PROOFS OF DEFINABILITY OF SOME VARIETIES AND SETS OF VARIETIES OF SEMIGROUPS

B. M. VERNIKOV

COMMUNICATED BY L. N. SHEVRIN

Abstract. We show that many important varieties and sets of varieties of semigroups may be defined by relatively simple and transparent first-order formulas in the lattice of all semigroup varieties.

1. Introduction

A subset A of a lattice $\langle L; \vee, \wedge \rangle$ is called *definable in L* if there exists a first-order formula $\Phi(x)$ with one free variable x in the language of lattice operations $\vee$ and $\wedge$ which *defines A in L*. This means that, for an element $a \in L$, the sentence $\Phi(a)$ is true if and only if $a \in A$. If A consists of a single element, we speak about definability of this element.

We denote the lattice of all semigroup varieties by **SEM**. A set of semigroup varieties X (or a single semigroup variety $\mathcal{X}$) is said to be *definable* if it is definable in **SEM**. In this situation we will say that the corresponding first-order formula *defines* the set X or the variety $\mathcal{X}$.

A number of deep results about definable varieties and sets of varieties of semigroups have been obtained in [8] by Ježek and McKenzie[1]. It has been conjectured there that every finitely based semigroup variety is definable up to duality. The conjecture is confirmed in [8] for locally finite finitely based varieties. However the article [8] contains no explicit first-order formulas that define any given locally finite finitely based variety. On their way to obtain the mentioned fundamental result, Ježek and McKenzie proved the definability of several important sets of semigroup varieties such as the sets of all finitely based, all locally finite, all finitely generated and all 0-reduced semigroup varieties. But the article [8] contains no explicit first-order formulas that define any of these sets of varieties. The task of writing an explicit formula that defines the set of all finitely based or the set of all locally finite or the set of all finitely

2000 *Mathematics Subject Classification.* Primary 20M07, secondary 08B15.

Key words and phrases. Semigroup, variety, lattice of varieties, first-order formula, definable set of varieties.

The work was partially supported by the Russian Foundation for Basic Research (grant No. 10-01-00524) and the Federal Education Agency of the Russian Federation (project No. 2.1.1/13995).

[1]We note that the paper [8], as well as the articles [6,9] mentioned in Section 5 below have dealt with the lattice of equational theories of semigroups, that is, the dual of **SEM** rather than the lattice **SEM** itself. When reproducing results from [6, 8, 9], we adapt them to the terminology of the present note.

generated varieties seems to be extremely difficult. But on the other hand, many traditionally considered sets of semigroup varieties (including the set of all 0-reduced varieties) and many important individual varieties (for instance, an arbitrary Abelian periodic group variety) can be defined by relatively simple first-order formulas. Such formulas come naturally from the structural theory of semigroup varieties.

Here we present explicit formulas that define several well-known sets of semigroup varieties and individual varieties. Each of these varieties and sets of varieties has appeared multiple number of times in many articles in semigroup theory.

We will denote the conjunction by & rather than $\wedge$ because the latter symbol stands for the meet in a lattice. Since the disjunction and the join in a lattice are denoted usually by the same symbol $\vee$, we use this symbol for the join and denote the disjunction by OR. Evidently, the relations $\le$, $\ge$, $<$ and $>$ in a lattice L can be expressed in terms of, say, meet operation $\wedge$ in L. So, we will freely use these four relations in formulas. Let $\Phi(x)$ be a first-order formula. For the sake of brevity, we put

$$\min_x\bigl\{\Phi(x)\bigr\} \;\rightleftharpoons\; \Phi(x)\,\&\,(\forall y)\,\bigl(y<x \longrightarrow \neg\Phi(y)\bigr)$$

and

$$\max_x\bigl\{\Phi(x)\bigr\} \;\rightleftharpoons\; \Phi(x)\,\&\,(\forall y)\,\bigl(x<y \longrightarrow \neg\Phi(y)\bigr).$$

Clearly, the formula $\min_x\bigl\{\Phi(x)\bigr\}$ [respectively $\max_x\bigl\{\Phi(x)\bigr\}$] defines the set of all minimal [maximal] elements of the set

$$\{a\in L \mid \text{ the sentence } \Phi(a) \text{ is true}\}.$$

For reader convenience, we recall some definitions and notation concerning semigroups, semigroup varieties and semigroup identities. A pair of identities $wx = xw = w$ where the letter x does not occur in the word w is usually written as the symbolic identity $w = 0$. This notation is justified because a semigroup with such identities has a zero element and all values of the word w in this semigroup are equal to zero. Identities of the form $w = 0$ as well as varieties given by identities of such a form are called *0-reduced*. A semigroup is called *periodic* if every its cyclic subsemigroup is finite (equivalently, if it satisfies the identity $x^n = x^{n+m}$ for some natural numbers n and m). A semigroup is called *nilpotent* [*nil-semigroup*] if it satisfies the identity $x_1x_2\cdots x_n = 0$ [respectively $x^n = 0$] for some natural n. A semigroup is called *completely regular* if it is a union of groups. A semigroup S is called *completely simple* if S does not contain proper ideals and the set of all its non-zero idempotents has a minimal element under the following natural order relation: $e \le f$ if and only if $ef = fe = e$. We adopt the usual agreement that an adjective indicating a property shared by all semigroups of a given variety is applied to the variety itself; the expressions like “completely regular variety”, “periodic variety”, “nil-variety” etc. are understood in this sense. A semigroup variety $\mathcal{V}$ is called *overcommutative* if it contains the variety of all commutative semigroups. Finally, a semigroup variety $\mathcal{V}$ is called *combinatorial* if all groups in $\mathcal{V}$ are singleton.

2. Atoms and chain varieties

Many important sets of semigroup varieties admit a characterization in the language of atoms of the lattice **SEM**. The set of all atoms of a lattice L with 0 is defined by the formula

$$\mathtt{A}(x) \ \rightleftharpoons\ (\exists y)\ \big((\forall z)\,(y \le z)\,\&\ \min_x\{x \ne y\}\big).$$

A description of all atoms of the lattice **SEM** is well known. To list these varieties, we need notation for several semigroup varieties. By $\operatorname{var}\Sigma$ we denote the semigroup variety given by the identity system Σ. Put:

$\mathcal{A}_n = \operatorname{var}\{x^n y = y,\, xy = yx\}$ — the variety of Abelian groups whose exponent divides n,

$\mathcal{SL} = \operatorname{var}\{x^2 = x,\, xy = yx\}$ — the variety of semilattices,

$\mathcal{LZ} = \operatorname{var}\{xy = x\}$ — the variety of left zero semigroups,

$\mathcal{RZ} = \operatorname{var}\{xy = y\}$ — the variety of right zero semigroups,

$\mathcal{ZM} = \operatorname{var}\{xy = 0\}$ — the variety of null semigroups.

The following lemma is well known (see the surveys [4, 16], for instance).

Lemma 2.1. *The varieties $\mathcal{A}_p$ (where p is a prime number), $\mathcal{SL}$, $\mathcal{LZ}$, $\mathcal{RZ}$, $\mathcal{ZM}$ and only they are atoms of the lattice* **SEM**. □

If $\mathcal{V}$ is a semigroup variety then we denote by $\overleftarrow{\mathcal{V}}$ the variety *dual to* $\mathcal{V}$, that is, the variety consisting of semigroups anti-isomorphic to members of $\mathcal{V}$. The map from the lattice **SEM** into itself given by the rule $\mathcal{V} \longmapsto \overleftarrow{\mathcal{V}}$ is an automorphism of **SEM**. Therefore if $\mathcal{V} \ne \overleftarrow{\mathcal{V}}$ then the variety $\mathcal{V}$ is not definable. The varieties $\mathcal{LZ}$ and $\mathcal{RZ}$ are dual to each other, whence they are not definable. We will say that a variety $\mathcal{V}$ is *definable up to duality* if $\mathcal{V} \ne \overleftarrow{\mathcal{V}}$ and the set $\{\mathcal{V}, \overleftarrow{\mathcal{V}}\}$ is definable. We are going to verify that all atoms of **SEM** except $\mathcal{LZ}$ and $\mathcal{RZ}$ are definable, while the varieties $\mathcal{LZ}$ and $\mathcal{RZ}$ are definable up to duality (see Proposition 2.4 and Theorem 5.7 below). To achieve this aim, we need some additional definitions, notation and results. Put

$$\mathtt{Neut}(x) \ \rightleftharpoons\ (\forall y, z)\ \big((x \vee y) \wedge (y \vee z) \wedge (z \vee x) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)\big).$$

An element x of a lattice L such that the sentence $\mathtt{Neut}(x)$ is true is called *neutral*. Neutral elements play a distinguished role in the lattice theory (see Section III.2 in [5], for instance). We denote by $\mathcal{T}$ the trivial semigroup variety, and by $\mathcal{SEM}$ the variety of all semigroups.

