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Codistributive Elements of the Lattice of Semigroup Varieties

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Abstract—We prove that if a semigroup variety is a codistributive element of the lattice **SEM** of all semigroup varieties, then it either coincides with the variety of all semigroups or is a variety of semigroups with completely regular square. We completely classify strongly permutable varieties that are codistributive elements of **SEM**. We prove that a semigroup variety is a costandard element of the lattice **SEM** if and only if it is a neutral element of this lattice. In view of results obtained earlier, this gives a complete description of costandard elements of the lattice **SEM**.

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1. INTRODUCTION AND STATEMENTS OF THE MAIN RESULTS

The lattice of all semigroup varieties has been actively studied for more than four decades. The recent review [1] is dedicated to a systematic summary of the state-of-the-art in this area.

In the theory of lattices one pays much attention to the study of special elements of various types. Let us recall definitions of those of them which will be used below. An element x of a lattice $\langle L; \vee, \wedge\rangle$ is said to be *distributive*, if

$$\forall y, z \in L:\ x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z);$$

standard, if

$$\forall y, z \in L:\ (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z);$$

modular, if

$$\forall y, z \in L:\ y \le z \longrightarrow (x \vee y) \wedge z = (x \wedge z) \vee y;$$

upper-modular, if

$$\forall y, z \in L:\ y \le x \longrightarrow x \wedge (y \vee z) = y \vee (x \wedge z);$$

neutral, if for any elements $y, z \in L$ elements x, y, and z generate a distributive sublattice in L.

Codistributive, *costandard*, and *lower-modular* elements are defined as dual to distributive, standard, and upper-modular ones, respectively.

See, for example, the monograph, [2], § III.2 for a wide information about [co]distributive, [co]standard, and neutral elements, which shows the natural character and importance of their study. It is evident that any neutral element is standard and costandard, each [co]standard element is modular, and each [co]distributive element is lower-modular [upper-modular]. It is also well known that each [co]standard element is [co]distributive (the proof of this fact can be found, for example, in [2], theorem III.2.3).

For brevity we will use adjectives that denote types of special elements of lattices for semigroup varieties representing elements of the corresponding types in the lattice of all semigroup varieties. Just

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in this sense we will speak about *codistributive, costandard, neutral* etc. varieties. In papers [3–12] one studied modular, upper-modular, and lower-modular varieties of semigroups. In addition, in [5] one has completely described neutral varieties. Note that § 14 in [1] is dedicated to the statement of results of the most part of the mentioned papers. However, [co]distributive and [co]standard semigroup varieties were not considered in the literature till now.

The full description of distributive semigroup varieties was obtained by the author and V.Yu. Shaprynskii in the recent paper [13]. Results of the mentioned paper easily imply that a semigroup variety is standard if and only if it is distributive [1)]. In this paper we study codistributive and costandard semigroup varieties. We obtain a sufficiently strong necessary condition for the codistributivity of a variety, we describe codistributive varieties in a wide particular case, and we fully describe costandard varieties. In order to formulate the main results of the paper we need some definitions and denotations.

Recall that a semigroup variety $\mathcal{V}$ is said to be a *semigroup variety with completely regular square*, if the square of any semigroup in $\mathcal{V}$ is a completely regular semigroup or, equivalently, if $\mathcal{V}$ satisfies the identity $xy = (xy)^{n+1}$ for some natural n. We denote by $\mathcal{T}$, $\mathcal{SL}$, $\mathcal{ZM}$, and $\mathcal{SEM}$ the trivial variety, the variety of semilattices, the variety of semigroups with zero multiplication, and the variety of all semigroups, respectively.

The next theorem is the first of three main results of this paper.

Theorem 1.1. *If a semigroup variety $\mathcal{V}$ is codistributive, then either $\mathcal{V} = \mathcal{SEM}$ or $\mathcal{V}$ is a semigroup variety with completely regular square.*

A semigroup variety which satisfies the identity

$$x_1x_2\cdots x_n = x_{1\alpha}x_{2\alpha}\cdots x_{n\alpha}, \tag{1}$$

where α is a permutation on the set $\{1, 2, \ldots, n\}$ such that $1\alpha \neq 1$ and $n\alpha \neq n$, is said to be *strictly commutative*. One says that a semigroup variety $\mathcal{V}$ is a *variety of degree* n, if all nilsemigroups in $\mathcal{V}$ are nilpotent of degree $\leq n$, and n is the minimal number with this property. A variety different from that of degree $\leq n$ is called a *variety of degree* $> n$. A semigroup variety is said to be *proper*, if it differs from the variety $\mathcal{SEM}$. Theorem 1.1 in this paper and lemma 1 in [14] imply that any proper codistributive variety has degree ≤ 2. In particular, the following result shows that for strictly commutative varieties the inverse assertion is also valid.

