On modular elements of the lattice of semigroup varieties*

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Abstract

A semigroup variety is called *modular* if it is a modular element of the lattice of all semigroup varieties. We obtain a strong necessary condition for a semigroup variety to be modular. In particular, we prove that every modular nil-variety may be given by 0-reduced identities and substitutive identities only. (An identity $u = v$ is called *substitutive* if the words u and v depend on the same letters and v may be obtained from u by renaming of letters.) We completely determine all commutative modular varieties and obtain an essential information about modular varieties satisfying a permutable identity.

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Introduction

The class of all varieties of semigroups forms a lattice under the following naturally defined operations: for varieties $\mathcal{X}$ and $\mathcal{Y}$, their *join* $\mathcal{X} \vee \mathcal{Y}$ is the variety generated by the set-theoretical union of $\mathcal{X}$ and $\mathcal{Y}$ (as classes of semigroups), while their *meet* $\mathcal{X} \wedge \mathcal{Y}$ coincides with the set-theoretical intersection of $\mathcal{X}$ and $\mathcal{Y}$. Special elements of different types in lattices of varieties of semigroups and universal algebras have been examined in several articles (see [3, 4, 9, 14, 17], for instance). Here we continue these investigations.

An element x of a lattice $\langle L; \vee, \wedge\rangle$ is called *modular* if

$$\forall\, y, z \in L\colon \quad y \le z \longrightarrow (x \vee y) \wedge z = (x \wedge z) \vee y,$$

and *upper-modular* if

$$\forall\, y, z \in L\colon \quad y \le x \longrightarrow (z \vee y) \wedge x = (z \wedge x) \vee y.$$

Lower-modular elements are defined dually to upper-modular ones. An element $x \in L$ is called *neutral* if, for all $y, z \in L$, the elements x, y and z generate a distributive sublattice of L. For convenience, we call a semigroup variety *modular*

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[*upper-modular, lower-modular, neutral*] if it is a modular [respectively upper-modular, lower-modular, neutral] element of the lattice $\mathbb{SEM}$ of all semigroup varieties. A number of results about varieties of these four types have been obtained in [4, 10–12, 14, 17]. In particular, some necessary conditions for a semigroup variety to be modular, upper-modular or lower-modular were found in [4, Proposition 1.6], [10, Theorem 1] and [11, Theorem 1] respectively; commutative upper-modular and commutative lower-modular varieties were completely determined in [10, Theorem 2] and [11, Theorem 2] respectively. Here we obtain an essentially stronger necessary condition for semigroup varieties to be modular than the mentioned result from [4], completely describe commutative modular varieties and obtain an essential information about modular varieties satisfying a permutable identity (see Theorems 2.5, 3.1 and 4.5 respectively).

The article consists of 4 sections. Section 1 contains some preliminary information. In Sections 2, 3 and 4 we prove Theorems 2.5, 3.1 and 4.5 respectively.

1 Preliminaries

We denote by $\mathcal{SL}$ the variety of all semilattices. The following three lemmas contain some properties of this variety that will be used in the sequel. The claim (i) of the following lemma is well known (see [2], for instance), while the claim (ii) is proved in [17, Proposition 2.4].

Lemma 1.1. *The variety $\mathcal{SL}$ is*:

(i) *an atom of the lattice* $\mathbb{SEM}$;

(ii) *a neutral element of the lattice* $\mathbb{SEM}$. □

Lemma 1.2 ([14, Corollary 1.5(i)])**.** *A semigroup variety $\mathcal{X}$ is a modular element of the lattice* $\mathbb{SEM}$ *if and only if so is the variety $\mathcal{X} \vee \mathcal{SL}$.* □

We denote by $\mathcal{T}$ the trivial semigroup variety. If $\mathcal{V}$ is a semigroup variety then $L(\mathcal{V})$ stands for the subvariety lattice of $\mathcal{V}$. The following claim is a part of the semigroup folklore. It may be extracted from results scattered in the articles [1, 5, 6, 8]. But its explicit proof was not published so far, as far as we know. We provide the proof for the sake of completeness.

Lemma 1.3. *If $\mathcal{X}$ is a semigroup variety such that $\mathcal{SL} \nsubseteq \mathcal{X}$ then the lattice $L(\mathcal{X} \vee \mathcal{SL})$ is isomorphic to the direct product of $L(\mathcal{X})$ and the* 2*-element chain consisting of the varieties $\mathcal{T}$ and $\mathcal{SL}$.*

Proof. Let $\mathcal{V} \subseteq \mathcal{X} \vee \mathcal{SL}$. Lemma 1.1(ii) implies that

$$\mathcal{V} = \mathcal{V} \wedge (\mathcal{X} \vee \mathcal{SL}) = (\mathcal{V} \wedge \mathcal{X}) \vee (\mathcal{V} \wedge \mathcal{SL}).$$

