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RESEARCH ARTICLE

Upper-modular and related elements of the lattice of commutative semigroup varieties

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Abstract We completely determine upper-modular, codistributive and costandard elements in the lattice of all commutative semigroup varieties. In particular, we prove that the properties of being upper-modular and codistributive elements in the mentioned lattice are equivalent. Moreover, in the nil-case the properties of being elements of all three types turn out to be equivalent.

Keywords Semigroup · Variety · Lattice of varieties · Upper-modular element · Codistributive element · Costandard element

1 Introduction

A remarkable attention in the theory of lattices is devoted to special elements of lattices. Recall definitions of several types of special elements. An element x of the lattice $\langle L; \vee, \wedge\rangle$ is called

$$\begin{array}{lll} \textit{distributive} \quad \text{if} & \forall y, z \in L\colon & x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z); \\ \textit{standard} \quad \text{if} & \forall y, z \in L\colon & (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z); \\ \textit{modular} \quad \text{if} & \forall y, z \in L\colon & y \le z \longrightarrow (x \vee y) \wedge z = (x \wedge z) \vee y; \\ \textit{upper-modular} \quad \text{if} & \forall y, z \in L\colon & y \le x \longrightarrow x \wedge (y \vee z) = y \vee (x \wedge z); \end{array}$$

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neutral if, for all $y, z \in L$, the elements x, y and z generate a distributive sublattice of L. *Codistributive*, *costandard* and *lower-modular* elements are defined dually to distributive, standard and upper-modular elements respectively.

Special elements play an important role in the general lattice theory (see [1, Sect. III.2], for instance). In particular, one can say that neutral elements are related with decompositions of a lattice into subdirect product of its intervals, while [co]distributive elements are connected with homomorphisms of a lattice into its principal filters [principal ideals]. Thus the knowledge of which elements of a lattice are neutral or [co]distributive gives essential information on the structure of the lattice as a whole.

There is a number of interrelations between the mentioned types of elements. It is evident that a neutral element is both standard and costandard; a [co]standard element is modular; a [co]distributive element is lower-modular [upper-modular]. It is well known also that a [co]standard element is [co]distributive (see [1, Theorem 253], for instance).

During last years, a number of articles appeared concerning special elements of the above mentioned types in the lattice *SEM* of all semigroup varieties and in certain its important sublattices, first of all, in the lattice *Com* of all commutative semigroup varieties. Briefly speaking, these articles contain complete descriptions of special elements of many types and essential information about elements of other types (including strong necessary conditions and descriptions in wide and important partial cases). These results are discussed in details in the recent survey [11]. Special elements of the lattice *Com* are examined in [4,5]. Results of these works give a complete description of neutral, standard, distributive or lower-modular elements of *Com* and a considerable information about its modular elements that reduces the problem of description of such elements to the nil-case. However, practically anything was unknown so far about costandard, codistributive or upper-modular elements of the lattice *Com*. A unique exception is a description of elements of these three types in the narrow and particular class of 0-reduced varieties that follows from [4, Proposition 2.3 and Theorem 1.2]. In particular, it was unknown whether the lattice *Com* contains costandard but not neutral elements, as well as upper-modular but not codistributive elements. Corresponding questions were formulated in [11] (see Questions 4.11 and 4.12 there). For the sake of completeness, we mention that there exist codistributive but not costandard elements in the lattice *Com*. This fact can be easily deduced from results of [4] (see [11, Sect.4.5]).

In this article, we completely determine costandard, codistributive or upper-modular elements in the lattice *Com*. In particular, we answer Questions 4.11 and 4.12 of [11]. Namely, we prove that, in this lattice, the properties of being upper-modular and codistributive elements are equivalent, but the properties of being costandard and neutral elements are not equivalent. Moreover, it turns out that all three properties we consider are equivalent for commutative nil-varieties. Note that these results extremely contrast with the situation in the lattice *SEM* where the properties of being upper-modular and codistributive elements are not equivalent (compare [8, Theorem 1.2] and [10, Theorem 1.2]) but the properties of being costandard and neutral elements are equivalent [10, Theorem 1.3].

To formulate the main results, we need some notation. We denote by $\mathcal{T}$, $\mathcal{SL}$ and $\mathcal{COM}$ the trivial variety, the variety of semilattices and the variety of all commutative semigroups respectively. If Σ is a system of semigroup identities then var Σ stands for the semigroup variety given by Σ. For a natural number m, we put

$$\mathcal{C}_m = \mathrm{var}\left\{x^m = x^{m+1},\ xy = yx\right\}.$$

In particular, $\mathcal{C}_1 = \mathcal{SL}$. For brevity, we put also $\mathcal{C}_0 = \mathcal{T}$. Note that a semigroup S satisfies the identity system $wx = xw = w$ where the letter x does not occur in the word w if and only if S contains a zero element 0 and all values of w in S equal 0. We adopt the usual convention of writing $w = 0$ as a short form of this system. The main results of the article are the following two theorems.

Theorem 1.1 *For a commutative semigroup variety $\mathcal{V}$, the following are equivalent:*

(a) *$\mathcal{V}$ is an upper-modular element of the lattice Com;*
(b) *$\mathcal{V}$ is a codistributive element of the lattice Com;*
(c) *one of the following holds:*
  (i) *$\mathcal{V} = \mathcal{COM}$,*
  (ii) *$\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$, and $\mathcal{N}$ is a commutative variety satisfying the identities*

$$x^2yz = 0, \tag{1.1}$$
$$x^2y = xy^2, \tag{1.2}$$

  (iii) *$\mathcal{V} = \mathcal{G} \vee \mathcal{M} \vee \mathcal{N}$ where $\mathcal{G}$ is an Abelian periodic group variety, $\mathcal{M}$ is one of the varieties $\mathcal{T}$, $\mathcal{SL}$ or $\mathcal{C}_2$, and $\mathcal{N}$ is a commutative variety satisfying the identity*

$$x^2y = 0. \tag{1.3}$$

Theorem 1.2 *For a commutative semigroup variety $\mathcal{V}$, the following are equivalent:*

(a) *$\mathcal{V}$ is a modular and upper-modular element of the lattice Com;*
(b) *$\mathcal{V}$ is a costandard element of the lattice Com;*
(c) *one of the claims* (i) *or* (ii) *of Theorem 1.1 holds.*

Note that semigroup varieties that are simultaneously modular and upper-modular elements in *SEM* are completely determined in [13, Theorem 1], while costandard elements of *SEM* are completely classified in [10, Theorem 1.3]. A comparison of these two results shows that the two properties we mention are not equivalent in *SEM*, in contrast with the situation in the lattice *Com*.

The article consists of four sections. In Sect. 2 we collect auxiliary results used in what follows, Sect. 3 is devoted to the proof of Theorems 1.1 and 1.2, and Sect. 4 contains several corollaries from the main results.

2 Preliminaries

We start with certain results about special elements in the lattice *Com* obtained earlier.

Proposition 2.1 ([4, Theorem 1.2]) *A commutative semigroup variety $\mathcal{V}$ is a neutral element of the lattice Com if and only if $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$, and $\mathcal{N}$ is a commutative variety satisfying the identity* (1.3).

Proposition 2.2 ([5, Theorem 1.4]) *If a commutative semigroup variety $\mathcal{V}$ is a modular element of the lattice Com then $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$, and $\mathcal{N}$ is a nil-variety.*

It is generally known that the variety $\mathcal{SL}$ is an atom of the lattice *SEM* and therefore, of the lattice *Com* (see the survey [6], for instance). Proposition 2.1 implies that this variety is neutral in *Com*. Combining these facts with [4, Corollary 2.1], we have the following

Lemma 2.3 *A commutative semigroup variety $\mathcal{V}$ is an upper-modular* [*costandard*] *element of the lattice Com if and only if the variety $\mathcal{V} \vee \mathcal{SL}$ has the same property.*

The subvariety lattice of a variety $\mathcal{X}$ is denoted by $L(\mathcal{X})$. The following lemma is a part of the semigroup folklore since the beginning of the 1970s. It follows from the fact that the variety $\mathcal{SL}$ is a neutral element of the lattice *SEM* proved in [16, Proposition 4.1].