Theorem 1.2. *For a strictly commutative semigroup variety $\mathcal{V}$ the following conditions are equivalent*:

a) *$\mathcal{V}$ is codistributive*;

b) *$\mathcal{V}$ is a variety of degree ≤ 2*;

c) *$\mathcal{V} = \mathcal{G} \vee \mathcal{X}$, where $\mathcal{G}$ is the variety of periodic Abelian groups, and $\mathcal{X}$ is one of varieties $\mathcal{T}$, $\mathcal{SL}$, $\mathcal{ZM}$, and $\mathcal{SL} \vee \mathcal{ZM}$.*

The next theorem is the third main result of this paper.

Theorem 1.3. *For a semigroup variety $\mathcal{V}$ the following conditions are equivalent*:

a) *$\mathcal{V}$ is costandard*;

b) *$\mathcal{V}$ is neutral*;

c) *$\mathcal{V}$ coincides with one of varieties $\mathcal{T}$, $\mathcal{SL}$, $\mathcal{ZM}$, $\mathcal{SL} \vee \mathcal{ZM}$, and $\mathcal{SEM}$.*

Since each neutral element is standard, Theorem 1.3 has the following corollary.

Corollary 1.1. Any costandard semigroup variety is standard.

Section 2 of this paper contains the necessary in what follows additional definitions, denotations, and some auxiliary results. Section 3 is dedicated to the proof of Theorems 1.1–1.3, and Section 4 contains final observations and open questions.

[1)] Really, it is easy to understand that if an element of a lattice is distributive and modular simultaneously, then it is standard, and in [13] one has shown that any distributive semigroup variety is modular.

2. PRELIMINARY INFORMATION

2.1. Special elements in abstract lattices. We need the following three theoretical-lattice observations.

Lemma 2.1. *If x is a codistributive element of a lattice L and w is its neutral element, then $x \vee w$ is a codistributive element in L.*

Proof. Let $y, z \in L$. Then

$$\begin{aligned}(x \vee w) \wedge (y \vee z) &= \big(x \wedge (y \vee z)\big) \vee \big(w \wedge (y \vee z)\big) && \text{(because } w \text{ is neutral)}\\ &= \big((x \wedge y) \vee (x \wedge z)\big) && \text{(because } x \text{ is codistributive}\\ &\quad \vee \big((w \wedge y) \vee (w \wedge z)\big) && \text{and } w \text{ is neutral)}\\ &= \big((x \wedge y) \vee (w \wedge y)\big)\\ &\quad \vee \big((x \wedge z) \vee (w \wedge z)\big)\\ &= \big((x \vee w) \wedge y\big) \vee \big((x \vee w) \wedge z\big) && \text{(because } w \text{ is neutral).}\end{aligned}$$

Thus, $(x \vee w) \wedge (y \vee z) = \big((x \vee w) \wedge y\big) \vee \big((x \vee w) \wedge z\big)$, i.e., the element $x \vee w$ is codistributive. □

For any element x of a lattice L we set $(x] = \{y \in L \mid y \leq x\}$.

Lemma 2.2. *Let x be a codistributive element of a lattice L. The lattice $(x]$ is distributive if and only if each its element is codistributive in L.*

Proof. The sufficiency is evident. Let us prove the necessity. Let $w \in (x]$ and $y, z \in L$. Then

$$\begin{aligned}w \wedge (y \vee z) &= \big(w \wedge (y \vee z)\big) \wedge x && \text{(because } w \wedge (y \vee z) \leq w \leq x)\\ &= \big(x \wedge (y \vee z)\big) \wedge w\\ &= \big((x \wedge y) \vee (x \wedge z)\big) \wedge w && \text{(because } x \text{ is codistributive)}\\ &= \big((x \wedge y) \wedge w\big) \vee \big((x \wedge z) \wedge w\big) && \text{(because } x \wedge y, x \wedge z, w \in (x]\\ &&& \text{and the lattice } (x] \text{ is distributive)}\\ &= \big((w \wedge y) \wedge x\big) \vee \big((w \wedge z) \wedge x\big)\\ &= (w \wedge y) \vee (w \wedge z) && \text{(because } w \wedge y, w \wedge z \leq w \leq x).\end{aligned}$$