Lemma 2.2 ([24, Proposition 2.4])**.** *The varieties $\mathcal{T}$, $\mathcal{SL}$, $\mathcal{ZM}$, $\mathcal{SL} \vee \mathcal{ZM}$, $\mathcal{SEM}$ and only they are neutral elements of the lattice* **SEM**. □

A semigroup variety $\mathcal{V}$ is called *chain* if the subvariety lattice of $\mathcal{V}$ is a chain. Clearly, each atom of **SEM** is a chain variety. The set of all chain varieties is definable by the formula

$$\mathtt{Ch}(x) \ \rightleftharpoons\ (\forall y, z)\,(y \le x \,\&\, z \le x \longrightarrow y \le z \text{ OR } z \le y).$$

Put

$$\mathcal{N}_k = \mathrm{var}\,\{x^2 = x_1x_2\cdots x_k = 0,\ xy = yx\}\ (k \text{ is a natural number}),$$
$$\mathcal{N}_\omega = \mathrm{var}\,\{x^2 = 0,\ xy = yx\},$$
$$\mathcal{N}_3^2 = \mathrm{var}\,\{x^2 = xyz = 0\},$$
$$\mathcal{N}_3^c = \mathrm{var}\,\{xyz = 0,\ xy = yx\}$$

(in particular $\mathcal{N}_1 = \mathcal{T}$ and $\mathcal{N}_2 = \mathcal{ZM}$). The following lemma is proved in [17].

Lemma 2.3. *The varieties $\mathcal{SL}$, $\mathcal{LZ}$, $\mathcal{RZ}$, $\mathcal{N}_k$, $\mathcal{N}_\omega$, $\mathcal{N}_3^2$, $\mathcal{N}_3^c$ and only they are non-group chain varieties of semigroups.* □

The problem of a complete classification of chain group varieties seems to be extremely difficult (see Subsection 11.6 in [16] for additional comments on this subject). But in the Abelian case the problem turns out to be trivial. The lattice of all Abelian periodic group varieties is evidently isomorphic to the lattice of natural numbers ordered by divisibility. This readily implies that non-trivial chain Abelian group varieties are varieties $\mathcal{A}_{p^k}$ with prime p and natural k, and only they. Fig. 1 shows the relative location of the chain varieties mentioned above in the lattice **SEM**.

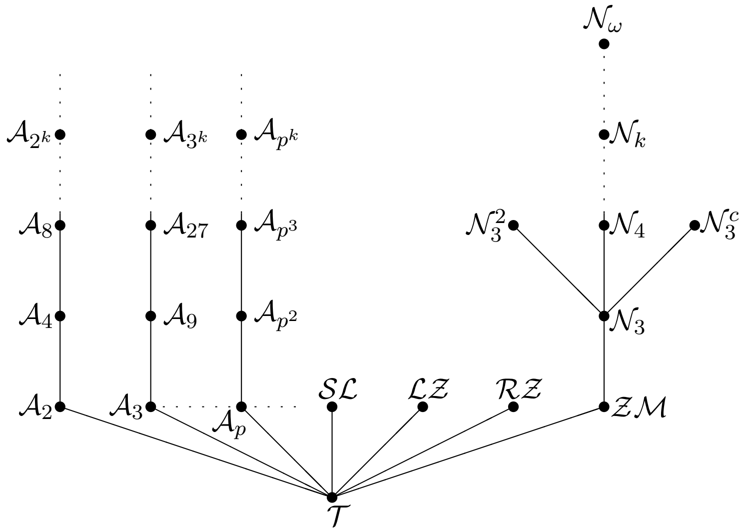

FIGURE 1. Non-group and Abelian group chain varieties

Combining above observations, it is easy to verify the following

Proposition 2.4. *The varieties $\mathcal{SL}$ and $\mathcal{ZM}$, and the set of varieties*

$$\{\mathcal{A}_p \mid p \text{ is a prime number}\}$$

are definable. The varieties $\mathcal{LZ}$ and $\mathcal{RZ}$ are definable up to duality.

Proof. By Lemma 2.1, all varieties mentioned in the proposition are atoms of **SEM**. By Lemma 2.2, the varieties $\mathcal{SL}$ and $\mathcal{ZM}$ are neutral elements in **SEM**, while $\mathcal{LZ}$, $\mathcal{RZ}$ and $\mathcal{A}_p$ are not. Fig. 1 shows that the varieties $\mathcal{ZM}$ and $\mathcal{A}_p$

are proper subvarieties of some chain varieties, while $\mathcal{SL}$, $\mathcal{LZ}$ and $\mathcal{RZ}$ are not. Therefore the formulas

$$\mathtt{SL}(x) \rightleftharpoons \mathtt{A}(x)\,\&\,\mathtt{Neut}(x)\,\&\,(\forall y)\,\bigl(\mathtt{Ch}(y)\,\&\,x \le y \longrightarrow x = y\bigr),$$
$$\mathtt{ZM}(x) \rightleftharpoons \mathtt{A}(x)\,\&\,\mathtt{Neut}(x)\,\&\,(\exists y)\,\bigl(\mathtt{Ch}(y)\,\&\,x < y\bigr)$$

define the varieties $\mathcal{SL}$ and $\mathcal{ZM}$ respectively, while the the formulas

$$\mathtt{LZ\text{-}and\text{-}RZ}(x) \rightleftharpoons \mathtt{A}(x)\,\&\,\neg\mathtt{Neut}(x)\,\&\,(\forall y)\,\bigl(\mathtt{Ch}(y)\,\&\,x \le y \longrightarrow x = y\bigr),$$
$$\mathtt{GrA}(x) \rightleftharpoons \mathtt{A}(x)\,\&\,\neg\mathtt{Neut}(x)\,\&\,(\exists y)\,\bigl(\mathtt{Ch}(y)\,\&\,x < y\bigr)$$

define the sets $\{\mathcal{LZ}, \mathcal{RZ}\}$ and $\{\mathcal{A}_p \mid p \text{ is a prime number}\}$ respectively. □

Note that in fact each of the group atoms $\mathcal{A}_p$ is individually definable (see Theorem 5.7 below). The definability of the varieties $\mathcal{SL}$ and $\mathcal{ZM}$ and the definability up to duality of the varieties $\mathcal{LZ}$ and $\mathcal{RZ}$ are mentioned in [8] (see Theorem 1.11 and Lemma 4.3 there) without any explicitly written formulas. Note also that the variety $\mathcal{ZM}$ can be defined by several different ways. For the sake of completeness, one can provide one more of them. It is a common knowledge that a semigroup variety is completely regular [a nil-variety] if and only if it does not contain the variety $\mathcal{ZM}$ [any atom of the lattice **SEM** except $\mathcal{ZM}$]. The lattice of completely regular semigroup varieties is modular (see [10–12] or Section 6 in [16]). In contrast, the lattice of nil-varieties does not satisfy any non-trivial lattice identity [3]. Combining these observations, we see that the variety $\mathcal{ZM}$ can be defined by the formula

$$\mathtt{ZM}'(x) \rightleftharpoons \mathtt{A}(x)\,\&\,(\forall y, z, t)\,\bigl(y, z, t \not\ge x\,\&\,z \le t \longrightarrow (y \vee z) \wedge t = (y \wedge t) \vee z\bigr).$$

Put $\mathcal{COM} = \operatorname{var}\{xy = yx\}$. As an immediate consequence of Lemma 2.4, we have the following

Proposition 2.5. *The variety $\mathcal{COM}$ is definable.*

Proof. It is well known that the join of the varieties $\mathcal{A}_p$ where p runs over the set of all prime numbers coincides with the variety $\mathcal{COM}$ (see [4], for instance). Therefore the formula

$$\mathtt{COM}(x) \rightleftharpoons \min_x\bigl\{(\forall y)\,\bigl(\mathtt{GrA}(y) \longrightarrow y \le x\bigr)\bigr\}$$

defines the variety $\mathcal{COM}$. □

The following general fact will be used in what follows.

