Identities determining varieties of semigroups with completely regular power*

R. A. R. Monzo and B. M. Vernikov

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Abstract

We provide a transparent syntactic algorithm to decide whether an identity defines a variety of semigroups with completely regular power.

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One of the most important classes of semigroup varieties is the class of *completely regular* varieties; that is, varieties all of whose members are *completely regular* semigroups (unions of groups). Such varieties were examined in a great many papers by many authors (see the monograph [5]). An essentially wider class is formed by semigroup varieties of finite degree. Recall that a semigroup variety $\mathcal{V}$ is called a *variety of finite degree* if all nilsemigroups in $\mathcal{V}$ are nilpotent; $\mathcal{V}$ is called a *variety of degree* n if nilpotency degrees of nilsemigroups in $\mathcal{V}$ are bounded by the number n and n is the least number with this property.

Clearly, completely regular varieties are nothing but varieties of degree 1. Semigroup varieties of finite degree were examined in several papers (cf. eg. [6–8]). In particular Tishchenko [7] characterized (in terms of forbidden subvarieties) some natural subclasses of the class of varieties of finite degree. Namely, he considered varieties $\mathcal{V}$ with the following property: for every member S of $\mathcal{V}$, there is a number n such that the semigroup S^n is completely regular. We call varieties with such a property *varieties of semigroups with completely regular power*. This class of varieties or its natural subclasses appear in the literature (eg. [1, 3, 10, 11]).

The objective of this paper is to give an answer to the following natural question: given an identity (or an identity system), how can one decide if the identity (respectively, the system) defines a variety of semigroups with completely regular power? Clearly, this question is a special case of the general problem of deducing identities, which is undecidable in general as was shown by Murskiĭ [4]. We show that, in contrast, the above question is decidable and provide a transparent syntactic algorithm. This algorithm provides a straightforward way of deciding, by visual inspection, whether any identity determines

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such a variety. For example, using the algorithm described after Theorem 5 below it is easy to see that the following identities in the the left column determine such a variety, while those similar identities from the right column do not:

$wzxy = wxzywyz$	$wzxy = wxzyxyz;$
$uvz = xuvz$	$uvz = vuvz;$
$x_2x_1^5x_3^2x_1x_3x_2 = x_3x_1x_2$	$x_3x_1^5x_3^2x_1x_3x_2 = x_3x_1x_2;$
$xywz = x^2ywz^2$	$xywz = x^2ywz;$
$wzxy = wx^3zy^2x^4wy^{18}z(yx)^2$	$wzxy = wx^3zy^2x^4zy^{18}z(yx)^2;$
$uvxy = y(uvx)^3y$	$uvxy = v(uvx)^3y.$

We need some definitions and notation. The free semigroup over a countably infinite alphabet $\{x_1, x_2, \ldots, x_n, \ldots\}$ will be denoted by F. The symbol $\equiv$ stands for the equality relation on F. If $w \in F$ and x is a letter then $c(w)$ denotes the set of letters occurring in w, $\ell(w)$ is the length of the word w, $\ell_x(w)$ denotes the number of occurrences of x in w, and $h(w)$ [respectively $t(w)$] stands for the first [the last] letter of w. A word w is called *linear* if $\ell_x(w) \leq 1$ for every letter x. An identity of the form

$$x_1x_2 \cdots x_n = x_{1\pi}x_{2\pi} \cdots x_{n\pi}$$

where π is a non-trivial permutation on the set $\{1, 2, \ldots, n\}$ is called *permutative*. A pair of identities $wx = xw = w$ where the letter x does not occur in the word w is usually written as the symbolic identity $w = 0$. (This notation is justified because a semigroup with such identities has a zero element and all values of the word w in this semigroup are equal to zero.) A semigroup variety given by an identity system Σ is denoted by $\operatorname{var}\Sigma$. Put $\mathcal{N} = \operatorname{var}\{x^2 = 0,\ xy = yx\}$.

Lemma 1. *For a semigroup variety $\mathcal{V}$, the following are equivalent:*

a) *$\mathcal{V}$ is a variety of finite degree;*

b) *$\mathcal{N} \nsubseteq \mathcal{V}$;*

c) *$\mathcal{V}$ satisfies an identity of the form $x_1x_2 \cdots x_n = w$ with $\ell(w) > n$;*

d) *$\mathcal{V}$ satisfies an identity of the form*

$$x_1x_2 \cdots x_n = x_1 \cdots x_{i-1}(x_i \cdots x_j)^{m+1}x_{j+1} \cdots x_n$$

for some natural n and m and some $1 \leq i \leq j \leq n$.

Proof. The equivalence of the claims a) and b) is proved in [6, Theorem 2], the implication a) $\longrightarrow$ d) is checked in [9, Proposition 2.11], the implication d) $\longrightarrow$ c) is evident, while the implication c) $\longrightarrow$ a) is verified in [6, Lemma 1]. □

Put $\mathcal{P} = \operatorname{var}\{xy = x^2y,\ x^2y^2 = y^2x^2\}$ and $\overleftarrow{\mathcal{P}} = \operatorname{var}\{xy = xy^2,\ x^2y^2 = y^2x^2\}$.