Therefore, $w \wedge (y \vee z) = (w \wedge y) \vee (w \wedge z)$, i.e., the element w is codistributive. □

Recall that a lattice L with 0 is said to be 0-*distributive*, if

$$\forall x, y, z \in L:\ x \wedge y = x \wedge z = 0 \longrightarrow x \wedge (y \vee z) = 0,$$

and it is called *atomic*, if for each $x \in L \setminus \{0\}$ there exists an atom $a \in L$ such that $a \leq x$.

Lemma 2.3. *Let L be an atomic 0-distributive lattice, let $a_1, a_2, \dots, a_k$ be atoms of L, and $x = \bigvee\limits_{i=1}^{k} a_i$. If the lattice $(x]$ is a finite Boolean algebra, then x is a codistributive element of the lattice L.*

Proof. For any element $w \in L$ we denote by $A(w)$ the set of all atoms $a \in L$ such that $a \leq w$. Since $(x]$ is a finite Boolean algebra, $w = \bigvee A(w)$ for all $w \leq x$ (here we assume that the union of the empty set of atoms equals zero). Let $b, c \in L$. Then

$$A(b \vee c) = A(b) \cup A(c) \ \text{ and } \ A(b \wedge c) = A(b) \cap A(c) \tag{2}$$

(the first equality follows from the 0-distributive property of the lattice L, and the second one is evident). Let $A(L)$ be the set of all atoms of the lattice L and let $y, z \in L$. Then

$$x \wedge (y \vee z) = \bigvee A\big(x \wedge (y \vee z)\big) \qquad \text{(because } x \wedge (y \vee z) \leq x)$$

$$
\begin{aligned}
&= \bigvee \big[A(x) \cap \big(A(y) \cup A(z)\big)\big] && \text{(due to (2))}\\
&= \bigvee \big[\big(A(x) \cap A(y)\big) && \text{(because the lattice of subsets}\\
&\qquad \cup \big(A(x) \cap A(z)\big)\big] && \text{of the set } A(L) \text{ is distributive)}\\
&= \bigvee A\big((x \wedge y) \vee (x \wedge z)\big) && \text{(due to (2))}\\
&= (x \wedge y) \vee (x \wedge z) && \text{(because } (x \wedge y) \vee (x \wedge z) \le x\text{).}
\end{aligned}
$$

Therefore, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. □

2.2. Special elements in the lattice of semigroup varieties. As was noted in Section 1, in the paper [5] one has fully described neutral varieties of semigroups. Let us formulate this description.

Proposition 2.1 ([5], proposition 4.1)**.** *Neutral semigroup varieties are only varieties $\mathcal{T}$, $\mathcal{SL}$, $\mathcal{ZM}$, $\mathcal{SL} \vee \mathcal{ZM}$, and $\mathcal{SEM}$.*

Lemma 2.1 and Proposition 2.1 immediately imply the following corollary.

Corollary 2.1. If a semigroup variety $\mathcal{V}$ is codistributive and $\mathcal{X}$ is one of varieties $\mathcal{SL}$, $\mathcal{ZM}$, and $\mathcal{SL} \vee \mathcal{ZM}$, then the variety $\mathcal{V} \vee \mathcal{X}$ is also codistributive.

Theorem 1 in the paper [9] immediately implies the following proposition.

Proposition 2.2. *If $\mathcal{V}$ is a proper upper-modular semigroup variety of degree > 2, then $\mathcal{V}$ is commutative.*

For a system of identities Σ we denote by $\mathrm{var}\,\Sigma$ the semigroup variety determined by Σ. We set $\mathcal{LZ} = \mathrm{var}\{xy = x\}$, $\mathcal{RZ} = \mathrm{var}\{xy = y\}$, $\mathcal{P} = \mathrm{var}\{xy = x^2y,\ x^2y^2 = y^2x^2\}$, and $\overleftarrow{\mathcal{P}} = \mathrm{var}\{xy = xy^2,\ x^2y^2 = y^2x^2\}$.