In view of Lemma 1.1(i), $\mathcal{V}$ is the join of some subvariety of $\mathcal{X}$ and one of the varieties $\mathcal{T}$ or $\mathcal{SL}$. It remains to verify that there exists a unique decomposition of $\mathcal{X}$ of such a form. Let $\mathcal{Y}_1, \mathcal{Y}_2 \in \{\mathcal{T}, \mathcal{SL}\}$ and $\mathcal{Z}_1, \mathcal{Z}_2 \subseteq \mathcal{X}$. Suppose that $\mathcal{Y}_1 \vee \mathcal{Z}_1 = \mathcal{Y}_2 \vee \mathcal{Z}_2$. We have to check that $\mathcal{Y}_1 = \mathcal{Y}_2$ and $\mathcal{Z}_1 = \mathcal{Z}_2$. Suppose that $\mathcal{Y}_1 \neq \mathcal{Y}_2$. We may assume without any loss that $\mathcal{Y}_1 = \mathcal{T}$ and $\mathcal{Y}_2 = \mathcal{SL}$. Then $\mathcal{Y}_1 \vee \mathcal{Z}_1 = \mathcal{Z}_1 \not\supseteq \mathcal{SL}$ while $\mathcal{Y}_2 \vee \mathcal{Z}_2 \supseteq \mathcal{SL}$, whence $\mathcal{Y}_1 \vee \mathcal{Z}_1 \neq \mathcal{Y}_2 \vee \mathcal{Z}_2$. Now let $\mathcal{Y}_1 = \mathcal{Y}_2$. If $\mathcal{Y}_1 = \mathcal{Y}_2 = \mathcal{T}$ then $\mathcal{Z}_1 = \mathcal{Y}_1 \vee \mathcal{Z}_1 = \mathcal{Y}_2 \vee \mathcal{Z}_2 = \mathcal{Z}_2$. Finally, if $\mathcal{Y}_1 = \mathcal{Y}_2 = \mathcal{SL}$ then the desired equality $\mathcal{Z}_1 = \mathcal{Z}_2$ follows from the following claim that was proved independently in [5] and [8]: if $\mathcal{X} \not\supseteq \mathcal{SL}$ then the map $\xi\colon L(\mathcal{X}) \longmapsto \mathbb{SEM}$ given by the rule $\xi(\mathcal{M}) = \mathcal{M} \vee \mathcal{SL}$ for every $\mathcal{M} \subseteq \mathcal{X}$ is one-to-one. □

We denote by F the free semigroup over a countably infinite alphabet. The equality relation on F is denoted by $\equiv$. If $u \in F$ then $c(u)$ stands for the set of letters occuring in u. Clearly, if $w \in F$ then a semigroup S satisfies the identity system $wu = uw = w$, where u runs over F, if and only if S contains a zero element 0 and all values of the word w in S equal 0. We adopt the usual convention of writing $w = 0$ as a short form of such a system and referring to the expression $w = 0$ as to a single identity. Such identities are called 0-*reduced*.

A semigroup S with 0 is said to be a *nilsemigroup* if, for every $s \in S$, there exists a positive integer n with $s^n = 0$. A semigroup variety $\mathcal{V}$ is called a *nil-variety* if each member of $\mathcal{V}$ is a nilsemigroup. We need the following two well known technical remarks about identities of nil-varieties.

Lemma 1.4. *Let $\mathcal{V}$ be a nil-variety.*

(i) *If $\mathcal{V}$ satisfies an identity $u = v$ with $c(u) \neq c(v)$ then $\mathcal{V}$ satisfies also the identity $u = 0$.*

(ii) *If $\mathcal{V}$ satisfies an identity of the form $u = v$ where u is a proper subword of v then $\mathcal{V}$ satisfies also the identity $u = 0$.* □

2 Arbitrary varieties

We denote by $\mathcal{SEM}$ the variety of all semigroups. The following result is in fact a reformulation of [4, Proposition 1.6][1].

Proposition 2.1. *If a semigroup variety $\mathcal{V}$ is a modular element of the lattice $\mathbb{SEM}$ then either $\mathcal{V} = \mathcal{SEM}$ or $\mathcal{V} = \mathcal{X} \vee \mathcal{N}$ where $\mathcal{X}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$, while $\mathcal{N}$ is a nil-variety.*

Proof. Suppose that $\mathcal{V}$ is a modular semigroup variety and $\mathcal{V} \neq \mathcal{SEM}$. Translating [4, Proposition 1.6] from the language of equational theories to the varietal language, we have $\mathcal{V} \subseteq \mathcal{SL} \vee \mathcal{M}$ for some nil-variety $\mathcal{M}$. Lemma 1.1(ii) implies that

$$\mathcal{V} = \mathcal{V} \wedge (\mathcal{SL} \vee \mathcal{M}) = (\mathcal{V} \wedge \mathcal{SL}) \vee (\mathcal{V} \wedge \mathcal{M}).$$

By Lemma 1.1(i), $\mathcal{V} \wedge \mathcal{SL}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$. Since the variety $\mathcal{N} = \mathcal{V} \wedge \mathcal{M}$ is a nil-variety, we are done. □

[1] One should note that the paper [4] has dealt with the lattice of equational theories of semigroups, that is, the dual of $\mathbb{SEM}$ rather than the lattice $\mathbb{SEM}$ itself. When reproducing results from [4], we adapt them to the terminology of the present article. Note that the definition of a modular element of a lattice is selfdual, whence modular elements of the lattice of equational theories precisely correspond to modular varieties.