Lemma 2 ([7, Theorem 2]). *A semigroup variety $\mathcal{V}$ is a variety of semigroups with completely regular power if and only if $\mathcal{V}$ is a variety of finite degree and $\mathcal{P}, \overleftarrow{\mathcal{P}} \nsubseteq \mathcal{V}$.* □

Lemma 3 ([2, Lemma 7]). *An identity $u = v$ holds in the variety $\mathcal{P}$ if and only if $c(u) = c(v)$ and either $h(u) \equiv h(v)$ and $\ell_{h(u)}(u) = \ell_{h(v)}(v) = 1$ or $\ell_{h(u)}(u) > 1$ and $\ell_{h(v)}(v) > 1$.* □

Lemmas 1–3 easily imply the following

Corollary 4. *For a semigroup variety $\mathcal{V}$ the following are equivalent*

a) *$\mathcal{V}$ is a variety of semigroups with completely regular power;*

b) *$\mathcal{V}$ satisfies an identity of the form $x_1x_2 \cdots x_n = (x_1 \cdots x_n)^{m+1}$ for some natural n and m;*

c) *$\mathcal{V}$ satisfies an identity of the form $x_1x_2 \cdots x_n = uv$ for some words u and v with $c(u) = c(v) = \{x_1, x_2, \dots, x_n\}$.* □

Theorem 5. *A non-trivial identity $u = v$ defines a variety of semigroups with completely regular power if and only if at least one of the words u and v is linear, the identity $u = v$ is not permutative, and either $c(u) \neq c(v)$ or the following holds:*

(i) *either $\ell_{h(u)}(u) = \ell_{h(v)}(v) = 1$ and $h(u) \not\equiv h(v)$ or $\ell_{h(u)}(u) = 1$ and $\ell_{h(v)}(v) > 1$ or $\ell_{h(u)}(u) > 1$ and $\ell_{h(v)}(v) = 1$;*

(ii) *either $\ell_{t(u)}(u) = \ell_{t(v)}(v) = 1$ and $t(u) \not\equiv t(v)$ or $\ell_{t(u)}(u) = 1$ and $\ell_{t(v)}(v) > 1$ or $\ell_{t(u)}(u) > 1$ and $\ell_{t(v)}(v) = 1$.*

Proof. Necessity. Suppose that the identity $u = v$ defines a variety of semigroups with completely regular power $\mathcal{V}$. If none of the words u and v is linear then $u = v$ holds in the variety $\mathcal{N}$, whence $\mathcal{N} \subseteq \mathcal{V}$. Then Lemma 1 implies that $\mathcal{V}$ is not a variety of finite degree, contradicting Lemma 2. Furthermore, if the identity $u = v$ is permutative then $\mathcal{V}$ contains the variety of all commutative semigroups, and therefore $\mathcal{V}$ is not periodic. But any variety of semigroups with completely regular power is periodic.

Thus at least one of the words u and v is linear and the identity $u = v$ is not permutative. If $c(u) \neq c(v)$ then we are done. Suppose now that $c(u) = c(v)$. Lemma 3 implies that if the claim (i) is false then $\mathcal{P} \subseteq \mathcal{V}$. In view of Lemma 2, this means that $\mathcal{V}$ is not a variety of semigroups with completely regular power. By symmetry, the same conclusion is true whenever the claim (ii) false. A contradiction shows that both the claims (i) and (ii) hold.

Sufficiency. Suppose that $\mathcal{V}$ satisfies an identity of the form $u = v$ such that at least one of the words u and v is linear, the identity $u = v$ is not permutative, and either $c(u) \neq c(v)$ or the claims (i) and (ii) hold. We may assume without any loss that $u \equiv x_1x_2 \cdots x_n$. First, we aim to verify that $\mathcal{V}$ is a variety of finite degree. Suppose that $c(v) \neq \{x_1, x_2, \dots, x_n\}$. Then there is a letter x that occurs in one of the parts of the identity $u = v$ but does not occur in another one. Substituting 0 for x in $u = v$, we obtain that every nilsemigroup in $\mathcal{V}$ satisfies the identity $x_1x_2\cdots x_n = 0$. We are done. Suppose now that $c(v) = \{x_1, x_2, \ldots, x_n\}$. Since $v \not\equiv x_{1\pi}x_{2\pi}\cdots x_{n\pi}$ for any non-trivial permutation π on the set $\{1, 2, \ldots, n\}$, we have that $\ell(v) > n$. Now Lemma 1 applies with the desirable conclusion.

Thus, $\mathcal{V}$ is a variety of finite degree. Since the claims (i) and (ii) hold, Lemma 3 and its dual imply that $\mathcal{P}, \overleftarrow{\mathcal{P}} \nsubseteq \mathcal{V}$. Now Lemma 2 applies with the conclusion that $\mathcal{V}$ is a variety of semigroups with completely regular power. $\square$

Theorem 5 permits us to formulate the following 'syntactic algorithm' for deciding whether an identity $u = v$ defines a variety of semigroups with completely regular power:

Step 1: Check to see that at least one side of the identity is linear.

Step 2: Check to make sure the identity is not permutative.

Step 3: If $c(u) \neq c(v)$ then we are finished.

Step 4: If $c(u) = c(v)$ and, say, u is linear then check to make sure that v is neither of the form xw where $x \equiv h(u)$ and $c(w) = c(u) \setminus \{x\}$ nor of the form $w'x$ where $x \equiv t(u)$ and $c(w') = c(u) \setminus \{x\}$.

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10 Albert Mansions, Crouch Hill, London N8 9RE, UK
E-mail address: `bobmonzo@talktalk.net`

Department of Mathematics and Mechanics, Ural State University, Lenina 51, 620083 Ekaterinburg, Russia
E-mail address: `bvernikov@gmail.com`