak{A}_q = \mathfrak{H}$ or $M \in \mathfrak{A}_q$. The first case is impossible. Thus $M \in \mathfrak{A}_q$. It is clear that $G/M \in \mathfrak{H} = \mathfrak{A}_p\mathfrak{A}_q$. Hence there exists a normal subgroup L/M of G/M such that $L/M \in \mathfrak{A}_p$ and $(G/M)/(L/M) \cong G/L \in \mathfrak{A}_q$. Then $L \in \mathfrak{A}_q\mathfrak{A}_p$. But varieties $\mathfrak{A}_q\mathfrak{A}_p$ and $\mathfrak{A}_p\mathfrak{A}_q$ are not comparable, and hence $\mathfrak{G} \not\supseteq \mathfrak{A}_q\mathfrak{A}_p$. Further $\mathfrak{A}_p \vee \mathfrak{A}_q$ is the gratest proper subvariety of $\mathfrak{A}_q\mathfrak{A}_p$ and hence $L \in \mathfrak{A}_p \vee \mathfrak{A}_q$. Put $L_p = \{x \in L \mid x^p = 1\}$. Then L_p is a characteristic subgroup of a normal subgroup L of G and hence L_p is a normal subgroup of G. But $L_p \not\supseteq M$ and M is the monolith of G. It means that $L_p = \{1\}$ and $L \in \mathfrak{A}_q$. Since $G/L \in \mathfrak{A}_q$ we see that $G \in \mathfrak{A}_q\mathfrak{A}_q$ and $\mathfrak{G} = \mathrm{var}\, G \subseteq \mathfrak{A}_q\mathfrak{A}_q$. But it is impossible because $\mathfrak{A}_q\mathfrak{A}_q \not\supseteq \mathfrak{A}_p$. The lemma is proved.

PROPOSITION 4. _Let_ $\mathfrak{N}$ _be a nilvariety of semigroups. The following are equivalent_:

a) $\mathfrak{N}$ _is h.a.d. in_ K;

b) $\mathfrak{N}$ _is hereditarily selfdual_;

c) $\mathfrak{N}$ _is a nilpotent chain variety_;

d) $\mathfrak{N} \in \{\mathfrak{N}_k, \mathfrak{N}_3^2, \mathfrak{N}_3^c\}$.

Proof. The equivalence of c) and d) follows from [15], the implications c) $\longrightarrow$ b) and b) $\longrightarrow$ a) are evident.

a) $\longrightarrow$ c). By Lemma 6 $\mathfrak{N}$ is nilpotent. Suppose that $\mathfrak{N}$ is not chain. By Corollary 2 of [15] $\mathfrak{N}$ contains a variety $\mathfrak{M}$ such that $L(\mathfrak{M})$ is the lattice dual to the lattice on the Figure 1. Lemma 8 shows that there no exist locally finite group varieties dual to $\mathfrak{M}$. Moreover result of [16] implies that there no exist nongroup varieties with such properties too. Hence $\mathfrak{N}$ is not h.a.d. in K. The proposition is proved.

THEOREM 3. Let $\mathfrak{V}$ be a semigroup variety and one of the following conditions holds:

(i) $\mathfrak{V}$ contains a nilsemigroup which is not a zero semigroup;

(ii) $\mathfrak{V}$ satisfies an identity of the kind $x_1 \ldots x_n = x_{1\tau} \ldots x_{n\tau}$ where τ is a permutation, $1\tau \neq 1$ and $n\tau \neq n$.

The following are equivalent:

a) $\mathfrak{V}$ is h.a.d. in K;

b) $\mathfrak{V}$ is hereditarily selfdual;

c) $\mathfrak{V} \subseteq \mathfrak{A}_r \vee \mathfrak{J} \vee \mathfrak{N}$ where $\mathfrak{N} \in \{\mathfrak{N}_k, \mathfrak{N}_3^2, \mathfrak{N}_3^c\}$.