Lemma 2.4 *If $\mathcal{V}$ is a semigroup variety with $\mathcal{V} \not\supseteq \mathcal{SL}$ then the lattice $L(\mathcal{V} \vee \mathcal{SL})$ is isomorphic to the direct product of the lattices $L(\mathcal{V})$ and $L(\mathcal{SL})$.*

It is evident that a semigroup variety $\mathcal{V}$ is periodic if and only if it satisfies an identity of the form $x^n = x^{n+m}$ for some natural n and m. Let n be the least number with such a property. Then a semigroup from $\mathcal{V}$ is a group [a nilsemigroup] if and only if it satisfies the identities $x^n y = yx^n = y$ [the identity $x^n = 0$]. This implies the following generally known fact.

Lemma 2.5 *A periodic semigroup variety contains the greatest group subvariety and the greatest nilsubvariety.*

If $\mathcal{V}$ is a periodic variety then the greatest group subvariety [the greatest nilsubvariety] of $\mathcal{V}$ will be denoted by $\mathrm{Gr}(\mathcal{V})$ [respectively, $\mathrm{Nil}(\mathcal{V})$].

We denote by F the free semigroup over a countably infinite alphabet. The symbol $\equiv$ stands for the equality relation on F. If $u \in F$ then we denote by $c(u)$ the set of letters occurring in u and by $\ell(u)$ the length of u. If $u, v \in F$ then we write $u \lhd v$ whenever $v \equiv a\xi(u)b$ for some (maybe empty) words a and b and some homomorphism ξ of F. The first claim in the following lemma is evident, while the second one follows from [12, Lemma 1.3(iii)].

Lemma 2.6 *Let $\mathcal{N}$ be a nil-variety of semigroups.*

(i) *If $\mathcal{N}$ satisfies an identity $u = v$ with $c(u) \neq c(v)$ then $\mathcal{N}$ also satisfies the identity $u = 0$.*

(ii) *If $\mathcal{N}$ is commutative and satisfies an identity $u = v$ with $u \lhd v$ then $\mathcal{N}$ also satisfies the identity $u = 0$.*

We need the following two technical corollaries from Lemma 2.6. Put $W = \{x^2y, xyx, yx^2, y^2x, yxy, xy^2\}$.

Corollary 2.7 *If a commutative nil-variety of semigroups $\mathcal{N}$ satisfies an identity of the form $u = v$ where $u \in \{x^2y, xy^2\}$ and $v \notin W$ then $\mathcal{N}$ also satisfies the identity* (1.3).

Proof Suppose at first that $u \equiv x^2y$. If $c(v) \neq \{x, y\}$ then $\mathcal{N}$ satisfies the identity (1.3) by Lemma 2.6(i). Let now $c(v) = \{x, y\}$. If $\ell(v) < 3$ then $v \lhd x^2y$ and Lemma 2.6(ii) implies that $\mathcal{N}$ satisfies the identity (1.3) again. If $\ell(v) = 3$ then $v \in W$ contradicting the hypothesis. Finally, if $\ell(v) > 3$ then it is easy to see that v equals in $\mathcal{COM}$ (and therefore, in $\mathcal{N}$) to a word v' such that $x^2y \lhd v'$. Now Lemma 2.6(ii) applies again, and we conclude that $\mathcal{N}$ satisfies the identity (1.3) as well.

The case when $u \equiv xy^2$ can be considered quite analogously with the conclusion that $\mathcal{N}$ satisfies the identity $xy^2 = 0$ that is equivalent to (1.3) modulo the commutative law. □

Corollary 2.8 *If a commutative nil-variety of semigroups $\mathcal{N}$ satisfies the identity* (1.2) *then $\mathcal{N}$ also satisfies the identity* (1.1).

Proof Substituting yz to y in the identity (1.2), we obtain $x^2yz = x(yz)^2 = xy^2z^2$. Since $x^2yz \lhd xy^2z^2$, it remains to refer to Lemma 2.6(ii). □

A semigroup variety $\mathcal{V}$ is called a *variety of degree n* if all nilsemigroups in $\mathcal{V}$ are nilpotent of degree $\leq n$ and n is the least number with such property. A variety is said to be a *variety of finite degree* if it has a degree n for some n; otherwise, a variety is called a *variety of infinite degree*. The following lemma follows from [8, Proposition 2.11] and [7, Theorem 2].

Lemma 2.9 *A commutative semigroup variety $\mathcal{V}$ is a variety of degree $\leq n$ if and only if it satisfies an identity of the form*

$$x_1x_2\dots x_n = (x_1x_2\dots x_n)^{t+1} \tag{2.1}$$

for some natural number t.

If $\mathcal{V}$ is a variety of finite degree then we denote the degree of $\mathcal{V}$ by $\deg(\mathcal{V})$; otherwise, we write $\deg(\mathcal{V}) = \infty$.

Corollary 2.10 *If $\mathcal{X}$ and $\mathcal{Y}$ are commutative semigroup varieties then*

$$\deg(\mathcal{X} \vee \mathcal{Y}) = \max\big\{\deg(\mathcal{X}), \deg(\mathcal{Y})\big\}.$$

Proof If at least one of the varieties $\mathcal{X}$ or $\mathcal{Y}$ has infinite degree then

$$\deg(\mathcal{X} \vee \mathcal{Y}) = \infty = \max\big\{\deg(\mathcal{X}), \deg(\mathcal{Y})\big\}.$$

Let now $\deg(\mathcal{X}) = k$ and $\deg(\mathcal{Y}) = m$. Lemma 2.9 implies that the varieties $\mathcal{X}$ and $\mathcal{Y}$ satisfy, respectively, the identities

$$x_1x_2 \dots x_k = (x_1x_2 \dots x_k)^{r+1}$$

and

$$x_1x_2 \dots x_m = (x_1x_2 \dots x_m)^{s+1}$$

for some r and s. Assume without loss of generality that $k \le m$. Substitute $x_k \dots x_m$ to x_k in the first of the two mentioned identities. We have that $\mathcal{X}$ satisfies the identity

$$x_1x_2 \dots x_m = (x_1x_2 \dots x_m)^{r+1}.$$

Then $\mathcal{X} \vee \mathcal{Y}$ satisfies the identity

$$x_1x_2 \dots x_m = (x_1x_2 \dots x_m)^{rs+1}.$$

Now Lemma 2.6(ii) applies with the conclusion that

$$\deg(\mathcal{X} \vee \mathcal{Y}) \le m = \max\{\deg(\mathcal{X}), \deg(\mathcal{Y})\}.$$

Since the unequality $\max\{\deg(\mathcal{X}), \deg(\mathcal{Y})\} \le \deg(\mathcal{X} \vee \mathcal{Y})$ is evident, we are done. □

The following statement follows from [14, Proposition 1] and results of [2].

Lemma 2.11 *If $\mathcal{V}$ is a commutative semigroup variety and $\mathcal{V} \ne \mathcal{COM}$ then $\mathcal{V} = \mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}$ for some Abelian periodic group variety $\mathcal{G}$, some $m \ge 0$ and some nil-variety $\mathcal{N}$.*

A semigroup variety $\mathcal{V}$ is called *combinatorial* if all groups in $\mathcal{V}$ are singletons.

Lemma 2.12 *If $\mathcal{G}$ is a periodic group variety and $\mathcal{F}$ is a combinatorial semigroup variety then* $\mathrm{Gr}(\mathcal{G} \vee \mathcal{F}) = \mathcal{G}$.