Proposition 2.3 ([9], theorem 2)**.** *If $\mathcal{V}$ is an upper-modular semigroup variety of degree ≤ 2, then one of the following conditions is fulfilled*:

(i) *$\mathcal{V} = \mathcal{K} \vee \mathcal{P}$, where $\mathcal{K}$ is a completely regular semigroup variety such that $\mathcal{K} \not\supseteq \mathcal{RZ}$*;

(ii) *$\mathcal{V} = \mathcal{K} \vee \overleftarrow{\mathcal{P}}$, where $\mathcal{K}$ is a completely regular semigroup variety such that $\mathcal{K} \not\supseteq \mathcal{LZ}$*;

(iii) *$\mathcal{V}$ is a semigroup variety with completely regular square.*

We also need the following information about modular varieties.

Proposition 2.4. *If a proper semigroup variety $\mathcal{V}$ is modular, then $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$, where $\mathcal{M}$ is one of varieties $\mathcal{T}$ and $\mathcal{SL}$, and $\mathcal{N}$ is a nilvariety.*

This assertion is proved (in another terminology and in a slightly weaker form) in [4] (proposition 1.6). In the form adduced above it is stated and verified in [7] (proposition 2.1).

2.3. Decomposition of some varieties into the union of subvarieties. Lemma 2 in the paper [15] and the proof of proposition 1 in the same paper imply the following lemma.

Lemma 2.4. *If a periodic semigroup variety $\mathcal{V}$ does not contain varieties $\mathcal{LZ}$, $\mathcal{RZ}$, $\mathcal{P}$, and $\overleftarrow{\mathcal{P}}$, then $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$, where $\mathcal{M}$ is a variety generated by a monoid, and $\mathcal{N}$ is a nilvariety.*

For any natural number m we set $\mathcal{C}_m = \mathrm{var}\{x^m = x^{m+1},\ xy = yx\}$. In particular, $\mathcal{C}_1 = \mathcal{SL}$. For convenience of exposition, we also set $\mathcal{C}_0 = \mathcal{T}$. Results of the paper [16] imply the next lemma.

Lemma 2.5. *If a semigroup variety $\mathcal{V}$ is generated by a commutative monoid, then $\mathcal{V} = \mathcal{G} \vee \mathcal{C}_m$ for some variety $\mathcal{G}$ of periodic Abelian groups and some $m \geq 0$.*

2.4. Identities of some varieties. We need the descriptions of identities which are fulfilled in varieties $\mathcal{RZ}$, $\mathcal{C}_2$, and $\mathcal{P}$.

Lemma 2.6. *The identity $v = w$ is fulfilled*

a) *in the variety $\mathcal{RZ}$ if and only if words v and w end with one and the same letter*;

b) *in the variety $\mathcal{C}_2$ if and only if words v and w depend on the same letters, and each of these letters occurs in v and w either once or more than one time*;

c) *in the variety $\mathcal{P}$ if and only if words v and w depend on the same letters, and either the last letter in each of words v and w occurs in this word more than one time or words v and w end with the same letter, which occurs in both words v and w exactly once.*

The assertions of this lemma about identities of varieties $\mathcal{RZ}$ and $\mathcal{C}_2$ are well-known and can be easily checked, and the assertion about identities of the variety $\mathcal{P}$ follows from lemma 7 in [17].

2.5. Some properties of the lattice of semigroup varieties. For brevity we denote the lattice of all semigroup varieties by **SEM**. The following assertion is well known (e.g., [1], § 1).

Lemma 2.7. *Atoms of the lattice* **SEM** *are only varieties of Abelian groups of an arbitrary simple exponent and varieties $\mathcal{LZ}$, $\mathcal{RZ}$, $\mathcal{SL}$, and $\mathcal{ZM}$.*

It is well-known that the lattice **SEM** is atomic. It is easy to understand that for an atomic lattice L the 0-distributive property is equivalent to the following condition: If a is an atom in L and $x, y \in L$, then the fact that $x \ngeq a$ and $y \ngeq a$ implies that $x \vee y \ngeq a$. Taking into account this observation, results of the paper [18] imply the following lemma.

Lemma 2.8. *The lattice* **SEM** *is* 0-*distributive.*

As usual, the symbol $L(\mathcal{V})$ stands for the lattice of subvarieties of the variety $\mathcal{V}$. Proposition 1 in the paper [19] implies the following lemma.