Note that Proposition 2.1 and Lemma 1.2 reduce the problem of description of modular varieties to the nil-case.

If Σ is a system of identities then $\operatorname{var}\Sigma$ stands for the variety of all semigroups satisfying Σ. The length of a word u is denoted by $\ell(u)$. We say that a word u is a *prefix* of a word v if there exists a (may be empty) word w such that $v \equiv uw$; if the word w is non-empty then u is said to be a *proper* prefix of v. For words u and v, we write $u \approx v$ if v may be obtained from u by renaming of letters. We call a non-trivial identity $u = v$ *substitutive* if $c(u) = c(v)$ and $u \approx v$. In [3], Ježek describes modular elements of the lattice of all varieties (more precisely, all equational theories) of any given type. In particular, it follows from [3, Lemma 6.3] that if a nil-variety $\mathcal{V}$ is a modular element of the lattice of all groupoid varieties then $\mathcal{V}$ may be given by 0-reduced and substitutive identities only. This does not imply directly the same conclusion for modular nil-varieties because a modular element of $\mathbb{SEM}$ need not be a modular element of the lattice of all groupoid varieties. Nevertheless, the following proposition shows that the 'semigroup analogue' of the mentioned result by Ježek is the case.

Proposition 2.2. *If a nil-variety $\mathcal{N}$ is a modular element of the lattice $\mathbb{SEM}$ and $\mathcal{N}$ satisfies a non-substitutive identity $u = v$ then $\mathcal{N}$ satisfies also the identity $u = 0$.*

Proof. If $c(u) \neq c(v)$ then Lemma 1.4(i) applies. Now let $c(u) = c(v)$. Since the identity $u = v$ is non-substitutive, we have $u \not\approx v$. We may assume without any loss that $\ell(u) \leq \ell(v)$. Let

$$\mathcal{Y} = \operatorname{var}\{u = u^2,\ v = v^2\} \ \text{ and } \ \mathcal{Z} = \operatorname{var}\{v = v^2\}.$$

The variety $\mathcal{N} \wedge \mathcal{Z}$ satisfies the identity $v = v^2$. Since $\mathcal{N} \wedge \mathcal{Z}$ is a nil-variety, Lemma 1.4(ii) implies that this variety satisfies the identities $u = v = 0$, and therefore the identity $u = u^2$. We have $\mathcal{N} \wedge \mathcal{Z} \subseteq \mathcal{Y}$, whence $(\mathcal{N} \wedge \mathcal{Z}) \vee \mathcal{Y} = \mathcal{Y}$. Since the variety $\mathcal{N}$ is modular and $\mathcal{Y} \subseteq \mathcal{Z}$, we have $(\mathcal{N} \vee \mathcal{Y}) \wedge \mathcal{Z} = \mathcal{Y}$. In particular, the variety $(\mathcal{N} \vee \mathcal{Y}) \wedge \mathcal{Z}$ satisfies the identity $u = u^2$. Hence there exists a deduction of this identity from the identities of the varieties $\mathcal{N} \vee \mathcal{Y}$ and $\mathcal{Z}$. In particular, there exists a word w such that $w \not\equiv u$ and the identity $u = w$ holds in one of the varieties $\mathcal{N} \vee \mathcal{Y}$ or $\mathcal{Z}$.

Suppose at first that $u = w$ holds in $\mathcal{Z}$. This means that there exists an endomorphism ξ of the semigroup F such that one of the following holds:

1) the word u contains a subword $a \equiv \xi(v)$ and the word w is obtained from u by a substitution of a^2 for a;

2) the word u contains a subword $a \equiv (\xi(v))^2$ and the word w is obtained from u by a substitution of $\xi(v)$ for a.

In the case 1) $\ell(u) \geq \ell(\xi(v)) \geq \ell(v) \geq \ell(u)$, whence $\ell(u) = \ell(\xi(v)) = \ell(v)$. Since $\xi(v)$ is a subword of u, we have $u \equiv \xi(v)$. Furthermore, the equality $\ell(\xi(v)) = \ell(v)$ implies that ξ is a permutation on the set $c(v)$. Thus $u \approx v$, a contradiction. In the case 2) $\ell(u) \geq \ell((\xi(v))^2) > \ell(\xi(v)) \geq \ell(v) \geq \ell(u)$, a contradiction. We see that none of the cases 1) and 2) is possible. Hence the identity $u = w$ holds in the variety $\mathcal{N} \vee \mathcal{Y}$.

