Preprint: May-September xx, 2005.
Submitted to Contr. Gen. Alg. 17

MODULAR ELEMENTS OF THE LATTICE OF SEMIGROUP VARIETIES. II

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Abstract. We completely determine all semigroup varieties that are both modular and upper-modular elements of the lattice of all semigroup varieties as well as nilsemigroup varieties that are upper-modular elements of this lattice.

Introduction and summary

This note continues the article [10]. An element x of a lattice $\langle L; \vee, \wedge\rangle$ is called *modular* if

$$\forall\, y, z \in L: \quad y \le z \longrightarrow (x \vee y) \wedge z = (x \wedge z) \vee y,$$

and *upper-modular* if

$$\forall\, y, z \in L: \quad y \le x \longrightarrow (z \vee y) \wedge x = (z \wedge x) \vee y.$$

Lower-modular elements are defined dually to upper-modular ones.

Semigroup varieties that are both modular and lower-modular elements of the lattice of all semigroup varieties were completely described in [10]. Here we consider the dual restriction. Besides that, we classify nilsemigroup varieties that are upper-modular elements of the lattice of all semigroup varieties.

In order to formulate our main results, we need some notation. We adopt the usual agreement of writing $w = 0$ as a short form of the identity system $wu = uw = w$ where u runs over the set of all words. By $\operatorname{var}\Sigma$ we denote the variety of all semigroups satisfying the identity system Σ. Put

$$\mathfrak{SL} = \operatorname{var}\{x^2 = x,\ xy = yx\},$$
$$\mathfrak{C} = \operatorname{var}\{x^2 y = 0,\ xy = yx\}.$$

We will denote the lattice of all semigroup varieties by $\mathbb{SEM}$. The first main result of this paper is the following

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1991 *Mathematics Subject Classification.* Primary 20M07; secondary 08B15.

Key words and phrases. Semigroup variety, lattice of varieties, modular element, upper-modular element, nil-variety.

The authors acknowledge support from the Russian Foundation for Basic Research, grant 05-01-00540.

Theorem 1. *A semigroup variety $\mathfrak{V}$ is both a modular and an upper-modular element of the lattice $\mathbb{SEM}$ if and only if either $\mathfrak{V}$ coincides with the class of all semigroups or $\mathfrak{V} \subseteq \mathfrak{SL} \vee \mathfrak{C}$.*

Recall that a semigroup variety is called a *nil-variety* if it satisfies the identity $x^n = 0$ for some n. Our second main result is the following

Theorem 2. *A nil-variety $\mathfrak{V}$ is an upper-modular element of the lattice $\mathbb{SEM}$ if and only if it satisfies the identities $x^2y = xy^2$ and $xy = yx$.*

The note is structured as follows. Section 1 contains all necessary preliminaries. In Section 2 the "only if" parts of both the theorems are proved. In Sections 3 and 4 we verify the "if" parts of respectively Theorems 2 and 1.

1. Preliminaries

We start with some information about special elements of abstract lattices. Recall that an element x of a lattice L is called *neutral* if, for any two elements $y, z \in L$, the sublattice of L generated by x, y and z is distributive. An element a of a lattice L with 0 is called an *atom* of L if a is a minimal non-zero element.

Lemma 1.1. *Let L be a lattice with 0 and let a be a neutral element of L. Then:*

(i) *if x is a modular element of L then so is $x \vee a$;*
(ii) *if a is an atom of L and x is an upper-modular element of L then $x \vee a$ is an upper-modular element of L too.*

Proof. Part (i) is proved in [10, Lemma 1.6(ii)]. Let us verify (ii). We have to check that

$$(z \vee y) \wedge (x \vee a) = (z \wedge (x \vee a)) \vee y \tag{1}$$

for every $y \in L$ such that $y \leq x \vee a$ and for an arbitrary $z \in L$. Since $y \leq x \vee a$ and a is neutral, we have

$$y = y \wedge (x \vee a) = (y \wedge x) \vee (y \wedge a). \tag{2}$$

Now consider two cases: $y \ngeq a$ and $y \geq a$.

***Case* 1:** $y \not\geq a$. Since a is an atom, we then have $y \wedge a = 0$, and from (2) we conclude that $y = y \wedge x \leq x$. We have

$$\begin{aligned}
(z \vee y) \wedge (x \vee a) &= ((z \vee y) \wedge x) \vee ((z \vee y) \wedge a) && \text{because } a \text{ is neutral}\\
&= ((z \vee y) \wedge x) \vee ((z \wedge a) \vee (y \wedge a)) && \text{because } a \text{ is neutral}\\
&= ((z \vee y) \wedge x) \vee (z \wedge a) && \text{because } y \wedge a = 0\\
&= ((z \wedge x) \vee y) \vee (z \wedge a) && \text{because } y \leq x \text{ and}\\
& && x \text{ is upper modular}\\
&= ((z \wedge x) \vee (z \wedge a)) \vee y\\
&= (z \wedge (x \vee a)) \vee y && \text{because } a \text{ is neutral.}
\end{aligned}$$

Thus, the desired equality (1) holds.

***Case* 2:** $y \geq a$. From (2) we then have

$$y = (y \wedge x) \vee a. \tag{3}$$

Therefore,

$$\begin{aligned}
(z \vee y) \wedge (x \vee a) &= (z \vee ((y \wedge x) \vee a)) \wedge (x \vee a) && \text{by (3)}\\
&= ((z \vee (y \wedge x)) \vee a) \wedge (x \vee a)\\
&= ((z \vee (y \wedge x)) \wedge x) \vee a && \text{because } a \text{ is neutral}\\
&= ((z \wedge x) \vee (y \wedge x)) \vee a && \text{because } y \wedge x \leq x \text{ and}\\
& && x \text{ is upper-modular}\\
&= ((z \wedge x) \vee (y \wedge x)) \vee (a \vee (z \wedge a)) && \text{by the absorbtion law}\\
&= ((z \wedge x) \vee (z \wedge a)) \vee ((y \wedge x) \vee a)\\
&= ((z \wedge x) \vee (z \wedge a)) \vee y && \text{by (3)}\\
&= (z \wedge (x \vee a)) \vee y && \text{because } a \text{ is neutral.}
\end{aligned}$$

Thus, the equality (1) holds in this case as well. □

Lemma 1.2. *Let L be a lattice with 0, $x \in L$, and let a be an atom and a neutral element of L. Then:*

(i) *if $x \vee a$ is a modular element of L then so is x;*
(ii) *if $x \vee a$ is an upper-modular element of L then so is x.*

Proof. Since a is an atom of L, we have that, for any $z \in L$, $z \not\geq a$ if and only if $z \wedge a = 0$. Because a is a neutral element of L, we have that, for any $b, c \in L$, if $b \wedge a = 0$ and $c \wedge a = 0$ then $(b \vee c) \wedge a = (b \wedge a) \vee (c \wedge a) = 0$. In other words,

$$\forall\, b, c \in L: \ \ b \not\geq a \ \ \& \ \ c \not\geq a \longrightarrow b \vee c \not\geq a. \tag{4}$$

Further, it is known that if e is a neutral element of a lattice L and the equalities $f \wedge e = g \wedge e$ and $f \vee e = g \vee e$ hold true for some elements $f, g \in L$ then $f = g$ (see [3, Theorem III.2.4], for instance). Therefore,

$$\forall\, b, c \in L: \;\; b \ngeq a \;\;\&\;\; c \ngeq a \;\;\&\;\; b \vee a = c \vee a \longrightarrow b = c. \tag{5}$$

Now we are well prepared to prove the claims (i) and (ii).

(i) Let $y, z \in L$ with $y \le z$. We may assume that $x \ngeq a$ because $x \vee a = x$ in the contrary case. We have to check that

$$(x \vee y) \wedge z = (x \wedge z) \vee y. \tag{6}$$

Now consider two cases: $z \ngeq a$ and $z \ge a$.

***Case* 1:** $z \ngeq a$. We have

$$\begin{aligned}
(x \vee y) \wedge z &= ((x \vee y) \wedge z) \vee (a \wedge z) && \text{because } a \wedge z = 0\\
&= ((x \vee y) \vee a) \wedge z && \text{because } a \text{ is neutral}\\
&= ((x \vee a) \vee y) \wedge z && \\
&= ((x \vee a) \wedge z) \vee y && \text{because } x \vee a \text{ is modular}\\
&= ((x \wedge z) \vee (a \wedge z)) \vee y && \text{because } a \text{ is neutral}\\
&= (x \wedge z) \vee y && \text{because } a \wedge z = 0.
\end{aligned}$$

We see that (6) holds whenever $z \ngeq a$.