Lemma 2.6. *If a countably infinite subset S of a lattice L is definable in L and forms a chain isomorphic to the chain of natural numbers under the order relation in L then every member of this set is definable in L.*

Proof. Let $S = \{s_n \mid n \in \mathbb{N}\}$, $s_1 < s_2 < \cdots < s_n < \cdots$ and let $\Phi(x)$ be the formula defining S in L. We are going to prove the definability of the element s_n for each n by induction on n. The induction base is evident because the element s_1 is definable by the formula $\min_x\bigl\{\Phi(x)\bigr\}$. Assume now that $n > 1$ and the element s_{n-1} is definable by some formula $\Psi(x)$. Then the formula

$$\min_x\bigl\{\Phi(x)\,\&\,(\exists y)\,\bigl(\Psi(y)\,\&\,y < x\bigr)\bigr\}$$

defines the element s_n. □

The fact that the variety $\mathcal{ZM}$ is definable is a partial case of the following

Proposition 2.7. *Every chain nil-variety of semigroups is definable.*

Proof. All considerations here are based on Lemma 2.3 and Fig. 1. The variety $\mathcal{N}_\omega$ is defined by the formula

$$\mathtt{N}_\omega(x) \;\rightleftharpoons\; \max_x\bigl\{\mathtt{Ch}(x)\,\&\,(\exists y,z,t)\,\bigl(\mathtt{ZM}(y)\,\&\,y<z<t<x\bigr)\bigr\}.$$

The formula

$$\mathtt{All\text{-}N_k}(x) \;\rightleftharpoons\; (\exists y)\,\bigl(\mathtt{N}_\omega(y)\,\&\,x<y\bigr)$$

defines the set of varieties $\{\mathcal{N}_k \mid k \in \mathbb{N}\}$. Now Lemma 2.6 successfully applies with the conclusion that the variety $\mathcal{N}_k$ is definable for each k. It remains to verify the definability of the varieties $\mathcal{N}_3^c$ and $\mathcal{N}_3^2$. Both these varieties (and only they) are chain varieties that contains $\mathcal{ZM}$ and are not contained in $\mathcal{N}_\omega$; besides that the variety $\mathcal{N}_3^c$ is commutative, while the variety $\mathcal{N}_3^c$ is not. Therefore the formulas

$$\mathtt{N_3^c}(x) \;\rightleftharpoons\; \mathtt{Ch}(x)\,\&\,(\exists y,z,t)\,\bigl(\mathtt{ZM}(y)\,\&\,y\le x\,\&\,\mathtt{N}_\omega(z)\,\&\,x\not\le z\,\&\,\mathtt{COM}(t)\,\&\,x\le t\bigr),$$
$$\mathtt{N_3^2}(x) \;\rightleftharpoons\; \mathtt{Ch}(x)\,\&\,(\exists y,z,t)\,\bigl(\mathtt{ZM}(y)\,\&\,y\le x\,\&\,\mathtt{N}_\omega(z)\,\&\,x\not\le z\,\&\,\mathtt{COM}(t)\,\&\,x\not\le t\bigr)$$

define the varieties $\mathcal{N}_3^c$ and $\mathcal{N}_3^2$ respectively. □

Note that every chain Abelian group variety $\mathcal{A}_{p^k}$ also is definable (see Theorem 5.7 below).

3. Main sublattices of the lattice SEM

The lattice **SEM** contains a number of wide and important sublattices (see Section 1 and Chapter 2 in [16]). In this section we aim to show that many of these sublattices are definable (as sets of varieties).

It is well known that every semigroup variety is either overcommutative or periodic. Thus the lattice **SEM** is the disjoint union of two big sublattices: the lattice of all periodic varieties and the lattice of all overcommutative varieties. One more important sublattice of **SEM** is the lattice of all commutative varieties. It is evident that the formulas

$$\mathtt{Per}(x) \;\rightleftharpoons\; (\exists y)\,\bigl(\mathtt{COM}(y)\,\&\,y\not\le x\bigr),$$
$$\mathtt{OC}(x) \;\rightleftharpoons\; \bigl(\forall y)\,(\mathtt{COM}(y)\longrightarrow y\le x\bigr),$$
$$\mathtt{Com}(x) \;\rightleftharpoons\; (\forall y)\,\bigl(\mathtt{COM}(y)\longrightarrow x\le y\bigr)$$

define the sets of all periodic varieties, all overcommutative varieties and all commutative varieties respectively. Thus we have the following

Theorem 3.1. *The sets of all periodic varieties, all overcommutative varieties and all commutative varieties of semigroups are definable.* □

As we have already mentioned in Section 2, many important sets of semigroup varieties admit a characterization in the language of atoms of the lattice **SEM**. Several facts of such a type are summarised in Table 1; all these facts are verified in [1].

Varieties of all types mentioned in Table 1 form sublattices in **SEM**. Combining the facts from Table 1 with Proposition 2.4, we have the following

A semigroup variety is	if and only if it does not contain the varieties
a completely regular variety	$\mathcal{ZM}$
a completely simple variety	$\mathcal{SL}, \mathcal{ZM}$
a periodic group variety	$\mathcal{LZ}, \mathcal{RZ}, \mathcal{SL}, \mathcal{ZM}$
a combinatorial variety	$\mathcal{A}_p$ for all prime p
a variety of idempotent semigroups	$\mathcal{ZM}$ and $\mathcal{A}_p$ for all prime p
a nil-variety	$\mathcal{LZ}, \mathcal{RZ}, \mathcal{SL}$ and $\mathcal{A}_p$ for all prime p

TABLE 1. A characterization of some sets of varieties

Theorem 3.2. *The sets of all completely regular varieties, all completely simple varieties, all periodic group varieties, all combinatorial varieties, all varieties of idempotent semigroups, and all nil-varieties of semigroups are definable.* □

Note that the definability of the set of all nil-varieties was mentioned in [20]. For convenience of references, we write in Table 2 formulas defining the sets of varieties listed in Theorem 3.2.

The set of all	is defined by the formula
completely regular varieties	$\mathtt{CR}(x) \;\rightleftharpoons\; (\forall y)\,\bigl(\mathtt{A}(y)\,\&\,y \le x \longrightarrow \neg\mathtt{ZM}(y)\bigr)$
completely simple varieties	$\mathtt{CS}(x) \;\rightleftharpoons\; (\forall y)\,\bigl(\mathtt{A}(y)\,\&\,y \le x \longrightarrow \neg\mathtt{SL}(y)\,\&\,\neg\mathtt{ZM}(y)\bigr)$
periodic group varieties	$\mathtt{Gr}(x) \;\rightleftharpoons\; (\forall y)\,\bigl(\mathtt{A}(y)\,\&\,y \le x \longrightarrow \mathtt{GrA}(y)\bigr)$
combinatorial varieties	$\mathtt{Comb}(x) \;\rightleftharpoons\; (\forall y)\,\bigl(\mathtt{A}(y)\,\&\,y \le x \longrightarrow \neg\mathtt{GrA}(y)\bigr)$
varieties of idempotent semigroups	$\mathtt{Idemp}(x) \;\rightleftharpoons\; (\forall y)\,\bigl(\mathtt{A}(y)\,\&\,y \le x \longrightarrow \neg\mathtt{GrA}(y)\,\&\,\neg\mathtt{ZM}(y)\bigr)$
nil-varieties	$\mathtt{Nil}(x) \;\rightleftharpoons\; (\forall y)\,\bigl(\mathtt{A}(y)\,\&\,y \le x \longrightarrow \mathtt{ZM}(y)\bigr)$

TABLE 2. Formulas defining some sets of varieties

One more interesting sublattice of the lattice **SEM** is the lattice of all 0-reduced varieties.