Let us verify that u is a proper prefix of w. Note that this claim is the analogue of [3, Lemma 6.2]. The identity $u = w$ holds in the variety $\mathcal{Y}$. Hence there exists a sequence of words $v_0, v_1, \dots, v_n$ such that $v_0 \equiv u$, $v_n \equiv w$, and, for each $i = 1, 2, \dots, n$, there exists an endomorphism ξ_i of the semigroup F such that one of the following holds:

1) the word v_{i-1} contains a subword a_i such that either $a_i \equiv \xi_i(u)$ or $a_i \equiv \xi_i(v)$, and the word v_i is obtained from v_{i-1} by a substitution of a_i^2 for a_i;

2) the word v_{i-1} contains a subword $a_i \equiv (\xi_i(u))^2$ [respectively $a_i \equiv (\xi_i(v))^2$] and the word v_i is obtained from v_{i-1} by a substitution of $\xi_i(u)$ [respectively $\xi_i(v)$] for a_i.

We may assume that the words $v_0, v_1, \dots, v_n$ are pairwise different. Now we are going to verify that u is a proper prefix of v_i for all $i = 1, 2, \dots, n$. We prove this claim by induction on i.

Induction basis. Let $i = 1$. Then $u_{i-1} \equiv u_0 \equiv u$. The case 2) is impossible here because $\ell((\xi_1(v))^2) \ge \ell((\xi_1(u))^2) > \ell(\xi_1(u)) \ge \ell(u)$. Therefore the case 1) holds. Since $\ell(\xi_1(v)) \ge \ell(v) \ge \ell(u)$ and $u \not\approx v$, the case $a_1 \equiv \xi_1(v)$ is impossible, whence $a_1 \equiv \xi_1(u)$. But a_1 is a subword of u and $\ell(\xi_1(u)) \ge \ell(u)$. Therefore $a_1 \equiv \xi_1(u) \equiv u$. Hence $v_1 \equiv u^2$ and we are done.

Induction step. Now let $1 < i \le n$. By the induction assumption, $v_{i-1} \equiv uv'_{i-1}$ for some non-empty word v'_{i-1}. If a_i is a subword of v'_{i-1} then clearly $v_i \equiv uv''_{i-1}$ for some non-empty word v''_{i-1} and we are done. Suppose that a_i is a subword of u. The same arguments as in the previous paragraph show that then the case 1) holds and $a_i \equiv \xi_i(u) \equiv u$. Therefore $v_i \equiv u^2 v'_{i-1}$ and we are done again. Finally, suppose that a_i is neither a subword of u nor a subword of v'_{i-1}. This means that there exist words b_1, b_2, b_3, b_4 such that $u \equiv b_1 b_2$, $v'_{i-1} \equiv b_3 b_4$, $a_i \equiv b_2 b_3$ and the words b_2, b_3 are non-empty. Clearly, $v_{i-1} \equiv b_1 b_2 b_3 b_4$. In the case 1) $v_i \equiv b_1(b_2 b_3)^2 b_4 \equiv ub_3 b_2 b_3 b_4$ and we are done. Finally, in the case 2) $v_{i-1} \equiv b_1 c^2 b_4$ and $v_i \equiv b_1 c b_4$ where c coincides with either $\xi_i(u)$ or $\xi_i(v)$. Note that $\ell(\xi_i(v)) \ge \ell(\xi_i(u)) \ge \ell(u)$, whence $\ell(c) \ge \ell(u)$. Since u is a prefix of $v_{i-1} \equiv b_1 c^2 b_4$ and $\ell(b_1) < \ell(u)$, we have that the word u is a prefix of $b_1 c$. If $\ell(u) < \ell(b_1 c b_4)$ then we are done. Suppose that $\ell(u) = \ell(b_1 c b_4)$. Since $\ell(c) \ge \ell(u)$, this means that the words b_1 and b_4 are empty and $\ell(u) = \ell(c)$. Thus u is a prefix of c and $\ell(u) = \ell(c)$, whence $u \equiv c$. Then $v_i \equiv b_1 c b_4 \equiv c \equiv u \equiv v_0$. But this is not the case because the words $v_0, v_1, \dots, v_n$ are pairwise different.

We have proved that u is a proper prefix of v_i for all $i = 1, 2, \dots, n$. In particular, u is a proper prefix of the word $w \equiv v_n$. Since a nil-variety $\mathcal{N}$ satisfies the identity $u = w$, Lemma 1.4(ii) applies with the desired conclusion that $u = 0$ holds in $\mathcal{N}$. □

We need some additional notation. Let u be a word and π a permutation on the set $c(u)$. We denote by $\pi[u]$ the word that is obtained from the word u by changing of every letter $x \in c(u)$ on the letter $x\pi$. Obviously, if $u = v$ is a substitutive identity then there exists a unique permutation π on the set $c(u)$

with $v \equiv \pi[u]$. If u is a word and x is a letter then we denote by $\ell_x(u)$ the number of occurences of x in u.