Lemma 2.9. *If a semigroup variety $\mathcal{V}$ is the union of a finite number of atoms of the lattice* **SEM**, *then the lattice $L(\mathcal{V})$ is a finite Boolean algebra.*

3. PROOF OF MAIN RESULTS

Proof of Theorem 1.1. Let $\mathcal{V}$ be a proper codistributive semigroup variety. Evidently, the variety $\mathcal{V}$ is upper-modular. Assume that $\mathcal{V}$ is a variety of degree > 2. Proposition 2.2 shows that the variety $\mathcal{V}$ is commutative. Therefore, varieties $\mathcal{P}$ and $\overleftarrow{\mathcal{P}}$ are not contained in $\mathcal{V}$. It is well-known that the variety $\mathcal{SL} \vee \mathcal{ZM}$ is the maximal proper subvariety of each of varieties $\mathcal{P}$ and $\overleftarrow{\mathcal{P}}$. Hence, $(\mathcal{V} \wedge \mathcal{P}) \vee (\mathcal{V} \wedge \overleftarrow{\mathcal{P}}) \subseteq \mathcal{SL} \vee \mathcal{ZM}$. In particular, $(\mathcal{V} \wedge \mathcal{P}) \vee (\mathcal{V} \wedge \overleftarrow{\mathcal{P}})$ is a variety of degree ≤ 2. On the other hand, it is well-known that $\mathcal{P} \vee \overleftarrow{\mathcal{P}}$ is a variety of degree 3. Since $\mathcal{V}$ is a variety of degree > 2, we obtain that $\mathcal{V} \wedge (\mathcal{P} \vee \overleftarrow{\mathcal{P}})$ is a variety of degree 3. Therefore, $\mathcal{V} \wedge (\mathcal{P} \vee \overleftarrow{\mathcal{P}}) \neq (\mathcal{V} \wedge \mathcal{P}) \vee (\mathcal{V} \wedge \overleftarrow{\mathcal{P}})$ in despite of the codistributive property of $\mathcal{V}$.

We have proved that $\mathcal{V}$ is a variety of degree ≤ 2. Assume that $\mathcal{V}$ is not a semigroup variety with a completely regular square. According to Proposition 2.3, $\mathcal{V}$ satisfies one of conditions (i) and (ii) in this proposition. In view of the symmetry we assume that condition (i) is fulfilled, i.e., $\mathcal{V} = \mathcal{K} \vee \mathcal{P}$, where $\mathcal{K}$ is a completely regular semigroup variety such that $\mathcal{K} \nsupseteq \mathcal{RZ}$. From Lemma 2.6 it easily follows that $\mathcal{C}_2 \vee \mathcal{RZ} \supseteq \mathcal{P}$. Therefore, $\mathcal{V} \wedge (\mathcal{C}_2 \vee \mathcal{RZ}) \supseteq \mathcal{P}$. On the other hand, it is well-known and can be easily verified that the maximal subvariety of degree ≤ 2 in the variety $\mathcal{C}_2$ is the variety $\mathcal{SL} \vee \mathcal{ZM}$. Hence,

$\mathcal{V} \wedge \mathcal{C}_2 \subseteq \mathcal{SL} \vee \mathcal{ZM}$. Since $\mathcal{K}, \mathcal{P} \not\supseteq \mathcal{RZ}$, from Lemmas 2.7 and 2.8 it follows that $\mathcal{V} = \mathcal{K} \vee \mathcal{P} \not\supseteq \mathcal{RZ}$, and therefore $\mathcal{V} \wedge \mathcal{RZ} = \mathcal{T}$. Combining these considerations, we get

$$(\mathcal{V} \wedge \mathcal{C}_2) \vee (\mathcal{V} \wedge \mathcal{RZ}) = (\mathcal{V} \wedge \mathcal{C}_2) \vee \mathcal{T} = \mathcal{V} \wedge \mathcal{C}_2 \subseteq \mathcal{SL} \vee \mathcal{ZM} \subset \mathcal{P} \subseteq \mathcal{V} \wedge (\mathcal{C}_2 \vee \mathcal{RZ}).$$

We have proved that $\mathcal{V} \wedge (\mathcal{C}_2 \vee \mathcal{RZ}) \neq (\mathcal{V} \wedge \mathcal{C}_2) \vee (\mathcal{V} \wedge \mathcal{RZ})$, i.e., $\mathcal{V}$ is not codistributive.