Proof The inclusion $\mathcal{G} \subseteq \mathrm{Gr}(\mathcal{G} \vee \mathcal{F})$ is evident. To verify the converse inclusion, we suppose that the variety $\mathcal{G}$ satisfies the identity $u = v$. Being combinatorial, the variety $\mathcal{F}$ satisfies the identity $x^n = x^{n+1}$ for some natural n. Therefore $\mathcal{G} \vee \mathcal{F}$ satisfies the identity $u^{n+1}v^n = u^n v^{n+1}$. If we reduce it on u^n from the left and on v^n from the right, we obtain that the identity $u = v$ holds in $\mathrm{Gr}(\mathcal{G} \vee \mathcal{F})$. □

It can be easily checked that the join of all varieties of the form $\mathcal{C}_m$ coincides with the variety $\mathcal{COM}$. Therefore, for a periodic semigroup variety $\mathcal{X}$ there exists the largest number $m \in \mathbb{N} \cup \{0\}$ with the property $\mathcal{C}_m \subseteq \mathcal{X}$. We denote this number by $m(\mathcal{X})$. The following two results can be easily deduced from results of [3], and also can be easily verified directly.

Lemma 2.13 *If $\mathcal{G}$ is a periodic group variety, $m \geq 0$ and $\mathcal{N}$ is a nil-variety of semigroups then $m(\mathcal{G} \vee \mathcal{C}_m \vee \mathcal{N}) = m$.*

Lemma 2.14 *If $\mathcal{X}$ and $\mathcal{Y}$ are periodic commutative semigroup varieties then $m(\mathcal{X} \vee \mathcal{Y}) = \max\{m(\mathcal{X}), m(\mathcal{Y})\}$.*

3 Proofs of main results

To prove both theorems, it suffices to verify the implications (a) $\longrightarrow$ (c) and (c) $\longrightarrow$ (b) because the implications (b) $\longrightarrow$ (a) in both theorems are evident.

The implication (a) $\longrightarrow$ (c) *of Theorem* 1.1. The article [8] contains, among others, the proof of the following fact: if a periodic commutative semigroup variety $\mathcal{V}$ is an upper-modular element of the lattice *SEM* then one of the claims (ii) or (iii) of Theorem 1.1 holds. Almost all varieties that appear in the corresponding fragment of [8] are commutative. The unique exception is a periodic group variety $\mathcal{G}$ that appear in the verification of the following fact: every nil-subvariety of $\mathcal{V}$ satisfies the identity (1.2). There are no the requirement that the variety $\mathcal{G}$ is Abelian in [8]. But if we add this requirement to arguments used in [8] then the proof remains valid. Thus, in actual fact, it is verified in [8] that if $\mathcal{V}$ is an upper-modular element of the lattice *Com* and $\mathcal{V} \neq \mathcal{COM}$ then $\mathcal{V}$ satisfies one of the claims (ii) or (iii) of Theorem 1.1.

The implication (a) $\longrightarrow$ (c) *of Theorem* 1.2. Let $\mathcal{V}$ be a modular and upper-modular element of the lattice *Com* and $\mathcal{V} \neq \mathcal{COM}$. Then we can apply Proposition 2.2 and conclude that $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$, and $\mathcal{N}$ is a nil-variety. The variety $\mathcal{N}$ is an upper-modular element in the lattice *Com* by Lemma 2.3. In view of the proved above implication (a) $\longrightarrow$ (c) of Theorem 1.1, we have that $\mathcal{N}$ satisfies the identities (1.1) and (1.2). Thus the claim (ii) of Theorem 1.1 fulfills.

The implication (c) $\longrightarrow$ (b) *of Theorem* 1.2. Let $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$, and $\mathcal{N}$ is a commutative variety satisfying the identities (1.1) and (1.2). We need to verify that $\mathcal{V}$ is costandard in *Com*. In view of Lemma 2.3, it suffices to check that the variety $\mathcal{N}$ is costandard in *Com*. Let $\mathcal{X}$ and $\mathcal{Y}$ be arbitrary commutative semigroup varieties. It suffices to verify that

$$(\mathcal{N} \vee \mathcal{Y}) \wedge (\mathcal{X} \vee \mathcal{Y}) \subseteq (\mathcal{N} \wedge \mathcal{X}) \vee \mathcal{Y}$$

because the converse inclusion is evident. If at least one of the varieties $\mathcal{X}$ or $\mathcal{Y}$ coincides with the variety $\mathcal{COM}$ then the desirable inclusion is evident. Thus we may assume that the varieties $\mathcal{X}$ and $\mathcal{Y}$ are periodic. Let $u = v$ be an arbitrary identity that is satisfied by $(\mathcal{N} \wedge \mathcal{X}) \vee \mathcal{Y}$. We need to verify that it holds in $(\mathcal{N} \vee \mathcal{Y}) \wedge (\mathcal{X} \vee \mathcal{Y})$. By the hypothesis, the identity $u = v$ holds in $\mathcal{Y}$ and there exists a deduction of this identity from the identities of the varieties $\mathcal{N}$ and $\mathcal{X}$. Let the sequence of words

$$u_0 \equiv u, u_1, \ldots, u_k \equiv v \tag{3.1}$$

be the shortest such deduction. If $k = 1$ then the identity $u = v$ holds in one of the varieties $\mathcal{N}$ or $\mathcal{X}$. Then it is satisfied by one of the varieties $\mathcal{N} \vee \mathcal{Y}$ or $\mathcal{X} \vee \mathcal{Y}$, whence by the variety $(\mathcal{N} \vee \mathcal{Y}) \wedge (\mathcal{X} \vee \mathcal{Y})$. Thus we may assume that $k > 1$. If the identity

$u = v$ holds in $\mathcal{N}$ then it holds in $\mathcal{N} \vee \mathcal{Y}$ and therefore, in $(\mathcal{N} \vee \mathcal{Y}) \wedge (\mathcal{X} \vee \mathcal{Y})$. Thus we may assume that $u = v$ fails in $\mathcal{N}$. In particular, at least one of the words u or v, say u, does not equal 0 in $\mathcal{N}$. Since $\mathcal{N}$ satisfies the identity (1.1), this means that u coincides with one of the words $x_1x_2 \dots x_n$ for some n, x^2, x^3 or x^2y. Further considerations are divided into three cases.

Case 1 $u \equiv x_1x_2 \dots x_n$. If $v \equiv x_{1\pi}x_{2\pi} \dots x_{n\pi}$ for some non-trivial permutation π on the set $\{1, 2, \dots, n\}$ then the identity $u = v$ holds in the variety $\mathcal{COM}$ and therefore, in the variety $(\mathcal{N} \vee \mathcal{Y}) \wedge (\mathcal{X} \vee \mathcal{Y})$. Otherwise, Lemma 2.6 applies with the conclusion that every nilsemigroup in $\mathcal{Y}$ satisfies the identity

$$x_1x_2 \dots x_n = 0. \tag{3.2}$$

This means that $\mathcal{Y}$ is a variety of degree $\leq n$. Now we can apply Lemma 2.9 and conclude that $\mathcal{Y}$ satisfies the identity

$$x_1x_2 \dots x_n = (x_1x_2 \dots x_n)^{r+1}$$

for some natural r and therefore, the identity

$$x_1x_2 \dots x_n = (x_1x_2 \dots x_n)^{r\ell+1}$$

for any natural ℓ. Thus the words $x_1x_2 \dots x_n$, $(x_1x_2 \dots x_n)^{r\ell+1}$ (for all ℓ) and v pairwise equal each to other in the variety $\mathcal{Y}$.