There are many ways to deduce a non-substitutive identity from a substitutive one. In view of Proposition 2.2, this means that, within a modular nil-variety, a substitutive identity implies numerous 0-reduced identities. In particular, we have the following

Corollary 2.3. *Let a nil-variety $\mathcal{V}$ be a modular element of the lattice* $\mathbb{SEM}$*. Suppose that $\mathcal{V}$ satisfies a substitutive identity $u = v$ and $x \in c(u)$.*

(i) *Let π be the permutation on the set $c(u)$ such that $v \equiv \pi[u]$. If $x \not\equiv x\pi$ then $\mathcal{V}$ satisfies the identities $ux = xu = 0$.*

(ii) *If $\ell_x(u) \neq \ell_x(v)$, y is a letter with $y \notin c(u)$ and u' is the word that is obtained by a substitution of either xy or yx for x in the word u then $\mathcal{V}$ satisfies the identity $u' = 0$.*

Proof. (i) The identities $ux = vx$ and $xu = xv$ follow from the identity $u = v$. In view of Proposition 2.2, it suffices to verify that these two identities are not substitutive. Arguing by contradiction, suppose that the identity $ux = vx$ is substitutive. Then $vx \equiv \sigma[ux]$ for some permutation σ on the set $c(ux) = c(u)$. It is evident that $v \equiv \sigma[u]$, whence $\sigma = \pi$. Therefore $x\sigma \not\equiv x$. On the other hand, $x\sigma \equiv x$ because x is the last letter in both the words ux and $\sigma[ux] \equiv vx$. We have a contradiction. Analogous arguments show that the identity $xu = xv$ is not substitutive as well.

(ii) Let $u' = v'$ be the identity that is obtained by a substitution of either xy or yx for x in the identity $u = v$. Then $\mathcal{V}$ satisfies the identity $u' = v'$. It is evident that both the parts of a substitutive identity have the same length. Therefore

$$\ell(u') = \ell(u) + \ell_x(u) = \ell(v) + \ell_x(u) \neq \ell(v) + \ell_x(v) = \ell(v'),$$

whence the identity $u' = v'$ is not substitutive. Now Proposition 2.2 applies. □

Proposition 2.2 gives a necessary condition for a nil-variety to be modular. It is interesting to compare it with the following sufficient condition that have been obtained independently in [4, Proposition 1.1] and [12, Corollary 3] (in fact, it immediately follows from [3, Proposition 2.2] but was not mentioned in [3] explicitly).

Proposition 2.4. *If a semigroup variety is given by 0-reduced identities only then it is a modular element of the lattice* $\mathbb{SEM}$*.* □

Note that the necessary condition given by Proposition 2.2 is not a sufficient one, while the sufficient condition given by Proposition 2.4 is not a necessary one (see Theorem 3.1 below).

The first main result of the article is the following

Theorem 2.5. *If a semigroup variety $\mathcal{V}$ is a modular element of the lattice* $\mathbb{SEM}$ *then either $\mathcal{V} = \mathcal{SEM}$ or $\mathcal{V} = \mathcal{X} \vee \mathcal{N}$ where $\mathcal{X}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$, while $\mathcal{N}$ is a nil-variety given by 0-reduced and substitutive identities only.*

Proof. Let $\mathcal{V}$ be a modular semigroup variety and $\mathcal{V} \neq \mathcal{SEM}$. By Proposition 2.1, $\mathcal{V} = \mathcal{X} \vee \mathcal{N}$ where $\mathcal{X}$ is one of the varieties $\mathcal{T}$ or $\mathcal{SL}$, while $\mathcal{N}$ is a nil-variety. Lemma 1.2 implies that $\mathcal{N}$ is a modular variety. Finally, Proposition 2.2 shows that $\mathcal{N}$ may be given by 0-reduced and substitutive identities only. $\square$

3 Commutative varieties

The second main result of the article is the following

Theorem 3.1. *For a commutative semigroup variety* $\mathcal{V}$ *the following are equivalent*:

a) $\mathcal{V}$ *is a modular element of the lattice* $\mathbb{SEM}$;

b) $\mathcal{V}$ *is a modular and an upper-modular element of the lattice* $\mathbb{SEM}$;

c) $\mathcal{V} = \mathcal{X} \vee \mathcal{N}$ *where* $\mathcal{X}$ *is one of the varieties* $\mathcal{T}$ *or* $\mathcal{SL}$, *while* $\mathcal{N}$ *is a variety satisfying the identities* $xy = yx$ *and* $x^2y = 0$.

Proof. The implication c) $\Longrightarrow$ b) is proved in [14, Theorem 1], while the implication b) $\Longrightarrow$ a) is evident. It remains to prove the implication a) $\Longrightarrow$ c). According to Proposition 2.1 and Lemma 1.2, it suffices to verify that a commutative modular nil-variety satisfies the identity $x^2y = 0$. This claim immediately follows from Proposition 2.2 and the fact that a commutative variety satisfies the identity $x^2y = yx^2$. (Note that we may apply here Corollary 2.3(i) rather than Proposition 2.2. Indeed, the commutative law is the identity $xy = \pi[xy]$ where π is the transposition on the set $\{x, y\}$. Since $x \not\equiv x\pi$, it remains to note that $x^2y \equiv x \cdot xy$.) $\square$

In particular, Theorem 3.1 shows that a commutative modular variety is upper-modular. For arbitrary varieties, this is not the case. For instance, the variety $\mathrm{var}\{x^3 = 0\}$ is modular by Proposition 2.4 but is not upper-modular by [10, Theorem 1] or [14, Theorem 2].