***Case* 2:** $z \ge a$. Then we have

$$\begin{aligned}
((x \vee y) \wedge z) \vee a &= ((x \vee y) \wedge z) \vee (a \wedge z) && \text{because } a \wedge z = a\\
&= ((x \vee y) \vee a) \wedge z && \text{because } a \text{ is neutral}\\
&= ((x \vee a) \vee y) \wedge z && \\
&= ((x \vee a) \wedge z) \vee y && \text{because } x \vee a \text{ is modular}\\
&= ((x \wedge z) \vee (a \wedge z)) \vee y && \text{because } a \text{ is neutral}\\
&= ((x \wedge z) \vee a) \vee y && \text{because } a \wedge z = a\\
&= ((x \wedge z) \vee y) \vee a. &&
\end{aligned}$$

We see that

$$((x \vee y) \wedge z) \vee a = ((x \wedge z) \vee y) \vee a. \tag{7}$$

Suppose at first that $y \ge a$. Since $z \ge a$, we have $(x \vee y) \wedge z \ge a$ and $(x \wedge z) \vee y \ge a$. Therefore, the equality (7) is equivalent to (6) in the case we consider.

Finally, let $y \ngeq a$. Recall that $x \ngeq a$. Applying (4) we have $x \vee y \ngeq a$, whence $(x \vee y) \wedge z \ngeq a$. Furthermore, $x \ngeq a$ implies $x \wedge z \ngeq a$. Applying (4) again we have $(x \wedge z) \vee y \ngeq a$. Now we may apply (5) and (7) concluding that (6) is valid. Thus, the equality (6) holds in any case.

(ii) Let $y, z \in L$ with $y \le x$. As in the proof of part (i), we may assume that $x \ngeq a$ because $x \vee a = x$ in the contrary case. Clearly, $y \le x$ implies $y \vee a \le x \vee a$. We have

$$\begin{aligned}
((z \vee y) \wedge x) \vee a &= ((z \vee y) \vee a) \wedge (x \vee a) && \text{because } a \text{ is neutral}\\
&= (z \vee (y \vee a)) \wedge (x \vee a) &&\\
&= (z \wedge (x \vee a)) \vee (y \vee a) && \text{because } x \vee a \text{ is upper-modular}\\
&= ((z \wedge x) \vee (z \wedge a)) \vee (y \vee a) && \text{because } a \text{ is neutral}\\
&= ((z \wedge x) \vee y) \vee ((z \wedge a) \vee a) &&\\
&= ((z \wedge x) \vee y) \vee a && \text{by the absorbtion law.}
\end{aligned}$$

We see that

$$((z \vee y) \wedge x) \vee a = ((z \wedge x) \vee y) \vee a. \tag{8}$$

Recall that $x \ngeq a$. This implies $z \wedge x \ngeq a$. Besides that, $y \ngeq a$ because $y \le x$. By (4) we conclude that $(z \wedge x) \vee y \ngeq a$. Furthermore, $x \ngeq a$ implies $(z \vee y) \wedge x \ngeq a$. Now we may apply (5) and (8) concluding that $(z \vee y) \wedge x = (z \wedge x) \vee y$, that is x is an upper-modular element. □

Combining Lemmas 1.1 and 1.2, we have

Proposition 1.3. *Let L be a lattice with 0, $x \in L$, and let a be an atom and a neutral element of L. Then:*

(i) *x is a modular element of L if and only if so is $x \vee a$;*
(ii) *x is an upper-modular element of L if and only if so is $x \vee a$.* □

Now we apply the above results to the lattice of semigroup varieties. The following lemma contains some properties of the variety $\mathfrak{SL}$ that are most important for this paper.

Lemma 1.4. *The variety $\mathfrak{SL}$ is:*

(i) *an atom of the lattice $\mathbb{SEM}$;*
(ii) *a neutral element of the lattice $\mathbb{SEM}$.* □

The claim (i) of this lemma is well known (see the survey [2], for instance). The statement (ii) is also known. It can be easily deduced from some remarks scattered over [1, 6, 7]; an explicit proof was given in [10, Proposition 2.4].

Lemma 1.4 and Proposition 1.3 immediately imply

Corollary 1.5. *Let $\mathfrak{M}$ be a semigroup variety.*

(i) *The variety $\mathfrak{M}$ is a modular element of the lattice $\mathbb{SEM}$ if and only if so is the variety $\mathfrak{M} \vee \mathfrak{SL}$.*
(ii) *The variety $\mathfrak{M}$ is an upper-modular element of the lattice $\mathbb{SEM}$ if and only if so is the variety $\mathfrak{M} \vee \mathfrak{SL}$.* □

2. Necessity

Modular elements of the lattice $\mathbb{SEM}$ have been studied by Ježek and McKenzie [5]. One should note that the paper [5] has dealt with the lattice of equational theories of semigroups, that is, the dual of $\mathbb{SEM}$ rather than the lattice $\mathbb{SEM}$ itself. However, the modular elements of the former lattice precisely correspond to the modular elements of $\mathbb{SEM}$. Indeed, the notion of a modular element is self-dual in the sense that a modular element of a lattice L is also modular in the dual of L (this readily follows from the definition or from [4, Proposition 2.1]). To reproduce a result from [5] concerning modular elements of the lattice $\mathbb{SEM}$, we need one definition. Following [10], we call a semigroup variety a *Rees* variety if it may be defined by a system of identities of the form $u = 0$. Clearly, every Rees variety is a nil-variety. We start the proof of Theorem 1 with the following result due to Ježek and McKenzie [5, Proposition 1.6] (we "translate" the original result from the language of equational theories to the language of varieties).

Proposition 2.1. *If a semigroup variety* $\mathfrak{V}$ *is a modular element of the lattice* $\mathbb{SEM}$ *then either* $\mathfrak{V}$ *coincides with the class of all semigroups or* $\mathfrak{V} \subseteq \mathfrak{SL} \vee \mathfrak{R}$ *for some Rees variety* $\mathfrak{R}$. □

This proposition easily implies

Corollary 2.2. *If a semigroup variety* $\mathfrak{V}$ *is a modular element of the lattice* $\mathbb{SEM}$ *then either* $\mathfrak{V}$ *coincides with the class of all semigroups or* $\mathfrak{V}$ *is a nil-variety or* $\mathfrak{V} = \mathfrak{SL} \vee \mathfrak{N}$ *for some nil-variety* $\mathfrak{N}$.

Proof. Suppose that $\mathfrak{V}$ differs from the class of all semigroups. By Proposition 2.1 $\mathfrak{V} \subseteq \mathfrak{SL} \vee \mathfrak{R}$ for some Rees variety $\mathfrak{R}$. Applying Lemma 1.4(ii), we get

$$\mathfrak{V} = \mathfrak{V} \wedge (\mathfrak{SL} \vee \mathfrak{R}) = (\mathfrak{V} \wedge \mathfrak{SL}) \vee (\mathfrak{V} \wedge \mathfrak{R}).$$

Put $\mathfrak{N} = \mathfrak{V} \wedge \mathfrak{R}$. Since the variety $\mathfrak{SL}$ is an atom of the lattice $\mathbb{SEM}$, the variety $\mathfrak{V} \wedge \mathfrak{SL}$ coincides with either $\mathfrak{SL}$ or the trivial variety. Therefore, either $\mathfrak{V} = \mathfrak{N}$ or $\mathfrak{V} = \mathfrak{SL} \vee \mathfrak{N}$. It remains to note that the variety $\mathfrak{N}$ is a nil-variety because of it is a subvariety of the nil-variety $\mathfrak{R}$. □

Let now $\mathfrak{V}$ be simultaneously a modular and an upper-modular element of the lattice $\mathbb{SEM}$. Of course, we may assume that $\mathfrak{V}$ differs from the class of all semigroups. By Corollaries 2.2 and 1.5, it suffices to verify that if $\mathfrak{V}$ is a nil-variety then $\mathfrak{V} \subseteq \mathfrak{C}$.

Throughout the rest of this section we assume that $\mathfrak{V}$ *is a nil-variety.*

We denote by F the free semigroup of a countable rank. The symbol $\equiv$ stands for the equality relation on F. If $u \in F$, then $c(u)$ denotes the set of all letters occurring in u, while $\ell(u)$ stands for the length of u. Let $u, v \in F$. We write $u \lhd v$ if $v \equiv a\xi(u)b$ for some endomorphism ξ of F and some $a, b \in F^1$ where F^1 is F with the empty word 1 adjoined. We need the following technical remarks about identities of nil-varieties.

Lemma 2.3. *Let $\mathfrak{N}$ be a nil-variety.*

(i) *If $\mathfrak{N}$ satisfies an identity $u = v$ with $c(u) \ne c(v)$, then $\mathfrak{N}$ satisfies also the identity $u = 0$.*
(ii) *If $\mathfrak{N}$ satisfies an identity of the form $u = vuw$ where $v, w \in F^1$ and at least one of the words v and w is non-empty, then it satisfies also the identity $u = 0$.*
(iii) *If $\mathfrak{N}$ satisfies an identity of the form $x_1x_2 \cdots x_n = u$ with $\ell(u) \ne n$, then it satisfies also the identity $x_1x_2 \cdots x_n = 0$.*
(iv) *If the variety $\mathfrak{N}$ is commutative and satisfies an identity $u = v$ where $\ell(u) < \ell(v)$ and $u \lhd v$, then $\mathfrak{N}$ satisfies also the identity $u = 0$.*

Proof. (i) We may assume that there is a letter $x \in c(v) \setminus c(u)$. Substituting 0 for x in the identity $u = v$, we obtain $u = 0$.