Theorem 3.3. *The set of all* 0*-reduced semigroup varieties is definable.*

Proof. Put

$$\mathtt{LMod}(x) \;\rightleftharpoons\; (\forall y, z)\,\bigl(x \le y \longrightarrow x \vee (y \wedge z) = y \wedge (x \vee z)\bigr).$$

An element x of a lattice L such that the sentence $\mathtt{LMod}(x)$ is true is called *lower-modular*. It is verified in [20] that a semigroup variety is 0-reduced if and only if it is a nil-variety and a lower-modular element of the lattice **SEM**. Therefore the formula

$$\mathtt{0\text{-}red}(x) \;\rightleftharpoons\; \mathtt{Nil}(x)\,\&\,\mathtt{LMod}(x)$$

defines the set of all 0-reduced varieties. □

Theorem 3.3 was verified in [8] (see Theorem 1.11 there) without an explicitly written formula defining the set of all 0-reduced varieties. The formula $\mathtt{0\text{-}red}(x)$

is given in [20], while some slightly more complex formula defining the set of all 0-reduced varieties was written earlier in [24].

Note that in Sections 4 and 7 we provide several other definable sublattices of **SEM**.

4. Varieties of finite degree

We call a semigroup variety $\mathcal{V}$ a *variety of finite degree* if all nil-semigroups in $\mathcal{V}$ are nilpotent; $\mathcal{V}$ is called a variety of *degree* k if nilpotency degrees of nilsemigroups in $\mathcal{V}$ are bounded by the number k and k is the least number with this property. In this section we show that the set of all semigroup varieties of finite degree and certain its important subsets and members are definable.

Theorem 4.1. *The sets of all semigroup varieties of finite degree and of all semigroup varieties of degree* k (*for an arbitrary natural number* k) *are definable.*

Proof. According to Theorem 2 of [14], $\mathcal{V}$ is a variety of finite degree if and only if $\mathcal{N}_\omega \nsubseteq \mathcal{V}$. It is easy to see also that $\mathcal{V}$ is a variety of degree $\le k$ if and only if $\mathcal{N}_{k+1} \nsubseteq \mathcal{V}$, whence $\mathcal{V}$ is a variety of degree k if and only if $\mathcal{N}_{k+1} \nsubseteq \mathcal{V}$ but $\mathcal{N}_k \subseteq \mathcal{V}$. Therefore the set of all varieties of finite degree is definable by the formula

$$\mathtt{FinDeg}(x) \;\rightleftharpoons\; (\forall y)\big(\mathtt{N}_\omega(y) \longrightarrow y \nleq x\big),$$

while the set of all varieties of degree k is defined by the formila

$$\mathtt{Deg_k}(x) \;\rightleftharpoons\; (\forall y, z)\big(\mathtt{N}_k(y) \,\&\, \mathtt{N}_{k+1}(z) \longrightarrow y \le x \,\&\, z \nleq x\big)$$

where $\mathtt{N}_m(x)$ (for any natural m) is the formula that defines the variety $\mathcal{N}_m$. $\square$

For a natural number k, we put $\mathcal{NILP}_k = \mathrm{var}\,\{x_1x_2\cdots x_k = 0\}$.

Theorem 4.2. *The set of all nilpotent semigroup varieties is definable. For an arbitrary natural number* k*, the variety* $\mathcal{NILP}_k$ *is definable.*

Proof. Evidently, a semigroup variety is nilpotent if and only if it is a nil-variety of finite degree. Therefore, the set of all nilpotent varieties is definable by the formula

$$\mathtt{Nilp}(x) \;\rightleftharpoons\; \mathtt{Nil}(x) \,\&\, \mathtt{FinDeg}(x).$$

The variety $\mathcal{NILP}_k$ is the largest nil-variety of degree k, whence the formula

$$\mathtt{NILP_k}(x) \;\rightleftharpoons\; \max_x\big\{\mathtt{Nil}(x) \,\&\, \mathtt{Deg_k}(x)\big\}$$

defines this variety. $\square$

As usual, we put $S^n = \{x_1x_2\cdots x_n \mid x_1, x_2, \dots, x_n \in S\}$ for any semigroup S and any natural number $n > 1$. A natural subset of varieties of finite degree is formed by *varieties of semigroups with completely regular power*, that is varieties $\mathcal{V}$ with the following property: for any member $S \in \mathcal{V}$, there is a natural number n such that the semigroup S^n is completely regular. A variety $\mathcal{V}$ is called a variety *with completely regular* k^{th} *power* if S^k is completely regular for any $S \in \mathcal{V}$ and k is the least number with this property. To prove that the set of all semigroup varieties with completely regular [k^{th}] power is definable, we need some additional information.

Put $\mathcal{P} = \mathrm{var}\,\{xy = x^2y,\, x^2y^2 = y^2x^2\}$. It is well known that the variety $\mathcal{P}$ is generated by the 3-element semigroup

$$P = \{e, a, 0\} = \langle a, e \mid e^2 = e,\, ea = a,\, ae = 0\rangle.$$

The semigroup P and the variety $\mathcal{P}$ frequently appear in articles devoted to different aspects of the theory of semigroup varieties.

Lemma 4.3. *A semigroup variety $\mathcal{V}$ of finite degree [of degree k] is a variety of semigroups with completely regular power [with completely regular k^{th} power] if and only if $\mathcal{P}, \overleftarrow{\mathcal{P}} \nsubseteq \mathcal{V}$.* □

This assertion is verified in [19] for varieties of semigroups with completely regular power, and its variant for varieties of semigroups with completely regular k^{th} power can be verified quite analogously.

Proposition 4.4. *The varieties $\mathcal{P}$ and $\overleftarrow{\mathcal{P}}$ are definable up to duality.*

Proof. It can be easily verified (and follows from results of [18], for instance) that the varieties $\mathcal{P}$, $\overleftarrow{\mathcal{P}}$ and only they have the property that any proper subvariety of a variety is contained in the variety $\mathcal{SL} \vee \mathcal{ZM}$. Therefore the formula

$$\texttt{P-and-}\overleftarrow{\texttt{P}}(x) \;\rightleftharpoons\; (\exists y, z)\,\big(\texttt{SL}(y)\,\&\,\texttt{ZM}(z)\,\&\,(\forall t)\,(t < x \longrightarrow t \le y \vee z)\big)$$

defines the set $\{\mathcal{P}, \overleftarrow{\mathcal{P}}\}$. □

The statement that the set of all completely regular varieties is definable (see Theorem 3.2) is generalized by the following

Theorem 4.5. *The set of all varieties of semigroups with completely regular power is definable. For every natural number k, the set of all varieties of semigroups with k^{th} completely regular power is definable.*

Proof. Lemma 4.3 immediately implies that the set of all varieties of semigroups with completely regular power and the set of all varieties of semigroups with k^{th} completely regular power are defined by the formulas

$$\texttt{CRPow}(x) \;\rightleftharpoons\; \texttt{FinDeg}(x)\,\&\,(\forall y)\,\big(\texttt{P-and-}\overleftarrow{\texttt{P}}(y) \longrightarrow y \nleq x\big),$$
$$\texttt{CRPow}_{\texttt{k}}(x) \;\rightleftharpoons\; \texttt{Deg}_{\texttt{k}}(x)\,\&\,(\forall y)\,\big(\texttt{P-and-}\overleftarrow{\texttt{P}}(y) \longrightarrow y \nleq x\big)$$

respectively. □

5. Commutative varieties

Here we are going to provide some series of definable varieties of commutative semigroups. To achieve this aim, we need some auxiliary facts. The following lemma follows from Lemma 2 of [23] and the proof of Proposition 1 of the same article.

Lemma 5.1. *If a periodic semigroup variety $\mathcal{V}$ does not contain the varieties $\mathcal{LZ}$, $\mathcal{RZ}$, $\mathcal{P}$ and $\overleftarrow{\mathcal{P}}$ then $\mathcal{V} = \mathcal{K} \vee \mathcal{N}$ where $\mathcal{K}$ is a variety generated by a monoid, while $\mathcal{N}$ is a nil-variety.* □

Let $C_{m,1}$ denote the cyclic monoid $\langle a \mid a^m = a^{m+1} \rangle$ and let $\mathcal{C}_m$ be the variety generated by $C_{m,1}$. It is clear that

$$\mathcal{C}_m = \mathrm{var}\,\{x^m = x^{m+1},\, xy = yx\}.$$

In particular, $C_{1,1}$ is the 2-element semilattice and $\mathcal{C}_1 = \mathcal{SL}$. For notation convenience we put also $\mathcal{C}_0 = \mathcal{T}$. The following lemma can be easily extracted from the results of [7].