Proof. a) $\longrightarrow$ c). By Theorem 2 $\mathfrak{V} \subseteq \mathfrak{A}_r \vee \mathfrak{J} \vee \mathfrak{N}$, where $\mathfrak{N}$ is a nilpotent h.a.d. variety. Moreover, the proof of Theorem 2 shows that $\mathfrak{V} = \mathfrak{G} \vee \mathfrak{K} \vee \mathfrak{N}$, where $\mathfrak{G} \subseteq \mathfrak{A}_r$ and $\mathfrak{K} \subseteq \mathfrak{J}$. In particular $\mathfrak{N} \subseteq \mathfrak{V}$ and $\mathfrak{N}$ is h.a.d. in K. It remains to account Proposition 4.

c) $\longrightarrow$ b) by Theorem 1.

b) $\longrightarrow$ a). It is sufficient to verify that if $\mathfrak{M} \subseteq \mathfrak{V}$ then $\mathfrak{M} \in$ K. If $\mathfrak{V}$ satisfies the condition (i) then each group of $\mathfrak{V}$ is abelian by Lemma 4. It is evident that in the case (ii) $\mathfrak{V}$ has the latest property too. We see that if $\mathfrak{M} \subseteq \mathfrak{V}$ then $G(\mathfrak{M})$ is abelian and hence $\mathfrak{M} \in$ K. It finishes the proof of the Theorem.

Theorem 3 and Proposition 4 permit to assume that an arbitrary semigroup variety h.a.d. in K is hereditarily selfdual. Unfortunately this conjecture is false. The corresponding examples show on the Figure 2 ($\mathfrak{E}_p$ is the variety of 3-Engel groups of prime exponent $p \geqslant 3$). The Figure 3 demonstrates an example of admitting duality but not h.a.d. in K semigroup variety (the proof of Proposition 4 implies that $\mathfrak{N}_3^2 \vee \mathfrak{N}_4$ admits no duality in class K).

FIGURE 2

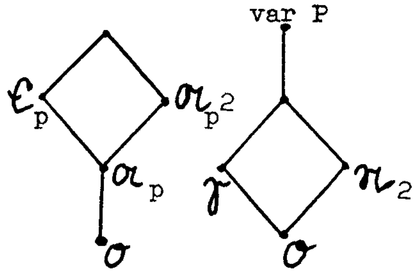
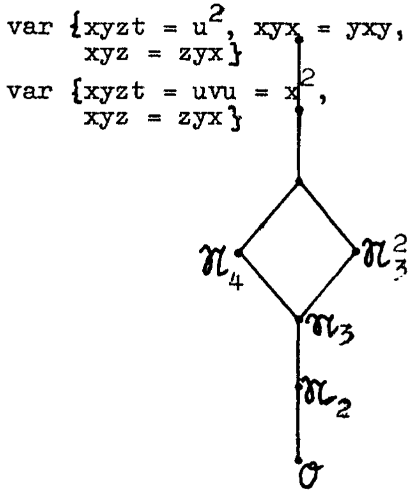


FIGURE 3



Added in proof. Recently the author has proved that if $\mathcal{N}$ is a nilpotent variety and $\mathcal{N} \nsubseteq \mathcal{N}^*$ then the variety $\mathcal{A}_r \vee \mathcal{N}$ is not h.a.d. It permits to obtain the following stronger version of Theorem 2: if $\mathcal{V}$ is a h.a.d. semigroup variety and one of the conditions (i) and (ii) holds then either $\mathcal{V} \subseteq \mathcal{A}_r \vee \mathcal{Y} \vee \mathcal{N}$ where $\mathcal{N}$ is a nilpotent h.a.d. variety and $\mathcal{N} \subseteq \mathcal{N}^*$ or $\mathcal{V} \subseteq \mathcal{Y} \vee \mathcal{N}$ where $\mathcal{N}$ is a nilpotent **h.a.d.** variety.

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