(ii) The identity $u = vuw$ implies $u = vuw = v^2uw^2 = \cdots = v^nuw^n = \cdots$. Since $\mathfrak{N}$ is a nil-variety and at least one of the words v and w is non-empty, there is n with either $v^n = 0$ or $w^n = 0$ in $\mathfrak{N}$. Therefore, $u = 0$ holds in $\mathfrak{N}$.

(iii) If $\ell(u) < n$, then $c(u) \ne \{x_1, x_2, \dots, x_n\}$ and the statement (i) applies. If $\ell(u) > n$, then the claim follows from [8, Lemma 1].

(iv) This claim is a partial case of [9, Lemma 1.3(iii)]. □

Recall that a word u is said to be an *isoterm in the variety* $\mathfrak{M}$ if no non-trivial identity of the form $u = v$ holds in $\mathfrak{M}$. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be arbitrary semigroup varieties and suppose that an identity $w_1 = w_2$ holds in the variety $\mathfrak{M}_1 \wedge \mathfrak{M}_2$. In this case there is a sequence of words $u_0, u_1, \dots, u_n$ such that $u_0 \equiv w_1$, $u_n \equiv w_2$ and, for every $i = 0, 1, \dots, n-1$, the identity $u_i = u_{i+1}$ holds in either $\mathfrak{M}_1$ or $\mathfrak{M}_2$. An arbitrary sequence of words with such properties will be called an $(\mathfrak{M}_1, \mathfrak{M}_2)$-*deduction* of the identity $w_1 = w_2$.

Proposition 2.4. *If $\mathfrak{V}$ is a nil-variety and $\mathfrak{V}$ is an upper-modular element of the lattice $\mathbb{SEM}$ then $\mathfrak{V}$ is commutative.*

Proof. Suppose that the commutative law fails in $\mathfrak{V}$ and denote by $\mathfrak{X}$ the subvariety of $\mathfrak{V}$ defined within $\mathfrak{V}$ by the identity $xy = yx$. Further, let $\mathfrak{G}$ be an arbitrary non-abelian periodic group variety. Clearly, $\mathfrak{G} \wedge \mathfrak{V}$ is the trivial variety, and therefore $(\mathfrak{G} \wedge \mathfrak{V}) \vee \mathfrak{X} = \mathfrak{X}$. Since $\mathfrak{V}$ is an upper-modular element of $\mathbb{SEM}$, this means that $(\mathfrak{G} \vee \mathfrak{X}) \wedge \mathfrak{V} = \mathfrak{X}$. The variety $\mathfrak{X}$ satisfies the commutative law. Therefore, there is a $(\mathfrak{G} \vee \mathfrak{X}, \mathfrak{V})$-deduction of the identity $xy = yx$. In particular, there is a word u with $u \not\equiv xy$ and the identity $xy = u$ holds in either $\mathfrak{G} \vee \mathfrak{X}$ or $\mathfrak{V}$. Suppose that $xy = u$ holds in $\mathfrak{V}$. If $u \not\equiv yx$ then either $c(u) \neq \{x,y\}$ or $\ell(u) \neq 2$. By the claims (i) and (iii) of Lemma 2.3 either $u \equiv yx$ or $xy = 0$ holds in $\mathfrak{V}$. Since the variety $\mathfrak{V}$ is non-commutative, both the cases are impossible. Therefore, the word xy is an isoterm in $\mathfrak{V}$. Whence the identity $xy = u$ is satisfied by the variety $\mathfrak{G} \vee \mathfrak{X}$. In particular, $xy = u$ holds in the nil-variety $\mathfrak{X}$. Using the same arguments as above, we have that either $u \equiv yx$ or $xy = 0$ holds in $\mathfrak{X}$. But the latter is not the case, and we have proved that the variety $\mathfrak{G} \vee \mathfrak{X}$ satisfies the commutative law. In particular, $xy = yx$ holds in the variety $\mathfrak{G}$, contradicting the choice of this variety. □

Put $W = \{x^2y, xyx, yx^2, y^2x, yxy, xy^2\}$.

Lemma 2.5. *If a commutative nil-variety $\mathfrak{M}$ satisfies an identity of the form $u = v$ where $u \in W$ then either $v \in W$ or $\mathfrak{M}$ satisfies the identity $u = 0$.*

Proof. If $c(v) \neq \{x,y\}$ then $u = 0$ in $\mathfrak{M}$ by Lemma 2.3(i). Let now $c(v) = \{x,y\}$. Let k (respectively, ℓ) be the number of occurences of the letter x (respectively, y) in v. Since the variety $\mathfrak{M}$ is commutative, it satisfies $v = x^ky^\ell$ and either $u = x^2y$ or $u = xy^2$. Suppose that $v \notin W$. Then either $k \geq 3$ or $\ell \geq 3$ or $k = \ell = 2$ or $k = \ell = 1$. Applying then Lemma 2.3(iv) we conclude that $\mathfrak{M}$ satisfies the identity $u = 0$. □

Proposition 2.6. *If $\mathfrak{V}$ is a nil-variety and $\mathfrak{V}$ is an upper-modular element of the lattice $\mathbb{SEM}$ then $\mathfrak{V}$ satisfies the identity $x^2y = xy^2$.*

Proof. Suppose that the identity $x^2y = xy^2$ is false in $\mathfrak{V}$ and denote by $\mathfrak{X}$ the subvariety of $\mathfrak{V}$ given within $\mathfrak{V}$ by this identity. Further, let $\mathfrak{G}$ be an arbitrary non-trivial periodic group variety. Clearly, $\mathfrak{G} \wedge \mathfrak{V}$ is the trivial variety, and therefore $(\mathfrak{G} \wedge \mathfrak{V}) \vee \mathfrak{X} = \mathfrak{X}$. Since $\mathfrak{V}$ is an upper-modular element of $\mathbb{SEM}$, this means that $(\mathfrak{G} \vee \mathfrak{X}) \wedge \mathfrak{V} = \mathfrak{X}$. The variety $\mathfrak{X}$ satisfies the identity $x^2y = xy^2$. Therefore, there is a $(\mathfrak{G} \vee \mathfrak{X}, \mathfrak{V})$-deduction of this identity. Let

$$x^2y \equiv u_0, u_1, \ldots, u_n \equiv xy^2$$

be an arbitrary such deduction. Put $W_1 = \{x^2y, xyx, yx^2\}$ and $W_2 = \{y^2x, yxy, xy^2\}$. Since $u_0 \in W_1$ and $u_n \notin W_1$, there is an index $i > 0$ such that $u_{i-1} \in W_1$ while $u_i \notin W_1$. The identity $u_{i-1} = u_i$ holds in one of the varieties $\mathfrak{G} \vee \mathfrak{X}$ and $\mathfrak{V}$. Suppose that $u_{i-1} = u_i$ in $\mathfrak{V}$. The variety $\mathfrak{V}$ is commutative by Proposition 2.4. Therefore, it satisfies all identities of the type $w_1 = x^2y$ with $w_1 \in W_1$ and $w_2 = xy^2$ with $w_2 \in W_2$. So, if $u_i \in W_2$ then $x^2y = yx^2$ in $\mathfrak{V}$. Furthermore, if $u_i \notin W_2$ then $u_i \notin W$. Now Lemma 2.5 applies and we conclude that $\mathfrak{V}$ satisfies the identity $x^2y = 0$. Therefore, $xy^2 = 0$ and $x^2y = xy^2$ in $\mathfrak{V}$. We prove that if $u_{i-1} = u_i$ holds in $\mathfrak{V}$ then $\mathfrak{V}$ satisfies the identity $x^2y = xy^2$. But this is not the case. Therefore, $u_{i-1} = u_i$ holds in $\mathfrak{G} \vee \mathfrak{X}$. In particular, $u_{i-1} = u_i$ in $\mathfrak{X}$. If $u_i \notin W_2$ then $u_i \notin W$, whence $x^2y = u_{i-1} = 0$ in $\mathfrak{X}$ by Lemma 2.5. But it is not the case. Therefore,

$u_i \in W_2$. This means that the variety $\mathfrak{G} \vee \mathfrak{X}$ satisfies the identity $u_{i-1} = u_i$ where $u_{i-1} \in W_1$ and $u_i \in W_2$. In particular, this identity holds true in the variety $\mathfrak{G}$. Recall that $\mathfrak{G}$ is a non-trivial group variety. Substituting 1 for y in the identity $u_{i-1} = u_i$, we obtain that $x^2 = x$ in $\mathfrak{G}$. Therefore, $\mathfrak{G}$ is the trivial variety, a contradiction. □

Propositions 2.4 and 2.6 imply necessity in Theorem 2.

We need the following easy observation.

Lemma 2.7. *Let $\mathfrak{M}$ be a nil-variety satisfying the identities $x^2y = xy^2$ and $xy = yx$. Then $\mathfrak{M}$ satisfies also the identity*

$$x^2yz = 0. \tag{9}$$

Proof. Substituting yz for y in $x^2y = xy^2$, we obtain that $\mathfrak{M}$ satisfies the identities $x^2yz = x(yz)^2 = xy^2z^2 = x^2yz^2$. Now Lemma 2.3(ii) applies. □

In [4], Ježek describes modular elements of the lattice of all varieties (more exactly, all equational theories) of any given type. In particular, [4, Lemma 6.3] implies that if a nil-variety $\mathfrak{V}$ satisfies the identity $x^2y = xy^2$ and $\mathfrak{V}$ is a modular element of the lattice of all groupoid varieties then $x^2y = 0$ holds in $\mathfrak{V}$. This does not imply directly the same conclusion for nil-varieties that are modular elements of $\mathbb{SEM}$ since a modular element of $\mathbb{SEM}$ need not be a modular element of the lattice of all groupoid varieties. Nevertheless, the following "semigroup analogue" of the mentioned result by Ježek is true.