Lemma 5.2. *If a periodic semigroup variety $\mathcal{V}$ is generated by a commutative monoid then $\mathcal{V} = \mathcal{G} \vee \mathcal{C}_m$ for some Abelian periodic group variety $\mathcal{G}$ and some $m \geq 0$.* □

Lemmas 5.1 and 5.2 immediately imply

Corollary 5.3. *If $\mathcal{V}$ is a commutative combinatorial semigroup variety then $\mathcal{V} = \mathcal{C}_m \vee \mathcal{N}$ for some $m \geq 0$ and some nil-variety $\mathcal{N}$.* □

Let now $\mathcal{V}$ be a commutative semigroup variety with $\mathcal{V} \neq \mathcal{COM}$. Lemmas 5.1 and 5.2 imply that $\mathcal{V} = \mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}$ for some Abelian periodic group variety $\mathcal{G}$, some $m \geq 0$ and some commutative nil-variety $\mathcal{N}$. Our aim in this section is to provide formulas defining the varieties $\mathcal{G}$ and $\mathcal{C}_m$.

It is well known that each periodic semigroup variety $\mathcal{X}$ contains its greatest nil-subvariety. We denote this subvariety by $\mathrm{Nil}(\mathcal{X})$. Put

$$\mathcal{D}_m = \mathrm{Nil}(\mathcal{C}_m) = \mathrm{var}\,\{x^m = 0,\, xy = yx\}$$

for every natural m. In particular, $\mathcal{D}_1 = \mathcal{T}$ and $\mathcal{D}_2 = \mathcal{N}_\omega$. Now we are well prepared to verify

Proposition 5.4. *For each $m \geq 0$, the variety $\mathcal{C}_m$ is definable.*

Proof. First, we are going to verify that the formula

$$\mathtt{All\text{-}C_m}(x) \;\rightleftharpoons\; \mathtt{Com}(x) \,\&\, \mathtt{Comb}(x) \,\&\, (\forall y, z)\,\big(\mathtt{Nil}(y) \,\&\, x = y \vee z \longrightarrow x = z\big)$$

defines the set of varieties $\{\mathcal{C}_m \mid m \geq 0\}$. Let $\mathcal{V}$ be a semigroup variety such that the sentence $\mathtt{All\text{-}C_m}(\mathcal{V})$ is true. Then $\mathcal{V}$ is commutative and combinatorial. Now Corollary 5.3 successfully applies with the conclusion that $\mathcal{V} = \mathcal{C}_m \vee \mathcal{N}$ for some $m \geq 0$ and some nil-variety $\mathcal{N}$. The fact that the sentence $\mathtt{All\text{-}C_m}(\mathcal{V})$ is true shows that $\mathcal{V} = \mathcal{C}_m$.

Let now $m \geq 0$. We aim to verify that the sentence $\mathtt{All\text{-}C_m}(\mathcal{C}_m)$ is true. It is evident that the variety $\mathcal{C}_m$ is commutative and combinatorial. Suppose that $\mathcal{C}_m = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{N}$ is a nil-variety. It remains to check that $\mathcal{N} \subseteq \mathcal{M}$. We may assume without any loss that $\mathcal{N} = \mathrm{Nil}(\mathcal{C}_m) = \mathcal{D}_m$. It is clear that $\mathcal{M}$ is a commutative and combinatorial variety. Corollary 5.3 implies that $\mathcal{M} = \mathcal{C}_r \vee \mathcal{N}'$ for some $r \geq 0$ and some nil-variety $\mathcal{N}'$. Then $\mathcal{N}' \subseteq \mathrm{Nil}(\mathcal{C}_m) = \mathcal{N}$, whence

$$\mathcal{C}_m = \mathcal{M} \vee \mathcal{N} = \mathcal{C}_r \vee \mathcal{N}' \vee \mathcal{N} = \mathcal{C}_r \vee \mathcal{N}.$$

It suffices to prove that $\mathcal{N} \subseteq \mathcal{C}_r$ because $\mathcal{N} \subseteq \mathcal{C}_r \vee \mathcal{N}' = \mathcal{M}$ in this case. The equality $\mathcal{C}_m = \mathcal{C}_r \vee \mathcal{N}$ implies that $\mathcal{C}_r \subseteq \mathcal{C}_m$, whence $r \leq m$. If $r = m$ then $\mathcal{N} \subseteq \mathcal{C}_r$, and we are done. Let now $r < m$. Then the variety $\mathcal{C}_m = \mathcal{C}_r \vee \mathcal{N}$ satisfies the identity $x^r y^m = x^{r+1} y^m$. Recall that the variety $\mathcal{C}_m$ is generated by a monoid. Substituting 1 for y in this identity, we obtain that $\mathcal{C}_m$ satisfies the identity $x^r = x^{r+1}$. Therefore $\mathcal{C}_m \subseteq \mathcal{C}_r$ contradicting the unequality $r < m$.

Thus we have proved that the set of varieties $\{\mathcal{C}_m \mid m \geq 0\}$ is definable by the formula $\mathtt{All\text{-}C_m}(x)$. Now Lemma 2.6 successfully applies with the conclusion that the variety $\mathcal{C}_m$ is definable for each m. □

The following assertion generalizes the fact that the variety $\mathcal{N}_\omega$ is definable (see Proposition 2.7).

Proposition 5.5. *For every natural number m, the variety $\mathcal{D}_m$ is definable.*

Proof. Put

$$\mathtt{Nil\text{-}part}(x,y) \ \rightleftharpoons\ \mathtt{Per}(x)\,\&\,y \leq x\,\&\,\mathtt{Nil}(y)\,\&\,(\forall z)\big(z \leq x\,\&\,\mathtt{Nil}(z) \longrightarrow z \leq y\big).$$

Clearly, for semigroup varieties $\mathcal{X}$ and $\mathcal{Y}$, the sentence $\mathtt{Nil\text{-}part}(\mathcal{X},\mathcal{Y})$ is true if and only if $\mathcal{X}$ is periodic and $\mathcal{Y} = \mathrm{Nil}(\mathcal{X})$. Let $\mathtt{C_m}$ be the formula defining the variety $\mathcal{C}_m$. The variety $\mathcal{D}_m$ is defined by the formula

$$\mathtt{D_m}(x) \ \rightleftharpoons\ (\exists y)\big(\mathtt{C_m}(y)\,\&\,\mathtt{Nil\text{-}part}(y,x)\big)$$

because $\mathcal{D}_m = \mathrm{Nil}(\mathcal{C}_m)$. □

To prove the definability of an arbitrary Abelian periodic group variety, we need some definitions, notation and an auxiliary result. We denote by **Com** the lattice of all commutative semigroup varieties. We call a commutative semigroup variety 0-*reduced in* **Com** if it may be given by the commutative law and some non-empty set of 0-reduced identities only. If $\mathcal{X}$ is a commutative nil-variety of semigroups then we denote by $\mathrm{ZR}(\mathcal{X})$ the least 0-reduced in **Com** variety that contains $\mathcal{X}$. Clearly, the variety $\mathrm{ZR}(\mathcal{X})$ is given by the commutative law and all 0-reduced identities that hold in $\mathcal{X}$. If u is a word and x is a letter then $c(u)$ denotes the set of all letters occurring in u, while $\ell_x(u)$ stands for the number of occurrences of x in u.

Lemma 5.6. *Let m and n be natural numbers with $m > 2$ and $n > 1$. The following are equivalent:*

(i) $\mathrm{Nil}(\mathcal{A}_n \vee \mathcal{X}) = \mathrm{ZR}(\mathcal{X})$ *for any variety* $\mathcal{X} \subseteq \mathcal{D}_m$*;*
(ii) $n \geq m - 1$.

Proof. (i)$\longrightarrow$(ii) Suppose that $n < m-1$. Let $\mathcal{X}$ be the subvariety of $\mathcal{D}_m$ given within $\mathcal{D}_m$ by the identity

$$x^{n+1}y = xy^{n+1}. \tag{1}$$

Since $n + 1 < m$, the variety $\mathcal{X}$ is not 0-reduced in **Com**. The identity (1) holds in the variety $\mathcal{A}_n \vee \mathcal{X}$, and therefore in the variety $\mathrm{Nil}(\mathcal{A}_n \vee \mathcal{X})$. But the latter variety does not satisfy the identity $x^{n+1}y = 0$ because this identity fails in $\mathcal{X}$. We see that the variety $\mathrm{Nil}(\mathcal{A}_n \vee \mathcal{X})$ is not 0-reduced in **Com**. Since the variety $\mathrm{ZR}(\mathcal{X})$ is 0-reduced in **Com**, we are done.