Further, one of the varieties $\mathcal{N}$ or $\mathcal{X}$ satisfies the identity $x_1x_2 \dots x_n = u_1$. If $v \equiv x_{1\pi}x_{2\pi} \dots x_{n\pi}$ for some non-trivial permutation π on the set $\{1, 2, \dots, n\}$ then the identity $u = u_1$ holds in both varieties $\mathcal{N}$ and $\mathcal{X}$. This contradicts the claim that (3.1) is the shortest deduction of the identity $u = v$ from the identities of the varieties $\mathcal{N}$ and $\mathcal{X}$. Repeating arguments from the previous paragraph, we may conclude that there exists a natural s such that the words $x_1x_2 \dots x_n$, $(x_1x_2 \dots x_n)^{s\ell+1}$ (for all ℓ) and v pairwise equal each to other in one of the varieties $\mathcal{N}$ or $\mathcal{X}$. Then the words $x_1x_2 \dots x_n$, $(x_1x_2 \dots x_n)^{rs+1}$ and v pairwise equal each to other in one of the varieties $\mathcal{N} \vee \mathcal{Y}$ or $\mathcal{X} \vee \mathcal{Y}$ and therefore, in the variety $(\mathcal{N} \vee \mathcal{Y}) \wedge (\mathcal{X} \vee \mathcal{Y})$. In particular, the variety $(\mathcal{N} \vee \mathcal{Y}) \wedge (\mathcal{X} \vee \mathcal{Y})$ satisfies the identity $u = v$.

Case 2 $u \equiv x^2$ or $u \equiv x^3$. One can verify the desirable statement in slightly more general situation when $u \equiv x^n$ for some n. (In actual fact, this statement is evident whenever $n > 3$ because the variety $\mathcal{N}$ satisfies the identity $x^4 = 0$. But our considerations below do not depend on n.) The identity $x^n = v$ holds in $\mathcal{Y}$. Then Lemma 2.6 implies that every nilsemigroup in $\mathcal{Y}$ satisfies the identity $x^n = 0$. Being periodic, the variety $\mathcal{Y}$ satisfies the identity $x^p = x^q$ for some natural p and q with $p < q$. Let p be the least number with such a property. In view of Lemma 2.6, each nilsemigroup in $\mathcal{Y}$ satisfies the identity $x^p = 0$. Clearly, p is the least number with such a property. Therefore $n \geq p$. Multiplying the identity $x^p = x^q$ on x^{n-p}, we see that $\mathcal{Y}$ satisfies the identity $x^n = x^{n+r}$ for some r and therefore, the identity $x^n = x^{n+r\ell}$ for every natural ℓ. Thus the words x^n, $x^{n+r\ell}$ (for all ℓ) and v pairwise equal each to other in the variety $\mathcal{Y}$.

Further, one of the varieties $\mathcal{N}$ or $\mathcal{X}$ satisfies the identity $x^n = u_1$. The same arguments as in the previous paragraph show that there exists a natural s such that the words x^n, $x^{n+s\ell}$ (for all ℓ) and v pairwise equal each to other in one of the varieties $\mathcal{N}$ or $\mathcal{X}$. Then the words x^n, x^{n+rs} and v pairwise equal each to other in one of the varieties $\mathcal{N} \vee \mathcal{Y}$ or $\mathcal{X} \vee \mathcal{Y}$ and therefore, in the variety $(\mathcal{N} \vee \mathcal{Y}) \wedge (\mathcal{X} \vee \mathcal{Y})$. In particular, the variety $(\mathcal{N} \vee \mathcal{Y}) \wedge (\mathcal{X} \vee \mathcal{Y})$ satisfies the identity $u = v$.

Case 3 $u \equiv x^2y$. This case is essentially more complex than the two previous ones. In view of Lemma 2.11, $\mathcal{X} = \mathcal{G}_1 \vee \mathcal{C}_{m_1} \vee \mathcal{N}_1$ and $\mathcal{Y} = \mathcal{G}_2 \vee \mathcal{C}_{m_2} \vee \mathcal{N}_2$ for some Abelian periodic group varieties $\mathcal{G}_1$ and $\mathcal{G}_2$, some $m_1, m_2 \geq 0$ and some nil-varieties $\mathcal{N}_1$ and $\mathcal{N}_2$. We may assume without loss of generality that $\mathcal{N}_1 = \mathrm{Nil}(\mathcal{X})$ and $\mathcal{N}_2 = \mathrm{Nil}(\mathcal{Y})$.

If the variety $\mathcal{N}$ satisfies the identity (1.3) then we are done by Proposition 2.1. Suppose now that the identity (1.3) fails in $\mathcal{N}$. Recall that (3.1) is the shortest deduction of the identity $u = v$ from the identities of the varieties $\mathcal{N}$ and $\mathcal{X}$. Hence, for every $i = 0, 1, \ldots, k-1$, the identity $u_i = u_{i+1}$ is false in $\mathcal{COM}$. This allows us to suppose that if u_i is a word of length 3 depending on letters x and y then $u_i \in \{x^2y, xy^2\}$. Put $S = \{x^2y, xy^2\}$. The words $u_0, u_1, \ldots, u_k$ are pairwise distinct, whence at most two of them lie in S. Recall that $u_0 \equiv x^2y \in S$. The identity $u = u_1$ is satisfied by one of the varieties $\mathcal{N}$ or $\mathcal{X}$. If it holds in $\mathcal{N}$ and $u_1 \notin S$ then Corollary 2.7 applies with the conclusion that $\mathcal{N}$ satisfies the identity (1.3). But this is not the case. Further, if the identity $u = u_1$ holds in $\mathcal{X}$ and $u_1 \in S$ then the identity $u_1 = u_2$ holds in $\mathcal{N}$ and $u_2 \notin S$. Then Corollary 2.7 applies again and we conclude that the variety $\mathcal{N}$ satisfies the identity (1.3). As we have already noted, this is not the case. Thus either the identity $u = u_1$ holds in $\mathcal{N}$ and $u_1 \in S$ or this identity holds in $\mathcal{X}$ and $u_1 \notin S$. Note that $u_2 \notin S$ in the first case because $u_0, u_1 \in S$ here. In both the cases, there exists an identity of the form $w_1 = w_2$ such that $w_1 \in S$, $w_2 \notin S$ and the identity holds in $\mathcal{X}$ (namely, the identity $u_1 = u_2$ in the first case, and the identity $u = u_1$ in the second one). Corollary 2.7 shows that $\mathcal{N}_1$ satisfies the identity (1.3). According to Proposition 2.1, this implies that the variety $\mathcal{N}_1$ is neutral in *Com*. We use this fact below without special references.

By the hypothesis, the identity $x^2y = v$ holds in the variety $\mathcal{Y}$. Then Corollary 2.7 implies that either the variety $\mathcal{N}_2$ satisfies the identity (1.3) or $v \in W$. In the second case, the identity $x^2y = v$ is equivalent to (1.2) because it fails in the variety $\mathcal{COM}$. Thus either $\mathcal{N}_2$ satisfies the identity (1.3) or $\mathcal{Y}$ satisfies the identity (1.2). Consider the second case. Corollary 2.8 implies that the identity (1.1) holds in $\mathcal{N}_2$ in this case. Besides that, substituting 1 to y in (1.2), we have that every monoid in $\mathcal{Y}$ is a band (in particular, each group in $\mathcal{Y}$ is singleton). We see that $\mathcal{G}_2 = \mathcal{T}$ and $m_2 \leq 1$ in the considerable case. Thus either $\mathcal{N}_2$ satisfies the identity (1.3) or $\mathcal{Y} = \mathcal{M} \vee \mathcal{N}_2$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$.