Proposition 2.8. *If a nil-variety $\mathfrak{V}$ is a modular element of the lattice $\mathbb{SEM}$ and satisfies the identities $x^2y = xy^2$ and $xy = yx$ then $x^2y = 0$ holds in $\mathfrak{V}$.*

Proof. By Lemma 2.7 $\mathfrak{V}$ satisfies the identity (9). Put $\mathfrak{X} = \operatorname{var}\{x^2y = (x^2y)^2\}$ and $\mathfrak{Y} = \operatorname{var}\{x^2y = (x^2y)^2,\ xy^2 = (xy^2)^2\}$. Clearly, $\mathfrak{Y} \subseteq \mathfrak{X}$. The variety $\mathfrak{V} \wedge \mathfrak{X}$ satisfies the identities $xy^2 = x^2y = (x^2y)^2$. Together with (9) this implies $xy^2 = 0$. In particular, $xy^2 = (xy^2)^2$ in $\mathfrak{V} \wedge \mathfrak{X}$, that is $\mathfrak{V} \wedge \mathfrak{X} \subseteq \mathfrak{Y}$. Thus, $(\mathfrak{V} \wedge \mathfrak{X}) \vee \mathfrak{Y} = \mathfrak{Y}$. Since $\mathfrak{V}$ is a modular element of the lattice $\mathbb{SEM}$, $(\mathfrak{V} \vee \mathfrak{Y}) \wedge \mathfrak{X} = \mathfrak{Y}$. In particular, $xy^2 = (xy^2)^2$ holds in $(\mathfrak{V} \vee \mathfrak{Y}) \wedge \mathfrak{X}$. Then there exists a $(\mathfrak{V} \vee \mathfrak{Y}, \mathfrak{X})$-deduction of the identity $xy^2 = (xy^2)^2$. This means that there is a word u such that $u \not\equiv xy^2$ and $xy^2 = u$ holds in either $\mathfrak{V} \vee \mathfrak{Y}$ or $\mathfrak{X}$. Clearly, the word xy^2 is an isoterm in the variety $\mathfrak{X}$. Therefore, $xy^2 = u$ holds in $\mathfrak{V} \vee \mathfrak{Y}$ and, in particular, in $\mathfrak{V}$. Applying Lemma 2.5 we conclude that either $u \in W$ or $xy^2 = 0$ in $\mathfrak{V}$. In the latter case $x^2y = 0$ in $\mathfrak{V}$ because $\mathfrak{V}$ is commutative. Let now $u \in W$. The identity $xy^2 = u$ holds in $\mathfrak{V} \vee \mathfrak{Y}$, and moreover in $\mathfrak{Y}$. But it is clear that all non-trivial identities of the type $xy^2 = u$ with $u \in W$ are false in $\mathfrak{Y}$. □

Of course, our proof of Proposition 2.8 (namely, the choice of the varieties $\mathfrak{X}$ and $\mathfrak{Y}$ in this proof) is inspired by the proof of [4, Lemma 6.3].

Propositions 2.4, 2.6 and 2.8 imply together that $\mathfrak{V} \subseteq \mathfrak{C}$. The necessity in Theorem 1 is proved.

3. Sufficiency in Theorem 2

By Lemma 2.7, if a nil-variety satisfies the identities $x^2y = xy^2$ and $xy = yx$ then it satisfies the identity (9) too. Put

$$\mathfrak{A} = \mathrm{var}\{x^2yz = 0,\ x^2y = xy^2,\ xy = yx\}.$$

In this section we have to verify that any subvariety of $\mathfrak{A}$ is an upper-modular element of the lattice $\mathbb{SEM}$.

Put $U = \{x^2, x^3, x^2y, x_1x_2\cdots x_n \mid n \in \mathbb{N}\}$ where $\mathbb{N}$ stands for the set of all natural numbers. It is evident that any subvariety of $\mathfrak{A}$ may be given in $\mathfrak{A}$ only by identities of the type $u = v$ or $u = 0$ where $u, v \in U$. The claims (i)–(iii) of Lemma 2.3 imply that if $u, v \in U$ and $u \not\equiv v$ then $u = v$ implies in $\mathfrak{A}$ the identity $u = 0$. Now it is very easy to check that the subvariety lattice of the variety $\mathfrak{A}$ has the form shown on Fig. 1, where

$$\begin{aligned}
\mathfrak{A}_n &= \mathrm{var}\{x^2yz = x_1x_2\cdots x_n = 0,\ x^2y = xy^2,\ xy = yx\}\ \ (n \geq 4),\\
\mathfrak{B} &= \mathrm{var}\{x^2yz = x^3 = 0,\ x^2y = xy^2,\ xy = yx\},\\
\mathfrak{B}_n &= \mathrm{var}\{x^2yz = x^3 = x_1x_2\cdots x_n = 0,\ x^2y = xy^2,\ xy = yx\}\ \ (n \geq 4),\\
\mathfrak{C}_n &= \mathrm{var}\{x^2y = x_1x_2\cdots x_n = 0,\ xy = yx\}\ \ (n \geq 3),\\
\mathfrak{D} &= \mathrm{var}\{x^2 = 0,\ xy = yx\},\\
\mathfrak{D}_n &= \mathrm{var}\{x^2 = x_1x_2\cdots x_n = 0,\ xy = yx\}\ \ (n \in \mathbb{N}).
\end{aligned}$$

Let $\mathfrak{X} \subseteq \mathfrak{A}$. We have to check that if $\mathfrak{Y} \subseteq \mathfrak{X}$ and $\mathfrak{Z}$ is an arbitrary semigroup variety then $(\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X} = (\mathfrak{Z} \wedge \mathfrak{X}) \vee \mathfrak{Y}$.

We need some definitions and notation. Recall that a semigroup S is called *nilpotent* if it satisfies an identity of the form $x_1x_2\cdots x_k = 0$. If k is the least number with such a property then S is said to be *nilpotent of index* k. A semigroup variety $\mathfrak{M}$ is called a *variety of a finite index* if there is a natural k such that every nilsemigroup from $\mathfrak{M}$ is nilpotent of index $\leq k$; the least k with this property is called the *index* of $\mathfrak{M}$. If $\mathfrak{M}$ is a variety of a finite index, we denote its index by $\mathrm{ind}(\mathfrak{M})$; otherwise we write $\mathrm{ind}(\mathfrak{M}) = \infty$. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be arbitrary semigroup varieties. It is clear that

$$(10)\qquad \begin{cases} \mathrm{ind}(\mathfrak{M}_1 \vee \mathfrak{M}_2) = \max\{\mathrm{ind}(\mathfrak{M}_1), \mathrm{ind}(\mathfrak{M}_2)\},\\ \mathrm{ind}(\mathfrak{M}_1 \wedge \mathfrak{M}_2) = \min\{\mathrm{ind}(\mathfrak{M}_1), \mathrm{ind}(\mathfrak{M}_2)\} \end{cases}$$

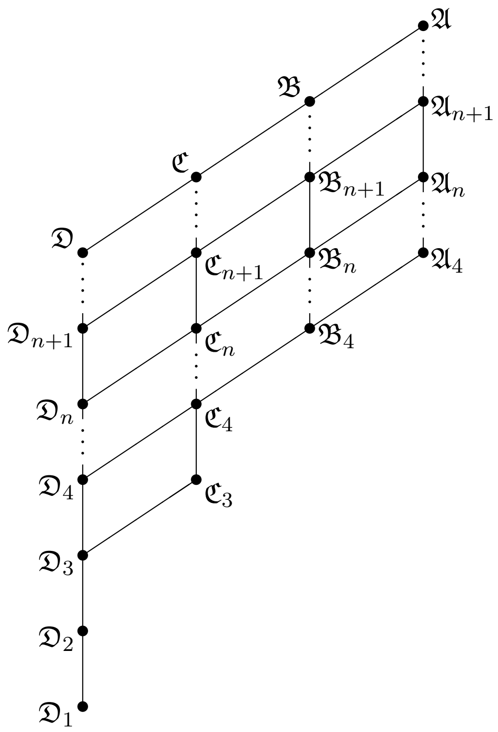

FIGURE 1. The subvariety lattice of the variety $\mathfrak{A}$

(we assume here that $k \le \infty$ for any $k \in \mathbb{N} \cup \{\infty\}$). For a variety $\mathfrak{M}$ with $\mathfrak{M} \subseteq \mathfrak{A}$, we define by $\overline{\mathfrak{M}}$ the least of the varieties $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$ that contains $\mathfrak{M}$. Fig. 1 shows that if $\mathfrak{M}_1, \mathfrak{M}_2 \subseteq \mathfrak{A}$ then

$$\mathfrak{M}_1 = \mathfrak{M}_2 \Longleftrightarrow \mathrm{ind}(\mathfrak{M}_1) = \mathrm{ind}(\mathfrak{M}_2) \text{ and } \overline{\mathfrak{M}_1} = \overline{\mathfrak{M}_2}.$$