(ii)$\longrightarrow$(i) Let $n \geq m - 1$ and $\mathcal{X} \subseteq \mathcal{D}_m$. One can verify that $\mathcal{A}_n \vee \mathcal{X} = \mathcal{A}_n \vee \mathrm{ZR}(\mathcal{X})$. Note that this equality immediately follows from Lemma 2.5 of [15] whenever $n \geq m$. We reproduce here the corresponding arguments for the sake of completeness. It suffices to check that $\mathcal{A}_n \vee \mathrm{ZR}(\mathcal{X}) \subseteq \mathcal{A}_n \vee \mathcal{X}$ because the opposite inclusion is evident. Suppose that the variety $\mathcal{A}_n \vee \mathcal{X}$ satisfies an identity $u = v$. We need to prove that this identity holds in $\mathcal{A}_n \vee \mathrm{ZR}(\mathcal{X})$. Since $u = v$ holds in $\mathcal{A}_n$, we have $\ell_x(u) \equiv \ell_x(v) (\mathrm{mod}\, n)$ for any letter x. If $\ell_x(u) = \ell_x(v)$ for all letters x then $u = v$ holds in $\mathcal{A}_n \vee \mathrm{ZR}(\mathcal{X})$ because this variety is commutative. Therefore we may assume that $\ell_x(u) \neq \ell_x(v)$ for some letter x. Then either $\ell_x(u) \geq n$ or $\ell_x(v) \geq n$. We may assume without any loss that $\ell_x(u) \geq n$.

Suppose that $n \geq m$. Then the identity $u = 0$ holds in the variety $\mathcal{D}_m$, whence it holds in $\mathcal{X}$. This implies that $v = 0$ holds in $\mathcal{X}$ too. Therefore the variety $\mathrm{ZR}(\mathcal{X})$ satisfies the identities $u = 0 = v$. Since the identity $u = v$ holds in $\mathcal{A}_n$, it holds in $\mathcal{A}_n \vee \mathrm{ZR}(\mathcal{X})$, and we are done.

It remains to consider the case $n = m - 1$. Let x be any letter such that $x \in c(u) \cup c(v)$ and $\ell_x(u) \neq \ell_x(v)$. If either $\ell_x(u) \geq m$ or $\ell_x(v) \geq m$, we go to the situation considered in the previous paragraph. Let now $\ell_x(u), \ell_x(v) < m$. Since $\ell_x(u) \geq n = m - 1$, $\ell_x(u) \equiv \ell_x(v) (\mathrm{mod}\, n)$ and $\ell_x(u) \neq \ell_x(v)$, we have $\ell_x(u) = n = m - 1$ and $\ell_x(v) = 0$. The latter equality means that $x \notin c(v)$. Substituting 0 for x in $u = v$, we obtain that the variety $\mathcal{X}$ satisfies the identity $v = 0$. We go to the situation considered in the previous paragraph again.

We have proved that $\mathcal{A}_n \vee \mathcal{X} = \mathcal{A}_n \vee \mathrm{ZR}(\mathcal{X})$. Therefore $\mathrm{ZR}(\mathcal{X}) \subseteq \mathrm{Nil}(\mathcal{A}_n \vee \mathcal{X})$. If the variety $\mathcal{X}$ satisfies an identity $u = 0$ then $u^{n+1} = u$ holds in $\mathcal{A}_n \vee \mathcal{X}$. This readily implies that $u = 0$ in $\mathrm{Nil}(\mathcal{A}_n \vee \mathcal{X})$. Hence $\mathrm{Nil}(\mathcal{A}_n \vee \mathcal{X}) \subseteq \mathrm{ZR}(\mathcal{X})$. Thus $\mathrm{Nil}(\mathcal{A}_n \vee \mathcal{X}) = \mathrm{ZR}(\mathcal{X})$. □

Now we are well prepared to prove the announced above

Theorem 5.7. *An arbitrary Abelian periodic group variety is definable.*

Proof. Abelian periodic group varieties are exhausted by the trivial variety and the varieties $\mathcal{A}_n$ with $n > 1$. The trivial variety is obviously definable. For brevity, put

$$\mathtt{Ab}(x) \rightleftharpoons \mathtt{Com}(x) \,\&\, \mathtt{Gr}(x),$$

$$\mathtt{Com\text{-}0\text{-}red}(x) \rightleftharpoons (\exists y, z)\big(\mathtt{COM}(y) \,\&\, \mathtt{0\text{-}red}(z) \,\&\, x = y \wedge z\big),$$

$$\mathtt{ZR}(x, y) \rightleftharpoons \mathtt{Com\text{-}0\text{-}red}(y) \,\&\, x \leq y \,\&\, (\forall z)\big(\mathtt{Com\text{-}0\text{-}red}(z) \,\&\, x \leq z \longrightarrow y \leq z\big).$$

The formula $\mathtt{Ab}(x)$ [respectively $\mathtt{Com\text{-}0\text{-}red}(x)$] defines the set of all Abelian periodic group varieties [respectively all 0-reduced in **Com** varieties] and, for semigroup varieties $\mathcal{X}$ and $\mathcal{Y}$, the sentence $\mathtt{ZR}(\mathcal{X}, \mathcal{Y})$ is true if and only if $\mathcal{Y} = \mathrm{ZR}(\mathcal{X})$. Let m be a natural number with $m > 2$. In view of Lemma 5.6, the formula

$$\mathtt{A}_{\geq \mathtt{m}-1}(x) \rightleftharpoons \mathtt{Ab}(x) \,\&\, (\forall y, z, t)\big(\mathtt{D_m}(y) \,\&\, z \leq y \,\&\, \mathtt{Nil\text{-}part}(x \vee z, t) \longrightarrow \mathtt{ZR}(z, t)\big)$$

defines the set of varieties $\{\mathcal{A}_n \mid n \geq m - 1\}$. Therefore the formula

$$\mathtt{A_n}(x) \rightleftharpoons \mathtt{A}_{\geq \mathtt{n}}(x) \,\&\, \neg \mathtt{A}_{\geq \mathtt{n+1}}(x)$$

defines the variety $\mathcal{A}_n$. □

Corollary 5.8. *A periodic semigroup variety generated by a commutative monoid is definable.*

Proof. Let $\mathcal{V}$ be a variety generated by some commutative monoid. According to Lemma 5.2, $\mathcal{V} = \mathcal{A}_n \vee \mathcal{C}_m$ for some $n \geq 1$ and $m \geq 0$. It is easy to check that the parameters n and m in this decomposition are defined uniquely. Therefore the formula

$$(\exists y, z)\, \bigl(\mathtt{A_n}(y) \mathbin{\&} \mathtt{C_m}(z) \mathbin{\&} x = y \vee z\bigr)$$

defines the variety $\mathcal{V}$ (we assume here that $\mathtt{A_1}$ is the evident formula defining the variety $\mathcal{A}_1 = \mathcal{T}$). □

It was proved in [9] that the set of all Abelian periodic group varieties and each Abelian group variety are definable in the lattice **Com**. Moreover, some characterization of all commutative semigroup varieties definable in the lattice **Com** was found in [6]. Proposition 2.5 readily implies that a commutative semigroup variety is definable in **SEM** whenever it is definable in **Com**. Thus Theorem 5.7 follows from results of [9]. However the articles [6, 9] contain no explicit first-order formulas that define the set of all Abelian periodic group varieties or any given Abelian periodic group variety or any other commutative variety in the lattice **Com**.