Put $\mathcal{Z}_1 = (\mathcal{N} \wedge \mathcal{X}) \vee \mathcal{Y}$ and $\mathcal{Z}_2 = (\mathcal{N} \vee \mathcal{Y}) \wedge (\mathcal{X} \vee \mathcal{Y})$. In view of Lemma 2.11, it suffices to verify that $\mathrm{Gr}(\mathcal{Z}_1) = \mathrm{Gr}(\mathcal{Z}_2)$, $m(\mathcal{Z}_1) = m(\mathcal{Z}_2)$ and $\mathrm{Nil}(\mathcal{Z}_1) = \mathrm{Nil}(\mathcal{Z}_2)$. Clearly, the variety $\mathcal{C}_m \vee \mathcal{U}$ is combinatorial whenever $m \geq 0$ and $\mathcal{U}$ is an arbitrary nil-variety. Using Lemma 2.12, we have

$$\begin{aligned}
\mathrm{Gr}(\mathcal{Z}_1) &= \mathrm{Gr}\big(\mathcal{G}_2 \vee \mathcal{C}_{m_2} \vee \mathcal{N}_2 \vee (\mathcal{N} \wedge \mathcal{X})\big) = \mathcal{G}_2,\\
\mathrm{Gr}(\mathcal{Z}_2) &= \mathrm{Gr}\big((\mathcal{N} \vee \mathcal{Y}) \wedge (\mathcal{X} \vee \mathcal{Y})\big) = \mathrm{Gr}(\mathcal{N} \vee \mathcal{Y}) \wedge \mathrm{Gr}(\mathcal{X} \vee \mathcal{Y})
\end{aligned}$$

$$\begin{aligned}&= \mathrm{Gr}(\mathcal{G}_2 \vee \mathcal{C}_{m_2} \vee \mathcal{N}_2 \vee \mathcal{N}) \wedge \mathrm{Gr}(\mathcal{G}_1 \vee \mathcal{G}_2 \vee \mathcal{C}_{m_1} \vee \mathcal{C}_{m_2} \vee \mathcal{N}_1 \vee \mathcal{N}_2)\\ &= \mathcal{G}_2 \wedge (\mathcal{G}_1 \vee \mathcal{G}_2) = \mathcal{G}_2.\end{aligned}$$

Thus $\mathrm{Gr}(\mathcal{Z}_1) = \mathrm{Gr}(\mathcal{Z}_2)$. Further, using Lemma 2.13, we have

$$\begin{aligned}m(\mathcal{Z}_1) &= m\big((\mathcal{N} \wedge \mathcal{X}) \vee \mathcal{Y}\big) = m\big(\mathcal{G}_2 \vee \mathcal{C}_{m_2} \vee \mathcal{N}_2 \vee (\mathcal{N} \wedge \mathcal{X})\big) = m_2,\\ m(\mathcal{Z}_2) &= m\big((\mathcal{N} \vee \mathcal{Y}) \wedge (\mathcal{X} \vee \mathcal{Y})\big) = \min\big\{m(\mathcal{N} \vee \mathcal{Y}), m(\mathcal{X} \vee \mathcal{Y})\big\}\\ &= \min\big\{m(\mathcal{G}_2 \vee \mathcal{C}_{m_2} \vee \mathcal{N}_2 \vee \mathcal{N}), m(\mathcal{G}_1 \vee \mathcal{G}_2 \vee \mathcal{C}_{m_1} \vee \mathcal{C}_{m_2} \vee \mathcal{N}_1 \vee \mathcal{N}_2)\big\}\\ &= \min\big\{m(\mathcal{G}_2 \vee \mathcal{C}_{m_2} \vee \mathcal{N}_2 \vee \mathcal{N}), m(\mathcal{G}_1 \vee \mathcal{G}_2 \vee \mathcal{C}_{\max\{m_1,m_2\}} \vee \mathcal{N}_1 \vee \mathcal{N}_2)\big\}\\ &= \min\big\{m_2, \max\{m_1, m_2\}\big\} = m_2.\end{aligned}$$

Thus $m(\mathcal{Z}_1) = m(\mathcal{Z}_2)$.

It remains to check that $\mathrm{Nil}(\mathcal{Z}_1) = \mathrm{Nil}(\mathcal{Z}_2)$. Put

$$\mathcal{I} = \mathrm{var}\{x^2y = xy^2,\ x^2yz = 0,\ xy = yx\}.$$

As we have seen above, the varieties $\mathcal{N}_1$ and $\mathcal{N}_2$ satisfy the identity (1.2) and therefore, the identity (1.1) (see Corollary 2.8). In other words, $\mathcal{N}_1, \mathcal{N}_2 \subseteq \mathcal{I}$. Simple calculations based on Lemma 2.6 show that proper subvarieties of the variety $\mathcal{I}$ are exhausted by the following varieties:

$$\begin{aligned}\mathcal{I}_n &= \mathrm{var}\{x^2yz = x_1x_2\dots x_n = 0,\ x^2y = xy^2,\ xy = yx\} \quad \text{where } n \geq 4,\\ \mathcal{J} &= \mathrm{var}\{x^2yz = x^3 = 0,\ x^2y = xy^2,\ xy = yx\},\\ \mathcal{J}_n &= \mathrm{var}\{x^2yz = x^3 = x_1x_2\dots x_n = 0,\ x^2y = xy^2,\ xy = yx\} \quad \text{where } n \geq 4,\\ \mathcal{K} &= \mathrm{var}\{x^2y = 0,\ xy = yx\},\\ \mathcal{K}_n &= \mathrm{var}\{x^2y = x_1x_2\dots x_n = 0,\ xy = yx\} \quad \text{where } n \geq 3,\\ \mathcal{L} &= \mathrm{var}\{x^2 = 0,\ xy = yx\},\\ \mathcal{L}_n &= \mathrm{var}\{x^2 = x_1x_2\dots x_n = 0,\ xy = yx\} \quad \text{where } n \in \mathbb{N}.\end{aligned}$$

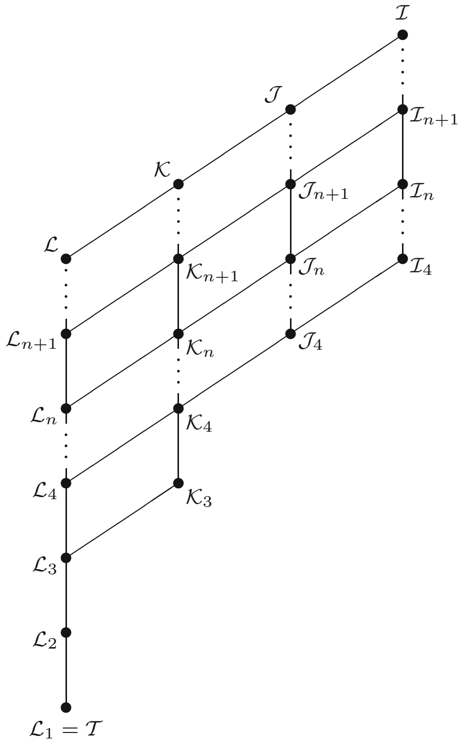
This implies that the lattice $L(\mathcal{I})$ has the form shown in Fig. 1.

Identities of the form $w = 0$ are called 0-*reduced*. For a commutative nil-variety of semigroups $\mathcal{V}$, we denote by $\mathrm{ZR}(\mathcal{V})$ the variety given by the commutative law and all 0-reduced identities that hold in $\mathcal{V}$. The exponent of a periodic group variety $\mathcal{H}$ is denoted by $\exp(\mathcal{H})$. To verify the equality $\mathrm{Nil}(\mathcal{Z}_1) = \mathrm{Nil}(\mathcal{Z}_2)$, we need two auxiliary facts.

Lemma 3.1 *Let $\mathcal{G}$ be a periodic group variety and $\mathcal{U}$ be a nil-variety of semigroups with $\mathcal{U} \subseteq \mathcal{I}$ and $\mathcal{U} \supseteq \mathrm{Nil}(\mathcal{C}_m)$ for some $m \leq 2$. Then*

(a) $\mathrm{Nil}(\mathcal{G} \vee \mathcal{C}_m \vee \mathcal{U}) \subseteq \mathrm{ZR}(\mathcal{U})$*;*
(b) *if $\mathcal{U} \subseteq \mathcal{K}$ then* $\mathrm{Nil}(\mathcal{G} \vee \mathcal{C}_m \vee \mathcal{U}) = \mathcal{U}$.