Therefore, we have to verify the following two equalities:

$$\mathrm{ind}((\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X}) = \mathrm{ind}((\mathfrak{Z} \wedge \mathfrak{X}) \vee \mathfrak{Y}), \tag{11}$$

$$\overline{(\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X}} = \overline{(\mathfrak{Z} \wedge \mathfrak{X}) \vee \mathfrak{Y}}. \tag{12}$$

Put $\mathrm{ind}(\mathfrak{X}) = k$, $\mathrm{ind}(\mathfrak{Y}) = \ell$ and $\mathrm{ind}(\mathfrak{Z}) = m$. According to (10), we have

$$\mathrm{ind}((\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X}) = \min\{\max\{m, \ell\}, k\},$$
$$\mathrm{ind}((\mathfrak{Z} \wedge \mathfrak{X}) \vee \mathfrak{Y}) = \max\{\min\{m, k\}, \ell\}.$$

Clearly, $\ell \le k$ because $\mathfrak{Y} \subseteq \mathfrak{X}$. It is then evident that $\min\{\max\{m, \ell\}, k\} = \max\{\min\{m, k\}, \ell\}$. The equality (11) is proved.

It remains to verify the equality (12). Clearly, it is equivalent to the following claim: if u is one of the words x^3, x^2y and x^2 then the variety $(\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X}$

satisfies the identity $u = 0$ if and only if the variety $(\mathfrak{Z} \wedge \mathfrak{X}) \vee \mathfrak{Y}$ does so. Obviously, $(\mathfrak{Z} \wedge \mathfrak{X}) \vee \mathfrak{Y} \subseteq (\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X}$. Thus, we have to check that $u = 0$ holds in $(\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X}$ whenever it is so in $(\mathfrak{Z} \wedge \mathfrak{X}) \vee \mathfrak{Y}$. Further considerations are naturally divided into two cases.

***Case* 1:** u is one of the words x^2 and x^3. We prove that if the variety $(\mathfrak{Z} \wedge \mathfrak{X}) \vee \mathfrak{Y}$ satisfies an identity of the form $x^n = 0$ for some n then $x^n = 0$ holds in the variety $(\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X}$ as well. (This is evident whenever $n > 3$ because $x^4 = 0$ in $\mathfrak{A}$, and moreover in $\mathfrak{X}$. But the proof we give below does not depend on n.) Suppose that $x^n = 0$ in $(\mathfrak{Z} \wedge \mathfrak{X}) \vee \mathfrak{Y}$. This means that $x^n = 0$ in $\mathfrak{Y}$ and there is a $(\mathfrak{Z}, \mathfrak{X})$-deduction of the identity $x^n = 0$. In particular, there is a word v such that $v \not\equiv x^n$ and $x^n = v$ holds in either $\mathfrak{Z}$ or $\mathfrak{X}$. Suppose that $x^n = v$ in $\mathfrak{X}$. Since $\mathfrak{X}$ is a nil-variety, the claims (i) and (ii) of Lemma 2.3 imply that $x^n = 0$ in $\mathfrak{X}$, and moreover in $(\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X}$. Let now $x^n = v$ in $\mathfrak{Z}$. If $c(v) = \{x\}$ then $x^n = v$ is an identity of the form

$$x^n = x^m \tag{13}$$

where $n \neq m$. Suppose that $c(v) \neq \{x\}$. If $\ell(v) \neq n$ then substituting x for each letter from $c(v) \setminus \{x\}$ in the identity $x^n = v$, we deduce from this identity an identity of the form (13) with $n \neq m$. Finally, if $\ell(v) = n$ then we obtain an identity of the same form by substitution x^2 for any letter from $c(v) \setminus \{x\}$ in $x^n = v$. Thus, in any case the variety $\mathfrak{Z}$ satisfies an identity of the form (13) with $n \neq m$. If $m < n$ then multiplying both the sides of this identity by x^{n-m} we obtain the identity $x^{2n-m} = x^n$. Clearly, $2n - m > n$. Thus, we may assume that $\mathfrak{Z}$ satisfies an identity of the form (13) for some $m > n$. Since $x^n = 0$ in $\mathfrak{Y}$, the variety $\mathfrak{Z} \vee \mathfrak{Y}$ also satisfies (13) for some $m > n$. Applying Lemma 2.3(ii) we have that any nil-subvariety of $\mathfrak{Z} \vee \mathfrak{Y}$ satisfies $x^n = 0$. In particular, it is so for the variety $(\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X}$.

***Case* 2:** $u \equiv x^2y$. Now we have to check that if the variety $(\mathfrak{Z} \wedge \mathfrak{X}) \vee \mathfrak{Y}$ satisfies the identity $x^2y = 0$ then the variety $(\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X}$ does so. We may assume that $\mathfrak{Z} \not\supseteq \mathfrak{X}$ because $(\mathfrak{Z} \wedge \mathfrak{X}) \vee \mathfrak{Y} = \mathfrak{X} = (\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X}$ otherwise. In particular, $\mathfrak{Z} \not\supseteq \mathrm{var}\{xy = yx\}$. As well known, this means that $\mathfrak{Z}$ consists of periodic semigroups or, equivalently, satisfies an identity of the form (13) with $n < m$ (see, e.g., [2]). Since $\mathfrak{Y}$ is a nil-variety, an identity of this form holds in the variety $\mathfrak{Z} \vee \mathfrak{Y}$. By Lemma 2.3(ii) this implies that any nil-subvariety of $\mathfrak{Z} \vee \mathfrak{Y}$ satisfies the identity $x^n = 0$. Hence there exists the (unique) greatest nil-subvariety of the variety $\mathfrak{Z} \vee \mathfrak{Y}$. We denote this subvariety by $\mathrm{Nil}(\mathfrak{Z} \vee \mathfrak{Y})$. It is evident that $\mathrm{Nil}(\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X} \subseteq (\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X}$. On the other hand, any semigroup from $(\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X}$ is a nilsemigroup (because $\mathfrak{X}$ is a nil-variety) and this nilsemigroup lies both in $\mathfrak{Z} \vee \mathfrak{Y}$ and $\mathfrak{X}$. Therefore, $(\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X} \subseteq \mathrm{Nil}(\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X}$. We see that

$$(\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X} = \mathrm{Nil}(\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X}. \tag{14}$$

As in Section 2 put

$$W = \{x^2y, xyx, yx^2, y^2x, yxy, xy^2\}.$$

The variety $(\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X}$ is commutative. Therefore, it suffices to verify that this variety satisfies an identity $w = 0$ for some word $w \in W$. By the hypothesis the variety $(\mathfrak{Z} \wedge \mathfrak{X}) \vee \mathfrak{Y}$ satisfies the identity $x^2y = 0$. This means that $x^2y = 0$ in $\mathfrak{Y}$ and there is a $(\mathfrak{Z}, \mathfrak{X})$-deduction of the identity $x^2y = 0$. Let $x^2y \equiv u_0, u_1, \dots, u_n, 0$ be an arbitrary such deduction. Suppose that $u_n \in W$. The identity $u_n = 0$ holds in one of the varieties $\mathfrak{Z}$ and $\mathfrak{X}$. Suppose that $u_n = 0$ in $\mathfrak{Z}$. Because $u_n \in W$ and the variety $\mathfrak{Y}$ is commutative and satisfies the identity $x^2y = 0$, we have $u_n = 0$ in $\mathfrak{Y}$. Therefore, $u_n = 0$ in $\mathfrak{Z} \vee \mathfrak{Y}$, and moreover in $(\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X}$. Obviously, the same is the case whenever $u_n = 0$ in $\mathfrak{X}$. Since $u_n \in W$, we are done.

Let now $u_n \notin W$. Since $u_0 \in W$, then there is an index $i > 0$ such that $u_i \notin W$ while $u_{i-1} \in W$. The identity $u_{i-1} = u_i$ holds in one of the varieties $\mathfrak{Z}$ and $\mathfrak{X}$. If $u_{i-1} = u_i$ in $\mathfrak{X}$ then Lemma 2.5 applies and we conclude that $u_{i-1} = 0$ in $\mathfrak{X}$, and moreover in $(\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X}$. Since $u_{i-1} \in W$, we are done.

Finally, suppose that $u_{i-1} = u_i$ in $\mathfrak{Z}$. Note that $u_{i-1} = 0$ in $\mathfrak{Y}$ because $x^2y = 0$ in $\mathfrak{Y}$, the variety $\mathfrak{Y}$ is commutative and $u_{i-1} \in W$. We are going to check that $u_{i-1} = 0$ in $(\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X}$. This suffices for our aims because $u_{i-1} \in W$. Suppose at first that there is a letter $z \in c(u_i) \setminus \{x, y\}$. Substitute u_{i-1}^2 for z in the identity $u_{i-1} = u_i$. We obtain an identity of the type $u_{i-1} = w_1 u_{i-1}^2 w_2$ for some (maybe empty) words w_1 and w_2. This identity holds in $\mathfrak{Z} \vee \mathfrak{Y}$ because $u_{i-1} = u_i$ in $\mathfrak{Z}$ and $u_{i-1} = 0$ in $\mathfrak{Y}$. By Lemma 2.3(ii) $u_{i-1} = 0$ holds in $\mathrm{Nil}(\mathfrak{Z} \vee \mathfrak{Y})$, and moreover in $\mathrm{Nil}(\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X}$. We are done by (14).