Theorem 3.1 and results of the article [15] imply that a commutative modular variety has a distributive subvariety lattice. Below we obtain a stronger result (see Corollary 4.7). One more corollary of Theorem 3.1 is the following

Corollary 3.2. *If a commutative semigroup variety is a modular element of the lattice* $\mathbb{SEM}$ *then so is every its subvariety.* $\square$

4 Permutable varieties

A simplest but very important particular type of substitutive identities is *permutable* identities, that is identities of the form

$$x_1x_2\cdots x_n = x_{1\pi}x_{2\pi}\cdots x_{n\pi} \tag{1}$$

where π is a non-trivial permutation on the set $\{1, 2, \ldots, n\}$. The number n is called a *length* of the identity (1). A semigroup variety is called *permutable* if it satisfies some permutable identity. Theorems 2.5 and 3.1 inspire the following

Problem 4.1. *Describe permutable varieties that are modular elements of the lattice* $\mathbb{SEM}$.

Corollary 2.3(i) implies that if a modular nil-variety $\mathcal{V}$ satisfies a permutable identity of length n then it satisfies also an identity of the form $u = 0$ for some word u of length $n+1$ depending on n letters (namely, if $\mathcal{V}$ satisfies an identity of the form (1) then it satisfies also the identities $x_i x_1 x_2 \cdots x_n = x_1 x_2 \cdots x_n x_i = 0$ for each i with $1 \le i \le n$ and $i\pi \ne i$). To obtain a stronger result (see Theorem 4.5 below), we need some additional notation and auxiliary results.

Let $\mathcal{V}$ be a semigroup variety and $u \in F$. We denote by $\mathbf{S}_u$ the full permutation group on the set $c(u)$ and put

$$\mathbf{Perm}_u(\mathcal{V}) = \bigl\{\pi \in \mathbf{S}_u \mid \mathcal{V} \text{ satisfies the identity } u = \pi[u]\bigr\}.$$

Obviously, $\mathbf{Perm}_u(\mathcal{V})$ is a subgroup in $\mathbf{S}_u$. We denote the subgroup lattice of a group G by $\mathrm{Sub}(G)$.

The following lemma is the semigroup analogue of [3, Lemma 6.10]. We provide its proof for the sake of completeness.

Lemma 4.2. *If a semigroup variety* $\mathcal{V}$ *is a modular element of the lattice* $\mathbb{SEM}$ *and* $u \in F$ *then the group* $\mathbf{Perm}_u(\mathcal{V})$ *is a modular element of the lattice* $\mathrm{Sub}(\mathbf{S}_u)$.

Proof. For brevity, put $V = \mathbf{Perm}_u(\mathcal{V})$. Let H and K be subgroups of $\mathbf{S}_u$ with $H \subseteq K$. We have to verify that $(V \vee H) \wedge K = (V \wedge K) \vee H$. It suffices to check that $(V \vee H) \wedge K \subseteq (V \wedge K) \vee H$ because the opposite inclusion is evident. Let $\pi \in (V \vee H) \wedge K$. We need to verify that $\pi \in (V \wedge K) \vee H$.

Let us denote by $\mathcal{H}$ [respectively by $\mathcal{K}$] the semigroup variety given by all identities of the form $u = \sigma[u]$ with $\sigma \in H$ [respectively $\sigma \in K$]. It is evident that $\mathcal{K} \subseteq \mathcal{H}$. Since $\mathcal{V}$ is modular, we have $(\mathcal{V} \wedge \mathcal{H}) \vee \mathcal{K} = (\mathcal{V} \vee \mathcal{K}) \wedge \mathcal{H}$. Further, $\pi \in V \vee H$, whence $\pi = \nu_1\eta_1\nu_2\eta_2 \cdots \nu_m\eta_m$ for some permutations $\nu_1, \nu_2, \dots, \nu_m \in V$ and $\eta_1, \eta_2, \dots, \eta_m \in H$. Put $u_0 \equiv u$, $v_i \equiv \nu_i[u_{i-1}]$ and $u_i \equiv \eta_i[v_i]$ for all $i = 1, 2, \dots, m$. Then $u_m \equiv \pi[u_0]$. For every $i = 1, 2, \dots, m$, the identities $u_{i-1} = v_i$ and $v_i = u_i$ hold in the varieties $\mathcal{V}$ and $\mathcal{H}$ respectively. Therefore $u_0 = u_m$ holds in $\mathcal{V} \wedge \mathcal{H}$. Besides that, $u_0 = u_m$ holds in $\mathcal{K}$ because $\pi \in K$. Therefore the variety $(\mathcal{V} \vee \mathcal{K}) \wedge \mathcal{H} = (\mathcal{V} \wedge \mathcal{H}) \vee \mathcal{K}$ also satisfies the identity $u_0 = u_m$. Hence there exists a sequence of words $w_0, w_1, \dots, w_k$ such that $w_0 \equiv u_0 \equiv u$, $w_k \equiv u_m \equiv \pi[u]$ and, for each $i = 0, 1, \dots, k-1$, the identity $w_i = w_{i+1}$ holds in one of the varieties $\mathcal{V} \vee \mathcal{K}$ or $\mathcal{H}$.