Fig. 1 The lattice $L(\mathcal{I})$



Proof (a) Put $\mathcal{Z} = \mathcal{G} \vee \mathcal{C}_m \vee \mathcal{U}$. Let $w = 0$ be an arbitrary 0-reduced identity that holds in the variety $\mathcal{U}$. Because $\mathcal{U} \subseteq \mathcal{I}$, we have that w is one of the words x^2yz, x^2y, x^3, x^2 or $x_1x_2 \dots x_n$ for some natural n (see Fig. 1). Put $r = \exp(\mathcal{G})$. If $w \in \{x^2yz, x^2y, x^3, x^2\}$ then the variety $\mathcal{Z}$ satisfies the identity $x^r w = w$. Then Lemma 2.6(ii) applies with the conclusion that the identity $w = 0$ holds in the variety $\mathrm{Nil}(\mathcal{Z})$. Suppose now that $w \equiv x_1x_2 \dots x_n$. In other words, $\mathcal{U}$ satisfies the identity (3.2). Because $\mathrm{Nil}(\mathcal{Z}) \supseteq \mathrm{Nil}(\mathcal{C}_m)$ and the variety $\mathrm{Nil}(\mathcal{C}_m)$ with $m > 1$ does not satisfy the identity (3.2), we have that $m \leq 1$ in this case. Then the variety $\mathcal{Z}$ satisfies the identity $x_1x_2 \dots x_n = x_1^{r+1}x_2 \dots x_n$. Using Lemma 2.6(ii) again, we have that the variety $\mathrm{Nil}(\mathcal{Z})$ satisfies the identity (3.2). We see that if a 0-reduced identity holds in $\mathcal{U}$ then it holds in $\mathrm{Nil}(\mathcal{Z})$ as well. We prove that $\mathrm{Nil}(\mathcal{Z}) \subseteq \mathrm{ZR}(\mathcal{U})$.

(b) Let now $\mathcal{U} \subseteq \mathcal{K}$. All subvarieties of the variety $\mathcal{K}$ are given within $\mathcal{COM}$ by 0-reduced identities only (see Fig. 1). Therefore $\mathrm{ZR}(\mathcal{U}) = \mathcal{U}$. Now the claim (a) implies that $\mathrm{Nil}(\mathcal{G} \vee \mathcal{C}_m \vee \mathcal{U}) \subseteq \mathcal{U}$. The converse inclusion is evident. □

Lemma 3.2 *If* $\mathcal{U}_1, \mathcal{U}_2 \subseteq \mathcal{I}$ *then* $\mathrm{ZR}(\mathcal{U}_1) \wedge \mathcal{U}_2 = \mathcal{U}_1 \wedge \mathcal{U}_2$.

Proof Put $\mathcal{Q} = \mathrm{var}\{x^2y = xy^2,\ xy = yx\}$. Then $\mathcal{U}_1 = \mathcal{Q} \wedge \mathrm{ZR}(\mathcal{U}_1)$ (see Fig. 1) and $\mathcal{U}_2 \subseteq \mathcal{I} \subseteq \mathcal{Q}$. Therefore $\mathcal{U}_1 \wedge \mathcal{U}_2 = \mathcal{Q} \wedge \mathrm{ZR}(\mathcal{U}_1) \wedge \mathcal{U}_2 = \mathrm{ZR}(\mathcal{U}_1) \wedge \mathcal{U}_2$. □

Now we start with the proof of the equality $\mathrm{Nil}(\mathcal{Z}_1) = \mathrm{Nil}(\mathcal{Z}_2)$. Note that if $m > 2$ then the variety $\mathrm{Nil}(\mathcal{C}_m) = \mathrm{var}\{x^m = 0,\ xy = yx\}$ does not satisfy the identity (1.1). Since $\mathrm{Nil}(\mathcal{C}_{m_1}) \subseteq \mathcal{N}_1 \subseteq \mathcal{X}$ and $\mathrm{Nil}(\mathcal{C}_{m_2}) \subseteq \mathcal{N}_2 \subseteq \mathcal{Y}$, we have $m_1, m_2 \leq 2$. Below we use this fact without special references.

Further, we note that $\mathcal{N} \wedge \mathcal{X} = \mathcal{N} \wedge \mathrm{Nil}(\mathcal{X}) = \mathcal{N} \wedge \mathcal{N}_1$, whence

$$\mathcal{Z}_1 = (\mathcal{N} \wedge \mathcal{N}_1) \vee \mathcal{Y}. \tag{3.3}$$

Suppose at first that the variety $\mathcal{N}_2$ satisfies the identity (1.3). Using the equality (3.3), we have

$$\mathcal{Z}_1 = (\mathcal{N} \wedge \mathcal{N}_1) \vee \mathcal{Y} = (\mathcal{N} \wedge \mathcal{N}_1) \vee \mathcal{N}_2 \vee \mathcal{G}_2 \vee \mathcal{C}_{m_2}$$

where $m_2 \leq 2$. Recall that $\mathcal{N}_1$ satisfies the identity (1.3). Now Lemma 3.1(b) with $\mathcal{U} = (\mathcal{N} \wedge \mathcal{N}_1) \vee \mathcal{N}_2$, $\mathcal{G} = \mathcal{G}_2$ and $m = m_2$ applies with the conclusion that

$$\mathrm{Nil}(\mathcal{Z}_1) = (\mathcal{N} \wedge \mathcal{N}_1) \vee \mathcal{N}_2. \tag{3.4}$$

Applying Proposition 2.1, we have

$$\mathrm{Nil}(\mathcal{Z}_1) = (\mathcal{N} \vee \mathcal{N}_2) \wedge (\mathcal{N}_1 \vee \mathcal{N}_2). \tag{3.5}$$

One can consider the variety $\mathrm{Nil}(\mathcal{Z}_2)$ now. Since $\mathcal{Z}_2 = (\mathcal{N} \vee \mathcal{Y}) \wedge (\mathcal{X} \vee \mathcal{Y})$, we have

$$\mathrm{Nil}(\mathcal{Z}_2) = \mathrm{Nil}(\mathcal{N} \vee \mathcal{Y}) \wedge \mathrm{Nil}(\mathcal{X} \vee \mathcal{Y}). \tag{3.6}$$

Further, $\mathrm{Nil}(\mathcal{N} \vee \mathcal{Y}) = \mathrm{Nil}(\mathcal{N} \vee \mathcal{N}_2 \vee \mathcal{G}_2 \vee \mathcal{C}_{m_2})$. Now we can apply Lemma 3.1(a) with $\mathcal{U} = \mathcal{N} \vee \mathcal{N}_2$, $\mathcal{G} = \mathcal{G}_2$ and $m = m_2$, and conclude that

$$\mathrm{Nil}(\mathcal{N} \vee \mathcal{Y}) \subseteq \mathrm{ZR}(\mathcal{N} \vee \mathcal{N}_2).$$

On the other hand,

$$\mathcal{X} \vee \mathcal{Y} = \mathcal{G}_1 \vee \mathcal{G}_2 \vee \mathcal{C}_{m_1} \vee \mathcal{C}_{m_2} \vee \mathcal{N}_1 \vee \mathcal{N}_2 = \mathcal{G}_1 \vee \mathcal{G}_2 \vee \mathcal{C}_{\max\{m_1, m_2\}} \vee \mathcal{N}_1 \vee \mathcal{N}_2.$$

Now Lemma 3.1(b) with $\mathcal{U} = \mathcal{N}_1 \vee \mathcal{N}_2$, $\mathcal{G} = \mathcal{G}_1 \vee \mathcal{G}_2$ and $m = \max\{m_1, m_2\}$ applies with the conclusion that $\mathrm{Nil}(\mathcal{X} \vee \mathcal{Y}) = \mathcal{N}_1 \vee \mathcal{N}_2$. Thus

$$\mathrm{Nil}(\mathcal{Z}_2) = \mathrm{Nil}(\mathcal{N} \vee \mathcal{Y}) \wedge \mathrm{Nil}(\mathcal{X} \vee \mathcal{Y}) \subseteq \mathrm{ZR}(\mathcal{N} \vee \mathcal{N}_2) \wedge (\mathcal{N}_1 \vee \mathcal{N}_2).$$

By the hypothesis, $\mathcal{N} \subseteq \mathcal{I}$. Now Lemma 3.2 with $\mathcal{U}_1 = \mathcal{N} \vee \mathcal{N}_2$ and $\mathcal{U}_2 = \mathcal{N}_1 \vee \mathcal{N}_2$ can be applied with the conclusion that

$$\mathrm{Nil}(\mathcal{Z}_2) \subseteq (\mathcal{N} \vee \mathcal{N}_2) \wedge (\mathcal{N}_1 \vee \mathcal{N}_2).$$

Because the converse inclusion is evident, we have

$$\mathrm{Nil}(\mathcal{Z}_2) = (\mathcal{N} \vee \mathcal{N}_2) \wedge (\mathcal{N}_1 \vee \mathcal{N}_2). \tag{3.7}$$

The equalities (3.5) and (3.7) imply that $\mathrm{Nil}(\mathcal{Z}_1) = \mathrm{Nil}(\mathcal{Z}_2)$.