6. Finitely universal varieties

Following [16], we call a semigroup variety *finitely universal* if the subvariety lattice of this variety contains an anti-isomorphic copy of the partition lattice over arbitrary finite set. The interest in varieties with this property is motivated by the well known fact that the subvariety lattice of a finitely universal variety does not satisfy any non-trivial lattice identity. It is known [2] that the variety $\mathcal{COM}$ is finitely universal. Moreover, it is easy to see that $\mathcal{COM}$ is a minimal finitely universal variety. Another known example of a minimal finitely universal variety is the variety

$$\mathcal{H} = \mathrm{var}\,\{x^2 = xyx = 0\},$$

see [22]. The question whether or not there exist minimal finitely universal varieties different from $\mathcal{COM}$ and $\mathcal{H}$ is open so far (see Section 12 of [16] for more detailed comments). In this connection, it is interesting to note that both the varieties $\mathcal{COM}$ and $\mathcal{H}$ are definable. The variety $\mathcal{COM}$ is definable by Proposition 2.5. Here we are going to demonstrate the definability of $\mathcal{H}$, using results from the literature. By the way, we provide some other examples of definable 0-reduced varieties. Put

$$\mathcal{E}_m = \mathrm{var}\,\{x^m = 0\} \ (m \text{ is a natural number}),$$
$$\mathcal{F} = \mathrm{var}\,\{x^2y = xyx = yx^2 = 0\}.$$

In particular, $\mathcal{E}_1 = \mathcal{T}$.

Proposition 6.1. *The varieties $\mathcal{E}_m$ (for any natural number m), $\mathcal{F}$ and $\mathcal{H}$ are definable.*

Proof. One can prove that the variety $\mathcal{E}_m$ is definable by the formula

$$\mathtt{E_m}(x) \ \rightleftharpoons\ \max_x\bigl\{\mathtt{Nil}(x) \mathbin{\&} (\exists y, z)\, \bigl(\mathtt{COM}(y) \mathbin{\&} \mathtt{D_m}(z) \mathbin{\&} x \wedge y = z\bigr)\bigr\}.$$

In other words, we are going to check that $\mathcal{E}_m$ is the greatest nil-variety $\mathcal{N}$ with the property $\mathcal{N} \wedge \mathcal{COM} = \mathcal{D}_m$. The equality $\mathcal{E}_m \wedge \mathcal{COM} = \mathcal{D}_m$ is evident. Let

$\mathcal{N}$ be a nil-variety with $\mathcal{N} \wedge \mathcal{COM} = \mathcal{D}_m$. Then the variety $\mathcal{N} \wedge \mathcal{COM}$ satisfies the identity $x^m = 0$. Therefore one of the varieties $\mathcal{N}$ and $\mathcal{COM}$ satisfies a non-trivial identity of the form $x^m = u$. It is evident that $\mathcal{COM}$ does not satisfy any non-trivial identity of such the form. Therefore the identity $x^m = u$ holds in $\mathcal{N}$. It is easy to see that this identity implies $x^m = 0$ in arbitrary nil-variety. Thus $\mathcal{N}$ satisfies the identity $x^m = 0$, that is $\mathcal{N} \subseteq \mathcal{E}_m$.

Put

$$\mathtt{Distr}(x) \;\rightleftharpoons\; (\forall y, z)\,\big(x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)\big).$$

An element x of a lattice L such that the sentence $\mathtt{Distr}(x)$ is true is called *distributive*. Distributive elements in the lattice **SEM** are completely determined in [21]. In particular, it is proved there that a nil-variety is a distributive element of **SEM** if and only if it is 0-reduced and satisfies the identities $x^2y = xyx = yx^2 = 0$. Therefore the formula

$$\mathtt{F}(x) \;\rightleftharpoons\; \max_x\big\{\mathtt{0\text{-}red}(x)\,\&\,\mathtt{Distr}(x)\big\}$$

defines the variety $\mathcal{F}$. Since $\mathcal{H} = \mathcal{E}_2 \wedge \mathcal{F}$, the formula

$$\mathtt{H}(x) \;\rightleftharpoons\; (\exists y, z)\,\big(\mathtt{E}_2(y)\,\&\,\mathtt{F}(z)\,\&\,x = y \wedge z\big)$$

defines the variety $\mathcal{H}$. □

7. Permutative varieties

An identity of the form

$$x_1x_2\cdots x_n = x_{1\alpha}x_{2\alpha}\cdots x_{n\alpha} \tag{2}$$

where α is a non-trivial permutation on the set $\{1, 2, \ldots, n\}$ is called *permutational*. The number n is called a *length* of this identity. A semigroup variety is called *permutative* if it satisfies some permutational identity. Permutative varieties are natural and important generalization of commutative ones. Here we are going to prove the definability of the set of all permutative varieties and certain its important members and subset.

Theorem 7.1. *The set of all permutative semigroup varieties is definable.*

Proof. By Proposition 2 of [14], a semigroup variety is permutative if and only if it does not contain the variety of completely simple semigroups over Abelian groups of exponent p for some prime p, and contains none of the minimal non-Abelian periodic group varieties, the varieties

$$\mathcal{LRB} = \operatorname{var}\{x = x^2,\, xyx = xy\},$$
$$\mathcal{RRB} = \operatorname{var}\{x = x^2,\, xyx = yx\}$$

and $\mathcal{H}$. The set of all minimal non-Abelian periodic group varieties is defined by the formula

$$\mathtt{JNAb}(x) \;\rightleftharpoons\; \min_x\big\{\mathtt{Gr}(x)\,\&\,\neg\mathtt{Com}(x)\big\}.$$

Put

$$\mathtt{Gr\text{-}part}(x,y) \;\rightleftharpoons\; \mathtt{Per}(x)\,\&\,y \le x\,\&\,\mathtt{Gr}(y)\,\&\,(\forall z)\,\big(z \le x\,\&\,\mathtt{Gr}(z) \longrightarrow z \le y\big).$$

For semigroup varieties $\mathcal{X}$ and $\mathcal{Y}$, the sentence $\mathtt{Gr\text{-}part}(\mathcal{X},\mathcal{Y})$ is true if and only if $\mathcal{X}$ is periodic and $\mathcal{Y}$ is the greatest group subvariety of $\mathcal{X}$. A variety of completely simple semigroups over groups of some prime exponent p is the largest completely simple variety $\mathcal{V}$ such that the largest group subvariety of $\mathcal{V}$ is $\mathcal{A}_p$. Therefore the set of all such varieties is defined by the formula

$$\texttt{All-CSA}_\texttt{p}(x) \ \rightleftharpoons\ \max_x\big\{\texttt{CS}(x)\,\&\,(\forall y)\,\big(\texttt{Gr-part}(x,y) \longrightarrow \texttt{GrA}(y)\big)\big\}.$$

It is well known that the lattice of all varieties of idempotent semigroups has the form shown in Fig. 2 where $\mathcal{I} = \mathrm{var}\{x = x^2\}$ (see [4] or [16], for instance). In particular, we see that $\mathcal{LRB}$ [respectively $\mathcal{RRB}$] is the largest variety of idempotent semigroups that does not contain the variety $\mathcal{RZ}$ [respectively $\mathcal{LZ}$]. Therefore the set $\{\mathcal{LRB}, \mathcal{RRB}\}$ is defined by the formula

$$\texttt{LRB-and-RRB}(x) \ \rightleftharpoons\ \max_x\big\{\texttt{Idemp}(x)\,\&\,(\exists y)\,\big(\texttt{LZ-and-RZ}(y)\,\&\,y \nleq x\big)\big\}.$$

Combining the above observations we have that the formula

$$\texttt{Perm}(x) \rightleftharpoons (\forall y)\big(y \le x \longrightarrow \neg\texttt{JNAb}(y) \& \neg\texttt{All-CSA}_\texttt{p}(y) \& \neg\texttt{LRB-and-RRB}(y) \& \neg\texttt{H}(y)\big)$$

defines the set of all permutative varieties. □

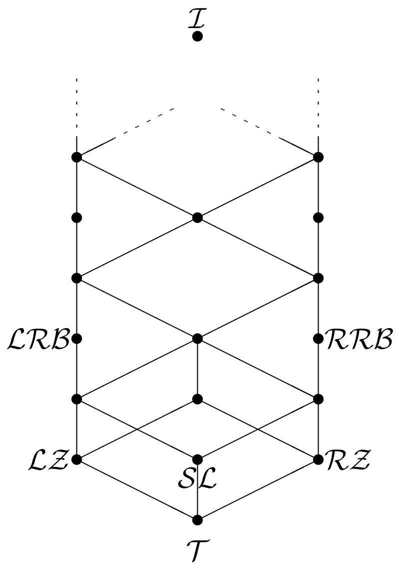

FIGURE 2. The lattice of varieties of idempotent semigroups

For a natural number $n > 1$, we denote by $\mathcal{PERM}_n$ the variety given by all permutational identities of length n. In particular, $\mathcal{PERM}_2 = \mathcal{COM}$. The fact that the variety $\mathcal{COM}$ is definable (see Proposition 2.5) is a partial case of the following

Proposition 7.2. *For an arbitrary natural number n, the variety $\mathcal{PERM}_n$ is definable.*

Proof. It is easy to see that $\mathcal{PERM}_n = \mathcal{COM} \vee \mathcal{NILP}_n$. Therefore the formula

$$\texttt{PERM}_\texttt{n}(x) \ \rightleftharpoons\ (\exists y, z)\,\big(\texttt{COM}(y)\,\&\,\texttt{NILP}_\texttt{n}(z)\,\&\,x = y \vee z\big)$$

defines the variety $\mathcal{PERM}_n$. □

A semigroup variety is called *strongly permutative* if it satisfies an identity of the form (2) with $1\alpha \neq 1$ and $n\alpha \neq n$. It is proved in [13] that a variety $\mathcal{V}$ is strongly permutative if and only if $\mathcal{V} \subseteq \mathcal{PERM}_n$ for some n.