Suppose that $w_i \not\approx u$ for some i. Let i be the least index with such a property. It is clear that $i > 0$. Then $w_{i-1} \approx u$ and the identity $w_{i-1} = w_i$ holds in one of the varieties $\mathcal{V} \vee \mathcal{K}$ or $\mathcal{H}$. Therefore it holds in one of the varieties $\mathcal{K}$ or $\mathcal{H}$. The choice of these two varieties shows that $w_i \approx w_{i-1} \approx u$, a contradiction with the choice of i.

Thus $w_i \approx u$ for all $i = 0, 1, \dots, k$. This means that, for every $i = 0, 1, \dots, k-1$, there exists a permutation $\sigma_i \in \mathbf{S}_u$ such that $w_{i+1} \equiv \sigma_i[w_i]$. It follows from the choice of the words $w_0, w_1, \dots, w_k$ that, for every $i = 0, 1, \dots, k-1$, the permutation σ_i lies in one of the groups $V \wedge K$ or H. Since $w_k \equiv \pi[w_0]$, we have $\pi = \sigma_0\sigma_1 \cdots \sigma_{k-1}$. Therefore $\pi \in (V \wedge K) \vee H$, as desired. $\square$

Let n be a positive integer. We denote by $\mathbf{S}_n$ the full permutation group on the set $\{1, 2, \dots, n\}$. For a semigroup variety $\mathcal{V}$, we put

$$\mathbf{Perm}_n(\mathcal{V}) = \bigl\{\pi \in \mathbf{S}_n \mid \mathcal{V} \text{ satisfies the identity } (1)\bigr\}.$$

Obviously, $\mathbf{S}_n \cong \mathbf{S}_{x_1 x_2 \cdots x_n}$ and $\mathbf{Perm}_n(\mathcal{V}) \cong \mathbf{Perm}_{x_1 x_2 \cdots x_n}(\mathcal{V})$. Lemma 4.2 implies the following

Corollary 4.3. *If a semigroup variety $\mathcal{V}$ is a modular element of the lattice* $\mathbb{SEM}$ *and n is a positive integer then the group $\mathbf{Perm}_n(\mathcal{V})$ is a modular element of the lattice* $\mathrm{Sub}(\mathbf{S}_n)$. $\square$

We denote by $\mathbf{A}_n$ the alternating subgroup of $\mathbf{S}_n$ and by $\mathbf{V}_4$ the Klein four group. The following fact immediately follows from [3, Propositions 3.1 and 3.8].

Lemma 4.4. *Let G be a non-singleton subgroup of the group $\mathbf{S}_n$ with $n \ge 4$. The group G is a modular element of the lattice* $\mathrm{Sub}(\mathbf{S}_n)$ *if and only if one of the following holds*:

(i) $n = 4$ *and* $G \supseteq \mathbf{V}_4$;

(ii) $n \ge 5$ *and* $G \supseteq \mathbf{A}_n$. $\square$

Let u be a word of length n, say $u \equiv y_1 y_2 \cdots y_n$ where $y_1, y_2, \dots, y_n$ are (not necessarily different) letters. For every $i = 1, 2, \dots, n$, we put $\lambda_i(u) \equiv y_i$. Clearly, if $u \approx v$ and $\lambda_i(u) \equiv \lambda_j(u)$ for some $1 \le i, j \le n$ then $\lambda_i(v) \equiv \lambda_j(v)$. Further, if $\pi \in \mathbf{S}_n$ then we denote by u^π the word $y_{1\pi} y_{2\pi} \cdots y_{n\pi}$. It is evident that the identity (1) implies the identity $u = u^\pi$. For every $i = 1, 2, \dots, n$, we put $\mathbf{Stab}_n(i) = \{\pi \in \mathbf{S}_n \mid i\pi = i\}$. Clearly, $\mathbf{Stab}_n(i)$ is a subgroup of $\mathbf{S}_n$.

The third main result of the article is the following

Theorem 4.5. *Let a semigroup variety $\mathcal{V}$ be a modular element of the lattice* $\mathbb{SEM}$ *and let $\mathcal{V}$ satisfy a non-trivial identity of the form* (1). *Then $\mathcal{V}$ satisfies also*:

(i) *all permutable identities of length* $n + 1$;

(ii) *all permutable identities of length n whenever $n \ge 5$ and the permutation π is odd.*

If, besides that, $\mathcal{V}$ is a nil-variety then it satisfies also:

(iii) *an arbitrary identity of the form $u = 0$, where u is a word of length n depending on $n - 1$ letters, whenever $n \ge 4$*;

(iv) *the identities $xyx = xy^2 = 0$ [respectively $x^2y = xy^2 = 0$, $x^2y = xyx = 0$, $x^2y = xyx = xy^2 = 0$] whenever $n = 3$ and $\pi = (12)$ [respectively $\pi = (13)$, $\pi = (23)$, $\pi = (123)$].*