It remains to consider the case when $\mathcal{N}_2$ does not satisfy the identity (1.3). Recall that $\mathcal{Y} = \mathcal{M} \vee \mathcal{N}_2$ where $\mathcal{M}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$ in this case. The equality (3.3) implies that

$$\mathcal{Z}_1 = (\mathcal{N} \wedge \mathcal{N}_1) \vee \mathcal{Y} = (\mathcal{N} \wedge \mathcal{N}_1) \vee \mathcal{N}_2 \vee \mathcal{M}$$

where $\mathcal{M}$ has the just mentioned sense. Lemma 2.4 implies that the equality (3.4) holds. This equality and Proposition 2.1 show that the equality (3.5) is true. Besides that, the equality (3.6) holds. Suppose that $\mathcal{M} = \mathcal{SL}$. Proposition 2.1 shows that

$$\begin{aligned}(\mathcal{N} \vee \mathcal{Y}) \wedge (\mathcal{X} \vee \mathcal{Y}) &= (\mathcal{N} \vee \mathcal{N}_2 \vee \mathcal{SL}) \wedge (\mathcal{X} \vee \mathcal{N}_2 \vee \mathcal{SL})\\ &= \big((\mathcal{N} \vee \mathcal{N}_2) \wedge (\mathcal{X} \vee \mathcal{N}_2)\big) \vee \mathcal{SL}.\end{aligned}$$

Now we can apply Lemma 2.4 and conclude that

$$\mathrm{Nil}(\mathcal{Z}_2) = \mathrm{Nil}\big((\mathcal{N} \vee \mathcal{N}_2) \wedge (\mathcal{X} \vee \mathcal{N}_2)\big).$$

Clearly, this equality holds whenever $\mathcal{M} = \mathcal{T}$ too. Thus it is valid always. Note that

$$\mathcal{X} \vee \mathcal{N}_2 = \mathcal{G}_1 \vee \mathcal{C}_{m_1} \vee \mathcal{N}_1 \vee \mathcal{N}_2.$$

Using Lemma 3.1(a) with $\mathcal{U} = \mathcal{N}_1 \vee \mathcal{N}_2$, $\mathcal{G} = \mathcal{G}_1$ and $m = m_1$, we have

$$\mathrm{Nil}(\mathcal{X} \vee \mathcal{N}_2) \subseteq \mathrm{ZR}(\mathcal{N}_1 \vee \mathcal{N}_2).$$

Since $\mathcal{N} \vee \mathcal{N}_2$ is a nil-variety, we have

$$\begin{aligned}\mathrm{Nil}(\mathcal{Z}_2) &= \mathrm{Nil}\big((\mathcal{N} \vee \mathcal{N}_2) \wedge (\mathcal{X} \vee \mathcal{N}_2)\big)\\ &= (\mathcal{N} \vee \mathcal{N}_2) \wedge \mathrm{Nil}(\mathcal{X} \vee \mathcal{N}_2)\\ &\subseteq (\mathcal{N} \vee \mathcal{N}_2) \wedge \mathrm{ZR}(\mathcal{N}_1 \vee \mathcal{N}_2).\end{aligned}$$

Now we can apply Lemma 3.2 with $\mathcal{U}_1 = \mathcal{N}_1 \vee \mathcal{N}_2$ and $\mathcal{U}_2 = \mathcal{N} \vee \mathcal{N}_2$ and conclude that the equality (3.7) holds. Because we prove above that the equality (3.5) is true, we have $\mathrm{Nil}(\mathcal{Z}_1) = \mathrm{Nil}(\mathcal{Z}_2)$.

We complete the proof of Theorem 1.2.

The implication (c) $\longrightarrow$ (b) *of Theorem* 1.1. It is evident that the variety $\mathcal{COM}$ is codistributive in *Com*. If the variety $\mathcal{V}$ satisfies the claim (ii) of Theorem 1.1 then Theorem 1.2 applies with the conclusion that $\mathcal{V}$ is costandard and therefore, is codistributive in *Com*. It remains to consider the case when $\mathcal{V}$ satisfies the claim (iii) of Theorem 1.1. So, let $\mathcal{V} = \mathcal{G} \vee \mathcal{M} \vee \mathcal{N}$ where $\mathcal{G}$ is an Abelian periodic group variety,

$\mathcal{M}$ is one of the varieties $\mathcal{T}$, $\mathcal{SL}$ or $\mathcal{C}_2$, and $\mathcal{N}$ is a commutative variety satisfying the identity (1.3).

Let $\mathcal{X}$ and $\mathcal{Y}$ be arbitrary commutative semigroup varieties. It remains to verify that $\mathcal{V} \wedge (\mathcal{X} \vee \mathcal{Y}) \subseteq (\mathcal{V} \wedge \mathcal{X}) \vee (\mathcal{V} \wedge \mathcal{Y})$ because the converse inclusion is evident. If at least one of the varieties $\mathcal{X}$ or $\mathcal{Y}$ coincides with the variety $\mathcal{COM}$ then the desirable inclusion is evident. Thus we may assume that the varieties $\mathcal{X}$ and $\mathcal{Y}$ are periodic. Put $\mathcal{Z}_1 = \mathcal{V} \wedge (\mathcal{X} \vee \mathcal{Y})$ and $\mathcal{Z}_2 = (\mathcal{V} \wedge \mathcal{X}) \vee (\mathcal{V} \wedge \mathcal{Y})$. The varieties $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are periodic. In view of Lemma 2.11, $\mathcal{Z}_1 = \mathcal{G}_1 \vee \mathcal{C}_{m_1} \vee \mathcal{N}_1$ and $\mathcal{Z}_2 = \mathcal{G}_2 \vee \mathcal{C}_{m_2} \vee \mathcal{N}_2$ for some Abelian periodic group varieties $\mathcal{G}_1$ and $\mathcal{G}_2$, some $m_1, m_2 \geq 0$ and some nil-varieties $\mathcal{N}_1$ and $\mathcal{N}_2$. We may assume without loss of generality that $\mathcal{G}_i = \mathrm{Gr}(\mathcal{Z}_i)$ and $\mathcal{N}_i = \mathrm{Nil}(\mathcal{Z}_i)$ for $i = 1, 2$. If $m > 2$ then the variety $\mathrm{Nil}(\mathcal{C}_m)$ does not satisfy the identity (1.3). Therefore $m_1, m_2 \leq 2$.