The obtained contradiction completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. The implication a) $\Rightarrow$ b) follows from Theorem 1.1.

b) $\Rightarrow$ c). Let $\mathcal{V}$ be a strictly commutable variety of degree ≤ 2. Evidently, $\mathcal{LZ}, \mathcal{RZ} \nsubseteq \mathcal{V}$ and Lemma 2.6 implies that $\mathcal{P}, \overleftarrow{\mathcal{P}} \nsubseteq \mathcal{V}$. It is clear that $\mathcal{V}$ is a periodic variety. Now from Lemma 2.4 we get $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$, where $\mathcal{M}$ is a variety generated by a monoid, and $\mathcal{N}$ is a nilvariety. Since $\mathcal{V}$ is a variety of degree ≤ 2, we conclude that $\mathcal{N} \subseteq \mathcal{ZM}$. Taking into account Lemma 2.7, we obtain that $\mathcal{N}$ is one of varieties $\mathcal{T}$ and $\mathcal{ZM}$.

Since each monoid that satisfies a nontrivial identity in the form (1) is commutative, so is the variety $\mathcal{M}$. Hence from Lemma 2.5 it follows that $\mathcal{M} = \mathcal{G} \vee \mathcal{C}_m$ for some variety of periodic Abelian groups $\mathcal{G}$ and some $m \geq 0$. From the fact that $\mathcal{C}_2$ is a variety of degree > 2 we get $m \leq 1$. Therefore, $\mathcal{M} = \mathcal{G} \vee \mathcal{K}$, where $\mathcal{K}$ is one of varieties $\mathcal{T}$ and $\mathcal{SL}$. We have proved that the variety $\mathcal{V}$ satisfies condition c).

c) $\Rightarrow$ a). Due to Corollary 2.1 it suffices to prove that an arbitrary variety $\mathcal{G}$ of periodic Abelian groups is codistributive. Let $\mathcal{Y}$ and $\mathcal{Z}$ be arbitrary semigroup varieties. It is well-known that any semigroup variety is either *super-commutative* (i.e., contains the variety $\mathcal{COM}$ of all commutative semigroups) or periodic. At first we assume that at least one of varieties $\mathcal{Y}$ and $\mathcal{Z}$ is super-commutative. For definiteness we assume that $\mathcal{Y} \supseteq \mathcal{COM}$. Then $\mathcal{Y} \vee \mathcal{Z} \supseteq \mathcal{Y} \supseteq \mathcal{COM} \supseteq \mathcal{G}$, whence

$$\mathcal{G} \wedge (\mathcal{Y} \vee \mathcal{Z}) = \mathcal{G} = \mathcal{G} \vee (\mathcal{G} \wedge \mathcal{Z}) = (\mathcal{G} \wedge \mathcal{Y}) \vee (\mathcal{G} \wedge \mathcal{Z}).$$

Therefore further we assume that varieties $\mathcal{Y}$ and $\mathcal{Z}$ are periodic. It is clear that $\mathcal{G} \wedge (\mathcal{Y} \vee \mathcal{Z})$ and $(\mathcal{G} \wedge \mathcal{Y}) \vee (\mathcal{G} \wedge \mathcal{Z})$ are varieties of periodic Abelian groups. Therefore it suffices to show that exponents of these varieties coincide. As is known, any periodic semigroup variety $\mathcal{X}$ contains the maximal group subvariety. We denote it by $\mathrm{Gr}(\mathcal{X})$. Let the symbol $\exp(\mathcal{H})$ stand for the exponent of the variety $\mathcal{H}$ of periodic groups. We set $r = \exp(\mathcal{G})$, $s = \exp\big(\mathrm{Gr}(\mathcal{Y})\big)$, and $t = \exp\big(\mathrm{Gr}(\mathcal{Z})\big)$. Evidently,

$$\exp\big(\mathcal{G} \wedge (\mathcal{Y} \vee \mathcal{Z})\big) = \mathrm{GCD}\big(r, \mathrm{LCM}(s,t)\big) = \mathrm{LCM}\big(\mathrm{GCD}(r,s), \mathrm{GCD}(r,t)\big) = \exp\big((\mathcal{G} \wedge \mathcal{Y}) \vee (\mathcal{G} \wedge \mathcal{Z})\big).$$

Therefore, $\mathcal{G} \wedge (\mathcal{Y} \vee \mathcal{Z}) = (\mathcal{G} \wedge \mathcal{Y}) \vee (\mathcal{G} \wedge \mathcal{Z})$. □