Proof. (i) In view of Proposition 2.1 and Lemma 1.2, we may assume that $\mathcal{V}$ is a nil-variety. If $n = 2$ then the claim is evident. Let $n = 3$. Results of [7] implies that $\mathbf{Perm}_4(\mathcal{V}) \supseteq \mathbf{Stab}_4(i)$ for some i (more precisely, $\mathbf{Perm}_4(\mathcal{V}) \supseteq \mathbf{Stab}_4(4)$ whenever $\pi = (12)$, $\mathbf{Perm}_4(\mathcal{V}) \supseteq \mathbf{Stab}_4(1)$ whenever $\pi = (23)$, and $\mathbf{Perm}_4(\mathcal{V}) = \mathbf{S}_4$ whenever π is one of the permutations (23) or (123)). On the other hand, $\mathbf{Perm}_4(\mathcal{V}) \supseteq \mathbf{V}_4$ by Lemma 4.4. Since the join of $\mathbf{Stab}_4(i)$ and $\mathbf{V}_4$ in the lattice $\mathrm{Sub}(\mathbf{S}_4)$ coincides with the whole group $\mathbf{S}_4$, we are done. Finally, if $n \geq 4$ then the desired conclusion immediately follows from Lemma 4.4 and [7, Theorem 1].

(ii) As in the proof of the previous claim, we may assume that $\mathcal{V}$ is a nil-variety. The desired conclusion immediatelly follows then from Lemma 4.4 and the fact that $\mathbf{A}_n$ is a coatom in the lattice $\mathrm{Sub}(\mathbf{S}_n)$.

(iii) Let u be a word of length n depending on $n-1$ letters, say $u \equiv y_1 y_2 \cdots y_n$, where $y_1, y_2, \dots, y_n$ are letters, $y_i \equiv y_j$ for some $1 \leq i < j \leq n$, and letters $y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n$ are pairwise different. Since $n \geq 4$, there exist numbers k and ℓ with $1 \leq k < \ell \leq n$ and $k, \ell \notin \{i, j\}$. Put $\pi = (ik)(j\ell)$. Corollary 4.3 and Lemma 4.4 imply that $\pi \in \mathbf{Perm}_n(\mathcal{V})$. Whence the variety $\mathcal{V}$ satisfies the identity $u = u^\pi$. Since

$$\lambda_i(u) \equiv y_i \equiv y_j \equiv \lambda_j(u) \quad \text{but} \quad \lambda_i(u^\pi) \equiv y_k \not\equiv y_\ell \equiv \lambda_j(u^\pi),$$

we have $u \not\approx u^\pi$. It remains to refer to Proposition 2.2.

(iv) The identity (1) with $n = 3$ and $\pi = (12)$ [respectively $\pi = (13)$, $\pi = (23)$, $\pi = (123)$] implies $xyx = yx^2$ [respectively $x^2y = yx^2$, $x^2y = xyx$, $x^2y = xyx = yx^2$]. Now Proposition 2.2 applies. □

To obtain a corollary of Theorem 4.5, we need some information about nil-varieties with distributive subvariety lattices. Such varieties were completely described in [16] (see also [13, Proposition 4.2] where the description was re-proved in a simpler and shorter way). In particular, this result immediately implies the following

Lemma 4.6. *Let $\mathcal{V}$ be a nil-variety.*

(i) *If the lattice $L(\mathcal{V})$ is distributive then $\mathcal{V}$ satisfies a permutable identity of length* 3.

(ii) *If $\mathcal{V}$ satisfies a permutable identity of length* 3 *and one of the identities $x^2y = 0$ or $xy^2 = 0$ then the lattice $L(\mathcal{V})$ is distributive.* □

Corollary 4.7. *Let a semigroup variety $\mathcal{V}$ be a modular element of the lattice $\mathbb{SEM}$. If $\mathcal{V}$ satisfies a permutable identity of length* 3 *then the lattice $L(\mathcal{V})$ is distributive.*

Proof. In view of Proposition 2.1 and Lemma 1.3, we may assume that $\mathcal{V}$ is a nil-variety. The desired conclusion follows then from Theorem 4.5(iv) and Lemma 4.6(ii). □

Analogues of Corollaries 3.2 and 4.7 for arbitrary semigroup varieties fail: an evident example is provided by the variety $\mathcal{SEM}$. Moreover, such analogoues are not the case even for *proper* semigroup varieties (that is, varieties $\mathcal{V}$ with $\mathcal{V} \neq \mathcal{SEM}$). Indeed, put $\mathcal{X} = \mathrm{var}\{x^3 = 0\}$ and $\mathcal{Y} = \mathrm{var}\{x^3 = 0,\ xy = yx\}$. The variety $\mathcal{X}$ is modular by Proposition 2.4. But its subvariety $\mathcal{Y}$ is not modular by Theorem 3.1 and the lattice $L(\mathcal{X})$ is not distributive by Lemma 4.6(i).

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