Let now $c(u_i) \subseteq \{x, y\}$. Recall that a word u is called *linear* if every letter occurs in u at most one time. We write $u \approx v$ if the word v may be obtained from the word u by renaming of letters. Suppose that the identity $u_i = 0$ is false in the variety $\mathfrak{Y}$. Because $x^2y = 0$ in $\mathfrak{Y}$, this means that either u_i is linear or $u_i \approx x^2$. Since $c(u_i) \subseteq \{x, y\}$, we have $u_i \in \{x, y, xy, yx, x^2, y^2\}$. Substitute x for y in the identity $u_{i-1} = u_i$. We obtain either $x^3 = x$ or $x^3 = x^2$. Each of these two identities implies $x^4 = x^2$. Thus, $x^4 = x^2$ in $\mathfrak{Z}$, and therefore $x^4y = x^2y$ in $\mathfrak{Z} \vee \mathfrak{Y}$. By Lemma 2.3(ii), we have $x^2y = 0$ in $\mathrm{Nil}(\mathfrak{Z} \vee \mathfrak{Y})$, and moreover in $\mathrm{Nil}(\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X}$. According to (14) $x^2y = 0$ in $(\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X}$.

Finally, let $u_i = 0$ in $\mathfrak{Y}$. Then we have $u_{i-1} = u_i$ in $\mathfrak{Y}$, and therefore in $\mathfrak{Z} \vee \mathfrak{Y}$. Recall that $c(u_i) \subseteq \{x, y\}$. If $c(u_i) \subset \{x, y\}$ then Lemma 2.3(i) applies and we conclude that $u_{i-1} = 0$ in $\mathrm{Nil}(\mathfrak{Z} \vee \mathfrak{Y})$, and moreover in $\mathrm{Nil}(\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X}$. Now the equality (14) applies. It remains to consider the case $c(u_i) = \{x, y\}$. Let k (respectively, ℓ) denote the number of occurrences of the letter x (respectively, y) in the word u_i. Since the variety $\mathfrak{X}$ satisfies the commutative law, $u_i = x^k y^\ell$ in $\mathfrak{X}$. If $k = \ell = 1$ then $u_i \in \{xy, yx\}$ and we may repeat literally the arguments from the previous paragraph. Furthermore, if either $k = 2$,

$\ell = 1$ or $k = 1$, $\ell = 2$ then $u_i \in W$ that contradicts the choice of the word u_i. Therefore, we may assume that either $k \geq 3$ or $\ell \geq 3$ or $k = \ell = 2$. The variety $\mathfrak{X}$ satisfies the identities $x^3y = 0$ and $x^2y^2 = 0$ because $\mathfrak{X} \subseteq \mathfrak{A}$. Hence $u_i = 0$ in $\mathfrak{X}$. Taking into account that $u_{i-1} = u_i$ in $\mathfrak{Z} \vee \mathfrak{Y}$, we obtain that the consequence $u_{i-1}, u_i, 0$ is a $(\mathfrak{Z} \vee \mathfrak{Y}, \mathfrak{X})$-deduction of the identity $u_{i-1} = 0$, whence this identity holds in $(\mathfrak{Z} \vee \mathfrak{Y}) \wedge \mathfrak{X}$.

The equality (12) is proved. Thus, we have completed the proof of sufficiency in Theorem 2. □

4. Sufficiency in Theorem 1

In this section we complete the proof of Theorem 1. It is evident that the variety of all semigroups is both a modular and an upper-modular element of the lattice $\mathbb{SEM}$. Let now $\mathfrak{X}$ be a semigroup variety with $\mathfrak{X} \subseteq \mathfrak{SL} \vee \mathfrak{C}$. Repeating literally the proof of Corollary 2.2, we conclude that either $\mathfrak{X} = \mathfrak{X}'$ or $\mathfrak{X} = \mathfrak{SL} \vee \mathfrak{X}'$ for some variety $\mathfrak{X}' \subseteq \mathfrak{C}$. According to Corollary 1.5 it suffices to verify that each subvariety of the variety $\mathfrak{C}$ is both a modular and an upper-modular element of the lattice $\mathbb{SEM}$. The "upper-modular half" of this claim immediately follows from Theorem 2. Thus, it remains to verify that any subvariety of the variety $\mathfrak{C}$ is a modular element of the lattice $\mathbb{SEM}$.

Let $\mathfrak{X} \subseteq \mathfrak{C}$. In other words, $\mathfrak{X}$ is one of the varieties $\mathfrak{C}$, $\mathfrak{C}_n$, $\mathfrak{D}$ and $\mathfrak{D}_n$ (see Fig. 1). We have to check that if $\mathfrak{Y}$ and $\mathfrak{Z}$ are arbitrary semigroup varieties with $\mathfrak{Y} \subseteq \mathfrak{Z}$ then $(\mathfrak{X} \vee \mathfrak{Y}) \wedge \mathfrak{Z} = (\mathfrak{X} \wedge \mathfrak{Z}) \vee \mathfrak{Y}$. It suffices to verify that $(\mathfrak{X} \vee \mathfrak{Y}) \wedge \mathfrak{Z} \subseteq (\mathfrak{X} \wedge \mathfrak{Z}) \vee \mathfrak{Y}$ since the opposite inclusion is evident. In other words, we have to prove that an identity $u = v$ holds in $(\mathfrak{X} \vee \mathfrak{Y}) \wedge \mathfrak{Z}$ whenever it does so in $(\mathfrak{X} \wedge \mathfrak{Z}) \vee \mathfrak{Y}$. Let $u = v$ be an identity that holds in $(\mathfrak{X} \wedge \mathfrak{Z}) \vee \mathfrak{Y}$. Then $u = v$ in $\mathfrak{Y}$ and there is an $(\mathfrak{X}, \mathfrak{Z})$-deduction of $u = v$. Let

$$u \equiv w_0, w_1, \ldots, w_n \equiv v \tag{15}$$

be the shortest $(\mathfrak{X}, \mathfrak{Z})$-deduction of the identity $u = v$. In particular, this means that there are no $i \in \{0, 1, \ldots, n-2\}$ with $w_i = w_{i+1} = w_{i+2}$ in one of the varieties $\mathfrak{X}$ and $\mathfrak{Z}$. Besides that, if $n > 1$ then there are no $i \in \{0, 1, \ldots, n-1\}$ such that $w_i = w_{i+1}$ in both the varieties $\mathfrak{X}$ and $\mathfrak{Z}$.

The case $n = 1$ is fairly simple. Indeed, if $n = 1$ then the identity $u = v$ holds in one of the varieties $\mathfrak{X}$ and $\mathfrak{Z}$. Since this identity holds in $\mathfrak{Y}$, we have that $u = v$ in one of the varieties $\mathfrak{X} \vee \mathfrak{Y}$ and $\mathfrak{Z}$, and moreover in $(\mathfrak{X} \vee \mathfrak{Y}) \wedge \mathfrak{Z}$.

Suppose now that $n = 2$. By symmetry we may assume that $u = w_1$ in $\mathfrak{X}$ and $w_1 = v$ in $\mathfrak{Z}$. Then $w_1 = v = u$ in $\mathfrak{Y}$. We see that $u = w_1$ in $\mathfrak{X} \vee \mathfrak{Y}$ and $w_1 = v$ in $\mathfrak{Z}$, so $u = v$ in $(\mathfrak{X} \vee \mathfrak{Y}) \wedge \mathfrak{Z}$.

Throughout the rest of this section we assume that $n \geq 3$.

First of all, let us consider some very special but important partial case. As we shall seen below, all other cases reduce to this one.

Lemma 4.1. *If* $n = 3$, $w_0 = w_1$ *in* $\mathfrak{Z}$, $w_1 = w_2$ *in* $\mathfrak{X}$ *and* $w_2 = w_3$ *in* $\mathfrak{Z}$ *then the identity* $u = v$ *holds in the variety* $(\mathfrak{X} \vee \mathfrak{Y}) \wedge \mathfrak{Z}$.