Proof of Theorem 1.3. Conditions b) and c) are equivalent due to Proposition 2.1. The implication b) $\Rightarrow$ a) is evident. Therefore it suffices to verify the implication a) $\Rightarrow$ c). Let $\mathcal{V}$ be a proper costandard semigroup variety. Evidently, $\mathcal{V}$ is codistributive and modular. By applying Proposition 2.4 we obtain $\mathcal{V} = \mathcal{M} \vee \mathcal{N}$, where $\mathcal{M}$ is one of varieties $\mathcal{T}$ and $\mathcal{SL}$, while $\mathcal{N}$ is a nilvariety. Theorem 1.1 gives $\mathcal{N} \subseteq \mathcal{ZM}$. Taking into account Lemma 2.7, we conclude that $\mathcal{N}$ is one of varieties $\mathcal{T}$ and $\mathcal{ZM}$. □

4. FINAL OBSERVATIONS

Lemmas 2.3, 2.8, and 2.9 imply the following remark.

Remark 4.1. The union of an arbitrary finite number of atoms of the lattice **SEM** is a codistributive variety.

Varieties indicated in Theorem 1.2 and Remark 4.1 exhaust all known to the author examples of proper codistributive varieties. Note that all these varieties have a distributive lattice of subvarieties. In fact, the lattice of varieties of periodic Abelian groups is distributive (which is well-known), lattices $L(\mathcal{SL})$ and $L(\mathcal{ZM})$ are 2-element chains (see Lemma 2.7), and $L(\mathcal{G} \vee \mathcal{SL} \vee \mathcal{ZM}) \cong L(\mathcal{G}) \times L(\mathcal{SL}) \times L(\mathcal{ZM})$ for an arbitrary group variety $\mathcal{G}$ (the latter fact immediately follows, for example, from Proposition 2.1). Hence follows the distributive property of lattices of subvarieties of varieties indicated in Theorem 1.2. An analogous property of varieties mentioned in Remark 4.1 immediately follows from Lemma 2.9. This reasoning gives rise to the following question.

Question 4.1. Does a proper codistributive semigroup variety with a non-distributive lattice of subvarieties exist?

Assume that the answer to this question is negative. Then due to Lemma 2.2 proper semigroup varieties the property "of being a codistributive variety" is inherited by subvarieties. For comparison we note that if a proper semigroup variety $\mathcal{V}$ is upper-modular, then the lattice $L(\mathcal{V})$ is modular and all subvarieties of the variety $\mathcal{V}$ are upper-modular ([9], corollaries 2 and 3).

It is easy to understand that there exist non-codistributive varieties of periodic groups. Really, since the lattice of varieties of periodic groups is modular, but not distributive, it contains a 5-element modular non-distributive sublattice M_3. Evidently, each of three pairwise incomparable elements of this lattice is a non-codistributive variety of periodic groups. We do not know whether an analogous assertion is valid for *combinatory* varieties (i.e., those that contain no nontrivial groups). Each combinatory semigroup variety with completely regular square (and, consequently, according to Theorem 1.1, each combinatory codistributive variety) is a semigroup variety with idempotent square, i.e., it satisfies the identity $xy = (xy)^2$. Hence, the following question is actual.

Question 4.2. Is an arbitrary semigroup variety with idempotent square codistributive?

A natural weakened variant of this question is the following one.

Question 4.3. Is an arbitrary variety of semigroups of idempotents codistributive?

As is shown in [20], the lattice $L(\mathrm{var}\{xy = (xy)^2\})$ is distributive. Taking into account Lemma 2.2, this means that for the positive answer to Question 4.2 [respectively, Question 4.3] it suffices to prove the codistributive property of the variety $\mathrm{var}\{xy = (xy)^2\}$ [respectively, the variety $\mathrm{var}\{x = x^2\}$]. Therefore, Question 4.3 is reduced to the following one: Is the equality

$$\mathcal{B} \wedge (\mathcal{X} \vee \mathcal{Y}) = (\mathcal{B} \wedge \mathcal{X}) \vee (\mathcal{B} \wedge \mathcal{Y}),$$

where $\mathcal{B} = \mathrm{var}\{x = x^2\}$, valid for arbitrary varieties $\mathcal{X}$ and $\mathcal{Y}$? In [21] (corollary 5.9) one has shown that this equality is fulfilled in the case when varieties $\mathcal{X}$ and $\mathcal{Y}$ are locally finite.

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