Theorem 7.3. *The set of all strongly permutative semigroup varieties is definable.*

Proof. Let $\mathcal{V}$ be a strongly permutative variety. Then

$$\mathcal{V} \subseteq \mathcal{PERM}_n = \mathcal{COM} \vee \mathcal{NILP}_n$$

for some n. Thus $\mathcal{V}$ is contained in the join of the variety $\mathcal{COM}$ and some nilpotent variety. Now, suppose that a variety $\mathcal{V}$ is contained in the join of $\mathcal{COM}$ and some nilpotent variety $\mathcal{N}$. Then $\mathcal{N} \subseteq \mathcal{NILP}_n$ for some n. Therefore $\mathcal{V} \subseteq \mathcal{COM} \vee \mathcal{NILP}_n = \mathcal{PERM}_n$, whence $\mathcal{V}$ is strongly permutative. We have proved that a variety is strongly permutative if and only if it is contained in the join of the variety $\mathcal{COM}$ and some nilpotent variety. Therefore the formula

$$\mathtt{StrPerm}(x) \ \rightleftharpoons\ (\exists y, z)\,\big(\mathtt{COM}(y)\,\&\,\mathtt{Nilp}(z)\,\&\,x \leq y \vee z\big)$$

defines the set of all strongly permutative varieties. □

At the conclusion, we note that there are many other semigroup varieties whose definability (or definability up to duality) may be confirmed by explicitly written formulas. We mention only one remarkable class of varieties of such a kind, namely the class of all varieties of idempotent semigroups. Indeed, the variety $\mathcal{I}$ is defined by the formula $\max_x\big\{\mathtt{Idemp}(x)\big\}$. Formulas defining the variety $\mathcal{SL}$ and defining up to duality the varieties $\mathcal{LZ}$, $\mathcal{RZ}$, $\mathcal{LRB}$ and $\mathcal{RRB}$ are given above. Let now $\mathcal{B}$ be a variety of idempotent semigroups with $\mathcal{B} \neq \mathcal{I}$. Then the lattice $L(\mathcal{B})$ is finite (see Fig. 2). Let n be the length of this lattice. Based on Fig. 2, it is easy to write (by induction on n) the formula that defines $\mathcal{B}$ whenever $\mathcal{B} = \overleftarrow{\mathcal{B}}$ and defines $\mathcal{B}$ up to duality whenever $\mathcal{B} \neq \overleftarrow{\mathcal{B}}$.

Acknowledgement. The author thanks Dr. Olga Sapir for many stimulating discussions and to the anonymous referee for a number of useful remarks.

References

[1] A. Ya. Aĭzenštat, *On some sublattices of the lattice of semigroup varieties*, Modern Algebra, Leningrad: Leningrad State Pedagogical Institute, No. 1 (1974), 3–15 [Russian].

[2] S. Burris and E. Nelson, *Embedding the dual of Π_m in the lattice of equational classes of commutative semigroups*, Proc. Amer. Math. Soc., **30** (1971), 37–39.

[3] S. Burris and E. Nelson, *Embedding the dual of Π_∞ in the lattice of equational classes of semigroups*, Algebra Universalis, **1** (1971), 248–254.

[4] T. Evans, *The lattice of semigroup varieties*, Semigroup Forum, **2** (1971), 1–43.

[5] G. Grätzer, General Lattice Theory, Birkhauser Verlag, Basel, 1998, 2nd ed.

[6] M. Grech, *Automorphisms of the lattice of equational theories of commutative semigroups*, Trans. Amer. Math. Soc., **361** (2009), 3435–3462.

[7] T. J. Head, *The lattice of varieties of commutative monoids*, Nieuw Arch. Wiskunde, **16** (1968), 203–206.

[8] J. Ježek and R. N. McKenzie, *Definability in the lattice of equational theories of semigroups*, Semigroup Forum, **46** (1993), 199–245.

[9] A. Kisielewicz, *Definability in the lattice of equational theories of commutative semigroups*, Trans. Amer. Math. Soc., **356** (2004), 3483–3504.

[10] F. J. Pastijn, *The lattice of completely regular semigroup varieties*, J. Austral. Math. Soc. Ser. A, **49** (1990), 24–42.

[11] F. J. Pastijn, *Commuting fully invariant congruences on free completely regular semigroups*, Trans. Amer. Math. Soc., **323** (1991), 79–92.

[12] M. Petrich and N. R. Reilly, *The modularity of the lattice of varieties of completely regular semigroups and related representations*, Glasgow Math. J., **32** (1990), 137–152.

[13] M. S. Putcha and A. Yaqub, *Semigroups satisfying permutation identities*, Semigroup Forum, **3** (1971), 68–73.

[14] M. V. Sapir and E. V. Sukhanov, *On varieties of periodic semigroups*, Izv. VUZ. Matem., No. 4 (1981), 48–55 [Russian; Engl. translation: Soviet Math. Iz. VUZ, **25**, No. 4 (1981), 53–63].

[15] V. Yu. Shaprynskiĭ, *Distributive and neutral elements of the lattice of commutative semigroup varieties*, Izv. Vuz. Matem., No. 7 (2011), 67–79 [Russian; Engl. translation: Russ. Math. Izv. VUZ, 55, No. 7 (2011), 56–67].

[16] L. N. Shevrin, B. M. Vernikov and M. V. Volkov, *Lattices of semigroup varieties*, Izv. VUZ. Matem., No. 3 (2009), 3–36 [Russian; Engl. translation: Russian Math. Iz. VUZ, **58**, No. 3 (2009), 1–28].

[17] E. V. Sukhanov, *Almost linear semigroup varieties*, Matem. Zametki, **32** (1982), 469–476 [Russian; Engl. translation: Math. Notes, **32** (1983), 714–717].

[18] E. V. Sukhanov, *Semigroup varieties of width* 2, Investig. of Algebraic Systems by Properties of their Subsystems, Sverdlovsk: Ural State University (1985), 148–157 [Russian].

[19] A. V. Tishchenko, *A remark on semigroup varieties of finite index*, Izv. VUZ. Matem., No. 7 (1990), 79–83 [Russian; Engl. translation: Soviet Math. Iz. VUZ, **34**, No. 7 (1990), 92–96].

[20] B. M. Vernikov, *Lower-modular elements of the lattice of semigroup varieties*, Semigroup Forum, **75** (2007), 554–566.

[21] B M. Vernikov and V. Yu. Shaprynskiĭ, *Distributive elements of the lattice of semigroup varieties*, Algebra and Logic, **49** (2010), 303–330 [Russian; Engl. translation: Algebra and Logic, **49** (2010), 201–220].

[22] B. M. Vernikov and M. V. Volkov, *Lattices of nilpotent semigroup varieties.* II, Proc. Ural State Univ., No. 10 (Ser. Matem., Mechan., No. 1) (1998), 13–33 [Russian].

[23] M. V. Volkov, *Semigroup varieties with modular subvariety lattices*, Izv. VUZ. Matem., No. 6 (1989), 51–60 [Russian; Engl. translation: Soviet Math. Iz. VUZ, **33**, No. 6 (1989), 48–58].

[24] M. V. Volkov, *Modular elements of the lattice of semigroup varieties*, Contrib. General Algebra, **16** (2005), 275–288.

Institute of Mathematics and Computer Science, Ural Federal University, Lenina 51, 620083 Ekaterinburg, Russia

E-mail address: bvernikov@gmail.com