Proof. Recall that the identity $u = v$ holds in $\mathfrak{Y}$, $u \equiv w_0$ and $v \equiv u_n$. Since $\mathfrak{Y} \subseteq \mathfrak{Z}$, we have $w_1 = w_0 = w_3 = w_2$ in $\mathfrak{Y}$. Therefore, the identity $w_1 = w_2$ holds in the variety $\mathfrak{X} \vee \mathfrak{Y}$. Since the identities $w_0 = w_1$ and $w_2 = w_3$ hold true in $\mathfrak{Z}$, we are done. □

Clearly, if u is a non-linear word and $u \not\approx x^2$ then the identity $u = 0$ holds in $\mathfrak{X}$. Put

$$Z = \{u \in F \mid u = 0 \text{ holds in } \mathfrak{X}\} =$$
$$= \begin{cases} \{u \in F \mid u \text{ is non-linear and } u \not\approx x^2\} & \text{if } \mathfrak{X} = \mathfrak{C}, \\ \{u \in F \mid \text{either } u \text{ is non-linear and } u \not\approx x^2 & \\ \qquad\qquad \text{or } u \text{ is linear and } \ell(u) \geq n\} & \text{if } \mathfrak{X} = \mathfrak{C}_n, \\ \{u \in F \mid u \text{ is non-linear}\} & \text{if } \mathfrak{X} = \mathfrak{D}, \\ \{u \in F \mid \text{either } u \text{ is non-linear} & \\ \qquad\qquad \text{or } u \text{ is linear and } \ell(u) \geq n\} & \text{if } \mathfrak{X} = \mathfrak{D}_n; \end{cases}$$
$$L = \{u \in F \mid u \text{ is linear and } u \notin Z\} =$$
$$= \begin{cases} \{u \in F \mid u \text{ is linear}\} & \text{if } \mathfrak{X} = \mathfrak{C} \text{ or } \mathfrak{X} = \mathfrak{D}, \\ \{u \in F \mid u \text{ is linear and } \ell(u) < n\} & \text{if } \mathfrak{X} = \mathfrak{C}_n \text{ or } \mathfrak{X} = \mathfrak{D}_n; \end{cases}$$
$$S = \{u \in F \mid u \notin Z \cup L\} = \begin{cases} \{u \in F \mid u \approx x^2\} & \text{if } \mathfrak{X} = \mathfrak{C} \text{ or } \mathfrak{X} = \mathfrak{C}_n, \\ \varnothing & \text{if } \mathfrak{X} = \mathfrak{D} \text{ or } \mathfrak{X} = \mathfrak{D}_n. \end{cases}$$

Clearly, every word belongs to exactly one of the sets Z, L and S. In particular, it is so for each of the words $w_0, w_1, \dots, w_n$. It is evident that if $w', w'' \in Z$ then $w' = w''$ in $\mathfrak{X}$. In the sequel we use this claim without any references.

Now we verify several properties of the sequence (15).

Lemma 4.2. *If* $w_i, w_j \in Z$ *for some* $0 \leq i < j \leq n$ *then* $j = i + 1$. *In particular, the sequence* (15) *contains at most two words from the set* Z.

Proof. Suppose that $w_i, w_j \in Z$ for some $0 \leq i, j \leq n$ and $j > i + 1$. Then $w_i = w_j$ in $\mathfrak{X}$, whence $w_0, w_1, \dots, w_i, w_j, \dots, w_n$ is an $(\mathfrak{X}, \mathfrak{Z})$-deduction of the identity $u = v$ that is shorter than (15). □

Lemma 4.3. *If* $w_i \in S$ *for some* $0 \leq i \leq n$ *then:*

(i) *either* $i = 0$ *or* $i = n$;
(ii) *if* $w_0 \in S$ (*respectively,* $w_n \in S$) *then the identity* $w_0 = w_1$ (*respectively,* $w_{n-1} = w_n$) *holds in the variety* $\mathfrak{Z}$.

Proof. Let $w_i \in S$, that is $w_i \approx x^2$. Suppose that $i > 0$ and the identity $w_{i-1} = w_i$ holds in $\mathfrak{X}$. The claims (i) and (ii) of Lemma 2.3 then imply that $x^2 = 0$ in $\mathfrak{X}$. Thus, $w_i = 0$ in $\mathfrak{X}$ that contradicts the claim $w_i \in S$. Therefore, if $i > 0$ then the identity $w_{i-1} = w_i$ holds in $\mathfrak{Z}$. In particular, if $w_n \in S$ then $w_{n-1} = w_n$ in $\mathfrak{Z}$. Analogous arguments show that if $i < n$ (in particular, if $i = 0$) then the identity $w_i = w_{i+1}$ holds in $\mathfrak{Z}$. The part (ii) is proved. Furthermore, if $0 < i < n$ then $w_{i-1} = w_i = w_{i+1}$ in $\mathfrak{Z}$. But this is impossible. The part (i) is proved as well. □

Lemma 4.4. *If $w_i \in L$ for some $0 < i < n$ then either $w_{i-1} \in L$ or $w_{i+1} \in L$.*

Proof. Arguing by contradiction, suppose that $w_{i-1}, w_{i+1} \in Z \cup S$. If $w_{i-1} \in S$ then $w_{i-1} = w_i$ holds in $\mathfrak{Z}$ by Lemma 4.3. Let now $w_{i-1} \in Z$. Then $w_{i-1} = 0$ holds in $\mathfrak{X}$ while the identity $w_i = 0$ is false in $\mathfrak{X}$. Therefore, the identity $w_{i-1} = w_i$ is false in $\mathfrak{X}$ too. This means that $w_{i-1} = w_i$ holds in $\mathfrak{Z}$ as well. Analogously, the identity $w_i = w_{i+1}$ holds in $\mathfrak{Z}$, that is $w_{i-1} = w_i = w_{i+1}$ in $\mathfrak{Z}$. But this is impossible. □

Lemma 4.5. *Let $w_i, w_{i+1} \in L$ for some $0 \le i \le n-1$. The identity $w_i = w_{i+1}$ holds in the variety $\mathfrak{X}$ if and only if $c(w_i) = c(w_{i+1})$.*

Proof. Let $w_i, w_{i+1} \in L$. If $c(w_i) = c(w_{i+1})$ then $w_i = w_{i+1}$ in $\mathfrak{X}$ because the variety $\mathfrak{X}$ is commutative. Let now $c(w_i) \ne c(w_{i+1})$. If $w_i = w_{i+1}$ in $\mathfrak{X}$ then $w_i = 0$ in $\mathfrak{X}$ by Lemma 2.3(i). But this contradicts $w_i \in L$. □

Lemma 4.6. *If $w_i, w_{i+1} \in L$ for some $0 \le i \le n-1$ then the sequence* (15) *contains no word from the set Z.*

Proof. Suppose that $i + 1 < n$ and $w_{i+2} \in Z$. Then $w_{i+2} = 0$ holds in $\mathfrak{X}$ while $w_{i+1} = 0$ is false in $\mathfrak{X}$. Therefore, the identity $w_{i+1} = w_{i+2}$ is false in $\mathfrak{X}$. This means that $w_{i+1} = w_{i+2}$ in $\mathfrak{Z}$, and therefore $w_i = w_{i+1}$ in $\mathfrak{X}$. The latter together with Lemma 4.5 imply that $c(w_i) = c(w_{i+1})$. Therefore, the word w_i may be obtained from w_{i+1} by an action of some permutation σ on indices of letters occurring in $c(w_{i+1})$. Substitute $x_{\sigma(j)}$ for x_j in the identity $w_{i+1} = w_{i+2}$ for all letters $x_j \in c(w_{i+1})$. We obtain an identity of the type $w_i = w'$ such that $w' \approx w_{i+2}$ and the identity $w_i = w'$ holds in $\mathfrak{Z}$. Since $w_{i+2} = 0$ in $\mathfrak{X}$, we have $w' = 0$ in $\mathfrak{X}$ too. Thus, $w' = w_{i+2}$ in $\mathfrak{X}$. Since $n \ge 3$, either $i > 0$ or $i + 2 < n$. If $i > 0$ then $w_{i-1} = w_i = w'$ in $\mathfrak{Z}$ and $w_0, w_1, \ldots, w_{i-1}, w', w_{i+2}, \ldots, w_n$ is an $(\mathfrak{X}, \mathfrak{Z})$-deduction of the identity $u = v$ that shorter than (15). Furthermore, if $i + 2 < n$ then $w' = w_{i+2} = w_{i+3}$ in $\mathfrak{X}$ and $w_0, w_1, \ldots, w_i, w', w_{i+3}, \ldots, w_n$ is an $(\mathfrak{X}, \mathfrak{Z})$-deduction of the identity $u = v$ that shorter than (15). We see that if $i + 1 < n$ then $w_{i+2} \notin Z$. By symmetry, if $i > 0$ then $w_{i-1} \notin Z$.