Clearly, it suffices to verify that $\mathcal{G}_1 = \mathcal{G}_2$, $m(\mathcal{Z}_1) = m(\mathcal{Z}_2)$ and $\mathcal{N}_1 \subseteq \mathcal{N}_2$. Put $q = \exp\bigl(\mathrm{Gr}(\mathcal{V})\bigr)$, $r = \exp\bigl(\mathrm{Gr}(\mathcal{X})\bigr)$ and $s = \exp\bigl(\mathrm{Gr}(\mathcal{Y})\bigr)$. Then

$$\exp(\mathcal{G}_1) = \gcd\bigl(q, \mathrm{lcm}(r, s)\bigr) \quad \text{and} \quad \exp(\mathcal{G}_2) = \mathrm{lcm}\bigl(\gcd(q, r), \gcd(q, s)\bigr).$$

Since the lattice of all natural numbers with the operations gcd and lcm is distributive, we have that $\exp(\mathcal{G}_1) = \exp(\mathcal{G}_2)$. This implies that $\mathcal{G}_1 = \mathcal{G}_2$ because the varieties $\mathcal{G}_1$ and $\mathcal{G}_2$ are Abelian.

Put $m(\mathcal{V}) = k$, $m(\mathcal{X}) = \ell$ and $m(\mathcal{Y}) = m$. It is clear that

$$m(\mathcal{E} \wedge \mathcal{F}) = \min\bigl\{m(\mathcal{E}), m(\mathcal{F})\bigr\}$$

for arbitrary periodic varieties $\mathcal{E}$ and $\mathcal{F}$. Combining this observation with Lemma 2.14, we have that

$$m(\mathcal{Z}_1) = \min\bigl\{k, \max\{\ell, m\}\bigr\} \quad \text{and} \quad m(\mathcal{Z}_2) = \max\bigl\{\min\{k, \ell\}, \min\{k, m\}\bigr\}.$$

This implies that $m(\mathcal{Z}_1) = m(\mathcal{Z}_2)$.

It remains to verify that $\mathcal{N}_1 \subseteq \mathcal{N}_2$. It is evident that $\mathcal{N}_1, \mathcal{N}_2 \subseteq \mathrm{Nil}(\mathcal{V})$. The variety $\mathcal{V}$ is commutative and satisfies the identity $x^2y = x^{r+2}y$ where $r = \exp(\mathcal{G})$. Lemma 2.6(ii) implies now that $\mathcal{N}_1$ and $\mathcal{N}_2$ satisfy the identity (1.3). This means that $\mathcal{N}_1, \mathcal{N}_2 \subseteq \mathcal{K}$. Every subvariety of the variety $\mathcal{K}$ is given within $\mathcal{K}$ either by the identity

$$x^2 = 0 \tag{3.8}$$

or by the identity (3.2) for some n or by these two identities simultaneously (see Fig. 1). Thus it suffices to verify that $\deg(\mathcal{Z}_1) = \deg(\mathcal{Z}_2)$ and the identity (3.8) holds in the variety $\mathcal{N}_1$ whenever it holds in $\mathcal{N}_2$.

Put $\deg(\mathcal{V}) = k$, $\deg(\mathcal{X}) = \ell$ and $\deg(\mathcal{Y}) = m$. It is evident that

$$\deg(\mathcal{E} \wedge \mathcal{F}) = \min\bigl\{\deg(\mathcal{E}), \deg(\mathcal{F})\bigr\}$$

for arbitrary semigroup varieties $\mathcal{E}$ and $\mathcal{F}$. Combining this observation with Corollary 2.10, we have that

$$\deg(\mathcal{Z}_1) = \min\bigl\{k, \max\{\ell, m\}\bigr\} \quad \text{and} \quad \deg(\mathcal{Z}_2) = \max\bigl\{\min\{k, \ell\}, \min\{k, m\}\bigr\}.$$

This implies that $\deg(\mathcal{Z}_1) = \deg(\mathcal{Z}_2)$.

Suppose now that $\mathcal{N}_2$ satisfies the identity (3.8). Being periodic, the variety $\mathcal{Z}_2$ satisfies the identity $x^n = x^m$ for some natural numbers n and m with $m > n$. Let n be the least number with such property. Then Lemma 2.6(ii) implies that the variety $\mathcal{N}_2 = \mathrm{Nil}(\mathcal{Z}_2)$ satisfies the identity $x^n = 0$ and n is the least number with such a property. Hence $n \le 2$. Thus the variety $\mathcal{Z}_2 = (\mathcal{V} \wedge \mathcal{X}) \vee (\mathcal{V} \wedge \mathcal{Y})$ satisfies the identity $x^2 = x^m$ for some $m > 2$. In particular, this identity holds in the variety $\mathcal{V} \vee \mathcal{X}$. Therefore there exists a deduction of this identity from the identities of the varieties $\mathcal{V}$ and $\mathcal{X}$. In particular, one of these varieties satisfies a non-trivial identity of the form $x^2 = w$. Now Lemma 2.6 implies that one of the varieties $\mathrm{Nil}(\mathcal{V})$ or $\mathrm{Nil}(\mathcal{X})$ satisfies the identity (3.8). If this identity holds in $\mathrm{Nil}(\mathcal{V})$ then it holds in the variety $\mathrm{Nil}\bigl(\mathcal{V} \wedge (\mathcal{X} \vee \mathcal{Y})\bigr) = \mathcal{N}_1$ too. Thus we may assume that the identity (3.8) is satisfied by the variety $\mathrm{Nil}(\mathcal{X})$. Analogously, using a deduction of the identity $x^2 = x^m$ from the identities of the varieties $\mathcal{V}$ and $\mathcal{Y}$, we can reduce our considerations to the case when the identity (3.8) holds in $\mathrm{Nil}(\mathcal{Y})$. The same arguments as we use at the beginning of this paragraph allows us to check that the varieties $\mathcal{X}$ and $\mathcal{Y}$ satisfy, respectively, the identities $x^2 = x^{q+2}$ and $x^2 = x^{r+2}$ for some natural numbers q and r. Therefore $\mathcal{X} \vee \mathcal{Y}$ satisfies the identity $x^2 = x^{qr+2}$. Then Lemma 2.6(ii) implies that the variety $\mathrm{Nil}(\mathcal{X} \vee \mathcal{Y})$ satisfies the identity (3.8). Then it holds in the variety $\mathrm{Nil}\bigl(\mathcal{V} \wedge (\mathcal{X} \vee \mathcal{Y})\bigr) = \mathcal{N}_1$ too.

We complete the proof of Theorem 1.1.

4 Corollaries

One can give several corollaries of main results. Theorem 1.1 and [8, Theorem 1.2] imply

Corollary 4.1 *A commutative semigroup variety $\mathcal{V}$ with $\mathcal{V} \ne \mathcal{COM}$ is an upper-modular element of the lattice Com if and only if it is an upper-modular element of the lattice SEM.*

Comparing Theorems 1.1 and 1.2, we have

Corollary 4.2 *For a commutative nil-variety of semigroups $\mathcal{V}$, the following are equivalent:*

(a) *$\mathcal{V}$ is an upper-modular element of the lattice Com;*
(b) *$\mathcal{V}$ is a codistributive element of the lattice Com;*
(c) *$\mathcal{V}$ is a costandard element of the lattice Com;*
(d) *$\mathcal{V}$ satisfies the identities* (1.1) *and* (1.2).

Theorem 1.1 implies

Corollary 4.3 *If a commutative semigroup variety $\mathcal{V}$ is an upper-modular element of the lattice Com and $\mathcal{V} \neq \mathcal{COM}$ then every subvariety of the variety $\mathcal{V}$ is an upper-modular element of the lattice Com.*

Note that the analog of this assertion for the lattice *SEM* is the case (see [9, Corollary 3]). Theorem 1.1 and results of [15] imply.

Corollary 4.4 *If a commutative semigroup variety $\mathcal{V}$ is an upper-modular element of the lattice Com and $\mathcal{V} \neq \mathcal{COM}$ then the lattice $L(\mathcal{V})$ is distributive.*

We do not know whether the analog of this fact in the lattice *SEM* is true. It is verified in [9, Corollary 2] that the following weaker statement is the case: if a variety $\mathcal{V}$ is an upper-modular element of the lattice *SEM* and $\mathcal{V}$ is not the variety of all semigroups then the lattice $L(\mathcal{V})$ is modular.

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