Let now $w_j, \ldots, w_i, \ldots, w_k$ (where $0 \le j \le i < k \le n$) be the maximal subsequence of the sequence (15) consisting of words from the set L. In other words, $w_j, \ldots, w_i, \ldots, w_k \in L$, $w_{k+1} \notin L$ whenever $k < n$, and $w_{j-1} \notin L$ whenever $j > 0$. Suppose that $k < n$. As we have proved above $w_{k+1} \notin Z$, whence $w_{k+1} \in S$. Applying Lemma 4.3(i) we have $k + 1 = n$. Analogously, if $j > 0$ then $j = 1$ and $w_0 \in S$. We see that the words from the set Z are absent in the sequence (15). □

Lemma 4.7. *If $w_i, w_{i+1}, w_{i+2} \in L$ for some $0 \le i \le n-2$ then either $c(w_i) = c(w_{i+1}) \ne c(w_{i+2})$ or $c(w_i) \ne c(w_{i+1}) = c(w_{i+2})$.*

Proof. Applying Lemma 4.5, we have $w_i = w_{i+1} = w_{i+2}$ in $\mathfrak{X}$ whenever $c(w_i) = c(w_{i+1}) = c(w_{i+2})$, and $w_i = w_{i+1} = w_{i+2}$ in $\mathfrak{Z}$ whenever $c(w_i) \ne c(w_{i+1}) \ne c(w_{i+2})$. Both the cases are impossible. □

Lemma 4.8. *If $w_i, w_{i+1}, w_{i+2}, w_{i+3} \in L$ and $c(w_i) = c(w_{i+1}) \ne c(w_{i+2}) = c(w_{i+3})$ for some $0 \le i \le n-3$ then there are words w' and w'' such that $w_i = w'$ in $\mathfrak{Z}$, $w' = w''$ in $\mathfrak{X}$ and $w'' = w_{i+3}$ in $\mathfrak{Z}$.*

Proof. By Lemma 4.5 the identities $w_i = w_{i+1}$ and $w_{i+2} = w_{i+3}$ hold true in $\mathfrak{X}$ while the identity $w_{i+1} = w_{i+2}$ holds in $\mathfrak{Z}$. Since $c(w_i) = c(w_{i+1})$ and $w_i, w_{i+1} \in L$, the word w_i may be obtained from w_{i+1} by an action of some permutation σ on the indices of the letters occurring in $c(w_{i+1})$. Substitute $x_{\sigma(j)}$ for x_j in the identity $w_{i+1} = w_{i+2}$ for all letters $x_j \in c(w_{i+1})$. We obtain an identity of the type $w_i = w'_{i+2}$ such that $w'_{i+2} \in L$, $c(w_i) \ne c(w'_{i+2})$ and the identity $w_i = w'_{i+2}$ holds in $\mathfrak{Z}$. There is a letter x such that either $x \in c(w_i) \setminus c(w'_{i+2})$ or $x \in c(w'_{i+2}) \setminus c(w_i)$. Substitute x^3 for x in the identity $w_i = w'_{i+2}$. If $x \in c(w_i) \setminus c(w'_{i+2})$ we obtain an identity of the type $w' = w'_{i+2}$ such that $w' \in Z$ (because $x^3 = 0$ in $\mathfrak{C}$, and moreover in $\mathfrak{X}$) and $w' = w'_{i+2}$ holds in $\mathfrak{Z}$. Clearly, $w_i = w'$ also is satisfied by $\mathfrak{Z}$. Furthermore, if $x \in c(w'_{i+2}) \setminus c(w_i)$ then the substitution $x \longmapsto x^3$ immediately deduces from $w_i = w'_{i+2}$ an identity of the type $w_i = w'$ such that $w' \in Z$ (by the aforementioned reason) and $w_i = w'$ in $\mathfrak{Z}$. Thus, in any case there is a word $w' \in Z$ such that $w_i = w'$ in $\mathfrak{Z}$. Analogously, there is a word $w'' \in Z$ such that the identity $w'' = w_{i+3}$ holds in $\mathfrak{Z}$. Because $w', w'' \in Z$, the identity $w' = w''$ holds in $\mathfrak{X}$. □

Now we are ready to complete the proof of sufficiency in Theorem 1. Further considerations are divided into four cases.

***Case* 1:** the sequence (15) contains no adjacent words from the set L. Lemmas 4.3(i) and 4.4 imply that $w_1, \ldots, w_{n-1} \in Z$ in this case. Combining this claim with Lemma 4.2 and the condition $n \ge 3$, we obtain that $n = 3$. Besides that, the identity $w_1 = w_2$ holds in $\mathfrak{X}$, and therefore the identities $w_0 = w_1$ and $w_2 = w_3$ hold in $\mathfrak{Z}$. Now Lemma 4.1 applies.

***Case* 2:** the sequence (15) contains adjacent words from the set L but does not contain three words in row from this set. Applying Lemma 4.6 we have that the sequence (15) contains no words from the set Z in this case. Then Lemma 4.3(i) together with the claim $n \geq 3$ implies that $n = 3$, $w_0, w_3 \in S$ and $w_1, w_2 \in L$. By Lemma 4.3(ii) the identities $w_0 = w_1$ and $w_2 = w_3$ hold true in $\mathfrak{Z}$. Therefore, $w_1 = w_2$ in $\mathfrak{X}$. Now Lemma 4.1 applies.

***Case* 3:** the sequence (15) contains three words in row from the set L but does not contain four words in row from this set. So, let $w_i, w_{i+1}, w_{i+2} \in L$ for some $0 \leq i \leq n-2$. According to Lemmas 4.7 and 4.5, either $c(w_i) = c(w_{i+1}) \neq c(w_{i+2})$, $w_i = w_{i+1}$ in $\mathfrak{X}$ and $w_{i+1} = w_{i+2}$ in $\mathfrak{Z}$ or $c(w_i) \neq c(w_{i+1}) = c(w_{i+2})$, $w_i = w_{i+1}$ in $\mathfrak{Z}$ and $w_{i+1} = w_{i+2}$ in $\mathfrak{X}$. By symmetry, it suffices to consider the former case. So, let $c(w_i) = c(w_{i+1}) \neq c(w_{i+2})$, $w_i = w_{i+1}$ in $\mathfrak{X}$ and $w_{i+1} = w_{i+2}$ in $\mathfrak{Z}$. Since $n \geq 3$, either $i+2 < n$ or $i > 0$. Suppose at first that $i+2 < n$. Then the identity $w_{i+2} = w_{i+3}$ holds in $\mathfrak{X}$. Since $w_{i+2} \in L$, the identity $w_{i+2} = 0$ is false in $\mathfrak{X}$. The claims (i) and (iii) of Lemma 2.3 imply now that $w_{i+3} \in L$. Therefore, the sequence (15) contains four words in row from the set L (namely, the words w_i, w_{i+1}, w_{i+2} and w_{i+3}). But this is impossible. Therefore, $i+2 = n$, whence $i > 0$. The identity $w_{i-1} = w_i$ holds in the variety $\mathfrak{Z}$. The case $w_{i-1} \in L$ is impossible because the sequence (15) contains no four words in row from the set L. Lemma 4.6 implies that $w_{i-1} \notin Z$. Therefore, $w_{i-1} \in S$. Then we can apply Lemma 4.3(i) and conclude that $i-1 = 0$. Now Lemma 4.1 applies.

***Case* 4:** the sequence (15) contains four words in row from the set L. So, let $w_i, w_{i+1}, w_{i+2}, w_{i+3} \in L$ for some $0 \leq i \leq n-3$. According to Lemma 4.7, this means that either $c(w_i) = c(w_{i+1}) \neq c(w_{i+2}) = c(w_{i+3})$ or $c(w_i) \neq c(w_{i+1}) = c(w_{i+2}) \neq c(w_{i+3})$. Consider two corresponding subcases.

***Subcase* 4.1:** $c(w_i) = c(w_{i+1}) \neq c(w_{i+2}) = c(w_{i+3})$. Then $w_i = w_{i+1}$ in $\mathfrak{X}$, $w_{i+1} = w_{i+2}$ in $\mathfrak{Z}$ and $w_{i+2} = w_{i+3}$ in $\mathfrak{X}$ by Lemma 4.5. Applying Lemma 4.8 we have that there are words w' and w'' such that $w_i = w'$ in $\mathfrak{Z}$, $w' = w''$ in $\mathfrak{X}$ and $w'' = w_{i+3}$ in $\mathfrak{Z}$. If $i > 0$ then $w_{i-1} = w_i = w'$ in $\mathfrak{Z}$. This means that $w_0, w_1, \ldots, w_{i-1}, w', w'', w_{i+3}, \ldots, w_n$ is an $(\mathfrak{X}, \mathfrak{Z})$-deduction of the identity $u = v$ that shorter than (15). Thus, $i = 0$. Analogous arguments imply that $i + 3 = n$. Now we can apply Lemma 4.1 for the sequence w_0, w', w'', w_{i+3}.

***Subcase* 4.2:** $c(w_i) \neq c(w_{i+1}) = c(w_{i+2}) \neq c(w_{i+3})$. By Lemma 4.5, $w_i = w_{i+1}$ in $\mathfrak{Z}$, $w_{i+1} = w_{i+2}$ in $\mathfrak{X}$ and $w_{i+2} = w_{i+3}$ in $\mathfrak{Z}$. Suppose that $i > 0$. Then the identity $w_{i-1} = w_i$ holds in $\mathfrak{X}$. By Lemmas 4.3 and 4.6 $w_{i-1} \notin S \cup Z$, whence $w_{i-1} \in L$. By Lemma 4.5 $c(w_{i-1}) = c(w_i)$. Now we can apply the same arguments as in Subcase 4.1 for the words $w_{i-1}, w_i, w_{i+1}, w_{i+2}$. The case $i + 3 < n$ can be considered quite analogously. Finally, if $i = 0$ and $i + 3 = n$, it remains to refer to Lemma 4.1.

We have completed the proof of sufficiency in Theorem 1. □

Acknowledgements. The authors are grateful to the anonymous referee for several useful remarks.

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